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A hybrid fixed point theorem for product of two operators in a lattice-ordered Banach algebra with applications to quadratic integral equations

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Abstract. We prove a hybrid fixed point theorem for the product of two operators in a lattice-ordered Banach algebra and apply to nonlinear hybrid quadratic integral equations of mixed type for proving the existence of maximal and minimal positive integrable solutions under certain mixed conditions of Lipschitzicity and monotonicity of the nonlinear functions. Our main existence result is illustrated with a numerical example as well as with an application to IVPs of nonlinear first-order discontinuous quadratically perturbed ordinary differential equations.

Keywords: lattice-ordered Banach algebra, hybrid fixed point principle, hybrid quadratic integral equation, extremal integrable solutions.

1 Introduction

It is well known that the existence theory of nonlinear problems constitutes the major and core part of the nonlinear analysis. Therefore, several operator theoretic techniques are developed and used from the subject of nonlinear functional analysis for the purpose. The existence theorems for nonlinear both continuous and discontinuous standard, that is, nonlinearly perturbed linear differential and integral equations are available in the literature and obtained under continuity, Caratheodory- and Chandrabhan-type conditions by using the classical analytic, topological, and algebraic fixed point principles, respectively. See Granas and Dugundji [16], Heikkilä and Lakshmikantham [18], Dhage [3–6] and references therein. But the case with nonlinear hybrid differential equations is different, and existence results have been obtained only for continuous hybrid differential and integral equations by using the hybrid fixed point principles on the lines of Krasnoselskii [19] and Dhage [4–6, 10, 11], and no existence result is so far proved for nonlinear discontinuous hybrid differential and integral equations. Therefore, it is of interest to obtain existence

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results for discontinuous hybrid differential and integral equations, which is the main motivation of the present work. In the present discussion, we establish a hybrid fixed point theorem for product of two operators in a lattice-ordered Banach algebra and apply to nonlinear discontinuous quadratically perturbed hybrid integral equations of mixed type for proving the existence of maximal and minimal integrable solutions. Before going to the main hybrid fixed point theorems, we give some preliminaries, which we needed in what follows.

2 Preliminaries

Let E denote a Banach algebra with norm $\|\cdot\|$. We introduce an partial-order relation $\preccurlyeq$ in E so that the partially ordered set $(E, \preccurlyeq)$ becomes a lattice. Thus, the triplet $(E, \|\cdot\|, \preccurlyeq)$ is called a partially lattice-ordered Banach algebra. If the partial-order relation $\preccurlyeq$ is defined by the positive cone K in the partially ordered Banach algebra E, then the triplet $(E, \|\cdot\|, \preccurlyeq)$ is simply called a lattice-ordered Banach algebra. Note that a nonempty closed convex subset K of the Banach algebra E is called a cone if it satisfies

(i) $K + K \subseteq K$,
(ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$ with $\lambda > 0$, and
(iii) $\{-K\} \cap K = \{0\}$.

Again, K is called positive cone if

(iv) $K \circ K \subseteq K$.

Moreover, if the norm $\|\cdot\|$ satisfies the property that $\|x\| \leqslant \|y\|$ whenever $x, y \in K$ with $x \preccurlyeq y$, then $(E, \|\cdot\|, K)$ is called a Banach lattice algebra. It is known that every Banach lattice algebra is lattice-ordered Banach algebra, but the converse is not necessarily true. Below we state a couple of classical fixed point theorems, which we need in what follows. To state first analytical fixed point theorem, we need the following definitions.

Definition 1. A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *$\mathcal{D}$-function* if it is upper semicontinuous and nondecreasing satisfying $\psi(0) = 0$. The set of all $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.

Definition 2. An operator $\mathcal{T} : E \to E$ is said to be a *$\mathcal{D}$-Lipschitzian* if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}} \in \mathfrak{D}$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leqslant \psi_{\mathcal{T}}\big(\|x - y\|\big)$$

for all $x, y \in E$. If $\psi_{\mathcal{T}}(r) = kr$, then $\mathcal{T}$ is called a *Lipschitz* operator on E with Lipschitz constant k. Again, if $0 \leqslant k < 1$, $\mathcal{T}$ is called a *linear contraction* operator on E with contraction constant k. Further, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a *nonlinear $\mathcal{D}$-contraction* on E with contraction $\mathcal{D}$-function $\psi_{\mathcal{T}}$.

Our first analytical or geometrical fixed point theorem is as follows.

Theorem 1. (*See* [5, 6].) *Let E be a Banach space, and let $\mathcal{A} : E \to E$ be a nonlinear $\mathcal{D}$-contraction. Then $\mathcal{A}$ has a unique fixed point ξ^*, and the sequence $\{\mathcal{A}^n(x)\}_{n=0}^{+\infty}$ of successive iterations converges to ξ^* for each $x \in E$.*

To state second algebraic fixed point theorem, we need the following definition.

