MAXIMUM LIKELIHOOD ESTIMATION
IN THE FRACTIONAL VASICEK MODEL

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Abstract. We consider the fractional Vasicek model of the form $dX_t = (\alpha - \beta X_t)\,dt + \gamma dB_t^H$, driven by fractional Brownian motion B^H with Hurst parameter $H \in (1/2, 1)$. We construct the maximum likelihood estimators for unknown parameters α and β, and prove their consistency and asymptotic normality.

Keywords: fractional Brownian motion, fractional Vasicek model, maximum likelihood estimation, strong consistency, asymptotic normality.

1. Introduction

The standard Vasicek model was proposed and studied by O. Vasicek [19] in 1977 for the purpose of interest rate modeling. It is described by the following stochastic differential equation

$$dX_t = (\alpha - \beta X_t)\,dt + \gamma dW_t, \tag{1.1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+$, and W is a standard Wiener process. From the financial point of view, β corresponds to the speed of recovery, the ratio α/β is the long-term average interest rate, and γ represents the stochastic volatility. Now the Vasicek model is widely used not only in finance, but also in various scientific areas such as economics, biology, physics, chemistry, medicine and environmental studies.

The present paper deals with the fractional Vasicek model of the form

$$dX_t = (\alpha - \beta X_t)\,dt + \gamma dB_t^H, \tag{1.2}$$

where the Wiener process W is replaced with B^H, a fractional Brownian motion with Hurst index $H \in (1/2, 1)$. This generalization of the model (1.1) enables one to model processes with long-range dependence. Such processes appear in finance, hydrology, telecommunication, turbulence and image processing. In particular, various financial applications of the fractional Vasicek model (1.2) can be found in the articles [3–9, 21].

The goal of the paper is to construct maximum likelihood estimators (MLEs) for the unknown parameters α and β and to establish their consistency and asymptotic normality. We mention that the least squares and ergodic-type estimators in the fractional Vasicek model have been recently studied in [16] and [20]. In [16] the strong consistency of these estimators was proved for the ergodic case $\beta > 0$, and the discretization of the ergodic-type estimators was considered. Note that in [20] a different parametrization was studied, namely

$$dX_t = \kappa(\mu - X_t)\,dt + \gamma dB_t^H,$$

and asymptotic theory for estimating only the persistent parameter κ was developed. The authors proved the strong consistency and asymptotic normality of the ergodic-type estimator for $\kappa > 0$. They also investigated the least squares estimator for the non-ergodic case $\kappa < 0$ and proved its convergence to the Cauchy distribution.

This paper is organized as follows. In Section 2 we describe the model and give necessary definitions. In Section 3 we formulate and prove the main results on consistency and asymptotic normality of MLEs. Some auxiliary results are proved in the appendix.

2. Model description

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a complete probability space. Let $B^H = \{B^H_t, \, t \geq 0\}$ be a fractional Brownian motion on this probability space, that is, a centered Gaussian process with covariance function

$$\mathbb{E} B^H_t B^H_s = \frac{1}{2} \left(s^{2H} + t^{2H} - |t-s|^{2H} \right).$$

Throughout the paper we assume that $H \in (1/2, 1)$. In what follows we consider the continuous (and even Hölder up to order H) modification of B^H_t that exists due to the Kolmogorov theorem.

We study the fractional Vasicek model, described by the stochastic differential equation

$$X_t = x_0 + \int_0^t (\alpha - \beta X_s) \, ds + \gamma B^H_t, \quad t \geq 0. \tag{2.1}$$

We assume that the parameters $x_0 \in \mathbb{R}$, $\gamma > 0$ and $H \in (1/2, 1)$ are known. The main goal is to estimate parameters $\alpha \in \mathbb{R}$ and $\beta > 0$ by continuous observations of a trajectory of X on the interval $[0, T]$. We shall consider three problems:

- estimation of α when β is known,

- estimation of β when α is known,

- estimation of unknown vector parameter $\theta = (\alpha, \beta)$.

The equation (2.1) has a unique solution, which is given by

$$X_t = x_0 e^{-\beta t} + \frac{\alpha}{\beta} \left(1 - e^{-\beta t} \right) + \gamma \int_0^t e^{-\beta(t-s)} dB^H_s, \quad t \geq 0. \tag{2.2}$$

where $\int_0^t e^{-\beta(t-s)} dB^H_s$ is a path-wise Riemann–Stieltjes integral. It exists due to [2, Prop. A.1].

Following [11], for $0 < s < t \leq T$ we define

$$\kappa_H = 2H\Gamma(3/2 - H)\Gamma(H + 1/2), \qquad \lambda_H = \frac{2H\Gamma(3 - 2H)\Gamma(H + 1/2)}{\Gamma(3/2 - H)},$$

$$k_H(t,s) = \kappa_H^{-1} s^{1/2 - H} (t-s)^{1/2 - H}, \qquad w^H_t = \lambda_H^{-1} t^{2 - 2H}, \qquad M^H_t = \int_0^t k_H(t,s) \, dB^H_s.$$

Then the process M^H is a Gaussian martingale, called the fundamental martingale, whose variance function $\langle M^H \rangle$ is the function w^H (see [17]). Moreover, the natural filtration of the martingale M^H coincides with the natural filtration of the fractional Brownian motion B^H.

