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Generalized solutions for singular double-phase elliptic equations under mixed boundary conditions*

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Abstract. In this article, we investigate at least one or two generalized solutions for double-phase singular elliptic equations with Hardy potential. We show the existence of at least one or two distinct generalized solutions under mixed boundary conditions via variational methods when the nonlinearity f satisfying suitable hypotheses.

Keywords: elliptic equations, double phase, Hardy potential, mixed boundary conditions.

1 Introduction

Elliptic equations in bounded domains with singular potential serve as a model in applied science, including the study of heat conduction in electrically conducting materials, singular minimal surfaces, and non-Newtonian fluids. In such cases, Dirichlet–Neumann-type mixed boundary conditions are employed to describe a multitude of engineering and physical phenomena, including the state of stress and strain on an elastic surface in mechanics and the solidification and melting of materials in industrial processes. The objective of this article is to identify at least one and two generalized solutions of this type for elliptic equations.

Recently, researches on the numbers of the existence of generalized solutions to nonlinear differential equations via variational methods have received wide attention (see, for example, [4, 5, 7–10]).

Khodabakhshi et al. [6] studied some singular elliptic equations involving p-Laplace operators subject to Dirichlet boundary conditions in a bounded domain with smooth

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boundary

$$-\Delta_p u + \frac{|u|^{p-2}u}{|x|^p} = \lambda f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

and the existence of at least two distinct generalized solutions for this type of singular elliptic problems was obtained.

In [11], we dealt with the existence of at least three weak solutions to the following elliptic equations with mixed boundary conditions:

$$-\operatorname{div}\mathbf{A}(x,\nabla u) + \frac{a(x)}{|x|^p}|u|^{p-2}u = \lambda f(x,u) \qquad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma_1,$$
$$\mathbf{A}(x,\nabla u)\cdot\nu = \mu g\big(x,\gamma(u)\big) \quad \text{on } \Gamma_2.$$

The existence of at least three solutions was obtained when f and g satisfy corresponding growth conditions.

In this paper, we focus on the existence of at least one or two nontrivial generalized solutions to the following singular elliptic equations with mixed boundary conditions:

$$\begin{aligned}
&-\Delta_p u - \Delta_q u + \frac{a(x)|u|^{p-2}u}{|x|^p} + \frac{b(x)|u|^{q-2}u}{|x|^q} = \lambda f(x,u) \quad \text{in } \Omega,\\
&u = 0 \quad \text{on } \Gamma_1,\\
&\big(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\big)\cdot\nu + c(x)|u|^{t-2}u = 0 \quad \text{on } \Gamma_2,
\end{aligned} \tag{1}$$

where

$$\Delta_p u = |\Delta u|^{p-2}\Delta u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big),$$
$$\Delta_q u = |\Delta u|^{q-2}\Delta u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big),$$

$1 < q < p < N$, Ω is an open bounded subset in $\mathbb{R}^N$ ($N \geqslant 3$) with smooth boundary $\partial\Omega$, ν is the outward normal vector field on $\partial\Omega$, Γ_1 and Γ_2 are two smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega$, $\overline{\Gamma_1} \cap \overline{\Gamma_2}$ is a $(N-2)$-dimensional submanifold of $\partial\Omega$, $\lambda > 0$, $a(x), b(x) \in L^\infty(\Omega)$ are positive functions, $a_0 = \operatorname{ess\,sup}_\Omega a(x)$, $b_0 = \operatorname{ess\,sup}_\Omega b(x)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

(f_1) $|f(x,u)| \leqslant M_1 + M_2|u|^{s-1}$, a.a. $(x,u) \in \Omega \times \mathbb{R}$, where $p < s < p* = Np/(N-p)$, M_1, M_2 are positive constants, and $p < t < p_* = (N-1)p/(N-p)$, $0 < c(x) \in L^\infty(\Gamma_2)$ with $c_0 = \operatorname{ess\,sup}_\Omega c(x)$.

The divergence operator in (1) is known as double-phase operator, which includes the general p-Laplacian operators. The aim of this paper is to present results of the existence of generalized solutions of problem (1) under very general assumptions on the nonlinear terms. The idea is to use corresponding critical point theorems applied to the energy functional of (1) in order to get the existence of bounded, nontrivial generalized solutions located within a precise interval.