Definition 3. A mapping $\mathcal{T}$ on a lattice $(L, \preccurlyeq)$ into itself is called *isotone increasing* if it preserve the partial-order relation $\preccurlyeq$, that is, if $x, y \in L$ with $x \preccurlyeq y$, then $\mathcal{T}x \preccurlyeq \mathcal{T}y$.

Theorem 2. (*See* [20].) *Let $(L, \preccurlyeq)$ be a partially ordered set, and let $T : L \to L$ be a mapping. Suppose that*

(a) *$\mathcal{T}$ is isotone increasing, and*
(b) *$(L, \preccurlyeq)$ is a complete lattice.*

Then $F_{\mathcal{T}} = \{u \in L\colon \mathcal{T}u = u\} \neq \emptyset$, and $(F_{\mathcal{T}}, \preccurlyeq)$ is a complete lattice.

Below in the present paper, we combine Theorems 1 and 2 and prove a hybrid fixed point theorem involving the product of two operators satisfying hybrid, that is, mixed Lipschitz- and isotonicity-type conditions in a partially lattice-ordered Banach algebra.

3 Hybrid fixed point principles

The hybrid fixed point theory involving the sum and product of two continuous operators using and without using the partial order appears in Dhage [10, 11] and references therein. In this section, we discuss the hybrid fixed point principles for product of the two continuous and discontinuous operators in a partially lattice-ordered Banach algebra. Throughout the rest of this paper, unless otherwise mentioned, let E denote the partially lattice-ordered Banach algebra. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two operators and consider the hybrid operator equation

$$\mathcal{A}x\,\mathcal{B}x = x. \tag{1}$$

If the operators $\mathcal{A}$ and $\mathcal{B}$ are positive and isotone increasing, then the operator $\mathcal{T}$ defined by $\mathcal{T}x = \mathcal{A}x\,\mathcal{B}x$ is isotone increasing, and by Theorem 2, the operator equation (1) has a solution, and the set of all solutions is a complete lattice. Note that an operator $\mathcal{T}$ on E is positive if $\mathcal{T}x \succeq 0$, $\mathcal{T}x \neq 0$ for all $x \in E$. Therefore, it is of interest to prove the solution of the operator equation (1) when both the operators $\mathcal{A}$ and $\mathcal{B}$ are not positive and isotone increasing on E.

Theorem 3. *Let $(E, \|\cdot\|, K)$ be a partially lattice-ordered Banach algebra, and let $\mathcal{A}, \mathcal{B} : E \to E$ be two nonlinear operators. Suppose that*

(a) *$\mathcal{A}$ is nonlinear $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,*
(b) *$(I/\mathcal{A})^{-1}$ exists, where $(I/\mathcal{A})x = x/(\mathcal{A}x)$, $\mathcal{A}x \neq 0$, and I is the identity operator on E,*
(c) *$(I/\mathcal{A})^{-1}\mathcal{B}$ is isotone increasing,*
(d) *$\mathcal{B}$ is bounded with bound $M_{\mathcal{B}} = \sup\{\|\mathcal{B}x\|,\ x \in E\}$,*

(e) $M_{\mathcal{B}}\,\psi_{\mathcal{A}}(r) < r$, $r > 0$, *and*
(f) $(E, \preccurlyeq)$ *is a complete lattice.*

Then $F_{\mathcal{AB}} = \{u \in E\colon \mathcal{A}u\,\mathcal{B}u = u\} \neq \emptyset$, *and* $(F_{\mathcal{AB}}, \preccurlyeq)$ *is a complete lattice.*

Proof. If the operator $\mathcal{A}$ or $\mathcal{B}$ is zero, then zero vector is the solution of the operator equation (1) trivially. Therefore, we assume that none of $\mathcal{A}$ and $\mathcal{B}$ is zero on E. Define an operator $\mathcal{T}$ on E by

$$\mathcal{T} = \left(\frac{I}{\mathcal{A}}\right)^{-1}\mathcal{B}. \tag{2}$$

We show that $\mathcal{T}$ is well defined and maps E into itself. Let $y \in E$ be fixed element and define an operator A_y on E by

$$\mathcal{A}_y(x) = \mathcal{A}x\,\mathcal{B}y.$$

Let $x_1, x_2 \in E$ be arbitrary. Then from (3) and hypothesis (a), it follows that

$$\bigl\|\mathcal{A}_y(x_1) - \mathcal{A}_y(x_2)\bigr\| \leqslant \bigl\|\mathcal{A}x_1 - \mathcal{A}x_2\bigr\|\bigl\|\mathcal{B}(y)\bigr\| \leqslant M_{\mathcal{B}}\,\psi_{\mathcal{A}}\bigl(\|x_1 - x_2\|\bigr),$$

where $\psi_{\mathcal{A}}$ is a $\mathcal{D}$-function satisfying $M_{\mathcal{B}}\psi_{\mathcal{A}}(r) < r$, $r > 0$. This shows that $\mathcal{A}_y$ is a nonlinear $\mathcal{D}$-contraction operator on E. Now, by an application of Theorem 1, there is a unique point $y' \in E$ such that

$$\mathcal{A}_y(y') = y' \quad \text{or} \quad \mathcal{A}y'\mathcal{B}y = y'. \tag{3}$$

