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Fixed point theorems for ξ-α-η-ΓF-fuzzy contraction with an application to neutral fractional integro-differential equation with nonlocal conditions

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Abstract. In this study, we define a new fuzzy contraction principle, namely, the concept of ξ-α-η-ΓF-mappings, and prove the existence and uniqueness of the fixed point for such class of mappings. To further demonstrate the validity of our results, we furnish an application to neutral fractional integro-differential equations with nonlocal conditions. The presented results unify, generalize, and enhance a number of prior findings in the literature.

Keywords: fixed point, fuzzy metric, contraction, fractional equation.

1 Introduction

The appeal of fuzzy sets theory has grown consistently since Zadeh's seminal work [24] in 1965. The theoretical concept of fuzzy sets has evolved into an essential and insightful modeling tool. As a result, theory, as well as applications in the areas of logic, topology, and analysis, have advanced extensively with many applications in the domains of computer science and engineering. One of the challenging problems in fuzzy topology is coming up with an appropriate and consistent definition of fuzzy metric. This issue has been dealt with by many investigators in a number of approaches. Kramosil and Michaelek [9] created the first formulation of fuzzy metric space (FMS), which was further improved by George and Veeramani [2], who also demonstrated that each fuzzy metric produces Hausdorff topology. An important theoretical advancement at the moment is the method for creating contraction mapping in fuzzy metric spaces. Grabiec [3] originally used the Banach and Edelstein principles to fuzzy metric spaces in 1988. Fuzzy contractive

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mappings were first initiated by Gregori and Sapena [4]. In recent years, a number of researchers attempted to generalize the Banach contraction idea by altering and adjusting the contraction criteria; see [1, 5, 7, 8, 10, 12–15, 17, 22].

Following that, various types of fuzzy contractive conditions were established by Tirado [21], Gregori et al. [4], and Mihet [11]. Wardowski [23] developed a new powerful contraction principle in usual metric spaces termed as F-contraction and demonstrated some related fixed point theorems. In a FMS, fuzzy F-contraction is a new type of condition that was just presented by Huang et al. [6]. This class is significantly simpler than the F-contraction since it just has one requirement that the function F be strictly increasing. Patel and Radenović [17] presented a family of functions called a class of Γ-functions, such as an implicit function, and presented a direct generalization of the fuzzy F-contraction that was established in [6] and established fixed-point theorems by taking into account α-ΓF-fuzzy contractive criteria in a complete FMS, the α-admissibility property, and a weaker continuity condition.

2 Preliminaries

In this section, we discuss some essential concepts in order to render our study self-contained.

Definition 1. (See [20].) An operation $\curlyvee : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], \curlyvee)$ is an Abelien topological monoid such that $c \curlyvee 1 = c$ for all $c \in [0,1]$ and $c \curlyvee a \leqslant \aleph \curlyvee \wp$ whenever $c \leqslant \aleph$ and $a \leqslant \wp$ for all $c, a, \aleph, \wp \in [0,1]$.

Example 1.

(i) $c \curlyvee_m a = \min\{c, a\}$;
(ii) $c \curlyvee_P a = c \cdot a$.

Definition 2. (See [2].) The triple $(\mathcal{K}, \vartheta, \curlyvee)$ is called a FMS if $\mathcal{K}$ is a nonempty set, $\curlyvee$ is a continuous t-norm, and ϑ is a fuzzy set on $\mathcal{K}^2 \times (0, +\infty)$ satisfying:

(MS1) $\vartheta(c, a, \Im) > 0$;
(MS2) $\vartheta(c, a, \Im) = 1$ if and only if $c = a$;
(MS3) $\vartheta(c, a, \Im) = \vartheta(a, c, \Im)$;
(MS4) $\vartheta(c, v, \Im) \curlyvee \vartheta(v, a, \wp) \leqslant \vartheta(c, a, \Im + \wp)$;
(MS5) $\vartheta(c, a, \cdot) : (0, +\infty) \to [0, 1]$ is continuous.

Here $c, a, v \in \mathcal{K}$ and $\Im, \wp > 0$.

One may consider the value of $\vartheta(c, a, \Im)$ as the degree of nearness of c and a w.r.t. the variable $\Im$.

Example 2. (See [2].) Assume that $(\mathcal{K}, d)$ is a metric space and let

$$\vartheta(c, a, \Im) = \frac{\lambda \Im^{\alpha}}{\lambda \Im^{\alpha} + \nu d(c, a)}, \quad \lambda, \nu, \alpha \in \mathbb{R}_+.$$

Then $(\mathcal{K}, \vartheta, \curlyvee_m)$ is a FMS.

Lemma 1. (*See* [3].) $\vartheta(c, a, \cdot)$ *is nondecreasing for all* c, a *in* $\mathcal{K}$.

Definition 3. (See [2].) Let $(\mathcal{K}, \vartheta, \Upsilon)$ be a FMS, and let $\{c_n\} \subseteq \mathcal{K}$ be a sequence in $\mathcal{K}$ and $a \in \mathcal{K}$. We say that:

(i) $\{c_n\}$ converges to $c \in \mathcal{V}$ if $\lim_{n\to+\infty} \vartheta(c_n, c, \Im) = 1$ for all $\Im > 0$;
(ii) $\{c_n\}$ is a Cauchy if for each $\wp \in (0, 1)$ and $\Im > 0$, there exists $n_0 \in \mathbb{N}$ such that $\vartheta(c_n, c_m, \Im) > 1 - \wp$ for each $n, m \geqslant n_0$;
(iii) $(\mathcal{K}, \vartheta, \Upsilon)$ is complete (CFMS) if all Cauchy sequence is convergent in $\mathcal{K}$.

Definition 4. (See [22].) Let $F : (0, +\infty) \to \mathbb{R}$ be a function fulfilling:

(F1) F is strictly increasing, i.e., $s < \Im$ implies $F(s) < F(\Im)$ for each $s, \Im > 0$;
(F2) for each sequence $\{s_n\}$ in $\mathbb{R}_+$, $\lim_{n\to+\infty} s_n = 0$ iff $\lim_{n\to+\infty} F(s_n) = -\infty$;
(F3) there exists $k \in (0, 1)$ such that $\lim_{s\to0^+} s^k \cdot F(s) = 0$.

$\boldsymbol{F}$ stands for the family of all functions F that fulfill the requirements (F1)–(F3).

Wardowski [22] initiated in 2012 the following contraction principle in a metric space $(\mathcal{K}, d)$.

Definition 5. (See [22].) A mapping $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ is called an $\mathcal{F}$-contraction on $\mathcal{K}$ if there exist $F \in \boldsymbol{F}$ and $\tau > 0$ such that for all $c, a \in \mathcal{K}$ with $d(\mathcal{G}c, \mathcal{G}a) > 0$, we have

$$\tau + F\big(d(\mathcal{G}c, \mathcal{G}a)\big) \leqslant F\big(d(c, a)\big).$$

In a recent paper, Huang et al. [6] developed the idea of fuzzy F-contraction in a FMS.

Definition 6. (See [6].) Let $(\mathcal{K}, \vartheta, \Upsilon)$ be a FMS, and let $F \in \Lambda_F$. The mapping $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ is called fuzzy F-contraction if there exists $\tau \in (0, 1)$ such that

$$\tau \cdot F\big(\vartheta(\mathcal{G}c, \mathcal{G}a, \Im)\big) \geqslant F\big(\vartheta(c, a, \Im)\big), \tag{1}$$

where Λ_F stands for the set of all functions $F : [0, 1] \to (0, +\infty)$ fulfilling: for all $s, \Im \in [0, 1]$, $s < \Im$ implies $F(s) < F(\Im)$. Thus, F is strictly increasing on $[0, 1]$.

Definition 7. (See [19].) Let $\mathcal{G}$ be a self-mapping on $\mathcal{K}$, and let $\alpha : \mathcal{K} \times \mathcal{K} \to [0, +\infty)$ be a function. $\mathcal{G}$ is an α-admissible mapping if

$$c, a \in \mathcal{K}, \quad \alpha(c, a) \geqslant 1 \quad \Longrightarrow \quad \alpha(\mathcal{G}c, \mathcal{G}a) \geqslant 1.$$

Definition 8. (See [18].) Let $\mathcal{G}$ be a self-mapping on $\mathcal{K}$, and let $\alpha, \eta : \mathcal{K} \times \mathcal{K} \to [0, +\infty)$ be two functions. $\mathcal{G}$ is an α-admissible mapping w.r.t. η if

$$c, a \in \mathcal{K}, \quad \alpha(c, a) \geqslant \eta(c, a) \quad \Longrightarrow \quad \alpha(\mathcal{G}c, \mathcal{G}a) \geqslant \eta(\mathcal{G}c, \mathcal{G}a).$$

If $\eta(c, a) = 1$, then Definition 8 yields Definition 7. Additionally, if we assume that $\alpha(c, a) = 1$ in Definition 8, then $\mathcal{G}$ is called η-subadmissible.

