Nonlinear Analysis: Modelling and Control, Vol. 29, No. 6, 1120–1126
https://doi.org/10.15388/namc.2024.29.37850

Periodic orbits for an autonomous version of the Duffing–Holmes oscillator

Guangfeng Dong[a, 1], **Jaume Llibre**[b, 2]

[a]Department of Mathematics, Jinan University,
Guangzhou, 510632, China
donggf@jnu.edu.cn

[b]Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona, Catalonia, Spain
jaume.llibre@uab.cat

Received: February 3, 2024 / **Revised:** November 4, 2024 / **Published online:** December 1, 2024

Abstract. In the autonomous Duffing–Holmes oscillator, the existence of periodic orbits was detected numerically. Using the Hopf bifurcation theory, we prove analytically that such periodic orbits exist. We also provide the exact bifurcation value where the Hopf bifurcation takes place.

Keywords: Hopf bifurcation, Duffing–Holmes oscillator, period orbit.

1 Introduction and statement of the results

A classical nonlinear differential system with chaotic dynamics is the Duffing–Holmes nonautonomous oscillator

$$\ddot{x} + b\dot{x} - x + x^3 = a \sin \omega t.$$

This oscillator has been studied intensively; see [1, 2, 4–15]. In article [11], the authors considered the following autonomous version of Duffing–Holmes oscillator:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x + ay - bz - x^3, \quad a, b, \in \mathbb{R}, \\ \dot{z} &= c(y - z), \quad c \in \mathbb{R}. \end{aligned} \tag{1}$$

They constructed a specific electrical circuit to imitate solutions of oscillator (1) and obtained the simulation and experimental results and the corresponding electronic device. The irregular behavior of time series of the differential system (1) can lead to chaos.

[1]The first author is partially supported by the China Scholarship Council (No. 202306780018).
[2]The author is partially supported by the Agencia Estatal de Investigación grant PID2022-136613NB-100, the H2020 European Research Council grant MSCA-RISE-2017-777911, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Real Acadèmia de Ciències i Arts de Barcelona.

In the reference [9], the authors provided the analysis of the differential system (1) around the specific values of parameters $a = 19/10$, $b = 5/2$, and $c \in (2.5, 3.85) \subset \mathbb{R}$. In this case, system (1) has exactly three equilibrium points at $(-1, 0, 0)$, $(0, 0, 0)$, and $(1, 0, 0)$. They examined the characteristics of these points under the change of the variable c from 2.55 to 3.85 and got some numerical results as follows. When $c \in (2.55, 3.55]$, all three equilibrium points are nonattractive, and there are periodic orbits around the two side equilibrium points $(-1, 0, 0)$ and $(1, 0, 0)$.

In this paper, using the Hopf bifurcation theory, we provide a proof for the existence of isolated periodic orbits (also called *limit cycles*) around the equilibrium points $(\pm 1, 0, 0)$ found numerically. Namely, we have the following main result.

Theorem 1. *Assume that* $a = 19/10$, $b = 5/2$, *and* $c \in (2.5, 3.85)$. *At the equilibrium point* $O_1 = (1, 0, 0)$, *the differential system* (1) *has a Hopf bifurcation at* $c_0 = (\sqrt{26049} + 57)/60 \approx 3.64$. *More precisely, on the center manifold of the equilibrium point* O_1, *the following statements hold*:

(i) O_1 *is a stable strong focus without any small limit cycle around* O_1 *for* $c > c_0$.
(ii) O_1 *is a stable weak focus of order* 1 *for* $c = c_0$.
(iii) O_1 *is an unstable strong focus with a small limit cycle around* O_1 *for* $c < c_0$.

Note that the differential system (1) is invariant under the change

$$(x, y, z) \mapsto (-x, -y, -z),$$

so the conclusion of Theorem 1 is also valid for the equilibrium point $O_{-1} = (-1, 0, 0)$.

