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ASYMPTOTIC PROPERTIES OF PARAMETER ESTIMATORS IN FRACTIONAL VASICEK MODEL

Stanislav Lohvinenko[1], Kostiantyn Ralchenko[2], Olga Zhuchenko[3]

Taras Shevchenko National University of Kyiv
Address: 64 Volodymyrs'ka Street, 01601, Kyiv, Ukraine
E-mail: [1]slavastas119@rambler.ru, [2]k.ralchenko@gmail.com, [3]ole4ka_zhuchenko@mail.ru

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Abstract. We consider the fractional Vasicek model of the form $dX_t = (\alpha - \beta X_t)\,dt + \gamma dB_t^H$, driven by fractional Brownian motion B^H with Hurst parameter $H \in (0,1)$. We construct three estimators for an unknown parameter $\theta = (\alpha, \beta)$ and prove their strong consistency.

Key words : fractional Brownian motion, fractional Vasicek model, parameter estimation, strong consistency, discretization.

1. Introduction

The fractional Vasicek model is described by the following stochastic differential equation

$$dX_t = (\alpha - \beta X_t)\,dt + \gamma dB_t^H, \tag{1.1}$$

where B^H is the fractional Brownian motion with Hurst index $H \in (0,1)$, and α, β and γ are positive constants. Recently this model has been used in various problems in mathematical finance, see [7,8,24]. When $H = 1/2$, the fractional Brownian motion is the Wiener process W, and the equation (1.1) becomes the well-known interest rate model

$$dX_t = (\alpha - \beta X_t)\,dt + \gamma dW_t,$$

proposed by Vasicek [22] in 1977. From the financial point of view, the parameters can be interpreted as follows: γ represents the stochastic volatility, the ratio α/β is the long-term average interest rate, and β represents the speed of recovery.

In paper [9] the parameter estimation problem for classical (Brownian) Vasicek model was investigated. The author proved that the maximum likelihood estimators for unknown α and β are given by

$$\widehat{\alpha} = \frac{(X_T - X_0)\int_0^T X_t^2\,dt - \int_0^T X_t\,dX_t \int_0^T X_t\,dt}{T\int_0^T X_t^2\,dt - \left(\int_0^T X_t\,dt\right)^2},$$

$$\widehat{\beta} = \frac{(X_T - X_0)\int_0^T X_t\,dt - T\int_0^T X_t\,dX_t}{T\int_0^T X_t^2\,dt - \left(\int_0^T X_t\,dt\right)^2},$$

and converge in L_2 to the true values of the parameters as $T \to \infty$. Using another approach, we prove the strong version of this result (with strong consistency instead of weak). Moreover, our methods allow us to generalize it to the fractional model (1.1) with $H > 1/2$. In this case, the stochastic integral $\int_0^T X_t\,dB_t^H$ is interpreted as a divergence-type integral as in [10]. This integral is the limit of the Riemann sums defined in terms of the Wick product (see [4]). Since the divergence-type integral is not suitable for simulation and discretization, we propose other estimators and prove their strong consistency. The discretized versions of these estimators are also considered.

It is worth mentioning that if $\alpha = 0$, then (1.1) is the fractional Ornstein–Uhlenbeck process introduced in [3]. The drift parameter estimation for this model has been studied by many authors, see [1, 5, 6, 10, 11, 13–16, 18–21, 23].

This paper is organized as follows. In Section 2 we give the necessary definitions and formulate our main consistency results. Section 3 is devoted to numerics. All proofs are given in Section 4.

2. Model description and main results

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a complete probability space. We consider the fractional Brownian motion $B^H = \{B_t^H,\, t \ge 0\}$ on this probability space, that is, the centered Gaussian process with covariance function

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

In what follows we consider the continuous (and even Hölder up to order H) modification that exists due to the Kolmogorov theorem. We study the fractional Vasicek model, described by the stochastic differential equation

$$X_t = x_0 + \int\limits_0^t (\alpha - \beta X_s)\, ds + \gamma B_t^H, \quad t \ge 0. \tag{2.1}$$

We assume that the parameters $x_0 \in \mathbb{R}$, $\gamma > 0$ and $H \in (0,1)$ are known. The parameters $\alpha \in \mathbb{R}$ and $\beta > 0$ are fixed but unknown.