Definition 9. (See [13].) The function $\xi : (0,1] \times (0,1] \to \mathbb{R}$ is an $\mathcal{FZ}$-simulation function if:

(ξ1) $\xi(1,1) = 0$;
(ξ2) $\xi(c,a) < 1/a - 1/c$ for all $c, a \in (0,1)$;
(ξ3) if $\{c_n\}$, $\{a_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to+\infty} c_n = \lim_{n\to+\infty} a_n < 1$, then $\lim_{n\to+\infty} \sup \xi(c_n, a_n) < 0$.

The set of all $\mathcal{FZ}$-simulation functions is denoted by $\mathcal{FZ}$.

Let $\Lambda_{\mathcal{FZ}}$ be the collection of all functions $\xi : (0,1] \times (0,1] \to \mathbb{R}$ fulfilling (ξ1), (ξ3), and ($\xi 2'$) instead of (ξ2):

($\xi 2'$) $\xi(c,a) \leqslant 1/a - 1/c$ for all $c, a \in (0,1)$.

Recently, Devi Patel et al. [17] initiated a new type of mappings like an implicit function called Γ-function.

Definition 10. (See [17].) Let Λ_Γ denote the set of all continuous functions $\Gamma : \mathbb{R}_+^4 \to \mathbb{R}$ fulfilling: for all $\Im_1, \Im_2, \Im_3, \Im_4 \in \mathbb{R}_+$ with $\max(\Im_1, \Im_2, \Im_3, \Im_4) = 1$, there exists $\tau \in (0,1)$ such that $\Gamma(\Im_1, \Im_2, \Im_3, \Im_4) = \tau$.

Example 3. (See [17].) Let $\Gamma_i : (\mathbb{R}_+)^4 \to \mathbb{R}$, $i = 1, 2$, be defined by

(i) $\Gamma_1(\Im_1, \Im_2, \Im_3, \Im_4) = \tau + \omega \log_e \max\{\Im_1, \Im_2, \Im_3, \Im_4\}$, where $\omega \in \mathbb{R}_+$;
(ii) $\Gamma_2(\Im_1, \Im_2, \Im_3, \Im_4) = \tau / \max\{\Im_1, \Im_2, \Im_3, \Im_4\}$.

Here $\Gamma_i \in \Lambda_\Gamma$ for $i = 1, 2$.

3 Main results

In this section, we define the concept of a ξ-α-η-ΓF-fuzzy contractive mapping.

Definition 11. Let $(\mathcal{K}, \vartheta, \curlyvee)$ be a FMS, and let a mapping $\mathcal{G} : \mathcal{K} \to \mathcal{K}$. Furthermore, suppose that $\alpha, \eta : \mathcal{K} \times \mathcal{K} \to [0, +\infty)$ are two functions. $\mathcal{G}$ is said to be a ξ-α-η-ΓF-fuzzy contractive mapping on $\mathcal{K}$ if for $c, a \in \mathcal{K}$ with $\eta(c, \mathcal{G}c) \leqslant \alpha(c,a)$ and $\vartheta(\mathcal{G}c, \mathcal{G}a, \Im) > 0$, we have

$$\begin{aligned} &\xi\big(\Gamma\big[\vartheta(c, \mathcal{G}c, \Im), \vartheta(a, \mathcal{G}a, \Im), \vartheta(c, \mathcal{G}a, \Im), \vartheta(a, \mathcal{G}c, \Im)\big] \\ &\quad \times F\big(\vartheta(\mathcal{G}c, \mathcal{G}a, \Im)\big), F\big(\vartheta(c, a, \Im)\big)\big) \geqslant 0, \end{aligned} \tag{2}$$

where $\Gamma \in \Lambda_\Gamma$, $\xi \in \Lambda_{\mathcal{FZ}}$, and $F \in \Lambda_F$.

We need the following lemma to prove our main results.

Lemma 2. (*See* [6].) *Let $(\mathcal{K}, \vartheta, \curlyvee)$ be a FMS, and let $\{c_n\}$ be a sequence in $\mathcal{K}$ such that for each $n \in \mathbb{N}$,*

$$\lim_{\Im \to 0^+} \vartheta(c_n, c_{n+1}, \Im) > 0,$$

and for any $\Im > 0$,

$$\lim_{n\to+\infty} \vartheta(c_n, c_{n+1}, \Im) = 1.$$

If $\{c_n\}$ *is not a Cauchy sequence in* $\mathcal{K}$, *then there exist* $\epsilon \in (0,1)$, $\Im_0 > 0$, *and two sequences of positive integers* $\{n_k\}$, $\{m_k\}$, $n_k > m_k > k$, $k \in \mathbb{N}$, *such that the sequences*

$$\bigl\{\vartheta(c_{m_k}, c_{n_k}, \Im_0)\bigr\},\ \bigl\{\vartheta(c_{m_k}, c_{n_k+1}, \Im_0)\bigr\},\ \bigl\{\vartheta(c_{m_k-1}, c_{n_k}, \Im_0)\bigr\},\\ \bigl\{\vartheta(c_{m_k-1}, c_{n_k+1}, \Im_0)\bigr\},\ \bigl\{\vartheta(c_{m_k+1}, c_{n_k+1}, \Im_0)\bigr\}$$

tend to $1-\wp$ *as* $k \to +\infty$

Theorem 1. *Let* $(\mathcal{K}, \vartheta, \curlyvee)$ *be a CFMS. Let* $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ *be a self-mapping satisfying the following assertions*:

(i) $\mathcal{G}$ *is* α*-admissible w.r.t.* η;
(ii) $\mathcal{G}$ *is a* ξ-α-η-ΓF*-fuzzy contractive mapping*;
(iii) *there exists* $c_0 \in \mathcal{K}$ *such that* $\alpha(c_0, \mathcal{G}c_0) \geqslant \eta(c_0, \mathcal{G}c_0)$;
(iv) $\mathcal{G}$ *is an* α-η*-continuous map.*

Then $\mathcal{G}$ *has a fixed point. Moreover,* $\mathcal{G}$ *has a unique fixed point whenever* $\alpha(c,a) \geqslant \eta(c,c)$ *for all* $c, a \in \mathrm{Fix}(\mathcal{G})$.

Proof. Let $c_0 \in \mathcal{K}$ such that $\alpha(c_0, \mathcal{G}c_0) \geqslant \eta(c_0, \mathcal{G}c_0)$. For $c_0 \in \mathcal{K}$, we define the sequence $\{c_n\}$ by $c_{n+1} = \mathcal{G}^n c_0 = \mathcal{G}c_n$ for all $n \in \mathbb{N}$. Now, since $\mathcal{G}$ is α-admissible w.r.t. η, then

$$\alpha(c_0, c_1) = \alpha(c_0, \mathcal{G}c_0) \geqslant \eta(c_0, \mathcal{G}c_0) = \eta(c_0, c_1),$$

by continuing this process, we have

$$\eta(c_{n-1}, c_n) \leqslant \alpha(c_{n-1}, c_n)$$

for all $n \in \mathbb{N}$. Furthermore, let $n_0 \in \mathbb{N}$ such that $c_{n_0} = \mathcal{G}c_{n_0}$, then c_{n_0} is fixed point of $\mathcal{G}$, and nothing to prove.