Define also three stochastic processes

$$P_H(t) = \frac{1}{\gamma} \frac{d}{dw^H_t} \int_0^t k_H(t,s) X_s \, ds, \qquad Q_H(t) = \frac{1}{\gamma} \frac{d}{dw^H_t} \int_0^t k_H(t,s)(\alpha - \beta X_s) \, ds, \qquad S_t = \frac{1}{\gamma} \int_0^t k_H(t,s) \, dX_s.$$

Note that by Lemma 4.1,

$$Q_H(t) = \frac{\alpha}{\gamma} - \beta P_H(t).$$

The process S is called a fundamental semimartingale [11]. It has the following properties.

Lemma 2.1 ([11, Theorem 1]). *For defined processes the following statements hold.*

1. *The process S is an $(\mathfrak{F}_t)$-semimartingale with the decomposition*

$$S_t = \int_0^t Q_H(s) \, dw^H_s + M^H_t. \tag{2.3}$$

2. *The process X admits the representation*

$$X_t = \int_0^t K_H(t,s) \, dS_s,$$

   where

$$K_H(t,s) = \gamma H(2H - 1) \int_s^t r^{H - 1/2} (r - s)^{H - 3/2} \, dr, \quad H \in (1/2, 1).$$

3. *Natural filtrations of processes S and X coincide.*

3. Main results

Applying the analog of the Girsanov formula for a fractional Brownian motion ([11, Theorem 3], see also [13]) and (2.3), one can obtain the following likelihood ratio

$$\Lambda_H(T) = \exp\left\{\int_0^T Q_H(t)\,dM_t^H + \frac{1}{2}\int_0^T (Q_H(t))^2\,dw_t^H\right\} = \exp\left\{\int_0^T Q_H(t)\,dS_t - \frac{1}{2}\int_0^T (Q_H(t))^2\,dw_t^H\right\}$$
$$= \exp\left\{\frac{\alpha}{\gamma}S_T - \beta\int_0^T P_H(t)\,dS_t - \frac{\alpha^2}{2\gamma^2}w_T^H + \frac{\alpha\beta}{\gamma}\int_0^T P_H(t)\,dw_t^H - \frac{\beta^2}{2}\int_0^T (P_H(t))^2\,dw_t^H\right\}.$$

(3.1)

Now we can construct MLEs.

3.1. MLE for α when β is known

Theorem 3.1. *Let $H > 1/2$ and β is known. The MLE for α is*

$$\widehat{\alpha}_T = \frac{S_T + \beta\int_0^T P_H(t)\,dw_t^H}{w_T^H}\gamma.$$

(3.2)

It is unbiased, strongly consistent and normal:

$$T^{1-H}\left(\widehat{\alpha}_T - \alpha\right) \overset{d}{=} \mathcal{N}\left(0, \lambda_H\gamma^2\right).$$

Proof. Let us maximize the likelihood ratio in (3.1) with respect to α. The first and the second partial derivatives are equal to

$$\frac{\partial\Lambda_H(T)}{\partial\alpha} = \frac{1}{\gamma}S_T - \frac{\alpha}{\gamma^2}w_T^H + \frac{\beta}{\gamma}\int_0^T P_H(T)\,dw_T^H, \qquad \frac{\partial^2\Lambda_H(T)}{\partial\alpha^2} = -\frac{1}{\gamma^2}w_T^H.$$

Hence, the MLE for α is given by (3.2).

By Lemma 2.1, the process S admits the representation:

$$S_T = \frac{\alpha}{\gamma}w_T^H - \beta\int_0^T P_H(t)\,dw_t^H + M_T^H.$$

Hence

$$\widehat{\alpha}_T = \frac{\frac{\alpha}{\gamma}w_T^H - \beta\int_0^T P_H(t)\,dw_t^H + M_T^H + \beta\int_0^T P_H(t)\,dw_t^H}{w_T^H}\gamma = \alpha + \gamma\frac{M_T^H}{w_T^H}.$$

Recall that the process M^H is a martingale with quadratic variation w^H. Since $w_T^H \to \infty$, as $T \to \infty$, by the strong law of large numbers for martingales [15, Theorem 2.6.10], we have

$$\frac{M_T^H}{w_T^H} \overset{a.s.}{\longrightarrow} 0, \quad \text{as } T \to \infty.$$

Hence, $\widehat{\alpha}_T \overset{a.s.}{\longrightarrow} \alpha$, as $T \to \infty$, which confirms the strong consistency of the estimator.

Since M^H is a Gaussian process with variance function w^H, it follows that

$$\frac{M_T^H}{\sqrt{w_T^H}} \overset{d}{=} \mathcal{N}(0,1).$$

Hence,

$$T^{1-H}\left(\widehat{\alpha}_T - \alpha\right) = \gamma T^{1-H}\frac{M_T^H}{w_T^H} = \gamma\sqrt{\lambda_H}\frac{M_T^H}{\sqrt{w_T^H}} \overset{d}{=} \mathcal{N}\left(0, \lambda_H\gamma^2\right). \qquad \square$$

3.2. MLE for β when α is known

Theorem 3.2. *Let $H > 1/2$ and α is known. The maximum likelihood estimator for β is*

$$\widehat{\beta}_T = \frac{\frac{\alpha}{\gamma} \int_0^T P_H(t)\,dw_t^H - \int_0^T P_H(t)\,dS_t}{\int_0^T (P_H(t))^2\,dw_t^H}. \tag{3.3}$$

