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Some Krasnosel'skii-type fixed point theorems for Meir–Keeler-type mappings

Ehsan Pourhadi[a], **Reza Saadati**[b], **Zoran Kadelburg**[c,1,2]

[a]International Center for Mathematical Modelling in Physics and Cognitive Sciences,
Mathematical Institute, Linnæus University,
SE-351 95, Växjö, Sweden
ehsan.pourhadi.extern@lnu.se

[b]Department of Mathematics, Iran University of Science and Technology,
Narmak, Tehran 16846-13114, Iran
rsaadati@eml.cc

[c]Faculty of Mathematics, University of Belgrade,
Studentski trg. 16, 11000 Beograd, Serbia
kadelbur@matf.bg.ac.rs

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Abstract. In this paper, inspired by the idea of Meir–Keeler contractive mappings, we introduce Meir–Keeler expansive mappings, say MKE, in order to obtain Krasnosel'skii-type fixed point theorems in Banach spaces. The idea of the paper is to combine the notion of Meir–Keeler mapping and expansive Krasnosel'skii fixed point theorem. We replace the expansion condition by the weakened MKE condition in some variants of Krasnosel'skii fixed point theorems that appear in the literature, e.g., in [T. Xiang, R. Yuan, A class of expansive-type Krasnosel'skii fixed point theorems, *Nonlinear Anal., Theory Methods Appl.*, 71(7–8):3229–3239, 2009].

Keywords: Meir–Keeler expansive, Meir–Keeler contractive, Krasnosel'skii fixed point theorem.

1 Introduction

As we know from the theory of fixed points, two classical and the most applicable results are Schauder's theorem and Banach contraction principle. In 1958, Krasnosel'skii combined them in order to consider the following fixed point problem (cf. [9] or [13, p. 31]):

$$Ay + By = y, \quad y \in \mathcal{M}, \tag{1}$$

where $\mathcal{M}$ is a subset of a Banach space X.

[1]Corresponding author.
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Theorem 1. *Let $\mathcal{M}$ be a nonempty, closed and convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into X so that*

(i) *$Ax + By \in \mathcal{M}$ for all $x, y \in \mathcal{M}$:*
(ii) *A is continuous and $A(\mathcal{M})$ is contained in a compact subset of X:*
(iii) *B is a contraction with constant $\alpha < 1$.*

Then there is $y \in \mathcal{M}$ with $Ay + By = y$.

Several real world problems and theoretical phenomena may be modelled in the setting of (1). Krasnosel'skii fixed point theorem, which reads as above, is an extensively utilizable tool for solving such equations. In particular, it is known that this theorem has a vast scope of applications to nonlinear integral equations of mixed type, as well as to differential and functional differential equations. It is worth mentioning that many problems in mathematical physics and population dynamics, which can be formulated in the form (1), may encounter the problem that the involved operator B does not satisfy the hypothesis of contraction. In this paper, we will try to overcome this problem by using a new condition for the mapping B.

There are diverse extensions of Krasnosel'skii fixed point theorem in the literature, and the operator B is often required to be contractive or expansive, or of any typical form involving control functions (see, e.g., [1, 3–8, 14, 17, 19] and the references therein). There are also a few results related with Meir–Keeler-type conditions used in Krasnosel'skii fixed point theorem (see, e.g., [11]).

The goal of this paper is to introduce the Meir–Keeler expansion and apply it as a novel constraint, which weakens the known expansion condition, in order to derive new results of Krasnosel'skii type.

Let us gather some auxiliary concepts, which will be utilized further on.

Definition 1. Let (X, d) be a metric space and $\mathcal{M} \subseteq X$. The mapping $B : \mathcal{M} \to X$ is said to be *expansive* if there exists a constant $\alpha > 1$ such that

$$d(Bx, By) \geqslant \alpha d(x, y), \quad x, y \in \mathcal{M}, \tag{2}$$

and it is called *contractive* if we utilize the inequality with opposite direction with a constant $\alpha < 1$. Further, we call B *strictly contractive* (resp. *strictly expansive*) if the corresponding inequality is strict and without α, that is, for any $x \neq y$ in X,

$$\begin{aligned} d(Bx, By) &< d(x, y) \quad \text{(strictly contractive)}, \\ d(x, y) &< d(Bx, By) \quad \text{(strictly expansive)}. \end{aligned}$$

In 1969, Meir and Keeler [10] established a fixed point theorem for self-mappings on a metric space (X, d) satisfying the following condition:

For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon \leqslant d(x, y) < \epsilon + \delta \quad \Longrightarrow \quad d(Bx, By) < \epsilon \tag{3}$$

for all $x, y \in X$. The mapping B satisfying (3) is called *Meir–Keeler contractive* (MKC for short). There are several results regarding the Meir–Keeler type contraction and its generalization (see, e.g., [12] and the references therein).