Assume that $c_n \neq c_{n+1}$ or $\vartheta(c_n, c_{n+1}, \Im) \in [0,1)$ for all $n \in \mathbb{N}$. Since $\mathcal{G}$ is an ξ-α-η-ΓF-contractive mapping and thus we use $c = c_{n-1}$ and $a = c_n$ in (2), we obtain

$$\begin{aligned} 0 &\leqslant \xi\bigl(\Gamma\bigl[\vartheta(c_{n-1}, \mathcal{G}c_{n-1}, \Im),\ \vartheta(c_n, \mathcal{G}c_n, \Im),\ \vartheta(c_{n-1},\ \mathcal{G}c_n, \Im),\ \vartheta(c_n, \mathcal{G}c_{n-1}, \Im)\bigr] \\ &\qquad \times F\bigl(\vartheta(\mathcal{G}c_{n-1}, \mathcal{G}c_n, \Im)\bigr), F\bigl(\vartheta(c_{n-1}, c_n, \Im)\bigr)\bigr) \\ &\leqslant F^{-1}\bigl(\vartheta(c_{n-1}, c_n, \Im)\bigr) \\ &\quad - \bigl(\Gamma\bigl[\vartheta(c_{n-1}, \mathcal{G}c_{n-1}, \Im),\ \vartheta(c_n, \mathcal{G}c_n, \Im),\ \vartheta(c_{n-1}, \mathcal{G}c_n, \Im),\ \vartheta(c_n, \mathcal{G}c_{n-1}, \Im)\bigr] \\ &\qquad \times F\bigl(\vartheta(\mathcal{G}c_{n-1}, \mathcal{G}c_n, \Im)\bigr)\bigr)^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} &\Gamma\bigl[\vartheta(c_{n-1}, \mathcal{G}c_{n-1}, \Im),\ \vartheta(c_n, \mathcal{G}c_n, \Im),\ \vartheta(c_{n-1}, \mathcal{G}c_n, \Im),\ \vartheta(c_n, \mathcal{G}c_{n-1}, \Im)\bigr] \\ &\qquad \times F\bigl(\vartheta(\mathcal{G}c_{n-1}, \mathcal{G}c_n, \Im)\bigr) \\ &\quad \geqslant F\bigl(\vartheta(c_{n-1}, c_n, \Im)\bigr). \end{aligned}$$

Since $\max(\vartheta(c_{n-1}, c_n, \Im), \vartheta(c_n, c_{n+1}, \Im), \vartheta(c_{n-1}, c_{n+1}, \Im), \vartheta(c_n, c_n, \Im)) = 1$, by definition of Γ-function, there exists $\tau \in (0, 1)$ such that

$$\Gamma\big(\vartheta(c_{n-1}, c_n, \Im), \vartheta(c_n, c_{n+1}, \Im), \vartheta(c_{n-1}, c_{n+1}, \Im), \vartheta(c_n, c_n, \Im)\big) = \tau.$$

Therefore,

$$\tau \cdot F\big(\vartheta(c_n, c_{n+1}, \Im)\big) \geqslant F\big(\vartheta(c_{n-1}, c_n, \Im)\big).$$

We have

$$F\big(\vartheta(c_n, c_{n+1}, \Im)\big) > \tau \cdot F\big(\vartheta(c_n, c_{n+1}, \Im)\big) \geqslant F\big(\vartheta(c_{n-1}, c_n, \Im)\big). \tag{3}$$

Since F is a strictly increasing function,

$$\vartheta(c_n, c_{n+1}, \Im) > \vartheta(c_{n-1}, c_n, \Im).$$

Thus, the sequence $\{\vartheta(c_n, c_{n+1}, \Im)\}$, $\Im > 0$, is a strictly increasing bounded from above, and thus sequence $\{\vartheta(c_n, c_{n+1}, \Im)\}$, $\Im > 0$, is convergent. In other words, there exists $a(\Im) \in [0, 1]$ such that

$$\lim_{n\to+\infty} \vartheta(c_n, c_{n+1}, \Im) = \ell(\Im) \tag{4}$$

for any $\Im > 0$ and $n \in \mathbb{N}$. It follows that

$$\vartheta(c_n, c_{n+1}, \Im) < \ell(\Im) \tag{5}$$

by (4) and (5), for any $\Im > 0$, we have

$$\lim_{n\to+\infty} F\big(\vartheta(c_n, c_{n+1}, \Im)\big) = F\big(\ell(\Im) - 0\big). \tag{6}$$

We have to show that $\ell(\Im) = 1$. Assume that $\ell(\Im) < 1$ for some $\Im > 0$, and by taking limit as $n \to +\infty$ in (3) and using (6), we obtain

$$F\big(\ell(\Im) - 0\big) \geqslant \tau \cdot F\big(\ell(\Im) - 0\big) \geqslant F\big(\ell(\Im) - 0\big).$$

This is a contradiction with $F(\ell(\Im) - 0) > 0$. Therefore,

$$\lim_{n\to+\infty} \vartheta(c_n, c_{n+1}, \Im) = 1.$$

Next, we have to prove that $\{c_n\}$ is a Cauchy sequence. Suppose that $\{c_n\}$ is not a Cauchy sequence. By using the Lemma 2, then there exist $\wp \in (0, 1)$, $\Im_0 > 0$, and sequences $\{c_{m_k}\}$ and $\{c_{n_k}\}$ such that

$$\lim_{k\to+\infty} \vartheta(c_{m_k}, c_{n_k}, \Im_0) = 1 - \wp. \tag{7}$$

Again, with $c = c_{m_k}$ and $a = c_{n_k}$ in (2), we have

$$\begin{aligned} 0 &\leqslant \xi\big(\Gamma\big[\vartheta(c_{m_k}, c_{m_k+1}, \Im),\, \vartheta(c_{n_k}, c_{n_k+1}, \Im),\, \vartheta(c_{m_k}, c_{n_k+1}, \Im),\, \vartheta(c_{n_k}, c_{m_k+1}, \Im)\big] \\ &\quad \times F\big(\vartheta(c_{m_k+1}, c_{n_k+1}, \Im)\big),\, F\big(\vartheta(c_{m_k}, c_{n_k}, \Im)\big)\big) \\ &\leqslant F^{-1}\big(\vartheta(c_{m_k}, c_{n_k}, \Im)\big) \\ &\quad - \big(\Gamma\big[\vartheta(c_{m_k}, c_{m_k+1}, \Im),\, \vartheta(c_{n_k}, c_{n_k+1}, \Im),\, \vartheta(c_{m_k}, c_{n_k+1}, \Im),\, \vartheta(c_{n_k}, c_{m_k+1}, \Im)\big] \\ &\quad \times F\big(\vartheta(c_{m_k+1}, c_{n_k+1}, \Im)\big)\big)^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} &\Gamma\big[\vartheta(c_{m_k}, c_{m_k+1}, \Im),\, \vartheta(c_{n_k}, c_{n_k+1}, \Im),\, \vartheta(c_{m_k}, c_{n_k+1}, \Im),\, \vartheta(c_{n_k}, c_{m_k+1}, \Im)\big] \\ &\quad \times F\big(\vartheta(c_{m_k+1}, c_{n_k+1}, \Im)\big) \\ &\quad \geqslant F\big(\vartheta(c_{m_k}, c_{n_k}, \Im)\big). \end{aligned}$$

Letting limit as $k \to +\infty$, we have

$$\begin{aligned} &\lim_{k\to+\infty} \Gamma\big[\vartheta(c_{m_k}, c_{m_k+1}, \Im),\, \vartheta(c_{n_k}, c_{n_k+1}, \Im),\, \vartheta(c_{m_k}, c_{n_k+1}, \Im),\, \vartheta(c_{n_k}, c_{m_k+1}, \Im)\big] \\ &\quad \times F\big(\vartheta(c_{m_k+1}, c_{n_k+1}, \Im)\big) \\ &\geqslant \lim_{k\to+\infty} F\big(\vartheta(c_{m_k}, c_{n_k}, \Im)\big), \end{aligned} \tag{8}$$

which means

$$\begin{aligned} &\Gamma\Big(1, 1, \lim_{k\to+\infty} \vartheta(c_{m_k}, c_{n_k+1}, \Im),\, \lim_{k\to+\infty} \vartheta(c_{n_k}, c_{m_k+1}, \Im)\Big) \\ &\quad \times \lim_{k\to+\infty} F\big(\vartheta(c_{m_k+1}, c_{n_k+1}, \Im)\big) \\ &\quad \geqslant \lim_{k\to+\infty} F\big(\vartheta(c_{m_k}, c_{n_k}, \Im)\big). \end{aligned}$$

Since

$$\max\Big(1,\, 1,\, \lim_{k\to+\infty} \vartheta(c_{m_k}, c_{n_k+1}, \Im),\, \lim_{k\to+\infty} \vartheta(c_{n_k}, c_{m_k+1}, \Im)\Big) = 1,$$

there exists $\tau \in (0, 1)$ such that

$$\Gamma\Big(1,\, 1,\, \lim_{k\to+\infty} \vartheta\big(c_{m_k}, c_{n_k+1}, \Im\big),\, \lim_{k\to+\infty} \vartheta(c_{n_k}, c_{m_k+1}, \Im)\Big) = \tau.$$

Using (7) and (8) implies that

$$\tau \cdot F\big((1 - \wp) - 0\big) \geqslant F\big((1 - \wp) - 0\big).$$

Additionally,

$$F\big((1 - \wp) - 0\big) \geqslant \tau \cdot F\big((1 - \wp) - 0\big) \geqslant F\big((1 - \wp) - 0\big).$$

This is a contraction with $F((1-\wp)-0)>0$. Thus, the sequence $\{c_n\}$ is a Cauchy sequence in $\mathcal{K}$. Since $(\mathcal{K},\vartheta,\curlyvee)$ is CFMS, then there exists $c^*\in\mathcal{K}$ such that

$$\lim_{n\to+\infty} c_n = c^*.$$

Let us prove that c^* is a fixed point of $\mathcal{G}$. Since $\mathcal{G}$ is an α-η-continuous and $\eta(c_{n-1},c_n)\leqslant\alpha(c_{n-1},c_n)$ for all $n\in\mathbb{N}$. Then $\lim_{n\to+\infty}\vartheta(\mathcal{G}c_n,\mathcal{G}c^*,\Im)=1$ implies $\vartheta(c^*,\mathcal{G}c^*,\Im)=1$, that is, $c^*=\mathcal{G}c^*$.