It is strongly consistent and asymptotically normal:

$$\sqrt{T}\left(\widehat{\beta}_T - \beta\right) \xrightarrow{d} \mathcal{N}(0, 2\beta)$$

or $\sqrt{\frac{T}{2\beta}}\left(\widehat{\beta}_T - \beta\right) \xrightarrow{d} \mathcal{N}(0,1).$

Proof. Let us maximize likelihood ratio in (3.1) with respect to β. We have

$$\frac{\partial \Lambda_H(T)}{\partial \beta} = -\int_0^T P_H(t)\,dS_t + \frac{\alpha}{\gamma}\int_0^T P_H(t)\,dw_t^H - \beta\int_0^T (P_H(T))^2\,dw_T^H, \qquad \frac{\partial^2 \Lambda_H(T)}{\partial \beta^2} = -\int_0^T (P_H(t))^2\,dw_t^H.$$

Hence the MLE for β is given by (3.3).

By Lemma 2.1,

$$dS_t = \frac{\alpha}{\gamma}\,dw_t^H - \beta P_H(t)\,dw_t^H + dM_t^H, \tag{3.4}$$

$$\int_0^T P_H(t)\,dS_t = \frac{\alpha}{\gamma}\int_0^T P_H(t)\,dw_t^H - \beta\int_0^T (P_H(t))^2\,dw_t^H + \int_0^T P_H(t)\,dM_t^H. \tag{3.5}$$

Hence

$$\widehat{\beta}_T = \beta - \frac{\int_0^T P_H(t)\,dM_t^H}{\int_0^T (P_H(t))^2\,dw_t^H}.$$

Since the process M^H is a martingale with quadratic variation w^H, the process $\int_0^{\cdot} P_H(t)\,dM_t^H$ is a martingale with quadratic variation $\int_0^{\cdot} (P_H(t))^2\,dw_t^H$. Taking into account the monotonicity of $\int_0^T (P_H(t))^2\,dw_t^H$ in upper bound T, we obtain from (4.7) the almost sure convergence

$$\int_0^T (P_H(t))^2\,dw_t^H \xrightarrow{\text{a.s.}} \infty, \quad \text{as } T \to \infty.$$

Therefore, by the strong law of large numbers for martingales [15, Theorem 2.6.10], we get the convergence $\widehat{\beta}_T \xrightarrow{\text{a.s.}} \beta$, as $T \to \infty$, which confirms the strong consistency of the estimator.

Applying Lemma 4.7, we obtain

$$\sqrt{T}\left(\widehat{\beta}_T - \beta\right) = -\sqrt{T}\frac{\int_0^T P_H(t)\,dM_t^H}{\int_0^T (P_H(t))^2\,dw_t^H} = -\frac{\frac{1}{\sqrt{T}}\int_0^T P_H(t)\,dM_t^H}{\frac{1}{T}\int_0^T (P_H(t))^2\,dw_t^H} \xrightarrow{d} \mathcal{N}(0, 2\beta). \qquad \square$$

Remark 3.3. If $\alpha = 0$, then the process X is the fractional Ornstein–Uhlenbeck process. In this case the MLE for β equals $\widehat{\beta}_T = -\frac{\int_0^T P_H(t)\,dS_t}{\int_0^T (P_H(t))^2\,dw_t^H}$. This MLE was first investigated in [12], where its strong consistency was established. Its asymptotic normality was proved in [18].

3.3. MLE for vector parameter (α, β)

Theorem 3.4. *Let $H > 1/2$. The MLEs for α and β equal*

$$\widetilde{\alpha}_T = \frac{\int_0^T P_H(t)\,dS_t \int_0^T P_H(t)\,dw_t^H - S_T \int_0^T (P_H(t))^2\,dw_t^H}{\left(\int_0^T P_H(t)\,dw_t^H\right)^2 - w_T^H \int_0^T (P_H(t))^2\,dw_t^H}\gamma, \tag{3.6}$$

$$\widetilde{\beta}_T = \frac{w_T^H \int_0^T P_H(t)\,dS_t - S_T \int_0^T P_H(t)\,dw_t^H}{\left(\int_0^T P_H(t)\,dw_t^H\right)^2 - w_T^H \int_0^T (P_H(t))^2\,dw_t^H}. \tag{3.7}$$

They are consistent and asymptotically normal:

$$T^{1-H}(\widetilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}(0, \lambda_H \gamma^2),$$

$$\sqrt{T}(\widetilde{\beta}_T - \beta) \xrightarrow{d} \mathcal{N}(0, 2\beta),$$

or $\frac{T^{1-H}}{\sqrt{\lambda_H}\gamma}(\widetilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}(0,1)$, $\sqrt{\frac{T}{2\beta}}(\widetilde{\beta}_T - \beta) \xrightarrow{d} \mathcal{N}(0,1)$.

