Nonlinear Analysis: Modelling and Control, Vol. 29, No. 6, 1080–1105
https://doi.org/10.15388/namc.2024.29.37848

Vilnius University Press

Results on integral inequalities for a generalized fractional integral operator unifying two existing fractional integral operators

Supriya Kumar Paul[a], **Lakshmi Narayan Mishra**[a,1], **Vishnu Narayan Mishra**[b]

[a]Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
rgumathsupriya@gmail.com; lakshminarayanmishra04@gmail.com;
lakshminarayan.mishra@vit.ac.in

[b]Department of Mathematics, Indira Gandhi National Tribal University,
Lalpur, Amarkantak, Anuppur 484 887, Madhya Pradesh, India
vishnunarayanmishra@gmail.com; vnm@igntu.ac.in

Received: November 13, 2023 / **Revised:** September 30, 2024 / **Published online:** December 1, 2024

Abstract. The main aim of this article is to design a novel framework to study a generalized fractional integral operator that unifies two existing fractional integral operators. To ensure the suitable selection of the operator and with the discussion of special cases, it is shown that our considered fractional integral generalizes the well-known Atangana–Baleanu fractional integral (AB-fractional integral) and the ABK-fractional integral. Conditions are stated for the generalized AB-fractional integral operator (GAB-fractional integral operator) to be bounded in the space $X_c^p(\gamma_1, \gamma_2)$. We also provide a fractional product-integration formula for this operator. Furthermore, we generalize the reverse Minkowski's inequality and the reverse Hölder-type inequality by utilizing the GAB-fractional integral operator. Additionally, some other types of integral inequalities are established, and several special cases are noted. The concepts in this article may influence further research in fractional calculus.

Keywords: Atangana–Baleanu fractional integral, generalized fractional integral, Minkowski's inequality, Hölder's inequality, fractional integral inequality.

1 Introduction

The theory of fractional calculus, a branch of mathematical analysis dealing with integrals and derivatives of arbitrary order using the gamma function, is one of the most significant mathematical tools for physical investigation, such as nonlinear oscillations of earthquakes, the fluid dynamic traffic model, computer networking, image processing, signals, biology, viscoelastic theory, and several others. The papers [10–12,21,25,27–29]

[1]Corresponding author

describe the advancement of fractional calculus and provide explanations of some of its wide applications in engineering and science.

An advantageous characteristic of this area is that there are many fractional operators, which allows researchers to select the most suitable operator in order to model the research issue. A major number of researchers have worked on definitions, theorems, models, and a variety of generalizations of existing results [3, 10, 15, 21, 23, 26].

One interesting area of study is the generalization of classical inequalities using fractional-order integral operators. Many authors have presented several types of integral inequality by means of fractional order. In the literature we found, a Gronwall-type inequality was presented by Jarad et al. [16] for the Atangana–Baleanu fractional derivative and Adjabi et al. [4] for generalized fractional operators. Nice outcomes were recently observed regarding Hermite–Hadamard-type inequalities [30], Ostrowski-type integral inequalities [5], and Minkowski's inequality [20] involving the Atangana–Baleanu fractional integral operator. In 2021, Kashuri [17] introduced a new generalized fractional integral operator, namely, ABK-fractional integral, and presented Hermite–Hadamard-type inequalities for this integral operator. Butt et al. [13] introduced some generalized integral inequalities for the ABK-fractional integrals.

Such types of generalizations motivate future research to introduce more new concepts to unify the fractional integral operators and obtain integral inequalities via such generalized operators. Integral inequalities and their applications play an important role in the theory of differential equations and applied mathematics [9].

In this paper, we introduce a generalized version of the AB-fractional integral and the ABK-fractional integral, which we name generalized AB-fractional integral (GAB-fractional integral). Conditions are stated for the GAB-fractional integral operator to be bounded in the space $X_c^p(\gamma_1, \gamma_2)$. We also provide a fractional product-integration formula for this operator. Taking into account the novel ideas, we establish a new version of reverse Hölder-type inequality and reverse Minkowski's inequality for the GAB-fractional integral, and we also introduce some certain types of integral inequalities related to this fractional integral that are advantageous to current research.

This article is structured as follows: In Section 2, some background information is given. The definition of GAB-fractional integrals (left-sided and right-sided) is given in Section 3. Reverse Minkowski's inequality and some corresponding integral inequalities for the GAB-fractional integrals are discussed in Section 4. Reverse Hölder-type inequality for the GAB-fractional integrals and some special cases are obtained in Section 5. In Section 6, some other types of inequalities for the GAB-fractional integrals are obtained. Finally, the conclusions and future work are given in Section 7.

2 Preliminaries

In this section, we provide some background information, which are useful for the presentation of our main results. We denote the set $\{\nu \in \mathbb{R}\colon\ \nu > 0\}$ by the notation $\mathbb{R}_+^*$ throughout the entire paper, where $\mathbb{R}$ is the set of real numbers.

Definition 1. (See [18].) Let $c \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$. The space $X_c^p(\gamma_1, \gamma_2)$ consists of those complex-valued Lebesgue measurable functions ζ on $[\gamma_1, \gamma_2]$ for which $\|\zeta\|_{X_c^p} < \infty$ with

$$\|\zeta\|_{X_c^p} = \left(\int_{\gamma_1}^{\gamma_2} \left| \nu^c \zeta(\nu) \right|^p \frac{\mathrm{d}\nu}{\nu} \right)^{1/p}, \quad 1 \leqslant p < \infty,$$

and

$$\|\zeta\|_{X_c^\infty} = \operatorname{ess} \sup_{\gamma_1 \leqslant \nu \leqslant \gamma_2} \left[\nu^c \left| \zeta(\nu) \right| \right], \quad p = \infty.$$

In particular, when $c = 1/p$ ($1 \leqslant p \leqslant \infty$), the space $X_c^p(\gamma_1, \gamma_2)$ coincides with the classical $L^p(\gamma_1, \gamma_2)$-space. Moreover, the Sobolev space $H^1(\gamma_1, \gamma_2)$ is defined as [14]

$$H^1(\gamma_1, \gamma_2) = \left\{ \zeta \colon \zeta \in L^2(\gamma_1, \gamma_2) \text{ and } \zeta' \in L^2(\gamma_1, \gamma_2) \right\}.$$

The generalized Mittag-Leffler function appears in the Atangana–Baleanu (AB) fractional derivative, which is known to have a nonlocal fractional derivative with nonsingular kernel [22, 24]. In [1, 8], Atangana and Baleanu developed two novel fractional-order derivatives depending on the definitions of Riemann–Liouville and Caputo fractional derivatives. The AB-fractional integrals in terms of the Riemann–Liouville fractional integral are stated as follows:

Definition 2. (See [2, 8].) The AB-fractional integral of a function $\zeta \in H^1(\gamma_1, \gamma_2)$ is given by

$${}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu} \zeta(\nu) = \frac{1-\sigma}{\mathsf{B}(\sigma)} \zeta(\nu) + \frac{\sigma}{\mathsf{B}(\sigma)\Gamma(\sigma)} \int_{\gamma_1}^{\nu} (\nu - \mu)^{\sigma - 1} \zeta(\mu) \, \mathrm{d}\mu, \quad \nu > \gamma_1,$$

where $\gamma_1 < \gamma_2$, $\sigma \in [0, 1]$, $\Gamma(\sigma)$ is the Gamma function and $\mathsf{B}(\sigma) > 0$ is the normalization function obeying $\mathsf{B}(0) = \mathsf{B}(1) = 1$.

Also, the opposite side of the AB-fractional integral is described as,

$${}^{\mathrm{AB}}_{\gamma_2}\mathcal{I}^{\sigma}_{\nu} \zeta(\nu) = \frac{1-\sigma}{\mathsf{B}(\sigma)} \zeta(\nu) + \frac{\sigma}{\mathsf{B}(\sigma)\Gamma(\sigma)} \int_{\nu}^{\gamma_2} (\mu - \nu)^{\sigma - 1} \zeta(\mu) \, \mathrm{d}\mu, \quad \nu < \gamma_2.$$

Remark 1. Since $\mathsf{B}(\sigma) > 0$ is positive, it immediately follows that the AB-fractional integral of a positive function is positive. It should be observed that, when σ is 0, we obtain the initial function, and if σ is 1, we recover the ordinary integral.