From (3) we obtain

$$\mathcal{B}y = \frac{y'}{\mathcal{A}y'} = \left(\frac{I}{\mathcal{A}}\right)y'. \tag{4}$$

Now, by hypothesis (b), $(I/\mathcal{A})^{-1}$ exists, so operating with $(I/\mathcal{A})^{-1}$ on both sides of the operator equation (4), we get

$$\left(\frac{I}{\mathcal{A}}\right)^{-1}\mathcal{B}y = y' \quad \text{or} \quad \mathcal{T}y = y'.$$

Therefore, the mapping $\mathcal{T}$ is well defined and maps E into itself. By hypotheses (c), $\mathcal{T}$ is isotone increasing on the complete lattice E into itself. Now the desired conclusion follows by an application of Theorem 2. □

Remark 1. We note that the operator $\mathcal{T} = (I/\mathcal{A})^{-1}\mathcal{B}$ is isotone increasing if the operators $\mathcal{A}$ and $\mathcal{B}$ are positive and isotone increasing on E, however, the converse may not be true. Now, the operator $\mathcal{T}$ is positive on E if $\mathcal{T}x \in K$ for each $x \in E$. To see the above assertion, let

$$\left(\frac{I}{\mathcal{A}}\right)^{-1}\mathcal{B}x = x_1 \quad \text{and} \quad \left(\frac{I}{\mathcal{A}}\right)^{-1}\mathcal{B}y = y_1. \tag{5}$$

Form first expression in (5) we obtain

$$\mathcal{B}x = \left(\frac{I}{\mathcal{A}}\right)x_1 \quad \Longrightarrow \quad \mathcal{B}x = \frac{x_1}{\mathcal{A}x_1}.$$

Therefore, we obtain

$$\mathcal{A}x_1\mathcal{B}x = x_1. \tag{6}$$

Similarly, we have

$$\mathcal{A}y_1\mathcal{B}y = y_1. \tag{7}$$

Let $x \preccurlyeq y$. Then from (6) and (7) we obtain

$$x_1 \preccurlyeq y_1 \iff \mathcal{A}x_1\mathcal{B}x \preccurlyeq \mathcal{A}y_1\,\mathcal{B}y,$$

and the later inequality holds if both $\mathcal{A}$ and $\mathcal{B}$ are positive and isotone increasing on E (see Dhage [4]).

Next, we prove a hybrid fixed point theorem for product of two operators in a partially lattice-ordered Banach algebra, which is applicable to nonlinear integral equations of mixed type for proving various aspects of the integrable solutions.

Theorem 4. *Let S be a nonempty subset of a lattice-ordered Banach algebra $(E, \|\cdot\|, K)$, and let $\mathcal{A} : E \to K$ and $\mathcal{B} : S \to K$ be two nonlinear isotone increasing operators. Suppose that*

(a) *$\mathcal{A}$ is nonlinear $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,*
(b) *$(I/\mathcal{A})^{-1}$ exists, where $(I/\mathcal{A})x = x/\mathcal{A}x$, $\mathcal{A}x \neq 0$, and I is the identity operator on E,*
(c) *$\mathcal{B}$ is bounded with bound $M_{\mathcal{B}} = \sup\{\|\mathcal{B}x\|,\, x \in E\}$,*
(d) *$M_{\mathcal{B}}\,\psi_{\mathcal{A}}(r) < r$, $r > 0$,*
(e) *$\mathcal{A}x\,\mathcal{B}y = x \;\Rightarrow\; x \in S$ for all $y \in S$, and*
(f) *$(S, \preccurlyeq)$ is a complete lattice.*

Then $F_{\mathcal{A}\mathcal{B}} = \{u \in E\colon \mathcal{A}u, \mathcal{B}u = u\} \neq \emptyset$, and $(F_{\mathcal{A}\mathcal{B}}, \preccurlyeq)$ is a complete lattice.