Let $c,a\in\operatorname{Fix}[\mathcal{G}]$ such that $c\neq a$, by Eq. (2),

$$\begin{aligned}
0 &\leqslant \xi\bigl(\Gamma\bigl[\vartheta(c,\mathcal{G}c,\Im),\,\vartheta(a,\mathcal{G}a,\Im),\,\vartheta(c,\mathcal{G}a,\Im),\,\vartheta(a,\mathcal{G}c,\Im)\bigr]\cdot F\bigl(\vartheta(\mathcal{G}c,\mathcal{G}a,\Im)\bigr),\\
&\qquad F\bigl(\vartheta(c,a,\Im)\bigr)\bigr)\\
&= \xi\bigl(\Gamma\bigl[\vartheta(c,c,\Im),\,\vartheta(a,a,\Im),\,\vartheta(c,a,\Im),\,\vartheta(a,c,\Im)\bigr]\cdot F\bigl(\vartheta(c,a,\Im)\bigr),\,F\bigl(\vartheta(c,a,\Im)\bigr)\bigr)\\
&\leqslant F^{-1}(\vartheta(c,a,\Im))\\
&\quad -\bigl(\Gamma\bigl[\vartheta(c,c,\Im),\,\vartheta(a,a,\Im),\,\vartheta(c,a,\Im),\,\vartheta(a,c,\Im)\bigr]\cdot F\bigl(\vartheta(c,a,\Im)\bigr)\bigr)^{-1}.
\end{aligned}$$

Thus,

$$\Gamma\bigl[\vartheta(c,c,\Im),\,\vartheta(a,a,\Im),\,\vartheta(c,a,\Im),\,\vartheta(a,c,\Im)\bigr]\cdot F\bigl(\vartheta(c,a,\Im)\bigr)\geqslant F\bigl(\vartheta(c,a,\Im)\bigr),$$

that is,

$$\Gamma\bigl[1,\,1,\,\vartheta(c,a,\Im),\,\vartheta(a,c,\Im)\bigr]\cdot F\bigl(\vartheta(c,a,\Im)\bigr)\geqslant F\bigl(\vartheta(c,a,\Im)\bigr).$$

Since $\max(1,1,\vartheta(c,a,\Im),\vartheta(a,c,\Im))=1$, there exists $\tau\in(0,1)$ such that

$$\Gamma\bigl(1,\,1,\,\vartheta(c,a,\Im),\,\vartheta(a,c,\Im)\bigr)=\tau.$$

Thus, we can deduce above

$$\tau\cdot F\bigl(\vartheta(c,a,\Im)\bigr)\geqslant F\bigl(\vartheta(c,a,\Im)\bigr).$$

This implies that

$$F\bigl(\vartheta(c,a,\Im)\bigr)>\tau\cdot F\bigl(\vartheta(c,a,\Im)\bigr)\geqslant F\bigl(\vartheta(c,a,\Im)\bigr),$$

which is a contradiction. Thus, $\mathcal{G}$ has a unique fixed point. □

We can deduce the following corollaries.

Corollary 1. *Let $(\mathcal{K},\vartheta,\curlyvee)$ be a CFMS. Let $\mathcal{G}:\mathcal{K}\to\mathcal{K}$ be a self-mapping satisfying the following assertions*:

(i) *$\mathcal{G}$ is α-admissible w.r.t. η*;
(ii) *if, for $c,a\in\mathcal{K}$ with $\alpha(c,a)\geqslant\eta(c,\mathcal{G}c)$ and $\vartheta(\mathcal{G}c,\mathcal{G}a,\Im)>0$, we have*

$$\xi\bigl(\tau\cdot F\bigl(\vartheta(\mathcal{G}c,\mathcal{G}a,\Im)\bigr),F\bigl(\vartheta(c,a,\Im)\bigr)\bigr)\geqslant 0,$$

where $c\neq a$, $\tau\in(0,1)$, $\xi\in\Lambda_{\mathcal{FZ}}$, and $F\in\Lambda_F$.;

(iii) *there exists* $c_0 \in \mathcal{K}$ *such that* $\alpha(c_0, \mathcal{G}c_0) \geqslant \eta(c_0, \mathcal{G}c_0)$;
(iv) $\mathcal{G}$ *is an* α-η-*continuous*;

Then $\mathcal{G}$ *has a fixed point. Moreover,* $\mathcal{G}$ *has a unique fixed point whenever* $\alpha(c, a) \geqslant \eta(c, c)$ *for all* $c, a \in \mathrm{Fix}(\mathcal{G})$.

Corollary 2. (*See* [17].) *Let* $(\mathcal{K}, \vartheta, \curlyvee)$ *be a CFMS. Let* $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ *be a self-mapping satisfying the following assertions*:

(i) $\mathcal{G}$ *is* α-*admissible w.r.t.* η;
(ii) *if, for* $c, a \in \mathcal{K}$ *with* $\alpha(c, a) \geqslant \eta(c, \mathcal{G}c)$ *and* $\vartheta(\mathcal{G}c, \mathcal{G}a, \Im) > 0$, *we have*

$$\tau \cdot F\big(\vartheta(\mathcal{G}c, \mathcal{G}a, \Im)\big) \geqslant F\big(\vartheta(c, a, \Im)\big),$$

where $c \neq a$, $\tau \in (0, 1)$, *and* $F \in \varLambda_F$;
(iii) *there exists* $c_0 \in \mathcal{K}$ *such that* $\alpha(c_0, \mathcal{G}c_0) \geqslant \eta(c_0, \mathcal{G}c_0)$;
(iv) $\mathcal{G}$ *is* α-η-*continuous.*

Then $\mathcal{G}$ *has a fixed point. Moreover,* $\mathcal{G}$ *has a unique fixed point whenever* $\alpha(c, a) \geqslant \eta(c, c)$ *for all* $c, a \in \mathrm{Fix}(\mathcal{G})$.

The following results are obtained by applying $\eta(c, a) = 1$ to Definition 11.

Definition 12. Let $(\mathcal{K}, \vartheta, \curlyvee)$ be a FMS, a mapping $\mathcal{G} : \mathcal{K} \to \mathcal{K}$, and $\alpha : \mathcal{K} \times \mathcal{K} \to [0, +\infty)$ be a function. $\mathcal{G}$ is said to be a ξ-α-ΓF-fuzzy contractive mapping on $\mathcal{K}$ if, for $c, a \in \mathcal{K}$ with $1 \leqslant \alpha(c, a)$ and $\vartheta(\mathcal{G}c, \mathcal{G}a, \Im) > 0$, we have

$$\begin{aligned}\xi\big(&\varGamma\big[\vartheta(c, \mathcal{G}c, \Im),\, \vartheta(a, \mathcal{G}a, \Im),\, \vartheta(c, \mathcal{G}a, \Im),\, \vartheta(a, \mathcal{G}c, \Im)\big] \\ &\times F\big(\vartheta\big(\mathcal{G}c, \mathcal{G}a, \Im\big)\big), F\big(\vartheta(c, c, \Im)\big)\big) \geqslant 0,\end{aligned}$$

where $\varGamma \in \varDelta_\varGamma$ and $F \in \varLambda_F$.

Theorem 2. *Let* $(\mathcal{K}, \vartheta, \curlyvee)$ *be a CFMS. Let* $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ *be a self-mapping satisfying the following assertions*:

(i) $\mathcal{G}$ *is* α-*admissible*;
(ii) $\mathcal{G}$ *is a* ξ-α-$\varGamma F$-*fuzzy contractive mapping*;
(iii) *there exists* $c_0 \in \mathcal{K}$ *such that* $\alpha(c_0, \mathcal{G}c_0) \geqslant 1$;
(iv) $\mathcal{G}$ *is* α-*continuous.*

Then $\mathcal{G}$ *has a fixed point. Moreover,* $\mathcal{G}$ *has a unique fixed point in* $\mathcal{K}$ *whenever* $\alpha(c, a) \geqslant 1$ *for all* $c, a \in \mathrm{Fix}(\mathcal{G})$.