2 Preliminaries and variational structure

Let Ω be an open bounded subset in $\mathbb{R}^N$ ($N \geqslant 3$), $1 < p < N$, and let

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega)\colon\ u|_{\Gamma_1} = 0\}$$

be the Sobolev space with the norm

$$\|u\| = \|\nabla u\|_p = \||\nabla u|\|_p = \biggl(\int\limits_\Omega |\nabla u|^p \,\mathrm{d}x\biggr)^{1/p},$$

where

$$|\nabla u| = \Biggl(\sum_{i=1}^N \biggl|\frac{\partial u}{\partial x_i}\biggr|^2\Biggr)^{1/2}.$$

Define the functional $\mathcal{I}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

where

$$\begin{aligned}\Phi(u) &:= \frac{1}{p}\biggl(\int\limits_\Omega |\nabla u|^p \,\mathrm{d}x + \int\limits_\Omega \frac{a(x)|u|^p}{|x|^p}\,\mathrm{d}x\biggr) + \frac{1}{q}\biggl(\int\limits_\Omega |\nabla u|^q \,\mathrm{d}x + \int\limits_\Omega \frac{b(x)|u|^q}{|x|^q}\,\mathrm{d}x\biggr),\\ \Psi(u) &:= \int\limits_\Omega F(x,u)\,\mathrm{d}x - \frac{1}{\lambda t}\int\limits_{\Gamma_2} c(x)|u|^t \,\mathrm{d}\sigma,\end{aligned}$$

and $F(x,u) = \int_0^u f(x,\tau)\,\mathrm{d}\tau$ for all $(x,u) \in \Omega \times \mathbb{R}$.

It is clear that Φ and Ψ are continuously Gâteaux differentiable with

$$\begin{aligned}\Phi'(u)[v] = &\int\limits_\Omega |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}x + \int\limits_\Omega |\nabla u|^{q-2}\nabla u \nabla v \,\mathrm{d}x\\ &+ \int\limits_\Omega \frac{a(x)|u|^{p-2}uv}{|x|^p}\,\mathrm{d}x + \int\limits_\Omega \frac{b(x)|u|^{q-2}uv}{|x|^q}\,\mathrm{d}x\end{aligned}$$

and

$$\Psi'(u)[v] = \int\limits_\Omega f(x,u)v \,\mathrm{d}x - \frac{1}{\lambda}\int\limits_{\Gamma_2} c(x)|u|^{t-2}uv \,\mathrm{d}\sigma.$$

We say that $u \in W_0^{1,p}(\Omega)$ is a generalized solution of problem (1) if

$$\mathcal{I}'_\lambda(u)[v] = \Phi'(u)[v] - \lambda\Psi'(u)[v] = 0 \quad \forall v \in W_0^{1,p}(\Omega).$$

Lemma 1. *The functional Φ' is Gâteaux differentiable, coercive, and strictly monotone in $W_0^{1,p}(\Omega)$.*

Proof. For any $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, one has

$$\begin{aligned}\Phi'(u)[u] &= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla u \,\mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla u \,\mathrm{d}x \\ &\quad + \int_\Omega \frac{a(x)|u|^{p-2}u}{|x|^p} u \,\mathrm{d}x + \int_\Omega \frac{b(x)|u|^{q-2}u}{|x|^q} u \,\mathrm{d}x \\ &\geqslant \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |\nabla u|^q \,\mathrm{d}x = \|u\|^p + \|u\|^q,\end{aligned}$$

thus

$$\liminf_{\|u\|\to\infty} \frac{\Phi'(u)[u]}{\|u\|} \geqslant \liminf_{\|u\|\to\infty} \frac{\|u\|^p + \|u\|^q}{\|u\|} = +\infty,$$

then Φ' is coercive thanks to $p, q > 1$.

According to (2.2) of [12], there is a constant C_p such that

$$\left(|x|^{p-2}x - |y|^{p-2}y,\, x - y\right) \geqslant \begin{cases} C_p|x-y|^p & \text{if } p \geqslant 2, \\ \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2, \end{cases}$$

where $(\cdot,\cdot)$ is the usual inner product.