Proof. If the operator $\mathcal{A}$ or $\mathcal{B}$ is zero, then zero vector is the solution of the operator equation (1). Therefore, we assume that none of $\mathcal{A}$ and $\mathcal{B}$ is zero on their respective domains of definition. Let $y \in S$ be fixed element and consider the operator $\mathcal{A}_y$ on E defined by

$$\mathcal{A}_y(x) = \mathcal{A}x\,\mathcal{B}y.$$

Then proceeding as in the proof of Theorem 3, it is proved that there exists a unique element $y' \in E$ such that

$$\mathcal{A}_y(y') = \mathcal{A}y'\mathcal{B}y = y'.$$

Now, by hypothesis (d), we have that $y' \in S$. Next, we consider an operator $\mathcal{T}$ on S defined by (2), which maps an element y of S into the unique fixed point y' of the operator $\mathcal{A}_y$, that is, $\mathcal{T}y = y'$. It is easy to see that $\mathcal{T}$ is well defined and maps S into itself. Now proceeding as in the proof of Theorem 3, it is shown that $\mathcal{T}$ is isotone increasing on S, and $(S, \preccurlyeq)$ is a complete lattice in view of hypothesis (e). Hence, the desired conclusion follows by an application of Theorem 2. □

Remark 2. Condition (e) is weaker than

(c′) $\mathcal{A}x\,\mathcal{B}y \in S$ for all $y \in S$, which is generally used in the hybrid fixed point theorems on the lines of Krasnoselskii [19] and Dhage [5]. Also, see Dhage [10] and references therein.

The following result easily follows from the definitions of positive cone K and the closure of the subsets of Banach algebra E.

Lemma 1. *A nonempty, closed, and bounded subset S of a complete lattice-ordered Banach algebra $(E, \|\cdot\|, K)$ is a complete lattice.*

Theorem 5. *Let S be a nonempty, closed, and bounded subset of a complete lattice-ordered Banach algebra $(E, \|\cdot\|, K)$, and let $\mathcal{A} : E \to K$ and $\mathcal{B} : S \to K$ be two nonlinear isotone increasing operators. Suppose that*

(a) *$\mathcal{A}$ is nonlinear $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,*
(b) *$(I/\mathcal{A})^{-1}$ exists, where $(I/\mathcal{A})x = x/\mathcal{A}x$, $\mathcal{A}x \neq 0$, and I is the identity operator on E,*
(c) *$\mathcal{B}$ is bounded with bound $M_{\mathcal{B}} = \sup\{\|\mathcal{B}x\|, x \in E\}$,*
(d) *$M_{\mathcal{B}}\, \psi_{\mathcal{A}}(r) < r$, $r > 0$, and*
(e) *$\mathcal{A}x\, \mathcal{B}y = x \Rightarrow x \in S$ for all $y \in S$.*

Then the hybrid operator equation (1) *has a positive solution in S, and the set of all positive solutions in S is a complete lattice.*

Proof. Since S is a closed and bounded subset of the complete lattice-ordered Banach algebra E, by Lemma 1, $(S, \preccurlyeq)$ is a complete lattice. Moreover, from hypothesis (d) it follows that $S \cap K \neq \emptyset$ because K is a positive cone in E. Thus, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4. Hence, the hybrid operator equation (1) has a positive solution in S and the set of all positive solutions in S is a complete lattice. □

The above Theorem 5 is an improvement of the following new Dhage-type hybrid fixed point theorem involving the product of two continuous and discontinuous operators in a lattice-ordered Banach algebra.

Corollary 1. *Let S be a nonempty, closed and bounded subset of a complete lattice-ordered Banach algebra $(E, \|\cdot\|, K)$ and let $\mathcal{A} : E \to K$ and $\mathcal{B} : S \to K$ be two nonlinear isotone increasing operators. Suppose that*

(a) *$\mathcal{A}$ is Lipschitz operator with Lipschitz constant α,*
(b) *$(I/\mathcal{A})^{-1}$ exists, where $(I/\mathcal{A})x = x/\mathcal{A}x$, $\mathcal{A}x \neq 0$, and I is the identity operator on E,*
(c) *$\mathcal{B}$ is bounded with bound $M_{\mathcal{B}} = \sup\{\|\mathcal{B}x\|, x \in E\}$,*
(d) *$\alpha M_{\mathcal{B}} < 1$, and*
(e) *$\mathcal{A}x\, \mathcal{B}y = x \Rightarrow x \in S$ for all $y \in S$.*

Then the hybrid operator equation (1) *has a positive solution in S, and the set of all positive solutions in S is a complete lattice.*

In the following section, we consider the nonlinear hybrid integral equation of quadratic type for the application of Corollary 1 under suitable conditions.

4 Hybrid quadratic integral equations

The origin of the quadratic differential equations appears in the works of Dhage [4] and discussed for different aspects of the solutions via operator theoretic techniques in Banach algebra developed by Dhage [4–7]. In the beginning, the development of the subject was slow, but recently, this topic has gained momentum and growing very rapidly. Several fixed point principles involving the sum and product of two and three operators in Banach algebras have been formulated by Dhage for this purpose, and since then, several nonlinear quadratic differential and integral equations have been studied in the literature for existence and other aspects of the solutions. Here we discuss a nonlinear hybrid quadratic integral equation (in short, HQIE) for minimal and maximal integrable positive solutions under mixed Lipschitz and Chandrabhan conditions on the nonlinearities involved in the problem.