In 2021, Kashuri [17] introduced the ABK-fractional integrals (left-sided and right-sided) in terms of the Katugampola fractional integrals [18], which are defined as follows:

Definition 3. (See [17].) Let $[\gamma_1, \gamma_2] \subset \mathbb{R}$ be a finite interval. Then the ABK-fractional integrals (left-sided and right-sided) of order $\sigma \in (0,1)$ of $\zeta \in X_c^p(\gamma_1, \gamma_2)$ are given by

$$\begin{aligned} &{}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}_\nu^\sigma \zeta(\nu) \\ &\quad = \frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu) + \frac{\sigma\beta^{1-\sigma}}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu} \left(\nu^\beta - \mu^\beta\right)^{\sigma-1}\mu^{\beta-1}\zeta(\mu)\,\mathrm{d}\mu, \quad \nu > \gamma_1 \geqslant 0, \end{aligned}$$

and

$$\begin{aligned} &{}^{\mathrm{ABK}_\beta}_{\gamma_2-}\mathcal{I}_\nu^\sigma \zeta(\nu) \\ &\quad = \frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu) + \frac{\sigma\beta^{1-\sigma}}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\nu}^{\gamma_2} \left(\mu^\beta - \nu^\beta\right)^{\sigma-1}\mu^{\beta-1}\zeta(\mu)\,\mathrm{d}\mu, \quad \nu < \gamma_2, \end{aligned}$$

where $\mathsf{B}(\sigma) > 0$ is the normalization function obeying $\mathsf{B}(0) = \mathsf{B}(1) = 1$, and $\beta > 0$.

Remark 2. (See [17].) Since $\mathsf{B}(\sigma) > 0$ is positive, it immediately follows that the ABK-fractional integral of a positive function is positive. Note that, when $\beta = 1$, we recover the AB-fractional integral.

Recently, Katugampola [6, 19] implemented a new fractional integral that unifies six fractional integrals, namely, Erdélyi–Kober, Katugampola, Riemann–Liouville, Hadamard, Liouville, and Weyl fractional integrals as follows:

Definition 4. (See [6, 19].) Let $\zeta \in X_c^p(\gamma_1, \gamma_2)$, $\sigma > 0$, and $\beta, \eta, \delta, k \in \mathbb{R}$. Then the generalized Katugampola fractional integrals (left-sided and right-sided) are defined by

$$\left({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\right)(\nu) = \frac{\beta^{1-\eta}\nu^k}{\Gamma(\sigma)}\int_{\gamma_1}^{\nu} \frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta - \mu^\beta)^{1-\sigma}}\zeta(\mu)\,\mathrm{d}\mu, \quad 0 \leqslant \gamma_1 < \nu < \gamma_2 \leqslant \infty,$$

and

$$\left({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_2-;\delta,k}\zeta\right)(\nu) = \frac{\beta^{1-\eta}\nu^{\beta\delta}}{\Gamma(\sigma)}\int_{\nu}^{\gamma_2} \frac{\mu^{k+\beta-1}}{(\mu^\beta - \nu^\beta)^{1-\sigma}}\zeta(\mu)\,\mathrm{d}\mu, \quad 0 \leqslant \gamma_1 < \nu < \gamma_2 \leqslant \infty,$$

respectively, if integrals exist.

Theorem 1. (*See* [19].) *Let* $\sigma > 0$, $1 \leqslant p \leqslant \infty$, $0 < \gamma_1 < \gamma_2 < \infty$, *and let* $\beta, c \in \mathbb{R}$ *be such that* $\beta \geqslant c$. *Then the operator* ${}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}$ *is bounded in* $X_c^p(\gamma_1, \gamma_2)$, *and*

$$\left\|{}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\right\|_{X_c^p} \leqslant \mathcal{K}\|\zeta\|_{X_c^p},$$

where

$$\mathcal{K} = \frac{\beta^{1-\eta}\gamma_2^{\beta(\sigma+\delta)+k}}{\Gamma(\sigma)}\int_1^{\gamma_2/\gamma_1} \frac{t^{c-\beta(\sigma+\delta)-1}}{(t^\beta - 1)^{1-\sigma}}\,\mathrm{d}t, \quad \beta \neq 0,\ k \in \mathbb{R},\ \delta \geqslant 0.$$

Theorem 2. (*See* [19].) *Let $\sigma > 0$, $1 \leqslant p \leqslant \infty$, $\eta \in \mathbb{R}$, $0 \leqslant \gamma_1 < \gamma_2 \leqslant \infty$, and let $\beta, c \in \mathbb{R}$ be such that $\beta \geqslant c$. Then for $\zeta, \mathcal{H} \in X_c^p(\gamma_1, \gamma_2)$, the fractional product-integration formula holds. That is,*

$$\int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\zeta(\nu)\big({}^{\beta}\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\mathcal{H}\big)(\nu)\,\mathrm{d}\nu = \int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\mathcal{H}(\nu)\big({}^{\beta}\mathcal{I}^{\sigma,\eta}_{\gamma_2-;\delta,k}\zeta\big)(\nu)\,\mathrm{d}\nu.$$

3 Generalized AB-fractional integral operator (GAB-fractional integral operator)

Motivated by the above literature, we introduce the GAB-fractional integrals (left-sided and right-sided) as follows:

Definition 5. Let $0 < \gamma_1 < \gamma_2 < \infty$, $1 \leqslant p \leqslant \infty$, $\delta \geqslant 0$, and let $\beta, \eta, k, c \in \mathbb{R}$ be such that $\beta \geqslant c$ and $\beta \neq 0$. Then the left-sided GAB-fractional integral of order $\sigma \in [0, 1]$ of a function $\zeta \in X_c^p(\gamma_1, \gamma_2)$ is defined by

$$\begin{aligned} &{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu) \\ &\quad= \frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu) + \frac{\sigma\beta^{1-\eta}\nu^{k}}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu} \frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\zeta(\mu)\,\mathrm{d}\mu, \quad \nu > \gamma_1 > 0, \end{aligned} \tag{1}$$

where $\mathsf{B} : [0, 1] \to (0, \infty)$ is the normalization function obeying $\mathsf{B}(0) = \mathsf{B}(1) = 1$.

Sometimes, especially in variational calculus, fractional integrals work in pairs [7]. Therefore, the corresponding right-sided GAB-fractional integral is defined by

$$\begin{aligned} &{}^{\mathrm{GAB}_\beta}_{\gamma_2-}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu) \\ &\quad= \frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu) + \frac{\sigma\beta^{1-\eta}\nu^{\beta\delta}}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\nu}^{\gamma_2} \frac{\mu^{k+\beta-1}}{(\mu^\beta-\nu^\beta)^{1-\sigma}}\zeta(\mu)\,\mathrm{d}\mu, \quad \nu < \gamma_2. \end{aligned}$$

Remark 3. When $\beta > 0$, since $\mathsf{B}(\sigma)$ is positive, it immediately follows that the GAB-fractional integrals of a positive function is positive. Also, when η is not an integer and $\beta < 0$, then $\beta^{1-\eta}$ is complex and can be treated using the theory of complex analysis, considering appropriate branches. It should be observed that, when $\eta = \sigma$, $k = 0$, and $\delta = 0$, we recover the ABK-fractional integrals. Moreover, when $k = 0$, $\delta = 0$, and $\beta = 1$, we recover the AB-fractional integrals. Furthermore, the interested researcher can discover a new nonlocal fractional derivative of it with a Mittag-Leffler nonsingular kernel, various formulas, and a variety of applications.

Now, we show that the GAB-fractional integral operator $\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big)$ is well defined on $X_c^p(\gamma_1, \gamma_2)$. We have the following theorem:

Theorem 3. *Let* $0 \leqslant \sigma \leqslant 1$, $0 < \gamma_1 < \gamma_2 < \infty$, $1 \leqslant p \leqslant \infty$, $\delta \geqslant 0$, *and let* $\beta, \eta, k, c \in \mathbb{R}$ *be such that* $\beta \geqslant c$ *and* $\beta \neq 0$. *Then the operator* $({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k})$ *is bounded in* $X^p_c(\gamma_1,\gamma_2)$, *and*

$$\big\|{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta\big\|_{X^p_c} \leqslant M\|\zeta\|_{X^p_c},$$

where

$$M = \frac{1-\sigma}{\mathsf{B}(\sigma)} + \frac{\sigma}{\mathsf{B}(\sigma)}\mathcal{K}$$

and

$$\mathcal{K} = \frac{\beta^{1-\eta}\gamma_2^{\beta(\sigma+\delta)+k}}{\Gamma(\sigma)}\int_1^{\gamma_2/\gamma_1}\frac{t^{c-\beta(\sigma+\delta)-1}}{(t^\beta-1)^{1-\sigma}}\,\mathrm{d}t.$$

Proof. To establish the result, we write Eq. (1) as follows:

$${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu) = \frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu) + \frac{\sigma}{\mathsf{B}(\sigma)}\big({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\big)(\nu),$$

where

$$\big({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\big)(\nu) = \frac{\beta^{1-\eta}\nu^k}{\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\zeta(\mu)\,\mathrm{d}\mu.$$

For $\sigma = 0$, the result is obvious. So, we have to show for $0 < \sigma \leqslant 1$.