The equation (2.1) has a unique solution, which is given by

$$X_t = x_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \gamma \int_0^t e^{-\beta(t-s)} dB_s^H, \quad t \ge 0. \tag{2.2}$$

where $\int_0^t e^{-\beta(t-s)} dB_s^H$ is a path-wise Riemann–Stieltjes integral. It exists for all $H \in (0,1)$, see [3, Prop. A.1].

Assume that we observe a trajectory of X continuously on the interval $[0,T]$. Let us introduce the following estimators of the unknown parameters:

$$\widehat{\alpha}_T^{(1)} = \frac{(X_T - X_0)\int_0^T X_t^2\, dt - \int_0^T X_t\, dX_t \int_0^T X_t\, dt}{T\int_0^T X_t^2\, dt - \left(\int_0^T X_t\, dt\right)^2}, \tag{2.3}$$

$$\widehat{\beta}_T^{(1)} = \frac{(X_T - X_0)\int_0^T X_t\, dt - T\int_0^T X_t\, dX_t}{T\int_0^T X_t^2\, dt - \left(\int_0^T X_t\, dt\right)^2}, \tag{2.4}$$

where, by (1.1),

$$\int_0^T X_t\, dX_t = \alpha \int_0^T X_t\, dt - \beta \int_0^T X_t^2\, dt + \gamma \int_0^T X_t\, dB_t^H. \tag{2.5}$$

For $H > 1/2$, we interpret the stochastic integral $\int_0^T X_t\, dB_t^H$ as a divergence-type (or Itô–Skorokhod integral), see [2,4] for details. It corresponds to the classical Itô integral if $H = 1/2$.

Theorem 2.1. *Let $H \in [\frac{1}{2}, 1)$. Then the estimators $\widehat{\alpha}_T^{(1)}$ and $\widehat{\beta}_T^{(1)}$ are strongly consistent.*

Remark 1. The case $H < 1/2$ can be reduced to the case $H > 1/2$ by the integral transformation of Jost [12, Cor. 5.2].

Since the discretization and simulation of $\widehat{\alpha}_T^{(1)}$ and $\widehat{\beta}_T^{(1)}$ when $H \ne 1/2$ is quite difficult, we introduce alternative estimators:

$$\widehat{\beta}_T^{(2)} = \left(\frac{1}{\gamma^2 H \Gamma(2H)\, T^2}\left(T\int_0^T X_t^2\, dt - \left(\int_0^T X_t\, dt\right)^2\right)\right)^{-\frac{1}{2H}},$$

$$\widehat{\alpha}_T^{(2)} = \frac{\widehat{\beta}_T^{(2)}}{T}\int_0^T X_t\, dt.$$

Theorem 2.2. *Let $H \in (0,1)$. Then the estimators $\widehat{\alpha}_T^{(2)}$ and $\widehat{\beta}_T^{(2)}$ are strongly consistent.*

In applications usually the observations cannot be continuous. The estimators $\widehat{\alpha}_T^{(2)}$ and $\widehat{\beta}_T^{(2)}$ can be discretized as follows. Let $h > 0$. Assume that a trajectory of X is observed at times $t_k = kh$, $k = 0, 1, \dots, n$. Define

$$\widehat{\beta}_n^{(3)} = \left(\frac{1}{\gamma^2 H\Gamma(2H)\, n^2} \left(n \sum_{k=0}^{n-1} X_{kh}^2 - \left(\sum_{k=0}^{n-1} X_{kh} \right)^2 \right) \right)^{-\frac{1}{2H}},$$

$$\widehat{\alpha}_n^{(3)} = \frac{\widehat{\beta}_n^{(3)}}{n} \sum_{k=0}^{n-1} X_{kh}.$$

Theorem 2.3. *Let $H \in (0,1)$. Then the estimators $\widehat{\alpha}_n^{(3)}$ and $\widehat{\beta}_n^{(3)}$ are strongly consistent.*

The proofs of Theorems 2.1–2.3 are given in Section 4.