Proof. The proof is similar to that of Theorem 1. □

Corollary 3. *Let* $(\mathcal{K}, \vartheta, \curlyvee)$ *be a CFMS. Let* $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ *be a self-mapping satisfying the following assertions*:

(i) $\mathcal{G}$ *is* α-*admissible*;

(ii) *if, for* $c, a \in \mathcal{K}$ *with* $\alpha(c, a) \geqslant 1$ *and* $\vartheta(\mathcal{G}c, \mathcal{G}a, \Im) > 0$*, we have*

$$\xi\big(\tau \cdot F\big(\vartheta(\mathcal{G}c, \mathcal{G}a, \Im)\big), F\big(\vartheta(c, a, \Im)\big)\big) \geqslant 0,$$

where $c \neq a, \tau \in (0,1)$ *and* $F \in \Lambda_F$;

(iii) *there exists* $c_0 \in \mathcal{K}$ *such that* $\alpha(c_0, \mathcal{G}c_0) \geqslant 1$;

(iv) $\mathcal{G}$ *is* α*-continuous.*

Then $\mathcal{G}$ *has a fixed point. Moreover,* $\mathcal{G}$ *has a unique fixed point in* $\mathcal{K}$ *whenever* $\alpha(c, a) \geqslant 1$ *for all* $c, a \in \mathrm{Fix}(\mathcal{G})$

In the next theorem, we omit the continuity hypothesis of $\mathcal{G}$.

Theorem 3. *Let* $(\mathcal{K}, \vartheta, \curlyvee)$ *be a CFMS. Let* $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ *be a self-mapping satisfying the following assertions*:

(i) $\mathcal{G}$ *is* α*-admissible w.r.t.* η;

(ii) $\mathcal{G}$ *is a* ξ-α-η-ΓF*-fuzzy contractive mapping*;

(iii) *there exists* $c_0 \in \mathcal{K}$ *such that* $\alpha(c_0, \mathcal{G}c_0) \geqslant \eta(c_0, \mathcal{G}c_0)$;

(iv) *if* $\{c_n\}$ *is a sequence in* $\mathcal{K}$ *such that* $\alpha(c_n, c_{n+1}) \geqslant \eta(c_n, c_{n+1})$ *with* $c_n \to c$ *as* $n \to +\infty$*, then*

$$\eta\big(\mathcal{G}c_n, \mathcal{G}^2c_n\big) \leqslant \alpha(\mathcal{G}c_n, c) \quad \textit{or} \quad \eta\big(\mathcal{G}^2c_n, \mathcal{G}^3c_n\big) \leqslant \alpha\big(\mathcal{G}^2c_n, c\big)$$

holds for all $n \in \mathbb{N}$.

Then $\mathcal{G}$ *has a fixed point. Moreover,* $\mathcal{G}$ *has a unique fixed point whenever* $\alpha(c, a) \geqslant \eta(c, c)$ *for all* $c, a \in \mathrm{Fix}(\mathcal{G})$.

Proof. Let $c_0 \in \mathcal{K}$ such that $\alpha(c_0, \mathcal{G}c_0) \geqslant \eta(c_0, \mathcal{G}c_0)$. Similarly to the proof of the Theorem 1, we can conclude that

$$\alpha(c_n, c_{n+1}) \geqslant \eta(c_n, c_{n+1}) \quad \text{and} \quad c_n \to c^* \quad \text{as } n \to +\infty,$$

where $c_{n+1} = \mathcal{G}c_n$. By assumption (iv), either

$$\eta\big(\mathcal{G}c_n, \mathcal{G}^2c_n\big) \leqslant \alpha(c_{n+1}, c^*) \quad \text{or} \quad \eta(\mathcal{G}^2c_n, \mathcal{G}^3c_n) \leqslant \alpha(\mathcal{G}^2c_n, c^*)$$

holds for all $n \in \mathbb{N}$. This implies that

$$\eta(c_{n+1}, c_{n+2}) \leqslant \alpha(c_{n+1}, c^*) \quad \text{or} \quad \eta(c_{n+2}, c_{n+3}) \leqslant \alpha(c_{n+2}, c^*)$$

holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ such that

$$\eta(c_{n_k}, c_{n_k+1}) \leqslant \alpha(c_{n_k}, c^*),$$

and by (2), we obtain

$$\begin{aligned}0 \leqslant \xi\big(&\Gamma\big[\vartheta(c_{n_k}, \mathcal{G}c_{n_k}, \Im), \vartheta(c^*, \mathcal{G}c^*, \Im), \vartheta(c_{n_k}, \mathcal{G}c^*, \Im), \vartheta(c^*, \mathcal{G}c_{n_k}, \Im)\big] \\ &\times F\big(\vartheta(\mathcal{G}c_{n_k}, \mathcal{G}c^*, \Im)\big), F\big(\vartheta(c_{n_{k_k}} c^*, \Im)\big)\big)\end{aligned}$$

$$\begin{aligned}&\leqslant F^{-1}(\vartheta(c_{n_k}c^*,\Im))\\&\quad-\big(\Gamma\big[\vartheta(c_{n_k},\mathcal{G}c_{n_k},\Im),\,\vartheta(c^*,\mathcal{G}c^*,\Im),\,\vartheta(c_{n_k},\mathcal{G}c^*,\Im)\,\vartheta(c^*,\mathcal{G}c_{n_k},\Im)\big]\\&\qquad\times F\big(\vartheta(\mathcal{G}c_{n_k},\mathcal{G}c^*,\Im)\big)\big)^{-1}.\end{aligned}$$

Thus,

$$\begin{aligned}&\Gamma\big[\vartheta(c_{n_k},\mathcal{G}c_{n_k},\Im),\,\vartheta(c^*,\mathcal{G}c^*,\Im),\,\vartheta(c_{n_k},\mathcal{G}c^*,\Im),\,\vartheta(c^*,\mathcal{G}c_{n_k},\Im)\big]\\&\qquad\times F\big(\vartheta(\mathcal{G}c_{n_k},\mathcal{G}c^*,\Im)\big)\\&\qquad\geqslant F\big(\vartheta(c_{n_k}c^*,\Im)\big),\end{aligned}$$

which implies for any $\Im>0$,

$$F\big(\vartheta(\mathcal{G}c_{n_k},\mathcal{G}c^*,\Im)\big)>\tau\cdot F\big(\vartheta(\mathcal{G}c_{n_k},\mathcal{G}c^*,\Im)\big)\geqslant F\big(\vartheta(c_{n_k},c^*,\Im)\big).$$

Since F is a strictly increasing function,

$$\vartheta(\mathcal{G}c_{n_k},\mathcal{G}c^*,\Im)>\vartheta(c_{n_k},c^*,\Im).$$

Taking limit as $k\to+\infty$ in the above inequality, we obtain $\vartheta(c^*,\mathcal{G}c^*,\Im)=1$, that is, $c^*=\mathcal{G}c^*$. The uniqueness of the fixed point can be proved as in the case of Theorem 1. □

Corollary 4. *Let $(\mathcal{K},\vartheta,\curlyvee)$ be a CFMS. Let $\mathcal{G}:\mathcal{K}\to\mathcal{K}$ be a self-mapping satisfying the following assertions*:

(i) *$\mathcal{G}$ is α-admissible w.r.t. η*;

(ii) *if, for $c,a\in\mathcal{K}$ with $\alpha(c,a)\geqslant\eta(c,\mathcal{G}c)$ and $\vartheta(\mathcal{G}c,\mathcal{G}a,\Im)>0$, we have*

$$\xi\big(\tau\cdot F\big(\vartheta(\mathcal{G}c,\mathcal{G}a,\Im)\big),F(\vartheta(c,a,\Im))\big)\geqslant 0,$$

where $c\neq a$, $\tau\in(0,1)$, $\xi\in\Lambda_{\mathcal{FZ}}$, and $F\in\Lambda_F$;

(iii) *there exists $c_0\in\mathcal{K}$ such that $\alpha(c_0,\mathcal{G}c_0)\geqslant\eta(c_0,\mathcal{G}c_0)$*;

(iv) *if $\{c_n\}$ is a sequence in $\mathcal{K}$ such that $\alpha(c_n,c_{n+1})\geqslant\eta(c_n,c_{n+1})$ with $c_n\to c$ as $n\to+\infty$, then*

$$\eta\big(\mathcal{G}c,\mathcal{G}^2c_n\big)\leqslant\alpha(\mathcal{G}c_n,c)\quad\text{or}\quad\eta\big(\mathcal{G}^2c_n,\mathcal{G}^3c_n\big)\leqslant\alpha\big(\mathcal{G}^2c_n,c\big)$$

holds for all $n\in N$.