Since $\zeta(\nu) \in X^p_c(\gamma_1,\gamma_2)$ and by Theorem 1, we get that the operator $({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k})$ is bounded in $X^p_c(\gamma_1,\gamma_2)$ with

$$\big\|{}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\big\|_{X^p_c} \leqslant \mathcal{K}\|\zeta\|_{X^p_c} \quad \text{for } 1 \leqslant p \leqslant \infty,$$

where

$$\mathcal{K} = \frac{\beta^{1-\eta}\gamma_2^{\beta(\sigma+\delta)+k}}{\Gamma(\sigma)}\int_1^{\gamma_2/\gamma_1}\frac{t^{c-\beta(\sigma+\delta)-1}}{(t^\beta-1)^{1-\sigma}}\,\mathrm{d}t.$$

Then for $1 \leqslant p < \infty$ and by using the triangle inequality, we get

$$\begin{aligned}\big\|{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta\big\|_{X^p_c} &= \left\|\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta + \frac{\sigma}{\mathsf{B}(\sigma)}\big({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\big)\right\|_{X^p_c} \\ &\leqslant \frac{1-\sigma}{\mathsf{B}(\sigma)}\|\zeta\|_{X^p_c} + \frac{\sigma}{\mathsf{B}(\sigma)}\big\|\big({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\big)\big\|_{X^p_c} \\ &\leqslant \frac{1-\sigma}{\mathsf{B}(\sigma)}\|\zeta\|_{X^p_c} + \frac{\sigma}{\mathsf{B}(\sigma)}\mathcal{K}\|\zeta\|_{X^p_c},\end{aligned}$$

which implies that

$$\big\|{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta\big\|_{X^p_c} \leqslant M\|\zeta\|_{X^p_c},$$

where

$$M = \frac{1-\sigma}{\mathsf{B}(\sigma)} + \frac{\sigma}{\mathsf{B}(\sigma)}\mathcal{K}. \tag{2}$$

For $p = \infty$, we have

$$\begin{aligned}\left|\nu^c\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu)\right)\right| &\leqslant \frac{1-\sigma}{\mathsf{B}(\sigma)}\left|\nu^c\zeta(\nu)\right| + \frac{\sigma}{\mathsf{B}(\sigma)}\left|\nu^c\left({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\right)(\nu)\right| \\ &\leqslant \frac{1-\sigma}{\mathsf{B}(\sigma)}\|\zeta\|_{X_c^\infty} + \frac{\sigma}{\mathsf{B}(\sigma)}\mathcal{K}\|\zeta\|_{X_c^\infty} \\ &= \left(\frac{1-\sigma}{\mathsf{B}(\sigma)} + \frac{\sigma}{\mathsf{B}(\sigma)}\mathcal{K}\right)\|\zeta\|_{X_c^\infty}.\end{aligned}$$

This agrees with above (2). This completes the proof. □

Corollary 1. *Let $0 \leqslant \sigma \leqslant 1$, $0 < \gamma_1 < \gamma_2 < \infty$, $1 \leqslant p \leqslant \infty$, $\delta \geqslant 0$, and let $\beta, \eta, k \in \mathbb{R}$ be such that $\beta \geqslant 1/p$ and $\beta \neq 0$. Then the operator $\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\right)$ is bounded in $L^p(\gamma_1,\gamma_2)$.*

Now we will establish the fractional product-integration formula for the GAB-fractional integral operator with the help of Theorem 2.

Theorem 4. *Let $0 < \sigma \leqslant 1$, $0 < \gamma_1 < \gamma_2 < \infty$, $1 \leqslant p \leqslant \infty$, $\delta \geqslant 0$, and let $\beta, \eta, k, c \in \mathbb{R}$ be such that $\beta \geqslant c$ and $\beta \neq 0$. Then for $\zeta, \mathcal{H} \in X_c^p(\gamma_1,\gamma_2)$, the fractional product-integration formula holds. That is,*

$$\int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\zeta(\nu)\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu)\right)\mathrm{d}\nu = \int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\mathcal{H}(\nu)\left({}^{\mathrm{GAB}_\beta}_{\gamma_2-}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu)\right)\mathrm{d}\nu.$$

Proof. To establish the result, we write Eq. (1) as follows:

$${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu) = \frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu) + \frac{\sigma}{\mathsf{B}(\sigma)}\left({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\right)(\nu),$$

where

$$\left({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\zeta\right)(\nu) = \frac{\beta^{1-\eta}\nu^k}{\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\zeta(\mu)\,\mathrm{d}\mu.$$

From Theorem 2 we have

$$\int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\zeta(\nu)\left({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\mathcal{H}\right)(\nu)\,\mathrm{d}\nu = \int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\mathcal{H}(\nu)\left({}^\beta\mathcal{I}^{\sigma,\eta}_{\gamma_2-;\delta,k}\zeta\right)(\nu)\,\mathrm{d}\nu.$$

Now,

$$\begin{aligned}
&\int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\zeta(\nu)\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu)\right)\mathrm{d}\nu \\
&= \int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\zeta(\nu)\left(\frac{1-\sigma}{\mathsf{B}(\sigma)}\mathcal{H}(\nu) + \frac{\sigma}{\mathsf{B}(\sigma)}\left({}^{\beta}\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\mathcal{H}\right)(\nu)\right)\mathrm{d}\nu \\
&= \frac{1-\sigma}{\mathsf{B}(\sigma)}\int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\zeta(\nu)\mathcal{H}(\nu)\,\mathrm{d}\nu + \frac{\sigma}{\mathsf{B}(\sigma)}\int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\zeta(\nu)\left({}^{\beta}\mathcal{I}^{\sigma,\eta}_{\gamma_1+;\delta,k}\mathcal{H}\right)(\nu)\,\mathrm{d}\nu \\
&= \frac{1-\sigma}{\mathsf{B}(\sigma)}\int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\zeta(\nu)\mathcal{H}(\nu)\,\mathrm{d}\nu + \frac{\sigma}{\mathsf{B}(\sigma)}\int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\mathcal{H}(\nu)\left({}^{\beta}\mathcal{I}^{\sigma,\eta}_{\gamma_2-;\delta,k}\zeta\right)(\nu)\,\mathrm{d}\nu \\
&= \int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\mathcal{H}(\nu)\left(\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu) + \frac{\sigma}{\mathsf{B}(\sigma)}\left({}^{\beta}\mathcal{I}^{\sigma,\eta}_{\gamma_2-;\delta,k}\zeta\right)(\nu)\right)\mathrm{d}\nu \\
&= \int_{\gamma_1}^{\gamma_2} \nu^{\beta-1}\mathcal{H}(\nu)\left({}^{\mathrm{GAB}_\beta}_{\gamma_2-}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu)\right)\mathrm{d}\nu.
\end{aligned}$$

This completes the proof. □

4 Reverse Minkowski inequality for the GAB-fractional integral operator

In this section, we establish the reverse Minkowski's inequality for the GAB-fractional integrals. Due to a similar treatment for the right-sided integral, we will only work with the left-sided integral in this instance.