3. Numerical illustrations

In this section we illustrate the quality of the estimators with the help of simulation experiments. We simulate the trajectories of the fractional Brownian motion at points $t = 0, \Delta, 2\Delta, 3\Delta, \dots$ and compute approximate values of the process X as the solution to the equation (2.1), using Euler's approximations. For each set of parameters we simulate 100 sample paths with the step $\Delta = 1/1000$. We study the performance of the estimators $\widehat{\alpha}_n^{(3)}$ and $\widehat{\beta}_n^{(3)}$ for various values of the parameters. It turns out that the influence of the values of x_0 and γ on the behavior of the estimators is quite small compared to the other parameters. Therefore, we consider equation (2.1) only for $x_0 = \gamma = 1$.

First, we choose $h = 0.1$ and compute the mean values of the estimators for various α, β and H. The results are reported in Tables 1–2. We see that the estimates converge to the true values of the parameters. Hence these simulation studies confirm the theoretical results. However the rate of convergence for $H = 0.9$ is not very high.

Then we investigate the quality of the estimators, depending on the discretization step h. We choose $\alpha = 1$ and $\beta = 2$ (the results for other pairs of α and β are similar). In Tables 3–4 the means and standard deviations of the estimators for various partitions of the interval $[0, T]$ for $T = 1\,000$ are given. We see that different discretization steps give quite similar results. Thus, the horizon of observations T is more important for the quality of the estimators than the discretization step h.

4. Proofs

First, we study the asymptotic behavior of the integrals $\int_0^T X_t\,dt$ and $\int_0^T X_t^2\,dt$ as $T \to \infty$. The next technical result is crucial for the proof of Theorems 2.1 and 2.2.

Lemma 4.1. *Let $H \in (0,1)$. Then*

$$\frac{1}{T}\int_0^T X_t\,dt \to \frac{\alpha}{\beta}, \tag{4.1}$$

$$\frac{1}{T}\int_0^T X_t^2\,dt \to \frac{\alpha^2}{\beta^2} + \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}}, \tag{4.2}$$

a. s., as $T \to \infty$.

Proof. Let us introduce the following notation:

$$R(t) = x_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right), \quad Z_t = \gamma \int_0^t e^{-\beta(t-s)}\, dB_s^H.$$

Table 1. The means of the estimator $\widehat{\alpha}_n^{(3)}$

H	α	β	n = 100	500	1000	5000	10000
0.1	1	2	1.0642	1.0178	1.0187	0.9970	0.9973
	1	1	1.7556	1.0780	1.0123	1.0164	1.0148
	1	0.5	1.0087	0.8875	0.9047	0.9752	0.9951
	−1	2	−0.6189	−0.8668	−0.9086	−0.9792	−0.9770
	0	2	0.1531	0.0299	0.0162	0.0035	0.0018
	2	1	1.9462	1.9401	1.9683	1.9947	2.0045
0.3	1	2	1.2004	1.0244	1.0065	1.0082	1.0053
	1	1	1.4134	1.0238	1.0245	1.0101	1.0061
	1	0.5	1.3496	1.0195	1.0070	1.0132	1.0066
	−1	2	−0.6903	−0.9190	0.9510	−0.9931	−0.9949
	0	2	0.1357	0.0298	0.0117	0.0016	0.0012
	2	1	2.4387	2.0756	2.0241	1.9817	1.9980
0.5	1	2	1.2381	1.0406	1.0341	0.9978	0.9990
	1	1	1.7457	1.1337	1.0658	1.0026	0.9991
	1	0.5	1.6204	1.1769	1.0789	1.0012	0.9942
	−1	2	−0.8823	−0.9722	−0.9757	−1.0033	−1.0042
	0	2	0.0611	0.0197	0.0036	0.0015	0.0024
	2	1	2.5232	2.1098	2.0798	2.0387	2.0224
0.7	1	2	1.3935	1.0925	1.0390	1.0087	1.0180
	1	1	1.7103	1.2139	1.1404	1.0282	1.0035
	1	0.5	2.0723	1.3336	1.2110	1.0371	1.0223
	−1	2	−0.9499	−1.0821	−1.0832	−1.0252	−1.0151
	0	2	0.1555	−0.0260	−0.0719	−0.0200	−0.0087
	2	1	2.7851	2.2869	2.1246	2.0194	1.9912
0.9	1	2	2.3254	1.6084	1.4988	1.3027	1.2422
	1	1	2.7157	1.6785	1.5052	1.2865	1.2189
	1	0.5	2.5873	1.9564	1.6997	1.3939	1.3375
	−1	2	−0.8844	−1.1992	−1.2273	−1.1261	−1.0929
	0	2	0.3684	0.0930	0.0108	−0.0184	−0.0390
	2	1	3.9860	3.1214	2.8917	2.5297	2.4750