Proof. Let us maximize likelihood ratio in (3.1) with respect to α and β simultaneously. Obviously, the system of equations

$$\begin{cases} \dfrac{\partial \Lambda_H(T)}{\partial \alpha} = \dfrac{1}{\gamma}S_T - \dfrac{\alpha}{\gamma^2}w_T^H + \dfrac{\beta}{\gamma}\int_0^T P_H(T)\,dw_T^H = 0 \\[2mm] \dfrac{\partial \Lambda_H(T)}{\partial \beta} = -\int_0^T P_H(t)\,dS_t + \dfrac{\alpha}{\gamma}\int_0^T P_H(t)\,dw_t^H - \beta\int_0^T (P_H(T))^2\,dw_T^H = 0 \end{cases}$$

has the solution given by (3.6)–(3.7). Now we check the second partial derivatives of $\Lambda_H(T)$:

$$\frac{\partial^2 \Lambda_H(T)}{\partial \alpha^2} = -\frac{1}{\gamma^2}w_T^H < 0, \qquad \frac{\partial^2 \Lambda_H(T)}{\partial \beta^2} = -\int_0^T (P_H(t))^2\,dw_t^H < 0,$$

$$\frac{\partial^2 \Lambda_H(T)}{\partial \alpha^2}\frac{\partial^2 \Lambda_H(T)}{\partial \beta^2} - \left(\frac{\partial^2 \Lambda_H(T)}{\partial \alpha \partial \beta}\right)^2 = \frac{1}{\gamma^2}w_T^H \int_0^T (P_H(t))^2\,dw_t^H - \frac{1}{\gamma^2}\left(\int_0^T P_H(t)\,dw_t^H\right)^2 > 0,$$

by the Cauchy–Schwarz inequality, which confirms maximization.

Applying the representation of the process S from Lemma 2.1 and formulas (3.4)–(3.5), we obtain

$$\widetilde{\alpha}_T = \alpha + \frac{\int_0^T P_H(t)\,dM_t^H \int_0^T P_H(t)\,dw_t^H - M_T^H \int_0^T (P_H(t))^2\,dw_t^H}{\left(\int_0^T P_H(t)\,dw_t^H\right)^2 - w_T^H \int_0^T (P_H(t))^2\,dw_t^H}\gamma,$$

$$\widetilde{\beta}_T = \beta + \frac{w_T^H \int_0^T P_H(t)\,dM_t^H - M_T^H \int_0^T P_H(t)\,dw_t^H}{\left(\int_0^T P_H(t)\,dw_t^H\right)^2 - w_T^H \int_0^T (P_H(t))^2\,dw_t^H}.$$

Hence due to Lemmas 4.6 and 4.7 and the properties of the martingale M^H we get

$$\sqrt{T}(\widetilde{\beta}_T - \beta) = \frac{\frac{1}{\sqrt{T}}\int_0^T P_H(t)\,dM_t^H - \frac{M_T^H}{\sqrt{T}}\frac{1}{w_T^H}\int_0^T P_H(t)\,dw_t^H}{\left(\frac{1}{\sqrt{Tw_T^H}}\int_0^T P_H(t)\,dw_t^H\right)^2 - \frac{1}{T}\int_0^T (P_H(t))^2\,dw_t^H} \xrightarrow{d} \mathcal{N}(0, 2\beta),$$

$$T^{1-H}(\widetilde{\alpha}_T - \alpha) = \frac{\frac{1}{\sqrt{T}}\int_0^T P_H(t)\,dM_t^H \frac{1}{\sqrt{Tw_T^H}}\int_0^T P_H(t)\,dw_t^H - \sqrt{\lambda_H}\frac{M_T^H}{\sqrt{w_T^H}}\frac{1}{T}\int_0^T (P_H(t))^2\,dw_t^H}{\left(\frac{1}{\sqrt{Tw_T^H}}\int_0^T P_H(t)\,dw_t^H\right)^2 - \frac{1}{T}\int_0^T (P_H(t))^2\,dw_t^H}\gamma \xrightarrow{d} \mathcal{N}(0, \lambda_H \gamma^2),$$

which confirms asymptotical normality of the estimators and consequently their (weak) consistency. $\square$

Remark 3.5. It is worth noting that for $H = 1/2$ the estimators $\widehat{\alpha}_T, \widehat{\beta}_T$ and $\left(\widetilde{\alpha}_T, \widetilde{\beta}_T\right)$ are nothing but the MLEs for classical Vasicek model (1.1), see [14, Example 1.35]. This means that for $H = 1/2$ formulas (3.2)–(3.3) and (3.6)–(3.7) transform to

$$\widehat{\alpha}_T = \frac{X_T - X_0 + \beta\int_0^T X_t\,dt}{T}, \qquad \widehat{\beta}_T = \frac{\alpha\int_0^T X_t\,dt - \int_0^T X_t\,dX_t}{\int_0^T X_t^2\,dt},$$

$$\widetilde{\alpha}_T = \frac{(X_T - X_0)\int_0^T X_t^2\,dt - \int_0^T X_t\,dX_t \int_0^T X_t\,dt}{T\int_0^T X_t^2\,dt - \left(\int_0^T X_t\,dt\right)^2}, \qquad \widetilde{\beta}_T = \frac{(X_T - X_0)\int_0^T X_t\,dt - T\int_0^T X_t\,dX_t}{T\int_0^T X_t^2\,dt - \left(\int_0^T X_t\,dt\right)^2}.$$

Remark 3.6. The problem of finding the bivariate asymptotic distribution of the estimator $\left(\widetilde{\alpha}_T, \widetilde{\beta}_T\right)$ is more involved and requires different tools. In particular, one should find the joint asymptotic distribution of the statistics S_T, $\int_0^T P_H(t)\,dS_t$, $\int_0^T P_H(t)\,dw_t^H$, and $\int_0^T (P_H(t))^2\,dw_t^H$. This will be done in our further work.