Theorem 2 [Meir–Keeler]. (*See* [10].) *Let* (X,d) *be a complete metric space. If* $B: X \to X$ *is a Meir–Keeler contraction, then* B *has a unique fixed point* $x^* \in X$; *moreover, for any* $x \in X$, *the sequence* $\{B^n x\}$ *converges to* x^*.

In the rest of this work, inspired by the idea of Meir–Keeler contraction and combining it with expansive mappings, we aim to introduce a new concept and obtain some variants of Krasnosel'skii fixed point theorem.

2 Krasnosel'skii-type fixed point theorems via MKE mappings

We start with the following definition.

Definition 2. Let (X,d) be a metric space and B be a self-mapping on X. Then B is called *Meir–Keeler expansive* (MKE for short) if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x, y \in X$,

$$\epsilon \leqslant d(Bx, By) < \epsilon + \delta \quad \Longrightarrow \quad d(x,y) < \epsilon. \tag{4}$$

Obviously, if B is an injective MKE mapping (resp. an injective MKC mapping) then it is strictly expansive (resp. strictly contractive) following the implication (4) (resp. (3)). Moreover, any MKC mapping is continuous, while MKE mapping might not be.

It is also worth mentioning that the MKE mappings are not necessarily injective. Indeed, consider a constant function, $Bx = c$ for all $x \in X$, which is obviously not injective. However, since the inequality $\epsilon \leqslant d(Bx, By) < \epsilon + \delta$ is never satisfied for positive ϵ, implication (4) is always true, and hence, these mappings are MKE.

Remark 1. If B is an MKE mapping, then B^n is of the same type for any $n \geqslant 1$. Indeed, suppose that, on the contrary, B^2 is not MKE, that is, for some $\epsilon_0 > 0$ and all $\delta > 0$, there are $x_\delta, y_\delta \in X$ satisfying the following:

$$\epsilon_0 \leqslant d\big(B^2 x_\delta, B^2 y_\delta\big) < \epsilon_0 + \delta \quad \text{but} \quad d(x_\delta, y_\delta) \geqslant \epsilon_0.$$

The first assertion above implies that $d(Bx_\delta, By_\delta) < \epsilon_0$, which, together with the second assertion above, yields that B is not MKE and that is a contradiction. Inductively, one can follow the same process for any $n \geqslant 1$ to conclude the desired claim.

Remark 2. If B is expansive with constant $\alpha > 1$, then B is an MKE mapping. Otherwise, there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, we have

$$\epsilon_0 \leqslant d(Bx_\delta, By_\delta) < \epsilon_0 + \delta \quad \text{and} \quad d(x_\delta, y_\delta) \geqslant \epsilon_0$$

for some $x_\delta, y_\delta \in X$. By taking $\delta \to 0$ one can see that

$$\begin{aligned} 0 < \epsilon_0 &= \liminf_{\delta \to 0} d(Bx_\delta, By_\delta) \leqslant \liminf_{\delta \to 0} d(x_\delta, y_\delta) \\ &\leqslant \frac{1}{\alpha} \liminf_{\delta \to 0} d(Bx_\delta, By_\delta), \end{aligned}$$

which is a contradiction. Similar statement holds for MKC mapping; that is, if B is contractive with constant $\alpha < 1$, then B is an MKC mapping.

We note that the converses of the statements mentioned above are not necessarily true.

From now on, we always suppose that (X, d) and $(E, \|\cdot\|)$ stand for a complete metric space and a Banach space, respectively. Now, let us present our first result.