Then Z_t is the fractional Ornstein–Uhlenbeck process [3], and $X_t = R(t) + Z_t$. Let us first prove the convergence (4.1). We have

$$\begin{aligned}\frac{1}{T}\int_0^T X_t\,dt &= \frac{1}{T}\left(x_0 - \frac{\alpha}{\beta}\right)\int_0^T e^{-\beta t}\,dt + \frac{1}{T}\int_0^T \frac{\alpha}{\beta}\,dt + \frac{1}{T}\int_0^T Z_t\,dt\\ &= \frac{1-e^{-\beta T}}{\beta T}\left(x_0 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} + \frac{1}{T}\int_0^T Z_t\,dt.\end{aligned}$$

It is evident that first term converges to zero as $T \to \infty$. Let us now for all $t \ge 0$ define

$$Y_t = \gamma \int_{-\infty}^t e^{-\beta(t-s)}\,dB_s^H = Z_t + e^{-\beta t}\xi, \tag{4.3}$$

where $\xi = \gamma\int_{-\infty}^0 e^{\beta s}\,dB_s^H$. The stochastic process $(Y_t, t \ge 0)$ is Gaussian, stationary and ergodic, see [3]. Then the ergodic theorem implies that

$$\frac{1}{T}\int_0^T Y_t\,dt \to \mathbb{E}\left[Y_0\right],$$

Table 2. The means of the estimator $\widehat{\beta}_n^{(3)}$

			n				
H	α	β	100	500	1000	5000	10000
0.1	1	2	3.2630	2.1294	2.0545	2.0262	2.0101
	1	1	1.7301	1.0760	1.0127	1.0169	1.0153
	1	0.5	0.5591	0.4532	0.4573	0.4886	0.4981
	−1	2	1.4871	1.7951	1.8515	1.9652	1.9565
	0	2	2.1417	1.9786	2.0124	1.9700	1.9732
	2	1	1.0122	0.9776	0.9884	0.9987	1.0026
0.3	1	2	2.3268	2.0219	1.9969	2.0143	2.0105
	1	1	1.3789	1.0153	1.0178	1.0090	1.0065
	1	0.5	0.7707	0.5231	0.5093	0.5081	0.5042
	−1	2	1.6328	1.8994	1.9379	1.9949	1.9974
	0	2	2.1014	2.0630	2.0413	2.0229	2.0188
	2	1	1.2871	1.0523	1.0209	0.9931	1.0004
0.5	1	2	2.4081	2.0548	2.0378	2.0020	2.0026
	1	1	1.7743	1.1330	1.0673	1.0093	1.0050
	1	0.5	0.8951	0.5926	0.5470	0.5031	0.4987
	−1	2	1.9414	1.9577	1.9760	1.9988	2.0030
	0	2	2.2001	2.0652	2.0417	1.9938	1.9991
	2	1	1.3200	1.0524	1.0383	1.0174	1.0110
0.7	1	2	2.6807	2.1830	2.0923	2.0329	2.0230
	1	1	1.7129	1.2071	1.1319	1.0315	1.0124
	1	0.5	1.2491	0.6900	0.6153	0.5293	0.5156
	−1	2	2.2504	2.1114	2.1137	2.0573	2.0346
	0	2	2.6171	2.2063	2.1128	2.0325	2.0146
	2	1	1.5952	1.1789	1.0849	1.0229	1.0134
0.9	1	2	3.8871	2.9823	2.7992	2.6075	2.3925
	1	1	2.6533	1.7323	1.5612	1.2967	1.2251
	1	0.5	1.5364	0.9604	0.8205	0.6851	0.6486
	−1	2	2.9430	2.7970	2.6851	2.4477	2.3809
	0	2	3.3641	2.9170	2.7449	2.4465	2.3968
	2	1	2.2990	1.6009	1.4815	1.2719	1.2374