4. Appendix

We start with the following simple lemma.

Lemma 4.1. *For any $H \in (0,1)$ the following equation holds:*

$$\int_0^t k_H(t,s)\,ds = w_t^H.$$

Proof. The proof is carried out by substitution $s = tz$.

$$\int_0^t k_H(t,s)\,ds = \kappa_H^{-1}\int_0^t s^{1/2-H}(t-s)^{1/2-H}\,ds = \kappa_H^{-1}t^{2-2H}\int_0^1 z^{1/2-H}(1-z)^{1/2-H}\,dz$$

$$= \kappa_H^{-1}t^{2-2H}\mathrm{B}(3/2-H,3/2-H) = \lambda_H^{-1}t^{2-2H} = w_t^H \qquad \square$$

Let us introduce the following process

$$U_t = \int_0^t e^{-\beta(t-s)}\,dB_s^H, \quad t \geq 0.$$

Then U is a fractional Ornstein–Uhlenbeck process (see [2]), which is the solution of

$$dU_t = -\beta U_t\,dt + dB_t^H, \quad U_0 = 0.$$

Maximum likelihood estimation for this process was widely studied in [12] and [18].

Now from (2.2) one can get

$$X_t = \frac{\alpha}{\beta} + \left(x_0 - \frac{\alpha}{\beta}\right)e^{-\beta t} + \gamma U_t. \tag{4.1}$$

Then applying (4.1) and Lemma 4.1, we get

$$\begin{aligned} P_H(t) &= \frac{1}{\gamma}\frac{d}{dw_t^H}\int_0^t k_H(t,s)X_s\,ds = \frac{1}{\gamma}\frac{d}{dw_t^H}\int_0^t k_H(t,s)\left[\frac{\alpha}{\beta} + \left(x_0 - \frac{\alpha}{\beta}\right)e^{-\beta s} + \gamma U_s\right]ds \\ &= \frac{\alpha}{\beta\gamma}\frac{d}{dw_t^H}\int_0^t k_H(t,s)\,ds + \frac{1}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right)\frac{d}{dw_t^H}\int_0^t k_H(t,s)e^{-\beta s}\,ds + \frac{d}{dw_t^H}\int_0^t k_H(t,s)U_s\,ds \\ &= \frac{\alpha}{\beta\gamma} + \frac{1}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right)V_H(t) + \widetilde{P}_H(t), \end{aligned} \tag{4.2}$$

where

$$\widetilde{P}_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)U_s\,ds, \qquad V_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)e^{-\beta s}\,ds.$$

Lemma 4.2. *Let $H > 1/2$. Then*

$$\int_0^t k_H(t,s)e^{-\beta s}\,ds = \frac{\sqrt{\pi}\,\Gamma(3/2-H)}{\kappa_H\beta^{1-H}}t^{1-H}e^{-\frac{\beta t}{2}}I_{1-H}\left(\frac{\beta t}{2}\right) = \frac{\Gamma(3/2-H)}{\kappa_H\beta^{3/2-H}}t^{1/2-H} + O\left(t^{-1/2-H}\right), \quad \text{as } t \to \infty,$$

where $I_\nu(z)$ is the modified Bessel function of the first kind.

Proof. By [1, formulas 9.6.18 and 9.7.1],

$$I_\nu(z) = \frac{z^\nu}{\sqrt{\pi}2^\nu\Gamma(\nu+1/2)}\int_{-1}^1 (1-u^2)^{\nu-1/2}e^{-zu}\,du,$$

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 + O\left(z^{-1}\right)\right), \quad \text{as } z \to \infty.$$

Therefore, by substitution $s = \frac{t}{2}(u+1)$, we get

$$\begin{aligned} \int_0^t k_H(t,s)e^{-\beta s}\,ds &= \kappa_H^{-1}\int_0^t s^{1/2-H}(t-s)^{1/2-H}e^{-\beta s}\,ds = \kappa_H^{-1}\left(\frac{t}{2}\right)^{2-2H}e^{-\frac{\beta t}{2}}\int_{-1}^1 (1-u^2)^{1/2-H}e^{-\frac{\beta t}{2}u}\,du \\ &= \frac{\sqrt{\pi}\,\Gamma(3/2-H)}{\kappa_H\beta^{1-H}}t^{1-H}e^{-\frac{\beta t}{2}}I_{1-H}\left(\frac{\beta t}{2}\right) = \frac{\Gamma(3/2-H)}{\kappa_H\beta^{3/2-H}}t^{1/2-H}\left(1 + O\left(t^{-1}\right)\right) \\ &= \frac{\Gamma(3/2-H)}{\kappa_H\beta^{3/2-H}}t^{1/2-H} + O\left(t^{-1/2-H}\right) \quad \text{as } t \to \infty. \qquad \square \end{aligned}$$