Theorem 5. *Let $\sigma \in (0,1]$, $\beta > 0$, $\eta, k \in \mathbb{R}$, $\delta \geqslant 0$, and $p \geqslant 1$. Let $0 < \gamma_1 < \gamma_2 < \infty$, and $\zeta, \mathcal{H} \in X^p_c(\gamma_1,\gamma_2)$ be two positive functions such that ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu) < \infty$ and ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu) < \infty$ for all $\nu > \gamma_1 > 0$. If $0 < \lambda \leqslant \zeta(\nu)/\mathcal{H}(\nu) \leqslant B$ for some $\lambda, B \in \mathbb{R}^*_+$ and for all $\nu \in [\gamma_1,\gamma_2]$, then*

$$\begin{aligned}
&\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu)\right)^{1/p} + \left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu)\right)^{1/p} \\
&\quad\leqslant \Theta\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta(\nu)+\mathcal{H}(\nu)\right)^p\right)^{1/p},
\end{aligned}$$

where $\Theta = (B(1+\lambda)+(B+1))/((1+\lambda)(B+1))$.

Proof. From the condition $\zeta(\nu)/\mathcal{H}(\nu) \leqslant B$ we get

$$\zeta(\nu) \leqslant B\left(\zeta(\nu)+\mathcal{H}(\nu)\right) - B\zeta(\nu),$$

which implies

$$\zeta(\nu) \leqslant \frac{B}{B+1}\big(\zeta(\nu) + \mathcal{H}(\nu)\big).$$

Taking the pth power of both sides, we get

$$\zeta^p(\nu) \leqslant \left(\frac{B}{B+1}\right)^p \big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p. \tag{3}$$

Multiplying both sides of (3) by $(1-\sigma)/\mathsf{B}(\sigma)$, we have

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta^p(\nu) \leqslant \left(\frac{B}{B+1}\right)^p \frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p. \tag{4}$$

Now, replacing ν by μ in inequality (3) and multiplying both sides by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta - \mu^\beta)^{1-\sigma})$, where $\mu \in (\gamma_1, \nu)$, we obtain

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta - \mu^\beta)^{1-\sigma}}\zeta^p(\mu)\\ &\quad\leqslant \left(\frac{B}{B+1}\right)^p \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta - \mu^\beta)^{1-\sigma}}\big(\zeta(\mu) + \mathcal{H}(\mu)\big)^p.\end{aligned} \tag{5}$$

Integrating both sides of (5) with respect to μ, we get

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu} \frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta - \mu^\beta)^{1-\sigma}}\,\zeta^p(\mu)\,\mathrm{d}\mu\\ &\quad\leqslant \left(\frac{B}{B+1}\right)^p \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu} \frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta - \mu^\beta)^{1-\sigma}}\big(\zeta(\mu) + \mathcal{H}(\mu)\big)^p\,\mathrm{d}\mu.\end{aligned} \tag{6}$$

Adding (4) and (6), we get

$$\begin{aligned}&\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta^p(\nu) + \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu} \frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta - \mu^\beta)^{1-\sigma}}\zeta^p(\mu)\,\mathrm{d}\mu\\ &\quad\leqslant \left(\frac{B}{B+1}\right)^p \Bigg[\frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p\\ &\qquad+ \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu} \frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta - \mu^\beta)^{1-\sigma}}\big(\zeta(\mu) + \mathcal{H}(\mu)\big)^p\,\mathrm{d}\mu\Bigg],\end{aligned}$$

i.e.,

$$ {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu) \leqslant \left(\frac{B}{B+1}\right)^p {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p. \tag{7}$$

Taking the $(1/p)$th power of both sides of (7), we obtain

$$\left({}_{\gamma_1+}^{\mathrm{GAB}_\beta}\mathcal{I}_{\nu,\delta,k}^{\sigma,\eta}\zeta^p(\nu)\right)^{1/p} \leqslant \frac{B}{B+1}\left({}_{\gamma_1+}^{\mathrm{GAB}_\beta}\mathcal{I}_{\nu,\delta,k}^{\sigma,\eta}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p\right)^{1/p}. \tag{8}$$

Again, by the condition $0 < \lambda \leqslant \zeta(\nu)/\mathcal{H}(\nu)$, we can write

$$\mathcal{H}^p(\nu) \leqslant \left(\frac{1}{1+\lambda}\right)^p \big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p. \tag{9}$$

Multiplying both sides of (9) by $(1-\sigma)/\mathsf{B}(\sigma)$, we have

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\mathcal{H}^p(\nu) \leqslant \left(\frac{1}{1+\lambda}\right)^p \frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p. \tag{10}$$

Now, replacing ν by μ in inequality (9) and multiplying both sides by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma})$, we obtain

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^p(\mu)\\ &\quad\leqslant \left(\frac{1}{1+\lambda}\right)^p \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p.\end{aligned} \tag{11}$$

Integrating both sides of (11) with respect to μ, we get

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^p(\mu)\,\mathrm{d}\mu\\ &\quad\leqslant \left(\frac{1}{1+\lambda}\right)^p \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p\,\mathrm{d}\mu.\end{aligned} \tag{12}$$

Adding (10) and (12), we get

$$\begin{aligned}&\frac{1-\sigma}{\mathsf{B}(\sigma)}\mathcal{H}^p(\nu)+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^p(\mu)\,\mathrm{d}\mu\\ &\quad\leqslant \left(\frac{1}{1+\lambda}\right)^p\Bigg[\frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p\\ &\qquad+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p\,\mathrm{d}\mu\Bigg],\end{aligned}$$

i.e.,

$${}_{\gamma_1+}^{\mathrm{GAB}_\beta}\mathcal{I}_{\nu,\delta,k}^{\sigma,\eta}\mathcal{H}^p(\nu) \leqslant \left(\frac{1}{1+\lambda}\right)^p {}_{\gamma_1+}^{\mathrm{GAB}_\beta}\mathcal{I}_{\nu,\delta,k}^{\sigma,\eta}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p. \tag{13}$$

Taking the $(1/p)$th power of both sides of (13), we obtain

$$\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu)\right)^{1/p} \leqslant \frac{1}{1+\lambda}\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta(\nu)+\mathcal{H}(\nu)\right)^p\right)^{1/p}. \tag{14}$$

Adding (8) and (14), we get

$$\begin{aligned}&\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu)\right)^{1/p} + \left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu)\right)^{1/p}\\ &\quad\leqslant \Theta\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta(\nu)+\mathcal{H}(\nu)\right)^p\right)^{1/p},\end{aligned}$$

where $\Theta = B(1+\lambda) + (B+1)/((1+\lambda)(B+1))$. □

Corollary 2. *If we take $\eta = \sigma$, $k = 0$, and $\delta = 0$ in Theorem 5, the reverse Minkowski's inequality for the ABK-fractional integral is as follows*:

$$\left({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\zeta^p(\nu)\right)^{1/p} + \left({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\mathcal{H}^p(\nu)\right)^{1/p} \leqslant \Theta\left({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\left(\zeta(\nu)+\mathcal{H}(\nu)\right)^p\right)^{1/p},$$

where $\Theta = B(1+\lambda) + (B+1)/((1+\lambda)(B+1))$.

Remark 4. Khan et al. [20] presented the reverse Minkowski's inequality for the AB-fractional integral. In this case, $k = 0$, $\delta = 0$, and $\beta = 1$ in Theorem 5.

5 Reverse Hölder's inequality for the GAB-fractional integral operator

In this section, we prove the reverse Hölder-like inequalities for the GAB-fractional integral operator. Due to a similar treatment for the right-sided integral, we will only work with the left-sided integral in this instance.