Given a closed and bounded interval $J = [0, T]$ in the real line $\mathbb{R}$, we consider the nonlinear HQIE of the type

$$x(t) = \big[f\big(t, x(t)\big)\big]\left(q(t) + \int_0^t g\big(s, x(s)\big)\,\mathrm{d}s\right), \quad t \in J, \tag{8}$$

where the functions $q : J \to \mathbb{R}$, $f, g : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain hybrid, that is, mixed conditions of Lipschitz and Chandrabhan to be specified later.

Definition 4. A function $x \in L^1(J, \mathbb{R})$ is said to be a solution of the HQIE (1) if it satisfies the equations in (1) on J, where $L^1(J, \mathbb{R})$ is the space of all Lebesgue integrable function defined on J.

We place problem (8) in the function space $BM(J, \mathbb{R})$ of real-valued measurable and bounded functions defined on J. The multiplication "$\cdot$" of two elements $x, y \in BM(J, \mathbb{R})$ is defined by

$$(x \cdot y)(t) = x(t)y(t), \quad t \in J.$$

We define a supremum norm $\|\cdot\|$ in $BM(J, \mathbb{R})$ by

$$\|x\|_{BM} = \sup_{t \in J} \big|x(t)\big|. \tag{9}$$

Clearly, $BM(J, \mathbb{R})$ is a Banach algebra with respect to the above multiplication and the norm in it. Next, we introduce an partial-order relation $\preccurlyeq$ in $BM(J, \mathbb{R})$ with the help of the positive cone K given by

$$K = \big\{x \in BM(J, \mathbb{R})\colon\ x(t) \geqslant 0 \text{ for almost all } t \in J\big\}.$$

Thus,

$$x \preccurlyeq y \iff y - x \in K, \tag{10}$$

r equivalently,

$$x \preccurlyeq y \iff x(t) \leqslant y(t) \quad \text{a.e. } t \in J. \tag{11}$$

The details of cones and partial-order relations appear in Guo and Lakshmikantham [17] and Heikkilä and Lakshmikantham [18]. It is known that $BM(J,\mathbb{R})$ is a complete lattice-ordered Banach algebra with respect to the norm and partial-order relation defined by (9) and (11), respectively. See Birkhoff [2] and Dhage [3] and references therein. Here we do not assume both the nonlinear functions involved in Eq. (8) to be continuous, but satisfy certain measurability and integrability conditions. We need the following definition in what follows (see Dhage [8] and Dhage and Patil [14]).

Definition 5. A function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to be *Chandrabhan* if

(i) the map $t \mapsto f(t,x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) the map $x \mapsto f(t,x)$ is monotone nondecreasing for almost every $t \in J$.

Furthermore, a Chandrabhan function $f(t,x)$ is called L^1_r-*Chandrabhan* if

(iii) there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$\left|f(t,x)\right| \leqslant h_r(t), \quad \text{a.e. } t \in J,$$

for all $x \in \mathbb{R}$ with $|x| \leqslant r$.

Again, a Chandrabhan function $f(t,x)$ is called $L^1_{\mathbb{R}}$-*Chandrabhan* if

(iii) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\left|f(t,x)\right| \leqslant h(t), \quad \text{a.e. } t \in J,$$

for all $x \in \mathbb{R}$.

Finally, a Chandrabhan function $f(t,x)$ is called $\mathbb{R}$-*Chandrabhan* if

(iv) there exists a number $M_f \in \mathbb{R}$ such that

$$\left|f(t,x)\right| \leqslant M_f, \quad \text{a.e. } t \in J,$$

for all $x \in \mathbb{R}$.

Remark 3. Note that the relation between the above types of functions goes as follows:

- $\mathbb{R}$-Chandrabhan $\Rightarrow$ $L^1_{\mathbb{R}}$-Chandrabhan $\Rightarrow$ L^1_r-Chandrabhan,

however, the reverse implication may not hold.

Proposition 1. *Let $f(t,x)$ be a Chandrabhan function on $J \times \mathbb{R}$. Then the superposition operator $\mathcal{F}$ given by $\mathcal{F}x(t) = f(t,x(t))$, $t \in J$, monotonically maps the space $L^1(J,\mathbb{R})$ into itself if and only if f satisfies the growth condition*

$$\left|f(t,x)\right| \leqslant a(t) + b|x|$$

for almost every $t \in J$ and $x \in \mathbb{R}$, where $a \in L^1(J,\mathbb{R})$, and $b \geqslant 0$ is a real number.

Proof. The proof of proposition is similar to a result that given in Krasnoselsdkii [19] for Carathéodory functions f on $J \times \mathbb{R}$ into $\mathbb{R}$. See also Banas [1] and references therein. □

In the special case when $b = 0$ in above Proposition 1, we obtain the following useful result for our further discussion.