2 Preliminary results

To detect the Hopf bifurcation, the key point is to calculate the first Lyapunov coefficient. There are several methods to complete the computation. In this paper, we employ the method of the book [3] that we describe in what follows.

Consider a differential system in $\mathbb{R}^n$ or $\mathbb{C}^n$ with an equilibrium at $\mathbf{x} = 0$

$$\begin{aligned} \dot{\mathbf{x}} &= F(\mathbf{x}), \\ F(\mathbf{x}) &= \big(F_i(\mathbf{x})\big)^{\mathrm{T}} = A\mathbf{x} + \frac{1}{2}B(\mathbf{x}, \mathbf{x}) + \frac{1}{6}C(\mathbf{x}, \mathbf{x}, \mathbf{x}) + O\big(\|\mathbf{x}\|^4\big), \end{aligned}$$

where A is the Jacobian matrix, and $B(\mathbf{x}, \mathbf{y}) = (B_i(\mathbf{x}, \mathbf{y}))^{\mathrm{T}}$, $C(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (C_i(\mathbf{x}, \mathbf{y}, \mathbf{z}))^{\mathrm{T}}$ are multilinear functions, $\mathbf{x} = (x_1, \dots, x_n)^{\mathrm{T}}$, $\mathbf{y} = (y_1, \dots, y_n)^{\mathrm{T}}$, $\mathbf{z} = (z_1, \dots, z_n)^{\mathrm{T}}$. In coordinates, we have

$$\begin{aligned} B_i(\mathbf{x}, \mathbf{y}) &= \sum_{j,k=1}^{n} \frac{\partial^2 F_i(\mathbf{u})}{\partial u_j \partial u_k}\bigg|_{u=0} x_j y_k, \\ C_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= \sum_{j,k,l=1}^{n} \frac{\partial^3 F_i(\mathbf{u})}{\partial u_j \partial u_k \partial u_l}\bigg|_{u=0} x_j y_k z_l, \quad i = 1, 2, \dots, n, \end{aligned} \tag{2}$$

where $\mathbf{u} = (u_1, \dots, u_n)^{\mathrm{T}}$.

Assume that A has the unique pair of complex eigenvalues $\lambda_1 = \omega_0\mathrm{i}$ and $\lambda_2 = -\omega_0\mathrm{i}$, $\omega_0 > 0$, with zero real parts. Let $\mathbf{q} = (q_i)^{\mathrm{T}}$ be a unit complex eigenvector corresponding to λ_1, and let $\mathbf{p}$ be the adjoint eigenvector, i.e.,

$$A\mathbf{q} = \lambda_1\mathbf{q}, \qquad \overline{\mathbf{q}}^{\mathrm{T}}\mathbf{q} = 1,$$
$$A^{\mathrm{T}}\mathbf{p} = \lambda_2\mathbf{p}, \qquad \overline{\mathbf{q}}^{\mathrm{T}}\mathbf{p} = 1.$$

Then the first Lyapunov coefficient $l_1(0)$ can be obtained by formula (5.34) of the book [3], i.e.,

$$l_1(0) = \frac{1}{2\omega_0}\operatorname{Re}\bigl(\overline{\mathbf{p}}^{\mathrm{T}}C(\mathbf{q},\mathbf{q},\overline{\mathbf{q}}) - 2\overline{\mathbf{p}}^{\mathrm{T}}B\bigl(\mathbf{q}, A^{-1}B(\mathbf{q},\overline{\mathbf{q}})\bigr)\bigr) + \frac{1}{2\omega_0}\operatorname{Re}\bigl(\overline{\mathbf{p}}^{\mathrm{T}}B\bigl(\overline{\mathbf{q}}, (2\omega_0\mathrm{i}I - A)^{-1}B(\mathbf{q},\mathbf{q})\bigr)\bigr),$$

where $\operatorname{Re}(\cdot)$ represents the real part of a complex number.