Theorem 6. *Let $\sigma \in (0,1]$, $\beta > 0$, $\eta, k \in \mathbb{R}$, $\delta \geqslant 0$, $p > 1$, $q > 1$, and $1/p + 1/q = 1$. Let $0 < \gamma_1 < \gamma_2 < \infty$, and $\zeta, \mathcal{H} \in X^p_c(\gamma_1,\gamma_2)$ be two positive functions such that ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu) < \infty$ and ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu) < \infty$ for all $\nu > \gamma_1 > 0$. If $0 < \lambda \leqslant \zeta(\nu)/\mathcal{H}(\nu) \leqslant B$ for some $\lambda, B \in \mathbb{R}^*_+$ and for all $\nu \in [\gamma_1,\gamma_2]$, then*

$$\begin{aligned}&\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu)\right)^{1/p}\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu)\right)^{1/q}\\ &\quad\leqslant \left(\frac{B}{\lambda}\right)^{1/(pq)}\left[{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\right)\right].\end{aligned}$$

Proof. By the given condition $\zeta(\nu)/\mathcal{H}(\nu) \leqslant B$, we get

$$\zeta^{1/q}(\nu) \leqslant B^{1/q}\mathcal{H}^{1/q}(\nu). \tag{15}$$

Multiplying (15) by $\zeta^{1/p}(\nu)$ and using the condition $1/p + 1/q = 1$, we get

$$\zeta(\nu) \leqslant B^{1/q}\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu). \tag{16}$$

Now, multiplying (16) by $(1-\sigma)/\mathsf{B}(\sigma)$, we obtain

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\,\zeta(\nu) \leqslant B^{1/q}\,\frac{1-\sigma}{\mathsf{B}(\sigma)}\,\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu). \tag{17}$$

Again, replacing ν by μ in inequality (16) and multiplying by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma})$, we obtain

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta(\mu)\\&\quad\leqslant B^{1/q}\,\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^{1/p}(\mu)\mathcal{H}^{1/q}(\mu),\quad \mu\in(\gamma_1,\nu).\end{aligned} \tag{18}$$

Integrating both sides of (18) with respect to μ from γ_1 to ν, we get

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta(\mu)\,\mathrm{d}\mu\\&\quad\leqslant B^{1/q}\,\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^{1/p}(\mu)\mathcal{H}^{1/q}(\mu)\,\mathrm{d}\mu.\end{aligned} \tag{19}$$

From (17) and (19) we get

$$\begin{aligned}&\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu)+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta(\mu)\,\mathrm{d}\mu\\&\quad\leqslant B^{1/q}\Bigg[\frac{1-\sigma}{\mathsf{B}(\sigma)}\,\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\\&\qquad+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^{1/p}(\mu)\mathcal{H}^{1/q}(\mu)\,\mathrm{d}\mu\Bigg],\end{aligned}$$

which implies that

$${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu) \leqslant B^{1/q}\big[{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\big)\big]. \tag{20}$$

Taking $(1/p)$th power of both sides of (20), we have

$$\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu)\big)^{1/p} \leqslant B^{1/(pq)}\big[{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\big)\big]^{1/p}. \tag{21}$$

By the given condition $\lambda\leqslant\zeta(\nu)/\mathcal{H}(\nu)$, we get

$$\mathcal{H}^{1/p}(\nu) \leqslant \lambda^{-1/p}\zeta^{1/p}(\nu). \tag{22}$$

Multiplying (22) by $\mathcal{H}^{1/q}(\nu)$ and using the condition $1/p + 1/q = 1$, we get

$$\mathcal{H}(\nu) \leqslant \lambda^{-1/p}\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu). \tag{23}$$

Now, multiplying (23) by $(1-\sigma)/\mathsf{B}(\sigma)$, we obtain

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\mathcal{H}(\nu) \leqslant \lambda^{-1/p}\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu). \tag{24}$$

Again, replacing ν by μ in inequality (23) and multiplying by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma})$, we obtain

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}(\mu)\\ &\quad\leqslant \lambda^{-1/p}\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^{1/p}(\mu)\mathcal{H}^{1/q}(\mu), \quad \mu\in(\gamma_1,\nu).\end{aligned} \tag{25}$$

Integrating both sides of (25) with respect to μ from γ_1 to ν, we get

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}(\mu)\,\mathrm{d}\mu\\ &\quad\leqslant \lambda^{-1/p}\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^{1/p}(\mu)\mathcal{H}^{1/q}(\mu)\,\mathrm{d}\mu.\end{aligned} \tag{26}$$

From (24) and (26) we get

$$\begin{aligned}&\frac{1-\sigma}{\mathsf{B}(\sigma)}\mathcal{H}(\nu) + \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}(\mu)\,\mathrm{d}\mu\\ &\leqslant \lambda^{-1/p}\left[\frac{1-\sigma}{\mathsf{B}(\sigma)}\,\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu) + \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^{1/p}(\mu)\mathcal{H}^{1/q}(\mu)\,\mathrm{d}\mu\right],\end{aligned}$$

which implies that

$${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu) \leqslant \lambda^{-1/p}\left[{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\right)\right]. \tag{27}$$

Taking $(1/q)$th power of both sides of (27), we have

$$\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu)\right)^{1/q} \leqslant \lambda^{-1/(pq)}\left[{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\right)\right]^{1/q}. \tag{28}$$

From (21) and (28) we obtain

$$\begin{aligned}&\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu)\right)^{1/p}\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu)\right)^{1/q}\\ &\quad\leqslant \lambda^{-1/(pq)}B^{1/(pq)}\left[{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\right)\right]^{1/p+1/q}.\end{aligned}$$

This implies

$$\begin{aligned}&\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu)\right)^{1/p}\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu)\right)^{1/q}\\&\quad\leqslant\left(\frac{B}{\lambda}\right)^{1/(pq)}\left[{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\right)\right].\end{aligned}$$ □

Corollary 3. *Let $\sigma \in (0,1]$, $\beta > 0$, $\eta, k \in \mathbb{R}$, $\delta \geqslant 0$, $p > 1$, $q > 1$, and $1/p + 1/q = 1$. Let $0 < \gamma_1 < \gamma_2 < \infty$, and $\zeta, \mathcal{H} \in X^p_c(\gamma_1,\gamma_2)$ be two positive functions such that ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu) < \infty$ and ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^q(\nu) < \infty$ for all $\nu > \gamma_1 > 0$. If $0 < \lambda \leqslant \zeta^q(\nu)/\mathcal{H}^q(\nu) \leqslant B$ for some $\lambda, B \in \mathbb{R}^*_+$ and for all $\nu \in [\gamma_1,\gamma_2]$, then*

$$\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu)\right)^{1/p}\left({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^q(\nu)\right)^{1/q}\leqslant\left(\frac{B}{\lambda}\right)^{1/(pq)}\left[{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta(\nu)\mathcal{H}(\nu)\right)\right].$$

Corollary 4. *If we take $\eta = \sigma$, $k = 0$, and $\delta = 0$ in Theorem 6, the following inequality for the ABK-fractional integral holds*:

$$\left({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\zeta(\nu)\right)^{1/p}\left({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\mathcal{H}(\nu)\right)^{1/q}\leqslant\left(\frac{B}{\lambda}\right)^{1/(pq)}\left[{}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\left(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\right)\right].$$

Remark 5. If we take $k = 0$, $\delta = 0$, and $\beta = 1$ in Theorem 6, the following inequality for the AB-fractional integral holds:

$$\left({}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\zeta(\nu)\right)^{1/p}\left({}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\mathcal{H}(\nu)\right)^{1/q}\leqslant\left(\frac{B}{\lambda}\right)^{1/(pq)}\left[{}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\left(\zeta^{1/p}(\nu)\mathcal{H}^{1/q}(\nu)\right)\right].$$

Khan et al. presented this inequality for the AB-fractional integral in [20].

6 Other types of integral inequalities

In this section, our discussion will be on some other types of fractional integral inequalities via the left-sided GAB-fractional integral.

Theorem 7. *Let $\sigma \in (0,1]$, $\beta > 0$, $\eta, k \in \mathbb{R}$, $p > 1$, $q > 1$, and $1/p + 1/q = 1$. Let $0 < \gamma_1 < \gamma_2 < \infty$, and let $\zeta, \mathcal{H} \in X^p_c(\gamma_1,\gamma_2)$ be two positive functions such that ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu) < \infty$, ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^q(\nu) < \infty$, ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu) < \infty$, and ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^q(\nu) < \infty$ for all $\nu > \gamma_1 > 0$. If $0 < \lambda \leqslant \zeta(\nu)/\mathcal{H}(\nu) \leqslant B$ for some $\lambda, B \in \mathbb{R}^*_+$ and for all $\nu \in [\gamma_1,\gamma_2]$, then*

$$\begin{aligned}&{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta(\nu)\mathcal{H}(\nu)\right)\\&\quad\leqslant\mathcal{P}^*\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta^p(\nu)+\mathcal{H}^p(\nu)\right)+\mathcal{Q}^*\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\left(\zeta^q(\nu)+\mathcal{H}^q(\nu)\right),\end{aligned}$$

where $\mathcal{P}^* = 2^{p-1}B^p/(p(B+1)^p)$, $\mathcal{Q}^* = 2^{q-1}/(q(1+\lambda)^q)$.