Theorem 3. *Assume that $\mathcal{M} \subseteq X$ is a closed set, and let $B : \mathcal{M} \to X$ be an injective MKE mapping with $B(\mathcal{M}) \supseteq \mathcal{M}$. Then the mapping B possesses a unique fixed point x^*; moreover, for any $x \in \mathcal{M}$, the sequence $\{B^n x\}$ converges to x^*.*

Proof. First of all, since B is one-to-one and MKE, thus $B^{-1} : B(\mathcal{M}) \to \mathcal{M}$ is well defined and MKC. Indeed, let ϵ, δ be such that MKE condition for the operator B is fulfilled, and let $u, v \in B(\mathcal{M})$ be such that $\epsilon \leqslant d(u, v) < \epsilon + \delta$. If $x, y \in M$ are (uniquely determined) elements of M such that $u = Bx$, $v = By$, then, since B is MKE, we get that $d(B^{-1}u, B^{-1}v) = d(x, y) < \epsilon$. In particular, the restriction $B^{-1}|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is an MKC mapping. Since $\mathcal{M}$ is closed and hence complete, applying the Meir–Keeler's theorem, we see that there exists a unique point $x \in B(\mathcal{M})$ such that $B^{-1}x = x$; equivalently, there is a unique $x^* \in \mathcal{M}$ with $Bx^* = x^*$, and the conclusion follows. □

Remark 3. We note that Theorem 3 improves a result of Xiang [16], which requires the mapping B to be expansive (as a stronger condition).

Corollary 1. *Suppose that the mapping $B : X \to X$ is MKE and bijective. Then there exists a unique point $x^* \in X$ such that $Bx^* = x^*$. Furthermore, for any $x \in X$, the sequence $\{B^n x\}$ converges to x^*.*

Corollary 2. (*See* [15].) *Assume that the mapping $B : X \to X$ is expansive and onto. Then there exists a unique point $x^* \in X$ such that $Bx^* = x^*$.*

Now, combining the Meir–Keeler fixed point theorem and Corollary 1, we derive the following result.

Theorem 4. *Assume that B is a self-mapping on X. If one of the following assumptions is fulfilled*:

(i) *the mapping B is MKC*; *or*
(ii) *the mapping B is MKE and bijective,*

then there exists a unique point $x^* \in X$ such that $Bx^* = x^*$. Furthermore, for any $x \in X$, the sequence $\{B^n x\}$ converges to x^*.

Example 1. Suppose that $\lambda \in \mathbb{R}$, and B is the onto self-map on $\mathbb{R}$ with usual metric given by $Bx = x^3 + \varphi(x) + \lambda$, where φ is differentiable with $\varphi'(x) \geqslant 1$.

We first note that φ is defined in a way to support the surjectivity of B. Obviously, B is not necessarily expansive (since there is no way to find an $\alpha > 1$ satisfying inequality (2)), but for $x > y$, we get

$$\begin{aligned} |Bx - By| &= x^3 - y^3 + \varphi(x) - \varphi(y) > \varphi(x) - \varphi(y) \\ &\geqslant x - y = |x - y|. \end{aligned} \tag{5}$$

In other words, B is strictly expansive and one-to-one. We claim that B is MKE, too. Suppose the contrary. Then for some $\epsilon_0 > 0$ and for all $\delta > 0$, we have

$$\epsilon_0 \leqslant d(Bx_\delta, By_\delta) < \epsilon_0 + \delta \quad \text{and} \quad d(x_\delta, y_\delta) \geqslant \epsilon_0$$

for some distinct elements $x_\delta, y_\delta \in \mathbb{R}$. By taking $\delta \to 0$ one can see that

$$0 < \epsilon_0 = \lim_{\delta\to 0} \left|x_\delta^3 - y_\delta^3 + \varphi(x_\delta) - \varphi(y_\delta)\right| \leqslant \liminf_{\delta\to 0} |x_\delta - y_\delta|,$$

which, together with (5), implies that

$$0 < \epsilon_0 = \lim_{\delta\to 0} \left|x_\delta^3 - y_\delta^3 + \varphi(x_\delta) - \varphi(y_\delta)\right| = \lim_{\delta\to 0} |x_\delta - y_\delta|.$$

That is, $\lim_{\delta\to 0} |x_\delta - y_\delta|$ exists. Now, mean value theorem states that

$$0 < \epsilon_0 = \lim_{\delta\to 0} |x_\delta - y_\delta| = \lim_{\delta\to 0} |x_\delta - y_\delta| \left| \sum_{i=0}^{2} x_\delta^i y_\delta^{2-i} + \varphi'(z_\delta) \right|$$

for some appropriate $z_\delta \in \mathbb{R}$, which shows a contradiction. We remark that for any odd n and $x_\delta \neq y_\delta \in \mathbb{R}$,

$$0 < \frac{x_\delta^n - y_\delta^n}{x_\delta - y_\delta} = \sum_{i=0}^{n-1} x_\delta^i y_\delta^{n-i-1}.$$

Therefore, B is MKE, and by Corollary 1 there exists a unique point $x^* \in \mathbb{R}$ such that $x^* = (x^*)^3 + \varphi(x^*) + \lambda$.