a. s., as $T \to \infty$. Using the fact that $\mathbb{E}\,[Y_0] = 0$, we deduce

$$\frac{1}{T}\int_0^T Z_t\,dt \to 0, \tag{4.4}$$

a. s., as $T \to \infty$, which directly implies the convergence (4.1).

Now let us look at

$$\frac{1}{T}\int_0^T X_t^2\,dt = \frac{1}{T}\int_0^T (R(t)+Z_t)^2\,dt = \frac{1}{T}\int_0^T R^2(t)\,dt + \frac{1}{T}\int_0^T Z_t^2\,dt + \frac{2}{T}\int_0^T R(t)Z_t\,dt.$$

Table 3. The estimator $\widehat{\alpha}_n^{(3)}$ for $\alpha = 1$, $\beta = 2$

	h	0.001	0.01	0.1	1
$H = 0.1$	Mean	0.9972	0.9972	0.9973	0.9977
	Std. dev.	0.0429	0.0431	0.0425	0.0520
$H = 0.3$	Mean	1.0046	1.0047	1.0053	1.0062
	Std. dev.	0.0346	0.0347	0.0347	0.0402
$H = 0.5$	Mean	0.9994	0.9994	0.9990	1.0018
	Std. dev.	0.0473	0.0474	0.0476	0.0531
$H = 0.7$	Mean	1.0179	1.0179	1.0180	1.0191
	Std. dev.	0.1380	0.1380	0.1379	0.1384
$H = 0.9$	Mean	1.2422	1.2422	1.2422	1.2428
	Std. dev.	0.4512	0.4512	0.4512	0.4514

Table 4. The estimator $\widehat{\beta}_n^{(3)}$ for $\alpha = 1$, $\beta = 2$

	h	0.001	0.01	0.1	1
$H = 0.1$	Mean	1.9931	1.9918	2.0101	2.0438
	Std. dev.	0.0850	0.0919	0.1589	0.4664
$H = 0.3$	Mean	2.0095	2.0098	2.0105	2.0390
	Std. dev.	0.0661	0.0657	0.0757	0.1471
$H = 0.5$	Mean	2.0020	2.0021	2.0026	2.0057
	Std. dev.	0.0621	0.0623	0.0630	0.0897
$H = 0.7$	Mean	2.0232	2.0232	2.0230	2.0185
	Std. dev.	0.0839	0.0839	0.0843	0.0936
$H = 0.9$	Mean	2.3925	2.3925	2.3925	2.3962
	Std. dev.	0.1731	0.1731	0.1731	0.1748

Take each term separately.

$$\begin{aligned}
\frac{1}{T}\int_0^T R^2(t)\,dt &= \frac{1}{T}\int_0^T \left(x_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)\right)^2 dt \\
&= \frac{x_0^2}{T}\int_0^T e^{-2\beta t} dt + \frac{2x_0\alpha}{\beta T}\int_0^T e^{-\beta t} dt - \frac{2x_0\alpha}{\beta T}\int_0^T e^{-2\beta t} dt + \frac{\alpha^2}{\beta^2 T}\int_0^T \left(1 - e^{-\beta t}\right)^2 dt \\
&= \frac{x_0^2\left(1 - e^{-2\beta T}\right)}{2\beta T} + \frac{2x_0\alpha\left(1 - e^{-\beta T}\right)}{\beta^2 T} - \frac{x_0\alpha\left(1 - e^{-2\beta T}\right)}{\beta^2 T} \\
&\quad + \frac{\alpha^2}{\beta^2}\left(1 - \frac{2\left(1 - e^{-\beta T}\right)}{\beta T} + \frac{1 - e^{-2\beta T}}{2\beta T}\right) \to \frac{\alpha^2}{\beta^2},
\end{aligned}$$

as $T \to \infty$.