Thus, for any $u, v \in W_0^{1,p}(\Omega)$ satisfying $u \neq v$, there exists positive constant C_p such that, if $p \geqslant 2$,

$$\begin{aligned}&\left(\Phi'(u) - \Phi'(v),\, u - v\right) \\ &\quad = \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)(\nabla u - \nabla v) \,\mathrm{d}x \\ &\qquad + \int_\Omega \left(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\right)(\nabla u - \nabla v) \,\mathrm{d}x \\ &\qquad + \int_\Omega \left(\frac{a(x)|u|^{p-2}}{|x|^p} u(u-v) - \frac{a(x)|v|^{p-2}}{|x|^p} v(u-v)\right) \mathrm{d}x \\ &\qquad + \int_\Omega \left(\frac{b(x)|u|^{q-2}}{|x|^p} u(u-v) - \frac{b(x)|v|^{q-2}}{|x|^p} v(u-v)\right) \mathrm{d}x \\ &\quad \geqslant C_p \int_\Omega |\nabla u - \nabla v|^p \,\mathrm{d}x = C_p\|u-v\|^p > 0.\end{aligned}$$

Similarly, if $1 < p < 2$, then

$$\left(\Phi'(u) - \Phi'(v),\, u - v\right) \geqslant \int_\Omega \frac{C_p|\nabla x - \nabla y|^2}{(|\nabla x| + |\nabla y|)^{2-p}} \,\mathrm{d}x > 0,$$

thus we have that Φ' is strictly monotone in $W_0^{1,p}(\Omega)$. □

Lemma 2. *The functional Φ' is a mapping of (S_+)-type, i.e., if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\overline{\lim}_{n\to\infty}(\Phi'(u_n) - \Phi'(u),\, u_n - u) \leqslant 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

Proof. Let $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, and let $\overline{\lim}_{n\to\infty}(\Phi'(u_n) - \Phi'(u),\, u_n - u) \leqslant 0$.

Noting that Φ' is strictly monotone in $W_0^{1,p}(\Omega)$, one has

$$0 < \big(\Phi'(u_n) - \Phi'(u),\, u_n - u\big) \leqslant \overline{\lim_{n\to\infty}} \big(\Phi'(u_n) - \Phi'(u),\, u_n - u\big) \leqslant 0,$$

while

$$\begin{aligned}
&\big(\Phi'(u_n) - \Phi'(u),\, u_n - u\big) \\
&\quad= \int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,\mathrm{d}x \\
&\qquad+ \int_\Omega \big(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u\big)(\nabla u_n - \nabla u)\,\mathrm{d}x \\
&\qquad+ \int_\Omega \left(\frac{a(x)|u_n|^{p-2}}{|x|^p} u_n(u_n - u) - \frac{a(x)|u|^{p-2}}{|x|^p} u(u_n - u)\right)\mathrm{d}x \\
&\qquad+ \int_\Omega \left(\frac{b(x)|u_n|^{q-2}}{|x|^p} u_n(u_n - u) - \frac{b(x)|u|^{q-2}}{|x|^p} u(u_n - u)\right)\mathrm{d}x.
\end{aligned}$$

Thus we get

$$\overline{\lim_{n\to\infty}} \int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,\mathrm{d}x \leqslant 0,$$

then

$$\begin{aligned}
0 &\leqslant C_p\|u_n - u\|^p \\
&\leqslant \int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,\mathrm{d}x \\
&\leqslant \overline{\lim_{n\to\infty}} \int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)(\nabla u_n - \nabla u)\,\mathrm{d}x \\
&\leqslant 0,
\end{aligned}$$

i.e., $u_n \to u$ in $W_0^{1,p}(\Omega)$. □

Lemma 3. *The functional Φ' is a homeomorphism.*

Proof. The strictly monotonicity of Φ' leads to the injectivity. Since Φ' is coercive, thus Φ' is a surjection, and Φ' has an inverse mapping.

Next, we prove that the inverse mapping $(\Phi')^{-1}$ is continuous.

Let $\tilde{f}_n, \tilde{f} \in W_0^{1,p}(\Omega)^*$, $\tilde{f}_n \to \tilde{f}$, we only need to prove $(\Phi')^{-1}(\tilde{f}_n) \to (\Phi')^{-1}(\tilde{f})$.