Corollary 4.3. *Let $H > 1/2$. Then it is easy to see that*

$$\int_0^T V_H(t)\,dw_t^H = O\left(T^{1/2-H}\right), \quad as\ T \to \infty.$$

Lemma 4.4. *Let $H > 1/2$. Then*

$$V_H(t) = \frac{\beta^{H-1}\sqrt{\pi}\,\Gamma(3-2H)}{(2-2H)\Gamma(3/2-H)}\left[(1-H)t^{H-1}e^{-\frac{\beta t}{2}}I_{1-H}\left(\frac{\beta t}{2}\right)\right.$$
$$\left.+\frac{\beta}{2}t^H e^{-\frac{\beta t}{2}}\left\{\frac{1}{2}I_{2-H}\left(\frac{\beta t}{2}\right)+\frac{1}{2}I_{-H}\left(\frac{\beta t}{2}\right)-I_{1-H}\left(\frac{\beta t}{2}\right)\right\}\right]$$
$$=\frac{\beta^{H-3/2}\Gamma(2-2H)}{\Gamma(1/2-H)}t^{H-3/2}+O\left(t^{H-5/2}\right), \quad as\ t\to\infty.$$

Proof. By [1, formulas 9.6.29 and 9.7.1],

$$\frac{d}{dz}I_\nu(z)=\frac{1}{2}\left(I_{\nu+1}(z)+I_{\nu-1}(z)\right),$$

$$I_\nu(z)=\frac{e^z}{\sqrt{2\pi z}}\left(1-\frac{4\nu^2-1}{8z}+O\left(z^{-2}\right)\right), \quad as\ z\to\infty$$

Then applying Lemma 4.2 we get:

$$V_H(t)=\frac{d}{dw_t^H}\int_0^t k_H(t,s)e^{-\beta s}\,ds=\frac{\lambda_H\sqrt{\pi}\,\Gamma(3/2-H)}{\kappa_H(2-2H)\beta^{1-H}t^{1-2H}}\frac{d}{dt}\left[t^{1-H}e^{-\frac{\beta t}{2}}I_{1-H}\left(\frac{\beta t}{2}\right)\right]$$
$$=\frac{\sqrt{\pi}\,\Gamma(3-2H)}{(2-2H)\beta^{1-H}\Gamma(3/2-H)}t^{2H-1}\left[(1-H)t^{-H}e^{-\frac{\beta t}{2}}I_{1-H}\left(\frac{\beta t}{2}\right)\right.$$
$$\left.-\frac{\beta}{2}t^{1-H}e^{-\frac{\beta t}{2}}I_{1-H}\left(\frac{\beta t}{2}\right)+\frac{\beta}{2}t^{1-H}e^{-\frac{\beta t}{2}}\frac{1}{2}\left(I_{2-H}\left(\frac{\beta t}{2}\right)+I_{-H}\left(\frac{\beta t}{2}\right)\right)\right]$$
$$=\frac{\beta^{H-1}\sqrt{\pi}\,\Gamma(3-2H)}{(2-2H)\Gamma(3/2-H)}\left[(1-H)t^{H-1}e^{-\frac{\beta t}{2}}I_{1-H}\left(\frac{\beta t}{2}\right)\right.$$
$$\left.+\frac{\beta}{2}t^H e^{-\frac{\beta t}{2}}\left(\frac{1}{2}I_{2-H}\left(\frac{\beta t}{2}\right)+\frac{1}{2}I_{-H}\left(\frac{\beta t}{2}\right)-I_{1-H}\left(\frac{\beta t}{2}\right)\right)\right]$$
$$=\frac{\beta^{H-1}\sqrt{\pi}\,\Gamma(3-2H)}{(2-2H)\Gamma(3/2-H)}\left[\frac{1-H}{\sqrt{\beta\pi}}t^{H-3/2}\left(1-\frac{4(1-H)^2-1}{4\beta t}+O\left(t^{-2}\right)\right)\right.$$
$$+\frac{\sqrt{\beta}}{2\sqrt{\pi}}t^{H-1/2}\left\{\frac{1}{2}\left(1-\frac{4(2-H)^2-1}{4\beta t}\right)+\frac{1}{2}\left(1-\frac{4H^2-1}{4\beta t}\right)\right.$$
$$\left.\left.-\left(1-\frac{4(1-H)^2-1}{4\beta t}\right)+O\left(t^{-2}\right)\right\}\right]$$
$$=\frac{\beta^{H-1}\sqrt{\pi}\,\Gamma(3-2H)}{(2-2H)\Gamma(3/2-H)}\left[\frac{1-H}{\sqrt{\beta\pi}}t^{H-3/2}-\frac{1}{2\sqrt{\beta\pi}}t^{H-3/2}+O\left(t^{H-5/2}\right)\right]$$
$$=\frac{\beta^{H-3/2}\Gamma(2-2H)}{\Gamma(1/2-H)}t^{H-3/2}+O\left(t^{H-5/2}\right), \quad as\ t\to\infty. \qquad \square$$

Lemma 4.5. *For any $\varepsilon > 0$ the following convergence holds:*

$$\frac{1}{\left(w_T^H\right)^{1/2+\varepsilon}}\int_0^T \widetilde{P}_H(t)\,dw_t^H \to 0, \quad as\ T\to\infty,$$

in $\mathbb{L}_2 = \mathbb{L}_2(\Omega, \mathfrak{F}, \mathbf{P})$.

