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Exponential synchronization of dynamical complex networks via random impulsive scheme*

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Abstract. This paper investigates the synchronization of a complex network based on a class of random impulsive differential equation systems. Based on the random impulsive strategy of Poisson distribution, a random impulsive dynamical network model is constructed. Using the Lyapunov principle, random process theory, linear matrix inequality method, and some basic analysis methods, we realize the global mean-square index synchronization of the model. We then get sufficient criteria for the synchronization. By presenting a numerical example, we verified the validity of the theoretical results.

Keywords: random impulse, synchronization control, dynamic network.

1 Introduction

As a ubiquitous phenomenon in nature and human society, impulsivity is caused by instantaneous extreme changes. The system with impulsivity can be studied using mathematical models with pulses. Differential equations are often used for mathematical modeling. When pulses occur in these systems, correspondingly, we can introduce the pulses into the model based on differential equations. Such systems are called impulsive differential equation systems. Sometimes, the timing of the pulse cannot be determined in advance. In other words, the timing of the pulse is random. Random impulsive differential equations can be used to study such systems. Stochastic differential equations have wide applications in describing random or uncertain factors and, as such, have advantages over ordinary differential equations in modeling real-world systems with uncertainty. A variety of results on the theory of stochastic differential equations have been obtained; see [11, 16, 18, 25] and the references therein. Stochastic impulsive differential equations

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also have important applications in science and engineering [9]. Many systems in nature are affected by impulses and random factors. Impulsive differential equations can be used in modeling such systems. In the literature, the impulsive differential equation has been extensively studied [1, 4, 5, 7, 10, 20, 21, 29]. The behavior of a complex network under interference and impulse can be described by a complex network model with random impulsive differential equations.

In the last few decades, complex networks have attracted extensive attention [12, 13, 17, 19, 23, 28]. Many studies have been carried out and published in the literature. The behavior of complex networks can be modeled and described by nonlinear dynamical systems as each node in the network can be considered as a nonlinear system. Complex networks are widely used in modeling real-world problems. For example, complex networks can be used to model disease transmissions [24, 28], and so on. In view of the fact that every node in the complex network follows the behaviors of nonlinear dynamics, the nonlinear dynamical system plays an important role in the study of complex networks. Nonlinear dynamical systems have received extensive attention. Investigation indicated that such systems exhibit complex dynamical behaviors [27]. Various control strategies have been proposed to realize the synchronization of nonlinear dynamic systems. Synchronization of chaotic systems based on integer-order differential equations and fractional-order differential equations have been studied in the literature. Synchronization in complex networks has a variety of applications in neuroscience, biological systems, business, and social sciences [22]. The synchronization of complex networks has also been studied in the literature [2,6,8,12,13,15,17,26,30]. Recent studies showed that the topology of a network is critical to the design of synchronization schemes [12,13].

Random pulses exist widely in nature and human society. Studying such pulses is crucial to understanding and controlling corresponding risks. However, in the literature, the synchronization of dynamical systems with random pulses are rarely studied. On the other hand, it is challenging to realize the synchronization of such systems with random pulses. In this article, we build a random impulse model of the network and study its exponential synchronization strategy. In this paper, we discuss complex network systems with random impulsive differential equations. Such systems have complex dynamical behaviors. We use the Lyapunov principle, stochastic process theory, and linear matrix inequality method to achieve the synchronization of the complex network model.

The paper is organized as follows. In Section 2, we present the definition of a class of random impulsive dynamical network model and give some preliminaries. In Section 3, we analyze the mean-square exponential synchronization of the network and give some sufficient conditions to achieve the synchronization. In Section 4, a numerical simulation is given to verify our theoretical results. The conclusion is drawn in Section 5.

2 Model description and preliminaries

Before presenting the model, we introduce some notations.

Notations. The following standard notations will be used throughout this paper. The notation $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{N\times N}$ denotes the n-dimentional real square matrices; for a real symmetric matrix M, we use $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ to denote its minimum and maximum eigenvalues; I is an identity matrix with compatible dimensions; the superscript T represents transpose; $\operatorname{diag}(b_1, b_2, \dots, b_n)$ stands for a diagonal matrix with diagonal elements $b_1, b_2, \dots, b_n$; for real matrices A and B, the notation $A \otimes B$ denotes the Kronecker product of the matrices; if not explicitly stated, $\|\cdot\|$ are assumed as the Euclidean norm.