For $n = 3$, consider the following differential system depending on one parameter α:

$$\dot{\mathbf{x}} = F(\mathbf{x}, \alpha), \quad \mathbf{x} \in \mathbb{R}^3,\ \alpha \in \mathbb{R}, \tag{3}$$

with an equilibrium point $\mathbf{x}_0 = \mathbf{0}$, whose eigenvalues are a pair of conjugate complex number $\lambda_{1,2} = u(\alpha) \pm v(\alpha)\mathrm{i}$ and one negative number λ_3 such that $u(0) = 0$ and $v(0) = \omega_0 > 0$.

If $u(\alpha)$ and the first Lyapunov coefficient $l_1(\alpha)$ satisfy that the derivative $u'(0) \neq 0$ and $l_1(0) \neq 0$, then there exists a smooth invariant manifold (called *center manifold*) depending on α for small $|\alpha|$. On the center manifold, the stability of the equilibrium changes when α changes passing through 0, which leads to the appearance or disappearance of the limit cycles. In fact, the differential system (3) is topologically equivalent to the following system:

$$\begin{aligned} \dot{x} &= \beta x - y + \sigma x\bigl(x^2 + y^2\bigr), \\ \dot{y} &= x + \beta y + \sigma y\bigl(x^2 + y^2\bigr), \\ \dot{z} &= -z, \end{aligned} \tag{4}$$

where $\beta = u(\alpha)/v(\alpha)$ and $\sigma = \operatorname{sign}(l_1(0)) = \pm 1$. In the invariant plane $z = 0$, the differential system (4) undergoes a supercritical (resp. subcritical) Hopf bifurcation when $\sigma = -1$ (resp. $\sigma = 1$). For supercritical Hopf bifurcation, the equilibrium point is a stable strong focus when $\beta < 0$, a stable weak focus when $\beta = 0$, and an unstable focus when $\beta > 0$, respectively. It is not difficult to see that there is a small limit cycle $x^2 + y^2 = \beta$ when $\beta > 0$ is sufficiently small, which tends to the equilibrium point as β decreases monotonically to 0 and disappears as $\beta < 0$. Note that a small limit cycle is invariant under the topological equivalence, so the differential system (3) also has a limit cycle when $\beta > 0$, which disappears as β decreases through 0.

The similar arguments show that when subcritical Hopf bifurcation takes place, the limit cycle occurs for $\beta < 0$ and disappears as β increases through 0. For more details on Hopf bifurcation, one can read Chapters 3 and 5 of the book [3].

3 Proof of Theorem 1

For the differential system (1), we perform a translation $(x, y, z) \mapsto (x-1, y, z)$ transforming the differential system (1) to the following system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -2x + ay - bz - 3x^2 - x^3,\\ \dot{z} &= c(y-z)\end{aligned} \tag{5}$$

and the equilibrium point $O_1 = (1, 0, 0)$ to the origin. For the differential system (5), denoting by $\mathbf{x} = (x, y, z)^{\mathrm{T}}$, we have

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -2 & a & -b \\ 0 & c & -c \end{pmatrix}$$

and

$$\begin{aligned}B(\mathbf{x}, \mathbf{x}) &= \bigl(0, -6x^2, 0\bigr)^{\mathrm{T}},\\ C(\mathbf{x}, \mathbf{x}, \mathbf{x}) &= \bigl(0, -6x^3, 0\bigr)^{\mathrm{T}}.\end{aligned}$$

When $a = 19/10$ and $b = 5/2$, the characteristic polynomial of A is

$$P(\lambda) = \lambda^3 - \left(\frac{19}{10} - c\right)\lambda^2 + \left(2 + \frac{3}{5}c\right)\lambda + 2c.$$