Indeed, let $(\Phi')^{-1}(\tilde{f}_n) = u_n$, $(\Phi')^{-1}\tilde{f} = u$, then $\Phi'(u_n) = \tilde{f}_n, \Phi'(u) = \tilde{f}$, thus u_n is bounded according to the coercivity of Φ'. Without loss of generality, we note $u_n \rightharpoonup u_0$, which implies

$$\lim_{n\to\infty} \big(\Phi'(u_n) - \Phi'(u),\, u_n - u_0\big) = \lim_{n\to\infty} \big(\tilde{f}_n - \tilde{f},\, u_n - u_0\big) = 0.$$

Thus $u_n \to u_0$ because that Φ' is of (S_+)-type, thus $\Phi'(u_n) \to \Phi'(u_0)$. Combining with $\Phi'(u_n) \to \Phi'(u)$, one has $\Phi'(u) = \Phi'(u_0)$, in view of that Φ' is an injection, one has $u = u_0$ and $u_n \to u$, that is, $(\Phi')^{-1}(\tilde{f}_n) \to (\Phi')^{-1}(\tilde{f})$, so $(\Phi')^{-1}$ is continuous. □

Condition (f_1) and the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, $1 \leqslant r < p^* = Np/(N-p)$, and $W_0^{1,p}(\Omega) \hookrightarrow L^t(\partial\Omega) \hookrightarrow L^t(\Gamma_2)$, $1 \leqslant t < p_* = (N-1)p/(N-p)$, imply the following compactness.

Lemma 4. *The functional $\Psi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is compact.*

The following critical points theorems are the main tools to obtain our results.

Theorem 1. (*See* [2, *Thm.* 3.2].) *Let X be a real Banach space, and let $\Phi, \Psi : X \to R$ be two continuous Gâteaux differentiable functions such that Φ is bounded from below, and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{\Phi(u)<r} \Psi(u) < +\infty$ and assume that for each*

$$\lambda \in \left(0, \frac{r}{\sup_{\Phi(u)\leqslant r} \Psi(u)}\right),$$

the functional $\Phi - \lambda\Psi$ satisfies the Palais–Smale condition, and it is unbounded from below. Then for each $\lambda \in (0, r/\sup_{\Phi(u)\leqslant r} \Psi(u))$, the functional $\Phi - \lambda\Psi$ admits at least two distinct critical points.

Theorem 2. (*See* [3, *Thm.* 2.1].) *Let X be a real Banach space, and let $\Phi, \Psi : X \to R$ be two continuous Gâteaux differentiable functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\bar{u} \in X$ with $0 < \Phi(\bar{u}) < r$ such that*

$$\frac{1}{r} \sup_{\Phi(u)\leqslant r} \Psi(u) < \frac{\Psi(\bar{u})}{\Phi(\bar{u})},$$

and for each $\lambda \in (\Phi(\bar{u})/\Psi(\bar{u}), r/\sup_{\Phi(u)\leqslant r} \Psi(u))$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the Palais–Smale condition, and it is unbounded from below. Then for each $\lambda \in (\Phi(\bar{u})/\Psi(\bar{u}), r/\sup_{\Phi(x)\leqslant r} \Psi(u))$, the functional I_λ admits at least two nonzero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3 Main results

In this section, two theorems about the existence of at least one or two nontrivial generalized solutions to problem (1) are obtained.

Theorem 3. *Suppose* (f_1) *holds. Furthermore, there are* $\mu > t$ *and* $R > 0$ *such that*

(f_2) $0 < \mu F(x,u) \leqslant f(x,u)u$ *for all* $x \in \Omega$, $|u| \geqslant R$.

Then problem (1) *has at least two solutions when*

$$\lambda \in \left(0, \frac{s}{sM_1c_1p^{1/p} + M_2c_sp^{s/p}}\right),$$

where c_1 *is the constant of the embedding* $W_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$, *and* c_s *is the constant of the embedding* $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$.

Proof. Note that

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega \frac{a(x)|u(x)|^p}{|x|^p}\right)\mathrm{d}x \\ &\quad + \frac{1}{q}\left(\int_\Omega |\nabla u|^q \,\mathrm{d}x + \int_\Omega \frac{a(x)|u(x)|^q}{|x|^q}\right)\mathrm{d}x \\ &\geqslant \frac{1}{p}\|u\|^p + \frac{1}{q}\|u\|^q,\end{aligned}$$

thus Φ is bounded from below.