Proof. From the given condition $\zeta(\nu)/\mathcal{H}(\nu) \leqslant B$ we obtain

$$\zeta(\nu) \leqslant \frac{B}{B+1}\big(\zeta(\nu) + \mathcal{H}(\nu)\big). \tag{29}$$

Taking the pth power of both sides of (29), we get

$$\zeta^p(\nu) \leqslant \left(\frac{B}{B+1}\right)^p \big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p. \tag{30}$$

Multiplying both sides of (30) by $(1-\sigma)/\mathsf{B}(\sigma)$, we have

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\,\zeta^p(\nu) \leqslant \left(\frac{B}{B+1}\right)^p \frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p. \tag{31}$$

Replacing ν by μ in inequality (30) and multiplying by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)\times(\nu^\beta-\mu^\beta)^{1-\sigma})$, we obtain

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\zeta^p(\mu)\\&\quad\leqslant \left(\frac{B}{B+1}\right)^p \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p, \quad \mu \in (\gamma_1, \nu).\end{aligned} \tag{32}$$

Integrating both sides of (32) with respect to μ from γ_1 to ν, we get

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^p(\mu)\,\mathrm{d}\mu\\&\quad\leqslant \left(\frac{B}{B+1}\right)^p \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p\,\mathrm{d}\mu.\end{aligned} \tag{33}$$

From (31) and (33) we get

$$\begin{aligned}&\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta^p(\nu) + \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^p(\mu)\,\mathrm{d}\mu\\&\quad\leqslant \left(\frac{B}{B+1}\right)^p \Bigg[\frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p\\&\qquad + \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p\,\mathrm{d}\mu\Bigg],\end{aligned}$$

which implies that

$${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu) \leqslant \left(\frac{B}{B+1}\right)^p {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p \tag{34}$$

Multiplying (34) by $1/p$, we get

$$\frac{1}{p}{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu) \leqslant \frac{1}{p}\left(\frac{B}{B+1}\right)^p {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p. \tag{35}$$

Again, the given condition $0 < \lambda \leqslant \zeta(\nu)/\mathcal{H}(\nu)$, we obtain

$$\mathcal{H}^q(\nu) \leqslant \left(\frac{1}{1+\lambda}\right)^q \big(\zeta(\nu)+\mathcal{H}(\nu)\big)^q. \tag{36}$$

Multiplying both sides of (36) by $(1-\sigma)/\mathsf{B}(\sigma)$, we have

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\mathcal{H}^q(\nu) \leqslant \left(\frac{1}{1+\lambda}\right)^q \frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^q. \tag{37}$$

Replacing ν by μ in inequality (36) and multiplying by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)\times(\nu^\beta-\mu^\beta)^{1-\sigma})$, we obtain

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^q(\mu)\\&\quad\leqslant \left(\frac{1}{1+\lambda}\right)^q \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^q.\end{aligned} \tag{38}$$

Integrating both sides of (38) with respect to μ from γ_1 to ν, we get

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^q(\mu)\,\mathrm{d}\mu\\&\quad\leqslant \left(\frac{1}{1+\lambda}\right)^q \frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^q\,\mathrm{d}\mu.\end{aligned} \tag{39}$$

From (37) and (39) we get

$$\begin{aligned}&\frac{1-\sigma}{\mathsf{B}(\sigma)}\mathcal{H}^q(\nu)+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^q(\mu)\,\mathrm{d}\mu\\&\quad\leqslant \left(\frac{1}{1+\lambda}\right)^q\Bigg[\frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^q\\&\qquad+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^q\,\mathrm{d}\mu\Bigg],\end{aligned}$$

which implies that

$${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^q(\nu) \leqslant \left(\frac{1}{1+\lambda}\right)^q {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^q. \tag{40}$$

Multiplying (40) by $1/q$, we get

$$\frac{1}{q}{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^q(\nu) \leqslant \frac{1}{q}\left(\frac{1}{1+\lambda}\right)^q {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^q. \tag{41}$$

From (35) and (41) we get

$$\begin{aligned}
&\frac{1}{p}{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu) + \frac{1}{q}{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^q(\nu) \\
&\quad\leqslant \frac{1}{p}\left(\frac{B}{B+1}\right)^p {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p \\
&\qquad + \frac{1}{q}\left(\frac{1}{1+\lambda}\right)^q {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^q.
\end{aligned} \tag{42}$$

Using Young's inequality,

$$\zeta(\nu)\mathcal{H}(\nu) \leqslant \frac{\zeta^p(\nu)}{p} + \frac{\mathcal{H}^q(\nu)}{q}, \tag{43}$$

and multiplying (43) by $(1-\sigma)/\mathsf{B}(\sigma)$, we get

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu)\mathcal{H}(\nu) \leqslant \frac{1-\sigma}{\mathsf{B}(\sigma)}\left(\frac{\zeta^p(\nu)}{p} + \frac{\mathcal{H}^q(\nu)}{q}\right). \tag{44}$$

Replacing ν by μ in inequality (43) and multiplying by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)\times(\nu^\beta-\mu^\beta)^{1-\sigma})$, we obtain for $\mu\in(\gamma_1,\nu)$,

$$\begin{aligned}
&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta(\mu)\mathcal{H}(\mu) \\
&\quad\leqslant \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{p\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\zeta^p(\mu) + \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{q\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^q(\mu).
\end{aligned} \tag{45}$$

Integrating both sides of (45) with respect to μ from γ_1 to ν, we get

$$\begin{aligned}
&\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta(\mu)\mathcal{H}(\mu)\,\mathrm{d}\mu \\
&\quad\leqslant \frac{\sigma\beta^{1-\eta}\nu^k}{p\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^p(\mu)\,\mathrm{d}\mu \\
&\qquad + \frac{\sigma\beta^{1-\eta}\nu^k}{q\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^q(\mu)\,\mathrm{d}\mu.
\end{aligned} \tag{46}$$

From (44) and (46) we obtain

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\zeta(\nu)\mathcal{H}(\nu)+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta(\mu)\mathcal{H}(\mu)\,\mathrm{d}\mu$$

$$\leqslant\frac{1-\sigma}{\mathsf{B}(\sigma)}\left(\frac{\zeta^p(\nu)}{p}+\frac{\mathcal{H}^q(\nu)}{q}\right)+\frac{\sigma\beta^{1-\eta}\nu^k}{p\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\zeta^p(\mu)\,\mathrm{d}\mu$$

$$+\frac{\sigma\beta^{1-\eta}\nu^k}{q\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\mathcal{H}^q(\mu)\,\mathrm{d}\mu.$$

This implies

$${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)\mathcal{H}(\nu)\big)\leqslant\frac{1}{p}\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu)+\frac{1}{q}\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^q(\nu).\tag{47}$$

From (42) and (47) we have

$$\begin{aligned}{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)\mathcal{H}(\nu)\big)\leqslant{}&\frac{1}{p}\left(\frac{B}{B+1}\right)^p{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p\\&+\frac{1}{q}\left(\frac{1}{1+\lambda}\right)^q{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^q.\end{aligned}\tag{48}$$

Using the elementary inequality

$$(\zeta+\mathcal{H})^m\leqslant2^{m-1}\big(\zeta^m+\mathcal{H}^m\big),\quad\zeta,\mathcal{H}\geqslant0,\ m>1,\tag{49}$$

for $m=p$ and multiplying (49) by $(1-\sigma)/\mathsf{B}(\sigma)$, we get

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p\leqslant2^{p-1}\frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta^p(\nu)+\mathcal{H}^p(\nu)\big).\tag{50}$$

Again, for $m=p$ and multiplying (49) by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma})$, we get

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p\\&\quad\leqslant2^{p-1}\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta^p(\mu)+\mathcal{H}^p(\mu)\big),\quad\mu\in(\gamma_1,\nu).\end{aligned}\tag{51}$$

Integrating both sides of (51) with respect to μ from γ_1 to ν, we get

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p\,\mathrm{d}\mu\\&\quad\leqslant2^{p-1}\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta^p(\mu)+\mathcal{H}^p(\mu)\big)\,\mathrm{d}\mu.\end{aligned}\tag{52}$$

Now, from (50) and (52) we get

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int\limits_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta(\mu)+\mathcal{H}(\mu)\big)^p\,\mathrm{d}\mu$$
$$\leqslant 2^{p-1}\Bigg[\frac{1-\sigma}{\mathsf{B}(\sigma)}\big(\zeta^p(\nu)+\mathcal{H}^p(\nu)\big)$$
$$+\frac{\sigma\beta^{1-\eta}\nu^k}{\mathsf{B}(\sigma)\Gamma(\sigma)}\int\limits_{\gamma_1}^{\nu}\frac{\mu^{\beta(\delta+1)-1}}{(\nu^\beta-\mu^\beta)^{1-\sigma}}\big(\zeta^p(\mu)+\mathcal{H}^p(\mu)\big)\,\mathrm{d}\mu\Bigg].$$