Assume that $P(\lambda)$ has a pair of nonreal eigenvalues $\lambda_{1,2} = \alpha \pm \omega\mathrm{i} \neq 0$, where $\omega \neq 0$ can be regarded as a function in α. Then $\alpha = 0$ if and only if $P(\pm\omega\mathrm{i}) = 0$, which is equivalent to the following equalities:

$$2c + \left(\frac{19}{10} - c\right)\omega^2 = 0, \qquad \left(2 + \frac{3}{5}c\right) - \omega^2 = 0.$$

Thus we obtain

$$30c^2 - 57c - 190 = 0. \tag{6}$$

That is, only when the positive number c takes the positive root

$$c_0 = \frac{\sqrt{26049} + 57}{60} \in (2.5, 3.85)$$

of Eq. (6), $P(\lambda)$ has a pair of imaginary eigenvalues $\lambda_{1,2} = \pm\omega_0\mathrm{i}$ and one real eigenvalue $\lambda_3 = (-\sqrt{26049} + 57)/60 < 0$, where $\omega_0 = \sqrt{\sqrt{26049} + 257}/10$.

One can get the unit complex eigenvector $\mathbf{q}$ corresponding to λ_1, i.e.,

$$\mathbf{q} = \begin{pmatrix} \dfrac{\sqrt{19}(1200 + \mathrm{i}(57\sqrt{\sqrt{26049}+257} - \sqrt{26049(\sqrt{26049}+257)}))}{5\sqrt{961-\sqrt{26049}}(\sqrt{26049}+57)} \\ \dfrac{4\sqrt{19}(57+\sqrt{26049}+6\mathrm{i}\sqrt{\sqrt{26049}+257})}{\sqrt{961-\sqrt{26049}}(\sqrt{26049}+57)} \\ \dfrac{4\sqrt{19}}{\sqrt{961-\sqrt{26049}}} \end{pmatrix}$$

and the corresponding adjoint eigenvector

$$\mathbf{p} = \begin{pmatrix} \frac{\sqrt{961-\sqrt{26049}}(\sqrt{26049}+57)(1200-\mathrm{i}(57\sqrt{\sqrt{26049}+257}-\sqrt{26049(\sqrt{26049}+257)}))}{960(-1129\sqrt{19}+19\sqrt{1371}(3+\mathrm{i}\sqrt{\sqrt{26049}+257}))} \\ -\frac{\sqrt{961-\sqrt{26049}}(\sqrt{26049}+57)(57+\sqrt{26049}-6\mathrm{i}\sqrt{\sqrt{26049}+257})}{96(-1129\sqrt{19}+19\sqrt{1371}(3+\mathrm{i}\sqrt{\sqrt{26049}+257}))} \\ \frac{25\sqrt{961-\sqrt{26049}}(\sqrt{26049}+57)}{16(-1129\sqrt{19}+19\sqrt{1371}(3+\mathrm{i}\sqrt{\sqrt{26049}+257}))} \end{pmatrix}.$$

So we have the following equations by formulae (2):

$$\begin{aligned} C(\mathbf{q},\mathbf{q},\overline{\mathbf{q}}) &= \bigl(0,-6q_1^2\overline{q_1},0\bigr)^{\mathrm{T}}, \\ B(\mathbf{q},\mathbf{q}) &= \bigl(0,-6q_1^2,0\bigr)^{\mathrm{T}}, \\ B(\mathbf{q},\overline{\mathbf{q}}) &= (0,-6q_1\overline{q_1},0)^{\mathrm{T}}, \\ B\bigl(\mathbf{q},A^{-1}B(\mathbf{q},\overline{\mathbf{q}})\bigr) &= \bigl(0,-6q_1^2\overline{q_1}A_{12}^{-1},0\bigr)^{\mathrm{T}}, \\ B\bigl(\overline{\mathbf{q}},(2\omega_0\mathrm{i}I-A)^{-1}B(\mathbf{q},\mathbf{q})\bigr) &= \bigl(0,-6q_1^2\overline{q_1}(2\omega_0\mathrm{i}I-A)_{12}^{-1},0\bigr)^{\mathrm{T}}, \end{aligned}$$