Then $\mathcal{G}$ has a fixed point. Moreover, $\mathcal{G}$ has a unique fixed point whenever $\alpha(c,a)\geqslant\eta(c,c)$ for all $c,a\in\mathrm{Fix}(\mathcal{G})$.

Corollary 5. (*See* [17].) *Let $(\mathcal{K},\vartheta,\curlyvee)$ be a CFMS. Let $\mathcal{G}:\mathcal{K}\to\mathcal{K}$ be a self-mapping satisfying the following assertions*:

(i) *$\mathcal{G}$ is α-admissible w.r.t. η*;

(ii) *if, for $c,a\in\mathcal{K}$ with $\alpha(c,a)\geqslant\eta(c,\mathcal{G}c)$ and $\vartheta(\mathcal{G}c,\mathcal{G}a,\Im)>0$, we have*

$$\tau\cdot F\big(\vartheta(\mathcal{G}c,\mathcal{G}a,\Im)\big)\geqslant F\big(\vartheta(c,a,\Im)\big),$$

where $c\neq a$, $\tau\in(0,1)$, and $F\in\Lambda_F$;

(iii) *there exists* $c_0 \in \mathcal{K}$ *such that* $\alpha(c_0, \mathcal{G}c_0) \geqslant \eta(c_0, \mathcal{G}c_0)$;
(iv) *if* $\{c_n\}$ *is a sequence in* $\mathcal{K}$ *such that* $\alpha(c_n, c_{n+1}) \geqslant \eta(c_n, c_{n+1})$ *with* $c_n \to c$ *as* $n \to +\infty$, *then*

$$\eta\big(\mathcal{G}c, \mathcal{G}^2 c_n\big) \leqslant \alpha(\mathcal{G}c_n, c) \quad \text{or} \quad \eta\big(\mathcal{G}^2 c_n, \mathcal{G}^3 c_n\big) \leqslant \alpha\big(\mathcal{G}^2 c_n, c\big)$$

holds for all $n \in N$.

Then $\mathcal{G}$ *has a fixed point. Moreover,* $\mathcal{G}$ *has a unique fixed point whenever* $\alpha(c, a) \geqslant \eta(c, c)$ *for all* $c, a \in \mathrm{Fix}(\mathcal{G})$.

When we consider $\eta(c, a) = 1$ in Theorem 3 and Corollary 5, we obtain the following.

Theorem 4. *Let* $(\mathcal{K}, \vartheta, \Upsilon)$ *be a CFMS. Let* $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ *be a self-mapping satisfying the following assertions*:

(i) $\mathcal{G}$ *is* α*-admissible*;
(ii) $\mathcal{G}$ *is a* ξ-α-ΓF*-fuzzy contractive mapping*;
(iii) *there exists* $c_0 \in \mathcal{K}$ *such that* $\alpha(c_0, \mathcal{G}c_0) \geqslant 1$;
(iv) *if* $\{c_n\}$ *is a sequence in* $\mathcal{K}$ *such that* $\alpha(c_n, c_{n+1}) \geqslant 1$ *with* $c_n \to c$ *as* $n \to +\infty$, *then* $\alpha(c_n, c) \geqslant 1$ *or* $\alpha(c_{n+1}, c) \geqslant 1$ *holds for all* $n \in \mathbb{N}$.

Then $\mathcal{G}$ *has a fixed point. Moreover,* $\mathcal{G}$ *has a unique fixed point whenever* $\alpha(c, a) \geqslant 1$ *for all* $c, a \in \mathrm{Fix}(\mathcal{G})$.

Proof. Let $c_0 \in \mathcal{K}$ such that $\alpha(c_0, \mathcal{G}c_0) \geqslant 1$. Similarly to Theorem 3, we can conclude that

$$\alpha(c_n, c_{n+1}) \geqslant 1 \quad \text{and} \quad c_n \to c^* \quad \text{as } n \to +\infty,$$

where $c_{n+1} = \mathcal{G}c_n$. By assumption (iv), $\alpha(\mathcal{G}c_n, c) \geqslant 1$ holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ such that $\alpha(c_{n_k}, c) \geqslant 1$, and by definition of ξ-α-ΓF-fuzzy contractive mapping, we deduce that

$$\begin{aligned} 0 &\leqslant \xi\big(\Gamma\big[\vartheta(c_{n_k}, \mathcal{G}c_{n_k}, \Im), \vartheta(c^*, \mathcal{G}c^*, \Im), \vartheta(c_{n_k}, \mathcal{G}c^*, \Im), \vartheta(c^*, \mathcal{G}c_{n_k}, \Im)\big] \\ &\qquad \times F\big(\vartheta(\mathcal{G}c_{n_k}, \mathcal{G}c^*, \Im)\big), F\big(\vartheta(c_{n_k}, c^*, \Im)\big)\big) \\ &\leqslant F^{-1}\big(\vartheta(c_{n_k}, c^*, \Im)\big) \\ &\quad - \big(\Gamma\big[\vartheta(c_{n_k}, \mathcal{G}c_{n_k}, \Im), \vartheta(c^*, \mathcal{G}c^*, \Im), \vartheta(c_{n_k}, \mathcal{G}c^*, \Im), \vartheta(c^*, \mathcal{G}c_{n_k}, \Im)\big] \\ &\quad \times F\big(\vartheta(\mathcal{G}c_{n_k}, \mathcal{G}c^*, \Im)\big)\big)^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} &\Gamma\big[\vartheta(c_{n_k}, \mathcal{G}c_{n_k}, \Im), \vartheta(c^*, \mathcal{G}c^*, \Im), \vartheta(c_{n_k}, \mathcal{G}c^*, \Im), \vartheta(c^*, \mathcal{G}c_{n_k}, \Im)\big] \\ &\qquad \times F\big(\vartheta(\mathcal{G}c_{n_k}, \mathcal{G}c^*, \Im)\big) \\ &\quad \geqslant F\big(\vartheta(c_{n_k}, c^*, \Im)\big). \end{aligned}$$

Then

$$F\big(\vartheta(\mathcal{G}c_{n_k}, \mathcal{G}c^*, \Im)\big) > \tau \cdot F\big(\vartheta(\mathcal{G}c_{n_k}, \mathcal{G}c^*, \Im)\big) \geqslant F\big(\vartheta(c_{n_k}, c^*, \Im)\big).$$

Since F is strictly increasing function,

$$\vartheta(\mathcal{G}c_{n_k}, \mathcal{G}c^*, \Im) > \vartheta(c_{n_k}, c^*, \Im).$$

Taking limit $k \to +\infty$ in above inequality, we find

$$\vartheta(c^*, \mathcal{G}c^*, \Im) = 1, \quad \text{that is,} \quad c^* = \mathcal{G}c^*.$$

Uniqueness follows from the above Theorem 1. □

Corollary 6. *Let $(\mathcal{K}, \vartheta, \curlyvee)$ be a CFMS. Let $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ be a self-mapping satisfying the following assertions*:

(i) *$\mathcal{G}$ is α-admissible*;

(ii) *if, for $c, a \in \mathcal{K}$ with $\alpha(c, a) \geqslant 1$ and $\vartheta(\mathcal{G}c, \mathcal{G}a, \Im) > 0$, we have*

$$\xi\big(\tau \cdot F\big(\vartheta(\mathcal{G}c, \mathcal{G}a, \Im)\big), F\big(\vartheta(c, a, \Im)\big)\big) \geqslant 0,$$

where $c \neq a, \tau \in (0, 1)$ and $F \in \Lambda_F$;

(iii) *there exists $c_0 \in \mathcal{K}$ such that $\alpha(c_0, \mathcal{G}c_0) \geqslant 1$*;

(iv) *if $\{c_n\}$ is a sequence in $\mathcal{K}$ such that $\alpha(c_n, c_{n+1}) \geqslant 1$ with $c_n \to c$ as $n \to +\infty$, then either $\alpha(\mathcal{G}c_n, c) \geqslant 1$ or $\alpha(\mathcal{G}^2c_n, c) \geqslant 1$ holds for all $n \in \mathbb{N}$.*

Then $\mathcal{G}$ has a fixed point Moreover, $\mathcal{G}$ has a unique fixed point whenever $\alpha(c, a) \geqslant 1$ for all $c, a \in \mathrm{Fix}(\mathcal{G})$.