Applying [16, Lemma 5.6] (see also [10, Lemma 3.3] for $H \ge 1/2$), we get

$$\frac{1}{T}\int_0^T Z_t^2\,dt \to \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}}, \tag{4.5}$$

a. s., as $T \to \infty$.

And finally, using (4.4), (4.5), and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\left|\frac{2}{T}\int_0^T R(t)Z_t\,dt\right| &= \frac{2}{T}\left|\int_0^T\left(\left(x_0-\frac{\alpha}{\beta}\right)e^{-\beta t}+\frac{\alpha}{\beta}\right)Z_t\,dt\right|\\ &\le \frac{2}{T}\left|x_0-\frac{\alpha}{\beta}\right|\int_0^T\left|e^{-\beta t}Z_t\right|dt+\left|\frac{2\alpha}{\beta T}\int_0^T Z_t\,dt\right|\\ &\le \frac{2}{T}\left|x_0-\frac{\alpha}{\beta}\right|\left(\int_0^T e^{-2\beta t}\,dt\int_0^T Z_t^2\,dt\right)^{\frac12}+\left|\frac{2\alpha}{\beta T}\int_0^T Z_t\,dt\right|\\ &= 2\left|x_0-\frac{\alpha}{\beta}\right|\left(\frac{1-e^{-2\beta T}}{2\beta T}\cdot\frac{1}{T}\int_0^T Z_t^2\,dt\right)^{\frac12}+\left|\frac{2\alpha}{\beta T}\int_0^T Z_t\,dt\right|\to 0,\end{aligned}$$

a. s., as $T\to\infty$. Thus, we obtain (4.2). □

Proof of Theorem 2.1. This proof follows the scheme from [10]. First, assume that $H>1/2$. Using the relationship between the divergence integral and the pathwise Riemann–Stieltjes integral (see Th. 3.12 and Eq. (3.6) of [4]), we can write

$$\begin{aligned}\int_0^T X_t\circ dB_t^H &= \int_0^T X_t\,dB_t^H + H(2H-1)\int_0^T\int_0^t D_sX_t(t-s)^{2H-2}\,ds\,dt\\ &= \int_0^T X_t\,dB_t^H+\gamma H(2H-1)\int_0^T\int_0^t u^{2H-2}e^{-\beta u}\,du\,dt.\end{aligned}$$

However, the pathwise integral equals

$$\begin{aligned}\gamma\int_0^T X_t\circ dB_t^H &= \int_0^T X_t\circ dX_t-\alpha\int_0^T X_t\,dt+\beta\int_0^T X_t^2\,dt\\ &= \frac12(X_T^2-x_0^2)-\alpha\int_0^T X_t\,dt+\beta\int_0^T X_t^2\,dt.\end{aligned}$$

Therefore,

$$\gamma\int_0^T X_t\,dB_t^H=\frac12(X_T^2-x_0^2)-\alpha\int_0^T X_t\,dt+\beta\int_0^T X_t^2\,dt-\gamma^2H(2H-1)\int_0^T\int_0^t u^{2H-2}e^{-\beta u}\,du\,dt.$$

Substituting this into (2.5), we obtain

$$\int_0^T X_t\,dX_t=\frac12(X_T^2-x_0^2)-\gamma^2H(2H-1)\int_0^T\int_0^t u^{2H-2}e^{-\beta u}\,du\,dt. \tag{4.6}$$

By [10, formula (3.8)], $\frac{Z_T^2}{T}\to 0$ a. s., as $T\to\infty$. Therefore,

$$\frac{X_T^2}{T}=\left(\frac{1}{T^{1/2}}\left(x_0e^{-\beta T}+\frac{\alpha}{\beta}\left(1-e^{-\beta T}\right)\right)+\frac{Z_T}{T^{1/2}}\right)^2\to 0 \tag{4.7}$$

a. s., as $T\to\infty$. It is easy to calculate

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^t u^{2H-2}e^{-\beta u}\,du\,dt=\beta^{1-2H}\Gamma(2H-1). \tag{4.8}$$

Combining (4.6)–(4.8), we get the convergence

$$\frac{1}{T}\int_0^T X_t\,dX_t\to-\beta^{1-2H}\gamma^2H\Gamma(2H). \tag{4.9}$$

Applying (4.1), (4.2), (4.7) and (4.9), we obtain from (2.3)–(2.4) that

$$\widehat{\alpha}_T^{(1)}\to\alpha,\quad \widehat{\beta}_T^{(1)}\to\beta,$$

a. s., as $T \to \infty$.