where q_1 is the first component of $\mathbf{q}$, $A_{12}^{-1} = -1/2$ is the element at the first row and the second column in A^{-1},

$$\begin{aligned} &(2\omega_0\mathrm{i}I-A)_{12}^{-1} \\ &\quad= \frac{25(-12\sqrt{\sqrt{26049}+257}+\mathrm{i}(\sqrt{26049}+57))}{3(771\sqrt{\sqrt{26049}+257}+3\sqrt{26049(\sqrt{26049}+257)}-50\mathrm{i}(\sqrt{26049}+57))} \end{aligned}$$

is the element at the first row and the second column in $(2\omega_0\mathrm{i}I-A)^{-1}$, and I is the identity matrix of order 3.

Finally, we obtain the first Lyapunov coefficient for $\alpha = 0$ as follows:

$$\begin{aligned} l_1(0) &= -\frac{1}{2\omega_0}\,\mathrm{Re}\bigl(6\overline{p_2}q_1^2\overline{q_1}\bigl(2+(2\omega_0\mathrm{i}I-A)_{12}^{-1}\bigr)\bigr) \\ &= -\frac{\sqrt{961-\sqrt{26049}}(\sqrt{26049}+57)^2}{383794\sqrt{26049}+67795058} \approx -0.0103959, \end{aligned}$$

where p_2 is the second component of $\mathbf{p}$. In addition, the derivative of $\alpha = \mathrm{Re}(\lambda_{1,2})$ with respect to c at $c = c_0$ is equal to

$$\begin{aligned} &\frac{(-8617329501\sqrt{2}\sqrt{19}+54580536746)\sqrt{1371}+168911609193\sqrt{2}-54483320262\sqrt{19}}{(24309705600\sqrt{2}\sqrt{19}-154613493438)\sqrt{1371}+51422299200\sqrt{2}-11733728814\sqrt{19}} \\ &\quad\approx -0.36 < 0. \end{aligned}$$

Thus the real part of nonreal eigenvalues $\lambda_{1,2}$ is positive for $c<c_0$ and negative for $c>c_0$. This means that a supercritical Hopf bifurcation takes place for parameter $\alpha = \mathrm{Re}(\lambda_{1,2})$

when $\alpha = 0$ at the equilibrium point $O_1 = (1, 0, 0)$. More precisely, on the center manifold, the equilibrium point O_1 is a stable strong focus without any period orbits around O_1 for $c > c_0$ and is an unstable strong focus for $c < c_0$ with a small limit cycle around O_1. While for $c = c_0$, it is a stable weak focus of order 1. The proof is finished.

The same Hopf bifurcation also takes place at the equilibrium point $O_{-1} = (-1, 0, 0)$ under the symmetry $(x, y, z) \to (-x, -y, -z)$. So there is also a limit cycle around O_{-1} when $c < c_0$. That is, the two limit cycles around $O_{\pm 1} = (\pm 1, 0, 0)$ respectively appear simultaneously as $c < c_0$ and disappear simultaneously as $c > c_0$.

4 Conclusions

In this paper, we prove the numerical results of [9] by applying the theory of Hopf bifurcation, which provide the existence of two period orbits surrounding the equilibrium points $(\pm 1, 0, 0)$ respectively when $c < 3.55$. Furthermore, we obtain the exact bifurcation value c_0, which is close to 3.64.

Author contributions. All authors (G.D. and J.L.) contributed equally to this work. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

Acknowledgment. The authors thank the reviewers for their useful suggestions and comments.

References

1. B.K. Jones, G. Trefan, The duffing oscillator: A precise electronic analog chaos demonstrator for the undergraduate laboratory, *Am. J. Phys.*, **69**(4):464–469, 2001, https://doi.org/10.1119/1.1336838.