Lemma 2. (*See* [7–9].) *If $f(t, x)$ is Chandrabhan, then the function $t \mapsto f(t, x(t))$ is measurable for each $x \in L^1(J, \mathbb{R})$. Moreover, if $f(t, x)$ is $L^1_{\mathbb{R}}$-Chandrabhan, then $f(\cdot, x(\cdot))$ is Lebesgue integrable on J for each $x \in L^1(J, \mathbb{R})$.*

Lemma 3. (*See* [2,15].) *A nonempty closed and bounded subset S of the complete Banach lattice $(L^1(J, \mathbb{R}), \preccurlyeq)$ is a complete lattice.*

In the following section, we prove our main existence result for maximal and minimal positive integrable solutions of the HQIE (8) on J via lattice theoretic hybrid fixed point theorem developed in this paper.

5 Existence of extremal solutions

We consider the following set of hypotheses in what follows.

(H1) The function q is measurable, positive, and bounded on J.
(H2) The function f is positive and $\mathbb{R}$-Chandrabhan on $J \times \mathbb{R}$.
(H3) The map $x \mapsto x/f(t, x)$ is bijective for each $t \in J$.
(H4) There exists a constant $\alpha > 0$ such that

$$\bigl|f(t, x) - f(t, y)\bigr| \leqslant \alpha|x - y|, \quad \text{a.e. } t \in J,$$

for all $x, y \in \mathbb{R}$.
(H5) The function g is positive and L^1_r-Chandrabhan on $J \times \mathbb{R}$.

Theorem 6. *Assume that hypotheses* (H1)–(H5) *hold. If $\alpha(\|q\|_{BM} + \|h_r\|_{L^1}) < 1$, then the HQIE* (8) *has maximal and minimal positive integrable solutions defined on J.*

Proof. Set $E = BM(J, \mathbb{R})$. Define a subset S of the complete lattice $(BM(J, \mathbb{R}), \preccurlyeq)$ by

$$S = \bigl\{x \in BM(J, \mathbb{R})\colon \|x\|_{BM} \leqslant r\bigr\}, \tag{12}$$

where

$$r = M_f\bigl(\|q\|_{BM} + \|h_r\|_{L^1}\bigr).$$

Clearly, S is a nonempty, closed, and bounded subset of the complete lattice Banach algebra $BM(J, \mathbb{R})$, and so, by Lemma 3, $(S, \preccurlyeq)$ is again a complete lattice. Define two operators $\mathcal{A}$ on E and $\mathcal{B}$ on S by

$$\mathcal{A}x(t) = f\bigl(t, x(t)\bigr), \quad t \in J,$$

and

$$\mathcal{B}x(t) = q(t) + \int_a^t g\bigl(s, x(s)\bigr)\,\mathrm{d}s, \quad t \in J. \tag{13}$$

Then the HQIE (8) is transformed into an operator equation as

$$\mathcal{A}x(t)\,\mathcal{B}x(t) = x(t), \quad t \in J.$$

We show that $\mathcal{A}$ defines a mapping $\mathcal{A}\colon BM(J,\mathbb{R}) \to K$. Now, by hypothesis (H2), the function f is positive and bounded on $J \times \mathbb{R}$. Moreover, $f(t,x)$ is monotone nondecreasing in x for each $t \in J$. So the superposition operator $\mathcal{A}$ defined by $\mathcal{A}x(t) = f(t,x(t))$ is monotone positive, nondecreasing and maps an element of $BM(J,\mathbb{R})$ into the cone K. As a result, we have that $\mathcal{A}x \in K$ for all $x \in BM(J,\mathbb{R})$. Similarly, the integral on right hand of Eq. (13) exists in view of hypothesis (H5). Moreover, the integral is continuous and hence measurable on J. The sum of two measurable positive functions is again measurable and positive function on J. Moreover,

$$\|\mathcal{B}x\| \leqslant \|q\|_{BM} + \|h_r\|_{L^1}$$

for all $x \in S$. Hence, $\mathcal{B}x \in K$ for all $x \in S$. Next, we show that $\mathcal{A}$ and $\mathcal{B}$ are positive and isotone increasing on their respective domains of definition. Let $x, y \in BM(J,\mathbb{R})$ be such that $x \preccurlyeq y$. Then we have

$$\mathcal{A}x(t) = f\bigl(t,x(t)\bigr) \leqslant f\bigl(t,y(t)\bigr) = \mathcal{A}y(t)$$

for almost every $t \in J$. This shows that $\mathcal{A}x \preccurlyeq \mathcal{A}y$ almost everywhere on J. Consequently, $\mathcal{A}$ is an isotone increasing on E. Similarly, it can be proved that the operator $\mathcal{B}$ is also an isotone increasing on S.