Now, recall

Theorem 5 [Schauder fixed point theorem]. (*See* [13].) *Let K be a nonempty, closed and convex subset of a Banach space E. Let A be a continuous self-mapping on K such that $A(K)$ is contained in a compact subset of E. Then A has at least one fixed point in K.*

Using the Schauder's fixed point theorem with the help of MKE mappings, we present a variant of Krasnosel'skii-type fixed point theorem as follows. We will use the following denotation, which has similarly appeared and applied in several papers in the literature (see, e.g., [2, 6, 18]):

$$\mathcal{F} = \mathcal{F}(K; A, B) = \{x \in K\colon\ x = Bx + Ay \text{ for some } y \in K\},$$

where A, B will be specified later.

Theorem 6. *Suppose that $K \subseteq E$ is a nonempty closed convex subset, and let A and B map K into E so that*

(i) *A is continuous and $A(K)$ is included in a compact subset of E*;
(ii) *B is an injective MKE mapping having a closed graph in $\mathcal{F}$, and if $\{x_n\}$ is a sequence in $\mathcal{F}$ with $(I - B)x_n \to y$, then $\{x_n\}$ contains a convergent subsequence*;
(iii) *$a \in A(K)$ implies that $B(K) + a \supseteq K$.*

Then there exists a point $x^* \in K$ with $Ax^* + Bx^* = x^*$.

Proof. Concentrating on (ii) and (iii), for any $a \in A(K)$, one can observe that the mapping $B + a : K \to E$ enjoys the assumptions of Theorem 3. Thus, the equation $Bx + a = x$ possesses a unique solution $x = \eta(a) \in K$, i.e., for any $a \in A(K)$, we have

$$B\eta(a) + a = \eta(a).$$

Now, we show that $\eta : A(K) \to K$ satisfying

$$B\eta(a) + a = \eta(a) \quad \forall a \in A(K)$$

is a continuous mapping. Indeed, let $\{x_n\}$ be a sequence in $A(K)$ with $x_n \to x$ in $A(K)$. Since B is strictly expansive, then it is injective and $I - B$ is invertible. Now, setting $y_n := (I - B)^{-1}x_n$ and $y := (I - B)^{-1}x$, we get that $(I - B)y_n = x_n$ and $(I - B)y = x$, which means $y_n, y \in K \cap \mathcal{F}$ and

$$(I - B)y_n \to (I - B)y.$$

One derives from this convergence and (ii) that $y_{n_k} \to y_0$ for some $y_0 \in K$, where $\{y_{n_k}\}$ is a subsequence of $\{y_n\}$. We then deduce that

$$By_{n_k} \to y_0 - (I - B)y. \tag{6}$$

Since B is closed in $\mathcal{F}$, it follows from (6) that $y_0 - (I - B)y = By_0$, and so $y = y_0$ since $I - B$ is injective. Now a standard argument shows that $y_n \to y$. The continuity of η is therefore proved. In another way, since A is continuous on K, it yields that $\eta \circ A : K \to K$ is also continuous and $\eta(A(K))$ is included in a compact subset of E. By the Schauder's fixed point theorem, there is an $x^* \in K$ so that $\eta(A(x^*)) = x^*$. From the equation $Bx + a = x$ we conclude that

$$B\bigl(\eta\bigl(A(x^*)\bigr)\bigr) + A(x^*) = \eta\bigl(A(x^*)\bigr),$$

that is, $Bx^* + Ax^* = x^*$, and the desired consequence follows. $\square$

Remark 4. Considering Theorem 6, we notice that any expansive mapping satisfies condition (ii). More precisely, as we know from Remark 2, any expansive mapping is MKE; moreover, for any expansive mapping B with constant $\alpha > 1$, we have

$$\bigl\|(I - B)x - (I - B)y\bigr\| \geqslant (\alpha - 1)\|x - y\|,$$

which shows that $(I - B)^{-1}$ is continuous and the second part of (ii) is fulfilled. Besides, in the case that B is expansive, B would also have the closed graph in $\mathcal{F}$. Indeed, letting $x_n \to x$ in $\mathcal{F}$ and $Bx_n \to y$ we see that $\|Ay_n - (x - y)\| \to 0$ for some $\{y_n\}$ in K, which, together with the fact that $(I - B)^{-1}$ is continuous, implies that

$$x_n = (I - B)^{-1}Ay_n \to (I - B)^{-1}(x - y).$$

This is equivalent to $x = (I - B)^{-1}(x - y)$ and so $x = By$. Hence, we have as an immediate consequence the following main result of Xiang and Yuan.