2. A. Kandangath, S. Krishnamoorthy, Y.C. Lai, J.A. Gaudet, Inducing chaos in electronic circuits by resonant perturbations, *IEEE Trans. Circuits Syst. I, Regul. Pap.*, **54**(5):1109–1119, 2007, https://doi.org/https://doi.org/10.1109/TCSI.2007.893510.

3. Yu.A. Kuznetsov, *Elements of Applied Bifurcation Theory,* 4th ed., Appl. Math. Sci., Springer, Cham, 4 edition, 2023, https://doi.org/10.1007/978-3-031-22007-4.

4. A. Namajūnas, A. Tamaševičius, An optoelectronic technique for estimating fractal dimensions from dynamical Poincaré maps, in *Proceedings of the Second IFIP Working Conference on Fractals in the Natural and Applied Sciences*, IFIP Trans. A, Comput. Sci. Technol., Vol. A-41, North-Holland, Amsterdam, 1994, pp. 289–293.

5. A. Namajūnas, A. Tamaševičius, Simple laboratory instrumentation for measuring pointwise dimensions from chaotic time series, *Rev. Sci. Instrum.*, **65**(9):3032–3033, 1994, https://doi.org/10.1063/1.1144599.

6. A. Namajūnas, A. Tamaševičius, A. Čenys, A.N. Anagnostopoulos, Whitening power spectra of chaotic signals, in *Proceedings of the 7th International Workshop on Nonlinear Dynamics of Electronic Systems, Rønne, Denmark, July 15–17, 1999*, World Scientific, Singapore, 1999, pp. 137–140.
7. A. Namajūnas, A. Tamaševičius, G. Mykolaitis, A. Čenys, Smoothing chaotic spectrum of nonautonomous oscillator, *Nonlinear Phenom. Complex Syst., Minsk*, **3**(2):188–191, 2000.
8. A. Namajūnas, A. Tamaševičius, G. Mykolaitis, A. Čenys, Spectra transformation of chaotic signals, *Lith. J. Phys.*, **40**:134–139, 2000.
9. F. Sadyrbaev, I. Samuilik, From Duffing equation to bio-oscillations, in C.M.A. Pinto, C.M. Ionescu (Eds.), *Computational and Mathematical Models in Biology*, Nonlinear Syst. Complex., Vol. 38, Springer, Cham, 2023, pp. 159–182, https://doi.org/10.1007/978-3-031-42689-6_7.
10. C.P. Silva, A.M. Young, High frequency anharmonic oscillator for the generation of broadband deterministic noise, Google Patents, 2000.
11. A. Tamaševičius, S. Bumelienė, R. Kirvaitis, G. Mykolaitis, E. Tamaševičiūtė, E. Lindberg, Autonomous Duffing–Holmes type chaotic oscillator, *Elektronika ir Elektrotechnika*, **93**(5):43–46, 2009.
12. A. Tamaševičius, G. Mykolaitis, V. Pyragas, K. Pyragas, Delayed feedback control of periodic orbits without torsion in nonautonomous chaotic systems: Theory and experiment, *Phys. Rev. E (3)*, **76**(2):026203, 2007, https://doi.org/10.1103/PhysRevE.76.026203.
13. A. Tamaševičius, E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, Stabilization of unstable periodic orbit in chaotic duffing-holmes oscillator by second order resonant negative feedback, *Lith. J. Phys.*, **47**(3):235–239, 2007.
14. A. Tamaševičius, E. Tamaševičiūtė, G. Mykolaitis, S. Bumelienė, Switching from stable to unknown unstable steady states of dynamical systems, *Phys. Rev. E (3)*, **78**(2):026205, 2008, https://doi.org/10.1103/PhysRevE.78.026205.
15. E. Tamaševičiūtė, A.V. Tamaševičius, G. Mykolaitis, S. Bumelienė, E. Lindberg, Analogue electrical circuit for simulation of the duffing-holmes equation, *Nonlinear Anal. Model. Control*, **13**(2):241–252, 2008, https://doi.org/https://doi.org/10.15388/NA.2008.13.2.14582.