Next, since by hypothesis (H3), the map $x \mapsto x/f(t,x)$ is bijective for each $t \in J$, the mapping $x \mapsto (I/\mathcal{A})x$ is also bijective, so the operator $(I/\mathcal{A})^{-1}$ exists on S, where I is the identity operator on E. Next, we show that $\mathcal{A}$ is a Lipschitz operator on $BM(J,\mathbb{R})$. Let $x, y \in BM(J,\mathbb{R})$. Then, by hypothesis (H3), we get

$$\begin{aligned}\bigl|\mathcal{A}x(t) - \mathcal{A}y(t)\bigr| &\leqslant \bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr| \\ &\leqslant \alpha\bigl|x(s) - y(s)\bigr| \leqslant \alpha\|x-y\|_{BM}.\end{aligned}$$

Taking the supremum over t on both sides of above inequality, we get

$$\|\mathcal{A}x - \mathcal{A}y\|_{BM} \leqslant \alpha\|x-y\|_{BM}$$

for all $x, y \in BM(J,\mathbb{R})$. This shows that $\mathcal{A}$ is a Lipschitz operator on $BM(J,\mathbb{R})$ with Lipschitz constant α. Next, we show that condition (d) of Corollary 1 is satisfied. Here we have

$$M_{\mathcal{B}} = \sup_{x \in S} \|\mathcal{B}x\| \leqslant \|q\|_{BM} + \|h\|_{L^1}.$$

Therefore, $\alpha M_{\mathcal{B}} \leqslant \alpha[\|q\|_{BM} + \|h\|_{L^1}] < 1$. Finally, we show that condition (e) of Corollary 1 is satisfied. Let $y \in BM(J,\mathbb{R})$ be arbitrary and consider the operator equation $x = \mathcal{A}x\,\mathcal{B}y$ for some $x \in BM(J,\mathbb{R})$. Then we have $x(t) = \mathcal{A}x(t)\,\mathcal{B}y(t)$ for all $t \in J$. Now, by hypotheses (H4) and (H5), we obtain

$$\begin{aligned}\bigl|x(t)\bigr| &= \bigl|\mathcal{A}x(t)\bigr|\bigl|\mathcal{B}y(t)\bigr| \leqslant \|\mathcal{A}x\|_{BM}\|\mathcal{B}y\|_{BM} \\ &\leqslant M_f\bigl[\|q\|_{BM} + \|h_r\|_{L^1}\bigr] \leqslant r\end{aligned}$$

for all $t \in J$. Taking the supremum over t in the above inequality, we obtain $\|x\|_{BM} \leqslant r$ whence $x \in S$. Thus, all conditions of Corollary 1 are satisfied. Hence, the HQIE (8) has a solution in $S \cap K \subset BM(J, \mathbb{R})$, and the set $F_{\mathcal{AB}}$ of all solutions is a complete lattice. As every function $x \in BM(J, \mathbb{R})$ is Lebesgue integrable, the HQIE (8) has a positive integrable solution, and the set of all such solutions is a complete lattice. Hence, $\wedge F_{\mathcal{AB}}$ and $\vee F_{\mathcal{AB}}$ both exist and are respectively the maximal and minimal positive integrable solutions of the HQIE (8) defined on J. This completes the proof. □

Example 1. Let $J = [0, 1] \subset \mathbb{R}$ and consider the HQIE of the type

$$x(t) = \left[f_1\left(t, x(t)\right)\right]\left(t^2 + \int_0^t \left[1 + \tanh x(s)\right] \mathrm{d}s\right) \tag{14}$$

for all $t \in [0, 1]$, where

$$f_1(t, x) = \begin{cases} 1 & \text{if } x \leqslant 0, \\ 1 + \frac{1}{4}\tan^{-1} x & \text{if } x > 0. \end{cases}$$

Set $f(t, x) = f_1(t, x)$, $q(t) = t^2$, and $g(t, x) = 1 + \tanh x$ for all $t \in [0, 1]$ and $x \in \mathbb{R}$. Now, the function q is positive and continuous, so it is measurable bounded on $[0, 1] \times \mathbb{R}$ with bound $\|q\|_{BM} = 1$. Next, the functions $f(t, x)$ and $g(t, x)$ are positive, continuous, and bounded on $[0, 1] \times \mathbb{R}$ with bounds $M_f = 3$ and $h \equiv M_g = 2$, respectively. Also, the functions $x \mapsto f_1(t, x) = f(t, x)$ and $x \mapsto 1 + \tanh x = g(t, x)$ are monotone nondecreasing for each $t \in [0, 1]$. Therefore, the functions f and g are $L^1_{\mathbb{R}}$-Chandrabhan on $[0, 1] \times \mathbb{R}$. Next, the map $\mapsto x/f_1(t, x)$ is bijective for each $t \in [0, 1]$. Furthermore, the function $f(t, x)$ satisfies Lipschitz condition on $[0, 1] \times \mathbb{R}$. To see this, let $x, y \in \mathbb{R}$. Then by Lagrange's mean value theorem, we obtain

$$\left|f(t, x) - f(t, y)\right| = \frac{1}{4}\left|\tan^{-1} x - \tan^{-1} y\right| \leqslant \frac{1}{4}|x - y|$$

for all $t \in [0, 1]$. Therefore, f satisfies the Lipschitz condition with Lipschitz constant $\alpha = 1/4$. Finally, we have

$$\alpha\left(\|q\|_{BM} + \|h\|_{L^1}\right) = \frac{1}{4}(1 + 2) = \frac{3}{4} < 1.$$

Thus, all hypotheses (H1)–(H5) of Theorem 6 are satisfied with $\alpha < 1/4$. Hence, the HQIE (14) has minimal and maximal positive integrable solutions defined on J in the closed subset S of $BM([0, 1], \mathbb{R})$ given by $S = \{x \in BM([0, 1], \mathbb{R})\colon \|x\|_{BM} \leqslant 9\}$.