2.1 The model

Consider a dynamical network consisting of N nodes [17] described as

$$\dot{x}_k(t) = g\bigl(x_k(t)\bigr) + \sum_{j=1}^{N} \mu_{kj} B x_j(t), \tag{1}$$

where $x_k(t) = (x_{k1}(t), x_{k2}(t), \dots, x_{kn}(t)) \in \mathbb{R}^n$ is the state variable of the kth node at time t for $k = 1, 2, \dots, N$, $g(x_k(t)) = (g(x_{k1}(t)), g(x_{k2}(t)), \dots, g(x_{kn}(t))) : \mathbb{R}^n \to \mathbb{R}^n$ is a vector function, and $B = \operatorname{diag}(b_1, b_2, \dots, b_n)$ is the inner coupling matrix, which shows the connection in the node. $U = (\mu_{kj}) \in \mathbb{R}^{N\times N}$ is the outer coupling matrix denoting the connection between nodes and is defined as follows. If there is a connection between node k and node j ($k \neq j$), then $\mu_{kj} = \mu_{jk} \neq 0$. Otherwise, $\mu_{kj} = \mu_{jk} = 0$. The diagonal elements are defined as $\mu_{kk} = -\sum_{j=1,\, j\neq k}^{N} \mu_{kj}$.

We use $s(t)$ to denote the synchronization state variable, which satisfies $\dot{s}(t) = g(s(t))$. Network (1) is said to achieve synchronization if $\lim_{t\to\infty} \|x_k(t) - s(t)\| = 0$. Especially, the network is said to achieve globally exponential synchronization if there are constants $C, \epsilon > 0$ such that for any initial values $x_k(0)$,

$$\bigl\|x_k(t) - s(t)\bigr\|^2 \leqslant C\mathrm{e}^{-\epsilon(t-t_0)}$$

holds for any $k = 1, 2, \dots, N$ [3].

Many researchers designed pinning impulsive controllers and applied them to complex network models to achieve synchronization. For example, Feng et al. [8] designed an impulsive controller $\Delta(x_k(T_\psi)) = b_\psi x_k(T_\psi^-)$ to realize the synchronization of complex networks. Leng and Wu [15] proposed an impulsive scheme to synchronize a time-delayed complex network model. The controlled network model is described as follows:

$$\dot{x}_k(t) = g\bigl(x_k(t)\bigr) + \sum_{j=1}^{N} \mu_{kj} B x_j(t), \quad t \neq T_\psi,$$
$$x_k\bigl(T_\psi^+\bigr) = x_k\bigl(T_\psi^-\bigr) + \vartheta_\psi\bigl(x_k\bigl(T_\psi^-\bigr) - s(T_\psi)\bigr), \quad t = T_\psi,$$

where $k = 1, 2, \dots, N$, $\psi = 1, 2, \dots$, the impulsive time series satisfies $0 = t_0 < t_1 < \cdots < t_\psi < \cdots$, and $t_\psi \to \infty$ as $\psi \to \infty$. Here ϑ_ψ represents the impulsive gain at time t_ψ, $x_k(T_\psi^+) = \lim_{t\to T_\psi^+} x_k(t)$, and $x_k(T_\psi^-) = \lim_{t\to T_\psi^-} x_k(t)$. We always assume that $x_k(t)$ is left-hand continuous at T_ψ, i.e., $x_k(T_\psi^-) = x_k(T_\psi)$. $t_\psi = T_\psi - T_{\psi-1} (\psi = 1, 2, \dots)$ denotes the ψth impulse waiting time. We will apply the above impulsive controller to the random impulsive network model.

In real-world applications, the impulsive effect often occurs at random moments such as the random forces on physical systems and noise caused by environmental uncertainty. In this paper, we consider the scenario, where the impulses occur at random times. We notice that $T_{\psi-1} \leqslant T_\psi$. Suppose the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is given. In the following, we present definitions of random impulse.

A poisson impulse is defined as follows.

Definition 1. Let $\{\tau_\psi\}_{\psi=1}^{\infty}$ be a sequence of independent and identically distributed random variables, each of which has an exponential distribution with parameter λ, i.e., $\mathbf{P}(\tau_\psi = m) = \lambda^m \mathrm{e}^{-\lambda}/m!$, $m = 0, 1, \dots$. Let τ_ψ define the time between the $(\psi - 1)$th impulse and the ψth impulse. This kind of impulse is called Poisson impulse.