This implies that

$$ {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p\leqslant 2^{p-1}\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta^p(\nu)+\mathcal{H}^p(\nu)\big). \tag{53}$$

Now, applying the similar process for $m=q$, we get

$$ {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^q\leqslant 2^{q-1}\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta^q(\nu)+\mathcal{H}^q(\nu)\big). \tag{54}$$

Using (53), (54), and (48), we get

$$ {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)\mathcal{H}(\nu)\big)$$
$$\leqslant\mathcal{P}^*\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta^p(\nu)+\mathcal{H}^p(\nu)\big)+\mathcal{Q}^*\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta^q(\nu)+\mathcal{H}^q(\nu)\big).$$

where $\mathcal{P}^*=2^{p-1}B^p/(p(B+1)^p)$, $\mathcal{Q}^*=2^{q-1}/(q(1+\lambda)^q)$. The result is proved. □

Corollary 5. *If we take $\eta=\sigma$, $k=0$, and $\delta=0$ in Theorem 7, the following inequality for the ABK-fractional integral holds*:

$$ {}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu)\mathcal{H}(\nu)\big)$$
$$\leqslant\mathcal{P}^*\,{}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\big(\zeta^p(\nu)+\mathcal{H}^p(\nu)\big)+\mathcal{Q}^*\,{}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\big(\zeta^q(\nu)+\mathcal{H}^q(\nu)\big),$$

where $\mathcal{P}^*=2^{p-1}B^p/(p(B+1)^p)$, $\mathcal{Q}^*=2^{q-1}/(q(1+\lambda)^q)$.

Remark 6. If we take $k=0$, $\delta=0$, and $\beta=1$ in Theorem 7, the following inequality for the AB-fractional integral holds:

$$ {}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu)\mathcal{H}(\nu)\big)\leqslant\mathcal{P}^*\,{}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\big(\zeta^p(\nu)+\mathcal{H}^p(\nu)\big)+\mathcal{Q}^*\,{}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\big(\zeta^q(\nu)+\mathcal{H}^q(\nu)\big),$$

where $\mathcal{P}^*=2^{p-1}B^p/(p(B+1)^p)$, $\mathcal{Q}^*=2^{q-1}/(q(1+\lambda)^q)$.

Khan et al. presented this result for the AB-fractional integral in [20].

Theorem 8. *Let $\sigma \in (0,1]$, $\beta > 0$, $\eta, k \in \mathbb{R}$, $\delta \geqslant 0$, and $p \geqslant 1$. Let $0 < \gamma_1 < \gamma_2 < \infty$, and $\zeta, \mathcal{H} \in X_c^p(\gamma_1,\gamma_2)$ be two positive functions such that ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta(\nu) < \infty$ and ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}(\nu) < \infty$, for all $\nu > \gamma_1 > 0$. If $0 < \lambda \leqslant \zeta(\nu)/\mathcal{H}(\nu) \leqslant B$ for some $\lambda, B \in \mathbb{R}_+^*$ and for all $\nu \in [\gamma_1,\gamma_2]$, then*

$$\begin{aligned}\frac{1}{B}\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)\mathcal{H}(\nu)\big) &\leqslant \frac{1}{(B+1)(\lambda+1)}\Big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^2\Big)\\ &\leqslant \frac{1}{\lambda}\,{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)\mathcal{H}(\nu)\big).\end{aligned}$$

Proof. Since

$$0 < \lambda \leqslant \frac{\zeta(\nu)}{\mathcal{H}(\nu)} \leqslant B, \tag{55}$$

this can be written as

$$\frac{1}{B} \leqslant \frac{\mathcal{H}(\nu)}{\zeta(\nu)} \leqslant \frac{1}{\lambda}. \tag{56}$$

Therefore, from (55) and (56) we get

$$\mathcal{H}(\nu)(\lambda+1) \leqslant \zeta(\nu)+\mathcal{H}(\nu) \leqslant \mathcal{H}(\nu)(B+1), \tag{57}$$

and

$$\zeta(\nu)\,\frac{B+1}{B} \leqslant \zeta(\nu)+\mathcal{H}(\nu) \leqslant \zeta(\nu)\,\frac{\lambda+1}{\lambda}. \tag{58}$$

Using (57) and (58), we get

$$\frac{1}{B}\zeta(\nu)\mathcal{H}(\nu) \leqslant \frac{(\zeta(\nu)+\mathcal{H}(\nu))^2}{(B+1)(\lambda+1)} \leqslant \frac{1}{\lambda}\zeta(\nu)\mathcal{H}(\nu). \tag{59}$$

Multiplying (59) by $(1-\sigma)/\mathsf{B}(\sigma)$, we obtain

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\,\frac{1}{B}\zeta(\nu)\mathcal{H}(\nu) \leqslant \frac{1-\sigma}{\mathsf{B}(\sigma)}\,\frac{(\zeta(\nu)+\mathcal{H}(\nu))^2}{(B+1)(\lambda+1)} \leqslant \frac{1-\sigma}{\mathsf{B}(\sigma)}\,\frac{1}{\lambda}\zeta(\nu)\mathcal{H}(\nu). \tag{60}$$

Replacing ν by μ in inequality (59) and multiplying by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)\times(\nu^\beta-\mu^\beta)^{1-\sigma})$, where $\mu \in (\gamma_1,\nu)$, we obtain

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)\big(\nu^\beta-\mu^\beta\big)^{1-\sigma}}\,\frac{1}{B}\zeta(\mu)\mathcal{H}(\mu)\\ &\quad\leqslant \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\frac{(\zeta(\mu)+\mathcal{H}(\mu))^2}{(B+1)(\lambda+1)}\\ &\quad\leqslant \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\frac{1}{\lambda}\zeta(\mu)\mathcal{H}(\mu).\end{aligned} \tag{61}$$

Integrating (61) with respect to μ from γ_1 to ν, we get

$$\begin{aligned}
&\int_{\gamma_1}^{\nu} \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\frac{1}{B}\zeta(\mu)\mathcal{H}(\mu)\,\mathrm{d}\mu \\
&\quad\leqslant \int_{\gamma_1}^{\nu} \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\frac{(\zeta(\mu)+\mathcal{H}(\mu))^2}{(B+1)(\lambda+1)}\,\mathrm{d}\mu \\
&\quad\leqslant \int_{\gamma_1}^{\nu} \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta-\mu^\beta)^{1-\sigma}}\,\frac{1}{\lambda}\zeta(\mu)\mathcal{H}(\mu)\,\mathrm{d}\mu. \qquad (62)
\end{aligned}$$

Using (60) and (62), we get

$$\begin{aligned}
\frac{1}{B}{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)\mathcal{H}(\nu)\big) &\leqslant \frac{1}{(B+1)(\lambda+1)}\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^2\big) \\
&\leqslant \frac{1}{\lambda}{}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)\mathcal{H}(\nu)\big).
\end{aligned}$$

Corollary 6. *If we take $\eta=\sigma$, $k=0$, and $\delta=0$ in Theorem 8, the following inequality for the ABK-fractional integral holds*:

$$\begin{aligned}
\frac{1}{B}{}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu)\mathcal{H}(\nu)\big) &\leqslant \frac{1}{(B+1)(\lambda+1)}\big({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^2\big) \\
&\leqslant \frac{1}{\lambda}{}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu)\mathcal{H}(\nu)\big).
\end{aligned}$$

Remark 7. If we take $k=0$, $\delta=0$, and $\beta=1$ in Theorem 8, the following inequality for the AB-fractional integral holds:

$$\begin{aligned}
\frac{1}{B}{}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu)\mathcal{H}(\nu)\big) &\leqslant \frac{1}{(B+1)(\lambda+1)}\big({}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^2\big) \\
&\leqslant \frac{1}{\lambda}{}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu)\mathcal{H}(\nu)\big).
\end{aligned}$$

Khan et al. presented this inequality for the AB-fractional integral in [20].