6 An application

In this section, we apply our main existence theorem of the previous section to a hybrid quadratic differential equation for proving the existence of maximal and minimal positive solutions under suitable conditions.

Let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$ and consider the hybrid quadratic differential equation (in short HQDE) of the type

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x(t)}{f(t, x(t))}\right) = g\big(t, x(t)\big), \quad \text{a.e. } t \in J, \\ x(0) = x_0 > 0, \end{gathered} \tag{15}$$

where the function $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous, and the function $g : J \times \mathbb{R} \to \mathbb{R}$ is discontinuous, and they satisfy certain Lipschitz- and Chandrabhan-type hybrid conditions.

Definition 6. A function $x \in C(J, \mathbb{R})$ is said to be a solution of the HQDE (12) if

(i) the map $t \mapsto x(t)/f(t, x(t))$ is absolutely continuous on J, and
(ii) x satisfies the equations in (15),

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The differential problem (15) is well known and is a perturbed differential equation with a quadratic perturbation of second type (see Dhage [9]). The HQDE (15) has already been discussed for existence and existence of extremal solutions between the given lower and upper solutions via HFPTs due to Dhage [5] under continuity and Caratheodory-type conditions. See Dhage and O'Regan [13] and Dhage and Lakshmikantham [12] and references therein. Here we discuss the HQDE (15) for existence of maximal and minimal integrable solutions without assuming the existence of lower and upper solutions as well as without continuity conditions of any type.

Lemma 4. *Assume that hypothesis* (H3) *holds. If $h \in L^1(J, \mathbb{R})$, then the HQDE*

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x(t)}{f(t, x(t))}\right) = h(t), \quad \textit{a.e. } t \in J, \\ x(0) = x_0 > 0 \end{gathered}$$

is equivalent to the nonlinear quadratic integral equation

$$x(t) = \big[f\big(t, x(t)\big)\big]\left(\frac{x_0}{f(0, x_0)} + \int_0^t h(s)\,\mathrm{d}s\right), \quad t \in J.$$

Theorem 7. *Assume that hypotheses* (H1)–(H5) *hold. If*

$$\alpha\left(\left|\frac{x_0}{f(0, x_0)}\right| + \|h\|_{L^1}\right) < 1,$$

then the HQDE (15) *has maximal and minimal positive solutions defined on J.*

Proof. By Lemma 4, the HQDE (15) is equivalent to the nonlinear hybrid quadratic integral equation (in short, HQIE),

$$x(t) = \big[f\big(t, x(t)\big)\big]\left(\frac{x_0}{f(0, x_0)} + \int_0^t g\big(s, x(s)\big)\,\mathrm{d}s\right), \quad t \in J.$$

Set $E = BM(J, \mathbb{R})$. Let $q(t) = x_0/f(0, x_0)$. Then $q(t) > 0$ for all $t \in J$ because $x_0 > 0$ and f is positive on $J \times \mathbb{R}$. Also, q is measurable function on J. Now the desired conclusion follows by an application of Theorem 6. □

Example 2. Given a closed and bounded interval $J = [0, 1]$, consider the first-order ordinary HQDE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x(t)}{f_2(t, x(t))}\right) = 2 + \tan^{-1} x(t), \quad \text{a.e. } t \in [0, 1],$$
$$x(0) = 1,$$

where

$$f_2(t, x) = \begin{cases} \frac{1}{5} & \text{if } x \leqslant 0, \\ \frac{1}{5}(1 + \frac{x}{1+x}) & \text{if } x > 0. \end{cases}$$

Set $f(t, x) = f_2(t, x)$, $x_0 = 1$, and $g(t, x) = 2 + \tan^{-1} x$ for $t \in [0, 1]$ and $x \in \mathbb{R}$. Then it can be shown in an analogous way of Example 1 that the functions f and g satisfy all hypotheses (H1)–(H5) of Theorem 7, and hence the HQDE (15) has maximal and minimal positive soltions defined on $[0, 1]$.

Remark 4. The nonlinear discontinuous hybrid differential equations considered in this paper for applications of new hybrid fixed point theorem is a very simple one. However, other complex nonlinear hybrid differential equations with integer or fractional order may also be considered for the analysis of existence of extremal solutions. Some of the results in this directions will be reported elsewhere.

Conflicts of interest. The authors declare no conflicts of interest.

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