Remark 1. (See [14].) Denote $N(t)$ as the number of impulse until time t. Then, according to the stochastic process theory, $N(t)$ is a Poisson process.

Define a sequence of random variables $\{\xi_\psi\}_{\psi=1}^{\infty}$ by $\xi_0 = T_0$, $\xi_\psi = T_0 + \sum_{i=1}^{\psi}\tau_i$, $\psi = 1, 2, \dots$, where T_0 is the initial time. In this paper, we assume that $T_0 = 0$. We note that if τ_ψ is waiting time of the ψth impulse, then ξ_ψ represents the time the ψth impulse occurring.

Assign an arbitrary value t_ψ to each random variable τ_ψ. Then the increasing sequence of points $T_\psi = T_0 + \sum_{i=1}^{\psi} t_i$, $\psi = 1, 2, \dots$, are values of the random variables ξ_ψ. We thus get a pinning impulsive controlled model of network (1), given by

$$\begin{aligned} \dot{x}_k(t) &= g\big(x_k(t)\big) + \sum_{j=1}^{N} \mu_{kj} B x_j(t), \quad t \neq T_\psi, \\ x_k\big(T_\psi^+\big) &= x_k\big(T_\psi^-\big) + \vartheta_\psi\big(x_k\big(T_\psi^-\big) - s(T_\psi)\big), \quad t = T_\psi, \\ x_k(T_0) &= \rho_k, \end{aligned} \tag{2}$$

where ρ_k are constants. Denote the solution of (2) by $x(t; T_0, \{\rho_k\}_{k=1}^{N}, \{t_\psi\}_{\psi=1}^{\infty}) = (x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \dots, x_N^{\mathrm{T}}(t))^{\mathrm{T}}$. When the pulse time point is fixed, the solution of the differential equation obtained is definite. If all values of τ_ψ are taken with probability, the set of all solutions $x(t; T_0, \{\rho_k\}_{k=1}^{N}, \{t_\psi\}_{\psi=1}^{\infty})$ of network model (2) generates a stochastic process denoted by $x(t; T_0, \{\rho_k\}_{k=1}^{N}, \{\tau_\psi\}_{\psi=1}^{\infty})$, which is called the solution of network model (1) with random impulses at random times.

Definition 2. A stochastic process $x(t; T_0, \{\rho_k\}_{k=1}^{N}, \{\tau_\psi\}_{\psi=1}^{\infty})$, where ρ_k is real, is said to be the solution of random impulsive system

$$\begin{aligned} \dot{x}_k(t) &= g\big(x_k(t)\big) + \sum_{j=1}^{N} \mu_{kj} B x_j(t), \quad t \neq \xi_\psi, \\ x_k\big(\xi_\psi^+\big) &= x_k\big(\xi_\psi^-\big) + \vartheta_\psi\big(x_k\big(\xi_\psi^-\big) - s(\xi_\psi)\big), \quad t = \xi_\psi, \\ x_k(T_0) &= \rho_k \end{aligned} \tag{3}$$

if $x(t; T_0, \{\rho_k\}_{k=1}^{N}, \{t_\psi\}_{\psi=1}^{\infty})$ is the solution of network model (2) for any given value t_ψ of τ_ψ, where random variables ξ_ψ are defined above.

Furthermore, $x(t; T_0, \{\rho_k\}_{k=1}^N, \{t_\psi\}_{\psi=1}^\infty)$ is called a sample orbit of the stochastic process $x(t; T_0, \{\rho_k\}_{k=1}^N, \{\tau_\psi\}_{\psi=1}^\infty)$.

In this article, we consider the synchronization of network model (3). Here the most important part is to estimate the probability of the occurrence of just ψ impulses until time t.

For given $t \geqslant T_0$, let $A_\psi(t) = \{\omega \in \Omega\colon\ \xi_\psi(\omega) < t < \xi_{\psi+1}(\omega)\}$ be the events, where there are exactly ψ impulses occurring until time t. Consider the indicative function of $A_\psi(t)$ given by

$$\mathbf{1}_{A_\psi(t)}(\omega) = \begin{cases} 1, & \omega \in A_\psi(t), \\ 0, & \omega \notin A_\psi(t). \end{cases}$$

Remark 2. Denote $\mathbf{P}(A_\psi(t)) = \mathbf{E}(\mathbf{1}_{A_\psi(t)})$. We estimate the probability of $A_\psi(t)$ by calculating the mean value of $\mathbf{1}_{A_\psi(t)}$.