Proof. From [10, Lemma 5.4] we have for some $C_{H,\beta} > 0$

$$\mathbb{E}[U_s U_t] \le C_{H,\beta}|t-s|^{2H-2}.$$

Then

$$\mathbb{E}\left(\frac{1}{\left(w_T^H\right)^{1/2+\varepsilon}} \int_0^T \widetilde{P}_H(t)\, dw_t^H\right)^2 = \frac{1}{\left(w_T^H\right)^{1+2\varepsilon}} \mathbb{E}\left(\int_0^T k_H(T,t) U_t\, dt\right)^2$$

$$= \frac{1}{\left(w_T^H\right)^{1+2\varepsilon}} \int_0^T \int_0^T k_H(T,t) k_H(T,s) \mathbb{E}[U_s U_t]\, ds\, dt$$

$$\leq \frac{\lambda_H^{1+2\varepsilon} C_{H,\beta}}{\kappa_H^2 T^{(2-2H)(1+2\varepsilon)}} \int_0^T \int_0^T t^{1/2-H}(T-t)^{1/2-H} s^{1/2-H}(T-s)^{1/2-H} |t-s|^{2H-2}\, ds\, dt.$$

Due to [17, Lemma 2.2 (iv)] it holds that for $\mu \in (0,1)$ and $x \in (0,1)$,

$$\int_0^1 t^{-\mu}(1-t)^{-\mu} |x-t|^{2\mu-1}\, dt = \mathrm{B}(\mu, 1-\mu). \tag{4.3}$$

Substituting $s = uT$, $t = vT$, and applying (4.3) we obtain

$$\mathbb{E}\left(\frac{1}{\left(w_T^H\right)^{1/2+\varepsilon}} \int_0^T \widetilde{P}_H(t)\, dw_t^H\right)^2$$

$$\leq \frac{\lambda_H^{1+2\varepsilon} C_{H,\beta} T^{2-2H}}{\kappa_H^2 T^{(2-2H)(1+2\varepsilon)}} \int_0^1 \int_0^1 v^{1/2-H}(1-v)^{1/2-H} u^{1/2-H}(1-u)^{1/2-H} |v-u|^{2H-2}\, du\, dv$$

$$= \frac{\lambda_H^{1+2\varepsilon} C_{H,\beta}}{\kappa_H^2 T^{(2-2H)2\varepsilon}} \int_0^1 v^{1/2-H}(1-v)^{1/2-H} \left(\int_0^1 u^{1/2-H}(1-u)^{1/2-H} |v-u|^{2H-2}\, du\right) dv$$

$$= \frac{\lambda_H^{1+2\varepsilon} C_{H,\beta} \mathrm{B}(H-1/2, 3/2-H)}{\kappa_H^2 T^{(2-2H)2\varepsilon}} \int_0^1 v^{1/2-H}(1-v)^{1/2-H}\, dv$$

$$= \frac{\lambda_H^{1+2\varepsilon} C_{H,\beta} \mathrm{B}(H-1/2, 3/2-H) \mathrm{B}(3/2-H, 3/2-H)}{\kappa_H^2 T^{(2-2H)2\varepsilon}} \to 0, \quad \text{as } T \to \infty,$$

that concludes the proof of the lemma. $\qquad\square$

By [18, Proof of Th. 3], the next convergences hold:

$$\frac{1}{\sqrt{T}} \int_0^T \widetilde{P}_H(t)\, dM_t^H \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{2\beta}\right), \quad \text{as } T \to \infty, \tag{4.4}$$

$$\frac{1}{T} \int_0^T \left(\widetilde{P}_H(t)\right)^2 dw_t^H \xrightarrow{\mathbf{P}} \frac{1}{2\beta}, \quad \text{as } T \to \infty. \tag{4.5}$$

Hence we get the following results.

Lemma 4.6. *Let $H > 1/2$. Then it holds that:*

$$\frac{1}{w_T^H} \int_0^T P_H(t)\, dw_t^H \xrightarrow{\mathbf{P}} \frac{\alpha}{\beta\gamma}, \quad \text{as } T \to \infty.$$

Proof. Applying (4.2) we get:

$$\frac{1}{w_T^H} \int_0^T P_H(t)\, dw_t^H = \frac{1}{w_T^H} \int_0^T \left[\frac{\alpha}{\beta\gamma} + \frac{1}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right) V_H(t) + \widetilde{P}_H(t)\right] dw_t^H$$

$$= \frac{\alpha}{\beta\gamma} + \frac{1}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right) \frac{1}{w_T^H} \int_0^T V_H(t)\, dw_t^H + \frac{1}{w_T^H} \int_0^T \widetilde{P}_H(t)\, dw_t^H$$

Combining Corollary 4.3 and Lemma 4.5 with $\varepsilon = \frac{1}{2}$ concludes the proof. $\qquad\square$

Lemma 4.7. *Let $H > 1/2$. Then the following convergences hold:*

$$\frac{1}{\sqrt{T}} \int_0^T P_H(t) \, dM_t^H \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{2\beta}\right), \quad \text{as } T \to \infty, \tag{4.6}$$

$$\frac{1}{T} \int_0^T (P_H(t))^2 \, dw_t^H \xrightarrow{\mathbf{P}} \frac{1}{2\beta}, \quad \text{as } T \to \infty. \tag{4.7}$$