4 Application

In this section, we prove the existence theorem of solutions to nonlinear fractional differential equations as an application of Corollary 6. We address the problem of existence of solutions to the nonlinear fractional differential equation [16]

$$\begin{aligned}
&{}^{C}D^p\big\{{}^{C}D^q c(\Im) + f\big(\Im, c(\Im)\big)\big\} = g\big(\Im, c(\Im)\big) + \int_0^{\Im} K\big(\Im, s, c(s)\big)\,\mathrm{d}s, \\
&c(0) = \sum_{j=1}^{m} \beta_j c(\sigma_j), \qquad c(1) = \sum_{i=1}^{n} \alpha_i c(\xi_i),
\end{aligned} \tag{9}$$

where $0 < \sigma_j < \xi_i < 1$, $0 < p, q < 1$, $\beta_j, \alpha_i \in \mathbb{R}$, $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, n$. ${}^{C}D^p$, ${}^{C}D^q$ are the Caputo fractional derivatives, f, g, K, are given functions with $f, g \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and $K \in C(D \times \mathbb{R}, \mathbb{R})$, where $D = \{(\Im, s),\ \Im \in [0, 1],\ s \in [0, 1]\}$.

Taking into account $(\mathcal{K}, \|\cdot\|_\infty)$, where $\mathcal{K} = C([0, 1], \mathbb{R})$ is the Banach space of continuous function from $[0, 1]$ into $\mathbb{R}$ equipped with the supremum norm

$$\|c\|_\infty = \sup_{\Im \in [0,1]} \big|c(\Im)\big|.$$

Let $(\mathcal{K}, \vartheta, \curlyvee_p)$ be the CFMS, where ϑ is defined by

$$\vartheta(c, a, \Im) = \left(1 - \exp^{-\Im}\right)^{\|c-a\|_\infty} \quad \text{for all } c, a \in \mathcal{K} \text{ and } \Im > 0.$$

For a continuous function $g : \mathbb{R}_+ \to \mathbb{R}$, the fractional-order Caputo derivative β is given by

$$^{C}D^{\beta} g(\Im) = \frac{1}{\Gamma(n-\beta)} \int_0^{\Im} (\Im - s)^{n-\beta-1} g^{n}(s)\, \mathrm{d}s$$

($n - 1 < \beta < n$, $n = [\beta] + 1$), where $[\beta]$ denote the integer part of the real number β.
Now we state the following existence theorem.

Theorem 5. *Suppose that*:
(i) *There exist a function* $\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\tau \in (0, 1)$ *such that*

$$\begin{aligned}
\left|f\left(\Im, c(\Im)\right) - f\left(\Im, a(\Im)\right)\right| &\leqslant L_1|c - a|, \\
\left|g\left(\Im, c(\Im)\right) - g\left(\Im, a(\Im)\right)\right| &\leqslant L_2|c - a|, \\
\left|K\left(\Im, s, c(s)\right) - K\left(\Im, s, a(s)\right)\right| &\leqslant L_3|c - a|,
\end{aligned}$$

and $\tau \leqslant 1/(L\Lambda)$ *with* $L = \max\{L_1, L_2, L_3\}$ *and*

$$\begin{aligned}
\Lambda = {} & \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\
& + \bar{\lambda}_1 \left[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \right. \\
& \left. + \sum_{i=1}^{n} |\alpha_i| \left(\frac{\iota_i^{q}}{\Gamma(q+1)} + \frac{\iota_i^{p+q}}{\Gamma(q+p+1)} + \frac{\iota_i^{p+q+1}}{\Gamma(q+p+2)} \right) \right] \\
& + \bar{\lambda}_2 \sum_{j=1}^{m} |\beta_j| \left(\frac{\sigma_j^{q}}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)} \right),
\end{aligned}$$

where

$$\bar{\lambda}_1 = \frac{1}{|k|}\left(|\rho_1| + \frac{|\rho_2|}{\Gamma(q+1)} \right), \qquad \bar{\lambda}_2 = \frac{1}{|k|}\left(|\rho_3| + \frac{|\rho_4|}{\Gamma(q+1)} \right),$$

$$\rho_1 = \sum_{j=1}^{m} \frac{\beta_j \sigma_j^{q}}{\Gamma(q+1)}, \qquad \rho_2 = -1 + \sum_{j=1}^{m} \beta_j,$$

$$\rho_3 = \frac{1}{\Gamma(q+1)} - \sum_{i=1}^{n} \frac{\alpha_i \iota_i^{q}}{\Gamma(q+1)}, \qquad \rho_4 = 1 - \sum_{i=1}^{n} \alpha_i,$$

$k = \rho_2\rho_3 - \rho_1\rho_4 \neq 0$;

(ii) *There exists* $c_0 \in \mathcal{K}$ *such that* $\delta(c_0(\Im), \mathcal{G}c_0(\Im)) > 0$ *for all* $\Im \in [0,1]$, *where the operator* $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ *is defined by*

$$\begin{aligned}
&\mathcal{G}c(\Im) \\
&= \int_0^{\Im} \frac{(\Im - s)^{q+p-1}}{\Gamma(q+p)} g\bigl(s, c(s)\bigr)\,\mathrm{d}s + \int_0^{\Im} \frac{(\Im - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K\bigl(s, \iota, c(\iota)\bigr)\,\mathrm{d}\iota\,\mathrm{d}s \\
&\quad - \int_0^{\Im} \frac{(\Im - s)^{q-1}}{\Gamma(q)} f\bigl(s, c(s)\bigr)\,\mathrm{d}s - \lambda_1(\Im)\Biggl[\sum_{i=1}^{n} \alpha_i \Biggl(\int_0^{\eta_i} \frac{(\eta_i - s)^{q+p-1}}{\Gamma(q+p)} g\bigl(s, c(s)\bigr)\,\mathrm{d}s \\
&\quad + \int_0^{\eta_i} \frac{(\eta_i - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K\bigl(s, \iota, c(\iota)\bigr)\,\mathrm{d}\iota\,\mathrm{d}s - \int_0^{\eta_i} \frac{(\eta_i - s)^{q-1}}{\Gamma(q)} f\bigl(s, c(s)\bigr)\,\mathrm{d}s\Biggr) \\
&\quad - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g\bigl(s, c(s)\bigr)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K\bigl(s, \iota, c(\iota)\bigr)\,\mathrm{d}\iota\,\mathrm{d}s \\
&\quad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f\bigl(s, c(s)\bigr)\,\mathrm{d}s\Biggr] + \lambda_2(\Im)\sum_{j=1}^{m} \beta_j \Biggl[\int_0^{\sigma_j} \Biggl(\frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} f\bigl(s, c(s)\bigr) \\
&\quad - \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} g\bigl(s, c(s)\bigr) - \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K\bigl(s, \iota, c(\iota)\bigr)\,\mathrm{d}\iota\Biggr)\,\mathrm{d}s\Biggr],
\end{aligned}$$

where

$$\lambda_1(\Im) = \frac{1}{k}\biggl(\rho_1 - \frac{\rho_2 \Im^q}{\Gamma(q+1)}\biggr), \qquad \lambda_2(\Im) = \frac{1}{k}\biggl(\rho_3 - \frac{\rho_4 \Im^q}{\Gamma(q+1)}\biggr), \qquad \Im \in [0,1];$$

(iii) *For each* $\Im \in [0,1]$ *and* $c, a \in \mathcal{K}$, $\delta(c(\Im), a(\Im)) > 0$ *implies* $\delta(\mathcal{G}c(\Im), \mathcal{G}a(\Im)) > 0$;

(iv) *If* $\{c_n\}$ *is a sequence in* $\mathcal{K}$ *such that* $c_n \to c$ *in* $\mathcal{K}$ *and* $\delta(c_n, c_{n+1}) > 0$ *for all* $n \in \mathbb{N}$, *then* $\delta(c_n, c) > 0$ *for all* $n \in \mathbb{N}$. *Then* (9) *has at least one solution.*

Proof. It is well known that $c \in \mathcal{K}$ is a solution of (9) if and only if $c \in \mathcal{K}$ is a solution of the integral equation

$$\begin{aligned}
c(\Im) &= \int_0^{\Im} \frac{(\Im - s)^{q+p-1}}{\Gamma(q+p)} g\bigl(s, c(s)\bigr)\,\mathrm{d}s + \int_0^{\Im} \frac{(\Im - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K\bigl(s, \iota, \Im(\iota)\bigr)\,\mathrm{d}\iota\,\mathrm{d}s \\
&\quad - \int_0^{\Im} \frac{(\Im - s)^{q-1}}{\Gamma(q)} f\bigl(s, c(s)\bigr)\,\mathrm{d}s - \lambda_1(\Im)\Biggl[\sum_{i=1}^{n} \alpha_i \Biggl(\int_0^{\eta_i} \frac{(\eta_i - s)^{q+p-1}}{\Gamma(q+p)} g\bigl(s, c(s)\bigr)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{\eta_i} \frac{(\eta_i - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K\big(s, \iota, c(\iota)\big)\,\mathrm{d}\iota\,\mathrm{d}s - \int_0^{\eta_i} \frac{(\eta_i - s)^{q-1}}{\Gamma(q)} f\big(s, c(s)\big)\,\mathrm{d}s\Bigg) \\
&- \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g\big(s, c(s)\big)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K\big(s, \iota, c(\iota)\big)\,\mathrm{d}\iota\,\mathrm{d}s \\
&+ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f\big(s, c(s)\big)\,\mathrm{d}s\Bigg] + \lambda_2(\Im) \sum_{j=1}^m \beta_j \Bigg[\int_0^{\sigma_j} \Bigg(\frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} f\big(s, c(s)\big) \\
&- \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} g\big(s, c(s)\big) - \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K\big(s, \iota, c(\iota)\big)\,\mathrm{d}\iota \Bigg)\,\mathrm{d}s\Bigg],
\end{aligned}
$$

where $\Im \in [0,1]$.