Now let us consider the case $H = 1/2$. The process $M_t = \int_0^t X_s \, dB_s^{1/2}$, $t \ge 0$, is a martingale with quadratic variation $\langle M \rangle_t = \int_0^t X_s^2 \, ds$. Then

$$\frac{M_T}{\langle M \rangle_T} = \frac{\int_0^T X_t \, dB_t^{1/2}}{\int_0^T X_t^2 \, dt} \to 0$$

a. s., as $T \to \infty$, by the strong law of large numbers for martingales [17, Th. 2.6.10]. Therefore, using (2.5) and Lemma 4.1, we get

$$\frac{1}{T}\int_0^T X_t \, dX_t = \frac{\alpha}{T}\int_0^T X_t \, dt - \frac{\beta}{T}\int_0^T X_t^2 \, dt + \frac{\gamma}{T}\int_0^T X_t^2 \, dt \, \frac{\int_0^T X_t \, dB_t^{1/2}}{\int_0^T X_t^2 \, dt} \to \frac{\alpha^2}{\beta} - \beta\left(\frac{\alpha^2}{\beta^2} + \frac{\gamma^2}{2\beta}\right) = -\frac{\gamma^2}{2}$$

a. s., as $T \to \infty$. Hence, (4.9) holds for $H = 1/2$. The rest of the proof is similar to that for the case $H > 1/2$. □

Proof of Theorem 2.2. The proof follows from Lemma 4.1. □

Proof of Theorem 2.3. It suffices to show that

$$\frac{1}{n}\sum_{k=0}^{n-1} X_{kh} \to \frac{\alpha}{\beta}, \tag{4.10}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} X_{kh}^2 \to \frac{\alpha^2}{\beta^2} + \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}} \tag{4.11}$$

a. s., as $n \to \infty$.

Let us prove the convergence (4.10).

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} X_{kh} &= \frac{1}{n}\sum_{k=0}^{n-1} R(kh) + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh} = \frac{1}{n}\sum_{k=0}^{n-1} e^{-\beta kh}\left(x_0 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh} \\ &= \frac{1 - e^{-\beta nh}}{n(1 - e^{-\beta h})}\left(x_0 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}.\end{aligned}$$

The first term converges to zero as $n \to \infty$. Similarly to the proof of Lemma 4.1, the ergodic theorem implies that

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh} \to \mathbb{E}[Y_0] = 0, \tag{4.12}$$

a. s., as $n \to \infty$, where $(Y_t, t \ge 0)$ is the ergodic process defined by (4.3). Then

$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh} \to 0, \tag{4.13}$$

a. s., as $n \to \infty$. This implies the convergence (4.10).

Now consider the convergence (4.11). We have

$$\frac{1}{n}\sum_{k=0}^{n-1} X_{kh}^2 = \frac{1}{n}\sum_{k=0}^{n-1} (R(kh) + Z_{kh})^2 = \frac{1}{n}\sum_{k=0}^{n-1} R^2(kh) + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}^2 + \frac{2}{n}\sum_{k=0}^{n-1} R(kh)Z_{kh}. \tag{4.14}$$

Take each term separately.