Corollary 3. (*See* [17].) *Suppose $K \subseteq E$ is a nonempty closed convex subset. Let A, B map K into E so that*

(i) *A is continuous and $A(K)$ is included in a compact subset of E*;
(ii) *B is an expansive mapping*;
(iii) *$a \in A(K)$ implies $B(K) + a \supseteq K$.*

Then there exists a point $x^* \in K$ with $Ax^* + Bx^* = x^*$.

Corollary 4. *Suppose $K \subseteq E$ is a nonempty closed convex subset. Let A and B map K into E so that*

(i) *A is continuous and $A(K)$ is included in a compact subset of E*;
(ii) *B is MKE and bijective with a closed graph in $\mathcal{F}$, and if $\{x_n\}$ is a sequence in $\mathcal{F}$ with $(I - B)x_n \to y$, then $\{x_n\}$ contains a convergent subsequence.*

Then there exists a point $x^* \in K$ with $Ax^* + Bx^* = x^*$.

Theorem 7. *Suppose that $K \subseteq E$ is a nonempty closed convex subset. Let A and B map K into E so that*

(i) *A is continuous and $A(K)$ is included in a compact subset of E*;
(ii) *B is an injective MKE mapping and if $\{x_n\}$ is a sequence in $\mathcal{F}(K; A, B^{-1})$ with $(I - B^{-1})x_n \to y$, then $\{x_n\}$ contains a convergent subsequence*;
(iii) *$a \in A(K)$ implies that $K + a \subseteq K \subseteq B(K)$.*

Then there exists a point $x^* \in K$ with $B \circ (I - A)x^* = x^*$.

Proof. The proof is similar to the one given for Theorem 6. Following the lines of the proof of Theorem 3, one can see that $B^{-1} : B(K) \to K \subseteq B(K)$ is MKC and so is $B^{-1}|_K$. Moreover, for any fixed $a \in A(K)$, $B^{-1} + a : B(K) \to K$ has the same property. Thus, based on Meir–Keeler theorem, there exists a unique $x^* =: \eta(a)$ satisfying $B^{-1}x^* + a = x^*$; equivalently, $B^{-1}\eta(a) + a = \eta(a)$ for any $a \in A(K)$.

Now, we show that $\eta : A(K) \to K$ satisfying

$$B^{-1}\eta(a) + a = \eta(a) \quad \forall a \in A(K)$$

is continuous. Indeed, let $\{x_n\}$ be a sequence with $x_n \to x$ in $A(K)$. Since B^{-1} is strictly contractive, then $I - B^{-1}$ is invertible. Now, setting $y_n := (I - B^{-1})^{-1}x_n$ and $y := (I - B^{-1})^{-1}x$ implies that $(I - B^{-1})y_n = x_n$ and $(I - B^{-1})y = x$, which means $y_n, y \in K \cap \mathcal{F}(K; A, B^{-1})$ and

$$\bigl(I - B^{-1}\bigr)y_n \to \bigl(I - B^{-1}\bigr)y. \tag{7}$$

One derives from (7) and (ii) that $y_{n_k} \to y_0$ for some $y_0 \in K$, where $\{y_{n_k}\}$ is a subsequence of $\{y_n\}$. From the continuity of $I - B^{-1}$ we then deduce that

$$\bigl(I - B^{-1}\bigr)y_{n_k} \to \bigl(I - B^{-1}\bigr)y_0. \tag{8}$$

Since $I - B^{-1}$ is injective, (7) and (8) imply that $y = y_0$. Now a standard argument shows that $y_n \to y$. Therefore, $\eta : A(K) \to K$ is continuous.

On the other hand, since A is continuous on K, it yields that $\eta \circ A : K \to K$ is also continuous and $\eta(A(K))$ is included in a compact subset of E. By Schauder's fixed point theorem, there is an $x^* \in K$ so that $\eta(A(x^*)) = x^*$. From the equation $B^{-1}\eta(a) + a = \eta(a)$ we conclude that

$$B^{-1}\bigl(\eta\bigl(A(x^*)\bigr)\bigr) + A(x^*) = \eta\bigl(A(x^*)\bigr),$$

that is,

$$B^{-1}x^* + Ax^* = x^*,$$

and the desired result immediately follows. □

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