Hence, problem (9) is equivalent to find $c^* \in \mathcal{K}$, which is a fixed point of $\mathcal{G}$. Now, let $c, a \in \mathcal{K}$ such that $\delta(c(\Im), a(\Im)) > 0$ for all $\Im \in [0,1]$. By (1), we find

$$
\begin{aligned}
&\big\|\mathcal{G}c(\Im) - \mathcal{G}a(\Im)\big\|_\infty \\
&= \sup\Bigg[\int_s^{\Im} \frac{(\Im - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s \big|K\big(s, \iota, c(\iota)\big) - K\big(s, \iota, a(\iota)\big)\big|\,\mathrm{d}\iota\,\mathrm{d}s \\
&\quad + \int_0^{\Im} \frac{(\Im - s)^{q+p-1}}{\Gamma(q+p)} \big|g\big(s, c(s)\big) - g\big(s, a(s)\big)\big|\,\mathrm{d}s \\
&\quad + \int_0^{\Im} \frac{(\Im - s)^{q-1}}{\Gamma(q)} \big|f\big(s, c(s)\big) - f\big(s, a(s)\big)\big|\,\mathrm{d}s \\
&\quad + \big|\lambda_1(\Im)\big| \Bigg\{ \sum_{i=1}^n \alpha_i \int_0^{\eta_i} \Bigg(\frac{(\eta_i - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s \big|K\big(s, \iota, c(\iota)\big) - K\big(s, \iota, a(\iota)\big)\big|\,\mathrm{d}\iota \\
&\quad + \frac{(\eta_i - s)^{q+p-1}}{\Gamma(q+p)} \big|g\big(s, c(s)\big) - g\big(s, a(s)\big)\big| \\
&\quad + \frac{(\eta_i - s)^{q-1}}{\Gamma(q)} \big|f\big(s, c(s)\big) - f\big(s, a(s)\big)\big| \Bigg)\,\mathrm{d}s \\
&\quad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \big|f\big(s, c(s)\big) - f\big(s, a(s)\big)\big|\,\mathrm{d}s \\
&\quad + \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \big|g\big(s, c(s)\big) - g\big(s, a(s)\big)\big|\,\mathrm{d}s
\end{aligned}
$$

$$+\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s \bigl|K\bigl(s,\iota,c(\iota)\bigr)-K\bigl(s,\iota,a(\iota)\bigr)\bigr|\,\mathrm{d}\iota\,\mathrm{d}s\biggr\}$$
$$+\bigl|\lambda_2(\Im)\bigr|\sum_{j=1}^m \beta_j \int_0^{\sigma_j}\biggl(\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s \bigl|K\bigl(s,\iota,c(\iota)\bigr)-K\bigl(s,\iota,a(\iota)\bigr)\bigr|\,\mathrm{d}\iota$$
$$+\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}\bigl|f\bigl(s,c(s)\bigr)-f\bigl(s,c(s)\bigr)\bigr|$$
$$+\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\bigl|g\bigl(s,c(s)\bigr)-g\bigl(s,a(s)\bigr)\bigr|\biggr)\,\mathrm{d}s\biggr]$$
$$\leqslant L\Lambda\|c-a\|_\infty \leqslant \frac{\|c-a\|_\infty}{\tau}.$$

Thus, for each $c, a \in \mathcal{K}$ with $\delta(c(\Im)-a(\Im))>0$ for all $\Im\in[0,1]$, we have

$$\bigl\|\mathcal{G}c(\Im)-\mathcal{G}a(\Im)\bigr\|_\infty \leqslant \frac{\|c-a\|_\infty}{\tau}. \tag{10}$$

Moreover it is known that, for all $\Im\in[0,1]$, we have

$$0\leqslant \bigl(1-\exp^{-\Im}\bigr)\leqslant 1. \tag{11}$$

Using (10) and (11), we get

$$\bigl(1-\exp^{-\Im}\bigr)^{\tau\cdot\|\mathcal{G}c(\Im)-\mathcal{G}a(\Im)\|_\infty} \geqslant \bigl(1-\exp^{-\Im}\bigr)^{\|c-a\|_\infty},$$
$$\tau\cdot\log_{(1-\exp^{-\Im})}\bigl(1-\exp^{-\Im}\bigr)^{\|\mathcal{G}c(\Im)-\mathcal{G}a(\Im)\|_\infty} \geqslant \log_{(1-\exp^{-\Im})}\bigl(1-\exp^{-\Im}\bigr)^{\|c-a\|_\infty}.$$

Hence,

$$\bigl(\log_{(1-\exp^{-\Im})}\bigl(1-\exp^{-\Im}\bigr)^{\|c-a\|_\infty}\bigr)^{-1} \geqslant \bigl(\tau\cdot\log_{(1-\exp^{-\Im})}\bigl(1-\exp^{-\Im}\bigr)^{\|\mathcal{G}c(\Im)-\mathcal{G}a(\Im)\|_\infty}\bigr)^{-1}.$$

Therefore,

$$\bigl(\log_{(1-\exp^{-\Im})}\bigl(1-\exp^{-\Im}\bigr)^{\|c-a\|_\infty}\bigr)^{-1} - \bigl(\tau\cdot\log_{(1-\exp^{-\Im})}\bigl(1-\exp^{-\Im}\bigr)^{\|\mathcal{G}c(\Im)-\mathcal{G}a(\Im)\|_\infty}\bigr)^{-1}\geqslant 0.$$

Now, consider the functions $\xi:(0,1]\times(0,1]\to\mathbb{R}$ and $F:\mathbb{R}_+\to\mathbb{R}_+$ defined by $\xi(c,a)=1/a-1/c$ for all $a,c\in(0,1)$ and $F(z)=\log_{(1-\exp^{-\Im})}(z)$ for each $\Im>0$ such that $F\in\Lambda_F$. The above inequality implies that

$$\xi\bigl(\tau\cdot F\bigl(\vartheta(\mathcal{G}c,\mathcal{G}a,\Im)\bigr),F\bigl(\vartheta(c,a,\Im)\bigr)\bigr)\geqslant 0$$

for all $c, a \in \mathcal{K}$ with $\vartheta(\mathcal{G}c, \mathcal{G}a, \Im) > 0$. Therefore, $\mathcal{G}$ is an ξ-α-ΓF-contractive mapping. Then, by using assumption (iii) of Theorem 5, $\alpha(c, a) \geqslant 1$ implies $\delta(c(\Im), a(\Im)) > 0$, which implies $\delta(\mathcal{G}c(\Im), \mathcal{G}a(\Im))) > 0$, which gives that $\alpha(\mathcal{G}c, \mathcal{G}a) \geqslant 1$ for all $c, a \in \mathcal{K}$. Therefore, $\mathcal{G}$ is α-admissible. Applying assumption (ii) of Theorem 5, there exists $c_0 \in \mathcal{K}$ such that $\alpha(c_0, \mathcal{G}c_0) \geqslant 1$. Finally, by assumption (iv) of Theorem 5, if $\{c_n\}$ be a sequence in $\mathcal{K}$ such that $\alpha(c_n, c_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ implies $\delta(c_n, c_{n+1}) > 0$ for all $n \in \mathbb{N}$, then $\delta(c_n, c) > 0$ for all $n \in \mathbb{N}$ implies $\alpha(c_n, c) \geqslant 1$ for all $n \in \mathbb{N}$. Hence, condition (iv) of Corollary 6 holds true.

By Corollary 6, we conclude the existence of $c^* \in \mathcal{K}$ such that $c^* = \mathcal{G}c^*$ and c^* is the solution of problem (9). □

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