Let $\{u_n\} \in W_0^{1,p}(\Omega)$ such that $\{I_\lambda(u_n)\}$ is bounded, and let $I'_\lambda(u_n) \to 0$ as $n \to +\infty$, i.e., there is $m > 0$ independent from n such that

$$\left|I_\lambda(u_n)\right| \leqslant M,$$

and for n large enough, one has

$$\left|I'_\lambda(u_n)u_n\right| \leqslant \left\|I'_\lambda(u_n)\right\|_{(W_0^{1,p}(\Omega))^*}\|u_n\| \leqslant \|u_n\|. \tag{2}$$

Thus, when $|u_n| \geqslant R$, one has

$$\begin{aligned}&\mu I_\lambda(u_n) - I'_\lambda(u_n)u_n \\ &\quad = \frac{\mu}{p}\left(\int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega \frac{a(x)|u_n|^p}{|x|^p}\,\mathrm{d}x\right) + \frac{\mu}{q}\left(\int_\Omega |\nabla u_n|^q \,\mathrm{d}x + \int_\Omega \frac{b(x)|u_n|^q}{|x|^q}\,\mathrm{d}x\right) \\ &\qquad - \mu\int_\Omega F(x,u_n)\,\mathrm{d}x + \frac{\mu}{\lambda t}\int_{\Gamma_2} c(x)|u_n|^t\,\mathrm{d}\sigma \\ &\qquad - \int_\Omega |\nabla u_n|^p \,\mathrm{d}x - \int_\Omega \frac{a(x)|u_n|^p}{|x|^p}\,\mathrm{d}x - \int_\Omega |\nabla u_n|^q \,\mathrm{d}x - \int_\Omega \frac{b(x)|u_n|^q}{|x|^q}\,\mathrm{d}x \\ &\qquad + \int_\Omega f(x,u_n)u_n\,\mathrm{d}x - \frac{1}{\lambda}\int_{\Gamma_2} c(x)|u_n|^t\,\mathrm{d}\sigma \\ &\quad \geqslant \left(\frac{\mu}{p} - 1\right)\|u_n\|^p + \left(\frac{\mu}{q} - 1\right)\|u_n\|^q.\end{aligned}$$

Using (2), we get

$$\mu M + \|u_n\| \geqslant I_\lambda(u_n) - I'_\lambda(u_n)u_n \geqslant \left(\frac{\mu}{p} - 1\right)\|u_n\|^p + \left(\frac{\mu}{q} - 1\right)\|u_n\|^q,$$

which implies that $\{u_n\}$ is bounded in view of $\mu > t > p > q$.

Without loss of generality, we suppose $u_n \rightharpoonup u$, thus $\Psi'(u_n) \to \Psi'(u)$ because of the compactness of Ψ'. Combining with $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \to 0$, one has $\Phi'(u_n) \to \lambda\Psi'(u)$. Since Φ' is a homeomorphism, then $u_n \to u$. Thus one has that I_λ satisfies the Palais–Smale condition.

Next, we prove that I_λ is unbounded from below.

Noticing that $0 < \mu F(x,u) \leqslant f(x,u)u$ for all $x \in \bar{\Omega}$, $|u| \geqslant R$, one has that there exist constants $\alpha, \beta > 0$ such that

$$F(t,u) \geqslant \alpha|u|^\mu - \beta \quad \forall u \in \mathbb{R} \setminus \{0\}.$$

Now, let us recall the Hardy inequality (see [1, Lemma 2.1] for more details). If $1<p<N$, then

$$\int\limits_\Omega \frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x \leqslant \frac{1}{H_p}\int\limits_\Omega |\nabla u|^p\,\mathrm{d}x \quad \forall u \in W_0^{1,p}(\Omega),$$

where $H_p = ((N-p)/p)^p$ is the optimal constant.

Combining $q < p$ with the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$, one has

$$\int\limits_\Omega \frac{|u(x)|^q}{|x|^q}\,\mathrm{d}x \leqslant \frac{1}{H_q}\int\limits_\Omega |\nabla u|^q\,\mathrm{d}x \quad \forall u \in W_0^{1,p}(\Omega),$$

where $H_q = ((N-q)/q)^q$.