Proof. Due to [1, formula 9.6.7] we have

$$(V_H(t))^2 (2 - 2H)\lambda^{-1}t^{1-2H} = O\left(t^{1-2H}\right), \quad \text{as } t \to 0.$$

By the limit comparison test for improper integrals it follows from $\int_0^1 t^{1-2H} \, dt < \infty$ that

$$\int_0^1 (V_H(t))^2 \, dw_t^H < \infty.$$

From Lemma 4.4 we get

$$(V_H(t))^2 (2 - 2H)\lambda^{-1}t^{1-2H} = O\left(t^{-2}\right), \quad \text{as } t \to \infty.$$

Applying the limit comparison test for improper integrals again, we get from $\int_1^\infty t^{-2} \, dt < \infty$ that

$$\int_1^\infty (V_H(t))^2 \, dw_t^H < \infty.$$

Now it is easy to see that

$$\frac{1}{T} \int_0^T (V_H(t))^2 \, dw_t^H = \frac{1}{T} \int_0^1 (V_H(t))^2 \, dw_t^H + \frac{1}{T} \int_1^T (V_H(t))^2 \, dw_t^H \to 0, \quad \text{as } T \to \infty. \tag{4.8}$$

Hence applying equation (4.5) and the Cauchy–Schwarz inequality we obtain

$$\left| \frac{1}{T} \int_0^T V_H(t) \widetilde{P}_H(t) \, dw_t^H \right| \le \sqrt{\frac{1}{T} \int_0^T (V_H(t))^2 \, dw_t^H \frac{1}{T} \int_0^T \left(\widetilde{P}_H(t)\right)^2 \, dw_t^H} \xrightarrow{\mathbf{P}} 0, \quad \text{as } T \to \infty. \tag{4.9}$$

Due to (4.8) the following holds:

$$\mathbb{E}\left(\frac{1}{\sqrt{T}} \int_0^T V_H(t) \, dM_t^H \right)^2 = \frac{1}{T} \int_0^T (V_H(t))^2 \, dw_t^H \to 0, \quad \text{as } T \to \infty.$$

Hence

$$\frac{1}{\sqrt{T}} \int_0^T V_H(t) \, dM_t^H \xrightarrow{\mathbf{P}} 0, \quad \text{as } T \to \infty. \tag{4.10}$$

Then combining equations (4.2), (4.4), (4.10) and properties of the martingale M^H we get

$$\frac{1}{\sqrt{T}} \int_0^T P_H(t) \, dM_t^H = \frac{1}{\sqrt{T}} \int_0^T \left[\frac{\alpha}{\beta\gamma} + \frac{1}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right) V_H(t) + \widetilde{P}_H(t) \right] dM_t^H$$

$$= \frac{\alpha}{\beta\gamma} \frac{M_T^H}{\sqrt{T}} + \frac{1}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right) \frac{1}{\sqrt{T}} \int_0^T V_H(t) \, dM_t^H + \frac{1}{\sqrt{T}} \int_0^T \widetilde{P}_H(t) \, dM_t^H \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{2\beta}\right), \quad T \to \infty.$$

Finally, combining equations (4.2), (4.5), (4.8), (4.9), Corollary 4.3 and Lemma 4.5 we get

$$\frac{1}{T} \int_0^T (P_H(t))^2 \, dw_t^H = \frac{1}{T} \int_0^T \left[\frac{\alpha}{\beta\gamma} + \frac{1}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right) V_H(t) + \widetilde{P}_H(t) \right]^2 dw_t^H$$

$$= \left(\frac{\alpha}{\beta\gamma}\right)^2 \frac{w_T^H}{T} + \frac{1}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right) \frac{1}{T} \int_0^T (V_H(t))^2 \, dw_t^H + \frac{1}{T} \int_0^T \left(\widetilde{P}_H(t)\right)^2 \, dw_t^H$$

$$+ \frac{2\alpha}{\beta\gamma^2}\left(x_0 - \frac{\alpha}{\beta}\right) \frac{1}{T} \int_0^T V_H(t) \, dw_t^H + \frac{2\alpha}{\beta\gamma} \frac{1}{T} \int_0^T \widetilde{P}_H(t) \, dw_t^H$$

$$+ \frac{2}{\gamma}\left(x_0 - \frac{\alpha}{\beta}\right) \frac{1}{T} \int_0^T V_H(t) \widetilde{P}_H(t) \, dw_t^H \xrightarrow{\mathbf{P}} \frac{1}{2\beta}, \quad T \to \infty,$$

that concludes the proof of the lemma. $\qquad \square$

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DIDŽIAUSIOJO TIKĖTINUMO ĮVERTINIMAS TRUPMENINIAME VASICEKO MODELYJE

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Santrauka. Mes nagrinėjame trupmeninį Vasiceko modelį $dX_t = (\alpha - \beta X_t)\, dt + \gamma dB_t^H$, valdomą trupmeninio Brauno proceso B^H su Hursto parametru $H \in (1/2, 1)$. Sukonstruojami nežinomų parametrų α ir β didžiausiojo tikėtinumo įvertiniai ir įrodomas jų pagrįstumas bei asimptotinis normalumas.

Reikšminiai žodžiai: trupmeninis Brauno procesas, trupmeninis Vasiceko modelis, didžiausiojo tikėtinumo įvertinimas, stiprusis pagrįstumas, asimptotinis normalumas.