Theorem 9. *Let $\sigma\in(0,1]$, $\beta>0$, $\eta,k\in\mathbb{R}$, $\delta\geqslant 0$, and $p\geqslant 1$. Let $0<\gamma_1<\gamma_2<\infty$, and let $\zeta,\mathcal{H}\in X^p_c(\gamma_1,\gamma_2)$ be two positive functions such that ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu)<\infty$ and ${}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu)<\infty$ for all $\nu>\gamma_1>0$. If $0<\lambda_1\leqslant\zeta(\nu)\leqslant B_1$, $0<\lambda_2\leqslant\mathcal{H}(\nu)\leqslant B_2$ for all $\nu\in[\gamma_1,\gamma_2]$, then*

$$\begin{aligned}
&\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu)\big)^{1/p}+\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu)\big)^{1/p} \\
&\quad\leqslant \Theta\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu)+\mathcal{H}(\nu)\big)^p\big)^{1/p},
\end{aligned}$$

where $\Theta=B_1(\lambda_1+B_2)+B_2(\lambda_2+B_1)/((\lambda_1+B_2)(\lambda_2+B_1))$.

Proof. By the given condition $0 < \lambda_2 \leqslant \mathcal{H}(\nu) \leqslant B_2$, we get

$$\frac{1}{B_2} \leqslant \frac{1}{\mathcal{H}(\nu)} \leqslant \frac{1}{\lambda_2}. \tag{63}$$

Now, using (63) and the given condition $\lambda_1 \leqslant \zeta(\nu) \leqslant B_1$, we get

$$\frac{\lambda_1}{B_2} \leqslant \frac{\zeta(\nu)}{\mathcal{H}(\nu)} \leqslant \frac{B_1}{\lambda_2}. \tag{64}$$

From (64) we have

$$\mathcal{H}(\nu)\frac{\lambda_1}{B_2} \leqslant \zeta(\nu) \quad\Longrightarrow\quad \mathcal{H}(\nu) \leqslant \frac{B_2}{\lambda_1 + B_2}\bigl(\zeta(\nu) + \mathcal{H}(\nu)\bigr). \tag{65}$$

From (65) we get

$$\mathcal{H}^p(\nu) \leqslant \left(\frac{B_2}{\lambda_1 + B_2}\right)^p \bigl(\zeta(\nu) + \mathcal{H}(\nu)\bigr)^p. \tag{66}$$

Again, from (64) we have

$$\zeta(\nu)\frac{\lambda_2}{B_1} \leqslant \mathcal{H}(\nu) \quad\Longrightarrow\quad \zeta(\nu) \leqslant \frac{B_1}{\lambda_2 + B_1}\bigl(\zeta(\nu) + \mathcal{H}(\nu)\bigr). \tag{67}$$

From (67) we get

$$\zeta^p(\nu) \leqslant \left(\frac{B_1}{\lambda_2 + B_1}\right)^p \bigl(\zeta(\nu) + \mathcal{H}(\nu)\bigr)^p. \tag{68}$$

Multiplying (66) by $(1-\sigma)/\mathsf{B}(\sigma)$, we obtain

$$\frac{1-\sigma}{\mathsf{B}(\sigma)}\mathcal{H}^p(\nu) \leqslant \frac{1-\sigma}{\mathsf{B}(\sigma)}\left(\frac{B_2}{\lambda_1 + B_2}\right)^p \bigl(\zeta(\nu) + \mathcal{H}(\nu)\bigr)^p. \tag{69}$$

Replacing ν by μ in inequality (66) and multiplying by $\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}/(\mathsf{B}(\sigma)\Gamma(\sigma)\times(\nu^\beta - \mu^\beta)^{1-\sigma})$, we obtain

$$\begin{aligned}&\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta - \mu^\beta)^{1-\sigma}}\mathcal{H}^p(\mu)\\&\quad\leqslant \frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta - \mu^\beta)^{1-\sigma}}\left(\frac{B_2}{\lambda_1 + B_2}\right)^p \bigl(\zeta(\mu) + \mathcal{H}(\mu)\bigr)^p.\end{aligned} \tag{70}$$

Integrating (70) with respect to μ from γ_1 to ν, we get

$$\begin{aligned}&\int\limits_{\gamma_1}^{\nu}\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta - \mu^\beta)^{1-\sigma}}\mathcal{H}^p(\mu)\,\mathrm{d}\mu\\&\quad\leqslant \left(\frac{B_2}{\lambda_1 + B_2}\right)^p\int\limits_{\gamma_1}^{\nu}\frac{\sigma\beta^{1-\eta}\nu^k\mu^{\beta(\delta+1)-1}}{\mathsf{B}(\sigma)\Gamma(\sigma)(\nu^\beta - \mu^\beta)^{1-\sigma}}\bigl(\zeta(\mu) + \mathcal{H}(\mu)\bigr)^p\,\mathrm{d}\mu.\end{aligned} \tag{71}$$

From (69) and (71) it follows that

$$ {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu) \leqslant \left(\frac{B_2}{\lambda_1 + B_2}\right)^p {}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p \tag{72} $$

Taking $(1/p)$th power of both sides of (72), we get

$$ \big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu)\big)^{1/p} \leqslant \frac{B_2}{\lambda_1 + B_2}\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p\big)^{1/p}. \tag{73} $$

By the similar steps, from (68) we get

$$ \big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu)\big)^{1/p} \leqslant \frac{B_1}{\lambda_2 + B_1}\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p\big)^{1/p}. \tag{74} $$

Thus, from (73) and (74) we get

$$ \begin{aligned} &\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\zeta^p(\nu)\big)^{1/p} + \big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}\mathcal{H}^p(\nu)\big)^{1/p} \\ &\quad \leqslant \varTheta\big({}^{\mathrm{GAB}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma,\eta}_{\nu,\delta,k}(\zeta(\nu) + \mathcal{H}(\nu))^p\big)^{1/p}, \end{aligned} $$

where $\varTheta = (B_1(\lambda_1 + B_2) + B_2(\lambda_2 + B_1))/((\lambda_1 + B_2)(\lambda_2 + B_1))$. □

Corollary 7. *If we take* $\eta = \sigma$, $k = 0$, *and* $\delta = 0$ *in Theorem* 9, *the following inequality for the ABK-fractional integral holds*:

$$ \big({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\zeta^p(\nu)\big)^{1/p} + \big({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\mathcal{H}^p(\nu)\big)^{1/p} \leqslant \varTheta\big({}^{\mathrm{ABK}_\beta}_{\gamma_1+}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p\big)^{1/p}, $$

where $\varTheta = (B_1(\lambda_1 + B_2) + B_2(\lambda_2 + B_1))/((\lambda_1 + B_2)(\lambda_2 + B_1))$.

Corollary 8. *If we take* $k = 0$, $\delta = 0$, *and* $\beta = 1$ *in Theorem* 9, *the following inequality for the AB-fractional integral holds*:

$$ \big({}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\zeta^p(\nu)\big)^{1/p} + \big({}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\mathcal{H}^p(\nu)\big)^{1/p} \leqslant \varTheta\big({}^{\mathrm{AB}}_{\gamma_1}\mathcal{I}^{\sigma}_{\nu}\big(\zeta(\nu) + \mathcal{H}(\nu)\big)^p\big)^{1/p}, $$

where $\varTheta = (B_1(\lambda_1 + B_2) + B_2(\lambda_2 + B_1))/((\lambda_1 + B_2)(\lambda_2 + B_1))$.

7 Conclusions and future work

In this article, we introduced the left-sided and right-sided GAB-fractional integrals. In Theorem 3, we stated the conditions for the GAB-fractional integral operator to be bounded in the space $X^p_c(a, b)$. Also, a fractional product-integration formula for this operator is provided in Theorem 4. To develop the area of integral inequalities, we obtained the reverse Minkowski's inequality and reverse Hölder-type inequality for the left-sided GAB-fractional integral. Moreover, we established some other types of integral inequalities for the considered fractional integral, and several special cases were discussed. Theorems 5–8 that we presented are generalizations of the existing results obtained by Khan et al. [20] for the AB-fractional integral. Since the GAB-fractional integral is a generalized form of the existing integrals, we can do a lot of research based on this integral operator. In the future, one can explore more results in the area of integral inequalities for this generalized framework, such as the Ostrowski-, Chebyshev-, Gruss-type, etc.

Conflicts of interest. The authors declare no conflicts of interest.

Acknowledgment. The authors wish to express their gratitude to the Editor and the reviewers for their invaluable comments and suggestions.

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