Thus, for fixed $\tilde{u} \in W_0^{1,p}(\Omega) \setminus \{0\}$, one has

$$\begin{aligned}
I_\lambda(m\tilde{u}) &= \frac{m^p}{p}\left(\int\limits_\Omega |\nabla\tilde{u}|^p\,\mathrm{d}x + \int\limits_\Omega \frac{a(x)|\tilde{u}|^p}{|x|^p}\right)\mathrm{d}x \\
&\quad + \frac{m^q}{q}\left(\int\limits_\Omega |\nabla\tilde{u}|^q\,\mathrm{d}x + \int\limits_\Omega \frac{b(x)|\tilde{u}|^q}{|x|^q}\right)\mathrm{d}x \\
&\quad - \lambda\int\limits_\Omega F(x, m\tilde{u})\,\mathrm{d}x + \frac{1}{t}\int\limits_{\Gamma_2} c(x)|m\tilde{u}|^t\,\mathrm{d}\sigma \\
&\leqslant \frac{H_p + \|a\|_\infty}{pH_p}m^p\|\tilde{u}\|^p + \frac{H_q + \|b\|_\infty}{qH_q}m^q\|\tilde{u}\|^q - \lambda\beta|\Omega| \\
&\quad - \lambda\alpha m^\mu\int\limits_\Omega |\tilde{u}|^\mu\,\mathrm{d}x + \frac{m^t\|c\|_\infty}{t}\int\limits_{\Gamma_2} |\tilde{u}|^t\,\mathrm{d}\sigma,
\end{aligned}$$

which leads to $I_\lambda(m\tilde{u}) \to -\infty(m \to +\infty)$ because of $\mu > t > p > q$, thus the functional I_λ is unbounded from below.

For every $u \in \Phi^{-1}((-\infty, 1])$, one has $\Phi(u) \leqslant 1$ and $\|u\| \leqslant p^{1/p}$. Thus

$$\begin{aligned}\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) &= \sup_{u\in\Phi^{-1}((-\infty,1])} \left(\int_\Omega F(x,u)\,\mathrm{d}x - \frac{1}{\lambda t}\int_{\Gamma_2} c(x)|u|^t\,\mathrm{d}\sigma \right) \\ &\leqslant \sup_{\|u\|\leqslant p^{1/p}} \int_\Omega \left(M_1|u| + \frac{M_2}{s}|u|^s \right) \mathrm{d}x \\ &\leqslant M_1 c_1\|u\| + \frac{M_2 c_s}{s}\|u\|^s \leqslant M_1 c_1 p^{1/p} + \frac{M_2 c_s}{s} p^{s/p}.\end{aligned}$$

Therefore, the hypotheses of Theorem 1 are all verified. Thus, problem (1) has at least two distinct generalized solutions, possibly one being trivial solution. ☐

In the next part, we focus on the existence of two nontrivial generalized solutions. Put

$$\delta(x) = \sup\{\delta > 0\colon\ B(x,\delta) \subseteq \Omega\}$$

for all $x \in \Omega$. We can show that there exists $x_0 \in \Omega$ such that $B(x_0, d) \subseteq \Omega$, where

$$d = \sup_{x\in\Omega} \delta(x).$$

Theorem 4. *Assume that condition* (f_1) *and* (f_2) *hold,* $F(x,\xi) \geqslant 0$ *for all* $(x,\xi) \in B(x_0,d) \times [0,\delta]$. *In addition, suppose there are* C_δ *and* γ *with*

$$\sqrt[p]{C_\delta p \left| B(x_0,d) \setminus \bar{B}\left(x_0, \frac{d}{2}\right) \right|} < \gamma, \tag{3}$$

where

$$C_\delta = \left(\frac{H_p + \|a\|_\infty}{pH_p} \left(\frac{2\delta}{d}\right)^p + \frac{H_q + \|b\|_\infty}{qH_q} \left(\frac{2\delta}{d}\right)^q \right)$$

such that

$$\lambda_1 = \frac{psM_1c_1\gamma + pM_2c_s\gamma^s}{s\gamma^p} < \frac{\operatorname{ess\,inf}_{B(x_0,d/2)} F(x,\delta)|B(x_0,\frac{d}{2})|}{C_\delta|B(x_0,d) \setminus \bar{B}(x_0,\frac{d}{2})|} = \lambda_2.$$

Thus problem (1) *has at least two nontrivial solutions when* $\lambda \in (1/\lambda_2, 1/\lambda_1)$.