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} R^2(kh) &= \frac{1}{n}\sum_{k=0}^{n}\left(x_0 e^{-\beta kh} + \frac{\alpha}{\beta}\left(1 - e^{-\beta kh}\right)\right)^2 \\ &= \frac{x_0^2}{n}\sum_{k=0}^{n-1} e^{-2\beta kh} + \frac{2x_0\alpha}{\beta n}\sum_{k=0}^{n-1} e^{-\beta kh} - \frac{2x_0\alpha}{\beta n}\sum_{k=0}^{n-1} e^{-2\beta kh} + \frac{\alpha^2}{\beta^2 n}\sum_{k=0}^{n-1}\left(1 - e^{-\beta kh}\right)^2 \\ &= \frac{x_0^2\left(1 - e^{-2\beta nh}\right)}{n(1 - e^{-2\beta h})} + \frac{2x_0\alpha\left(1 - e^{-\beta nh}\right)}{\beta n(1 - e^{-\beta h})} - \frac{2x_0\alpha\left(1 - e^{-2\beta nh}\right)}{\beta n(1 - e^{-2\beta h})} \\ &\quad + \frac{\alpha^2}{\beta^2}\left(1 - \frac{2\left(1 - e^{-\beta nh}\right)}{n(1 - e^{-\beta h})} + \frac{1 - e^{-2\beta nh}}{n(1 - e^{-2\beta h})}\right) \to \frac{\alpha^2}{\beta^2},\end{aligned} \tag{4.15}$$

as $n \to \infty$. Applying again the ergodic theorem, we get

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}^2 \to \mathbb{E}\left[Y_0^2\right].$$

By the proof of [16, Lemma 5.6], $\mathbb{E}\left[Y_0^2\right] = \gamma^2 H\Gamma(2H)\beta^{-2H}$. Using the representation (4.3) and the convergence (4.12), we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}^2 = \frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}^2 - \frac{2\xi}{n}\sum_{k=0}^{n-1} Y_{kh} + \frac{\xi^2}{n}\sum_{k=0}^{n-1} e^{-2\beta kh} \to \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}}, \tag{4.16}$$

a. s., as $n \to \infty$.

Finally, we consider the third term in (4.14). By the Cauchy–Schwarz inequality, (4.13), and (4.16),

$$\begin{aligned}
\left|\frac{2}{n}\sum_{k=0}^{n-1} R(kh) Z_{kh}\right| &= \frac{2}{n}\left|\sum_{k=0}^{n-1}\left(\left(x_0 - \frac{\alpha}{\beta}\right)e^{-\beta kh} + \frac{\alpha}{\beta}\right) Z_{kh}\right| \\
&\le \frac{2}{n}\left|x_0 - \frac{\alpha}{\beta}\right|\left|\sum_{k=0}^{n-1}\right| e^{-\beta kh} Z_{kh}\Big| + \left|\frac{2\alpha}{n\beta}\sum_{k=0}^{n-1} Z_{kh}\right| \\
&\le \frac{2}{n}\left|x_0 - \frac{\alpha}{\beta}\right|\left(\sum_{k=0}^{n-1} e^{-2\beta kh}\sum_{k=0}^{n-1} Z_{kh}^2\right)^{\frac{1}{2}} + \left|\frac{2\alpha}{n\beta}\sum_{k=0}^{n-1} Z_{kh}\right| \\
&= 2\left|x_0 - \frac{\alpha}{\beta}\right|\left(\frac{1 - e^{-2\beta nh}}{n\left(1 - e^{-2\beta h}\right)} \cdot \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}^2\right)^{\frac{1}{2}} + \left|\frac{2\alpha}{n\beta}\sum_{k=0}^{n-1} Z_{kh}\right| \to 0,
\end{aligned}$$

a. s., as $n \to \infty$. Combining this with (4.14), (4.15), and (4.16), we get (4.11). □

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TRUPMENINIO VASICEKO MODELIO PARAMETRŲ ĮVERTINIŲ ASIMPTOTINĖS SAVYBĖS

Stanislav Lohvinenko, Kostiantyn Ralchenko, Olga Zhuchenko

Santrauka. Nagrinėjamas Vasiceko modelis, valdomas trupmeninio Brauno judesio B^H su Hursto indeksu $H \in (0,1)$, turintis pavidalą $dX_t = (\alpha - \beta X_t)\,dt + \gamma dB_t^H$. Nežinomam parametrui $\theta = (\alpha, \beta)$ sudaromi trys įvertiniai ir įrodomas jų stiprus suderinamumas.

Reikšminiai žodžiai: trupmeninis Brauno judesys, trupmeninis Vasiceko modelis, parametrų vertinimas, stiprus suderinamumas, diskretizavimas.