Proof. It is easy to verify that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Let

$$\bar{u}(x) = \begin{cases} 0, & x \in \bar{\Omega} \setminus \bar{B}(x_0,d), \\ \frac{2\delta}{d}(d - |x - x_0|), & x \in B(x_0,d) \setminus \bar{B}(x_0,\frac{d}{2}), \\ \delta, & x \in B(x_0,\frac{d}{2}), \end{cases}$$

thus,

$$\Psi(\bar{u}) = \int\limits_{\Omega} F(x,\bar{u})\,\mathrm{d}x - \frac{1}{\lambda t}\int\limits_{\Gamma_2} c(x)|\bar{u}|^t\,\mathrm{d}\sigma \geqslant \int\limits_{B(x_0,d/2)} F(x,\delta)\,\mathrm{d}x$$
$$\geqslant \left|B\left(x_0,\frac{d}{2}\right)\right| \operatorname*{ess\,inf}_{B(x_0,d/2)} F(x,\delta). \tag{4}$$

By the Hardy inequality and (3), one has

$$\begin{aligned}\Phi(\bar{u}) &= \frac{1}{p}\left(\int\limits_{\Omega} |\nabla\bar{u}|^p\,\mathrm{d}x + \int\limits_{\Omega}\frac{a(x)|\bar{u}|^p}{|x|^p}\,\mathrm{d}x\right) + \frac{1}{q}\left(\int\limits_{\Omega}|\nabla\bar{u}|^q\,\mathrm{d}x + \int\limits_{\Omega}\frac{b(x)|\bar{u}|^q}{|x|^q}\,\mathrm{d}x\right)\\ &\leqslant \left(\frac{1}{p} + \frac{\|a\|_\infty}{pH_p}\right)\|\bar{u}\|^p + \left(\frac{1}{q}+\frac{\|b\|_\infty}{qH_q}\right)\|\bar{u}\|^q\\ &\leqslant \left(\frac{H_p+\|a\|_\infty}{pH_p}\left(\frac{2\delta}{d}\right)^p + \frac{H_q+\|b\|_\infty}{qH_q}\left(\frac{2\delta}{d}\right)^q\right)\left|B(x_0,d)\setminus\bar{B}\left(x_0,\frac{d}{2}\right)\right|\\ &= C_\delta\left|B(x_0,d)\setminus\bar{B}\left(x_0,\frac{d}{2}\right)\right| < \frac{\gamma^p}{p}.\end{aligned} \tag{5}$$

Let $\gamma^p/p = r$, thus $0 < \Phi(\bar{u}) < r$.

For every $u \in \Phi^{-1}((-\infty,r])$, one has $\Phi(u)\leqslant r$, and $\|u\|\leqslant (pr)^{1/p}$. Thus

$$\begin{aligned}\sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) &= \sup_{u\in\Phi^{-1}((-\infty,r])}\left(\int\limits_{\Omega}F(x,u)\,\mathrm{d}x - \frac{1}{\lambda t}\int\limits_{\Gamma_2}c(x)|u|^t\,\mathrm{d}\sigma\right)\\ &\leqslant \sup_{\|u\|\leqslant(pr)^{1/p}}\int\limits_{\Omega}\left(M_1|u| + \frac{M_2}{s}|u|^s\right)\mathrm{d}x\\ &\leqslant M_1c_1\|u\| + \frac{M_2c_s}{s}\|u\|^s \leqslant M_1c_1\gamma + \frac{M_2c_s}{s}\gamma^s.\end{aligned} \tag{6}$$

In view of (4)–(6), one has

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{r} &\leqslant \frac{M_1c_1\gamma + \frac{M_2c_s}{s}\gamma^s}{\frac{\gamma^p}{p}}\\ &\leqslant \frac{\operatorname{ess\,inf}_{B(x_0,\frac{d}{2})}F(x,\delta)|B(x_0,\frac{d}{2})|}{C_\delta|B(x_0,d)\setminus\bar{B}(x_0,\frac{d}{2})|}\\ &= \frac{\Psi(\bar{u})}{\Phi(\bar{u})}.\end{aligned}$$

Therefore, the hypotheses of Theorem 2 are all verified. Thus, problem (1) has at least two nontrivial solutions. □

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