Lemma 1. *For ξ_ψ defined above, we then have*

$$\mathbf{P}\big(A_\psi(t)\big) = \mathbf{E}\big(\mathbf{1}_{A_\psi(t)}\big) = \frac{\lambda^\psi (t-T_0)^\psi}{\psi!} \mathrm{e}^{-\lambda(t-T_0)}.$$

Proof. By the probability theory we have

$$\begin{aligned}
\mathbf{E}(\mathbf{1}_{A_\psi(t)}) &= \underbrace{\int\int\cdots\int}_{\sum_{i=1}^{\psi}\tau_i \leqslant t-T_0 < \sum_{i=1}^{\psi+1}\tau_i} \lambda^{\psi+1}\mathrm{e}^{-\lambda\sum_{i=1}^{\psi+1}\tau_i}\,\mathrm{d}\tau_1\cdots\tau_{\psi+1} \\
&= \lambda^\psi \mathrm{e}^{-\lambda(t-T_0)} \underbrace{\int\int\cdots\int}_{\sum_{i=1}^{\psi}\tau_i \leqslant t-T_0} \mathrm{d}\tau_1\cdots\tau_\psi \\
&= \frac{\lambda^\psi (t-T_0)^\psi}{\psi!}\mathrm{e}^{-\lambda(t-T_0)},
\end{aligned}$$

where the last equation follows from

$$\underbrace{\int\int\cdots\int}_{\sum_{i=1}^{\psi}\tau_i \leqslant t-T_0} \mathrm{d}\tau_1\cdots\tau_\psi = \frac{(t-T_0)^\psi}{\psi!}.$$

This equation can be obtained by mathematical induction. □

3 Main results

In this section, we present the main results of this work. Considering the random impulsive network model (3), let $e_k(t) = x_k(t) - s(t)$ be the synchronization error variables.

Due to the diagonal elements of U are defined as $\mu_{kk} = -\sum_{j=1,\, j\neq k}^{N} \mu_{kj}$, we get

$$\sum_{j=1}^{N} \mu_{kj} B s_j(t) = 0.$$

Then the error system is

$$\begin{aligned} \dot{e}_k(t) &= \hat{g}\big(e_k(t)\big) + \sum_{j=1}^{N} \mu_{kj} B e_j(t), \quad t \neq \xi_\psi, \\ e_k\big(\xi_\psi^+\big) &= e_k\big(\xi_\psi^-\big) + \vartheta_\psi e_k\big(\xi_\psi^-\big), \quad t = \xi_\psi, \end{aligned}$$

where $\hat{g}(e_k(t)) = g(x_k(t)) - g(s(t))$. We then let $e(t) = (e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \dots, e_N^{\mathrm{T}}(t))^{\mathrm{T}}$ and $\delta_\psi = (1+\vartheta_\psi)^2$.

Assumption 1 [Lipschitz condition]. (See [30].) Suppose that there is a positive constant L such that

$$\begin{aligned} &\big(y(t) - x(t)\big)^{\mathrm{T}} \big(g\big(y(t)\big) - g\big(x(t)\big)\big) \\ &\quad \leqslant L\big(y(t) - x(t)\big)^{\mathrm{T}} \big(y(t) - x(t)\big) \end{aligned}$$

holds for any $x(t), y(t)$ with range in $\mathbb{R}^n$ and $t > T_0$.

Remark 3. (See [6].) A function $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is $\mathrm{QUAD}(\varDelta, \omega)$ if and only if

$$\begin{aligned} &(y-x)^{\mathrm{T}} \big(f(y,t) - f\big(x,t\big)\big) - (y-x)^{\mathrm{T}} \varDelta (y-x) \\ &\quad \leqslant -\omega (y-x)^{\mathrm{T}} (y-x) \end{aligned}$$

holds for any $x, y \in \mathbb{R}^n$, $t > 0$, where $\varDelta$ is an $n \times n$ matrix, and ω is a positive constant.

Obviously, function g in Assumption 1 is $\mathrm{QUAD}((L_\kappa)I, \kappa)$, where κ is a positive constant and I is an identity matrix with suitable dimension. The inner function of many systems satisfies $\mathrm{QUAD}(\varDelta, \omega)$. It is often supposed that $g(x)$ is QUAD.

Definition 3. (See [2].) The random impulsive network model (3) is said to achieve global mean-square exponential synchronization if there exist positive constant C and ϵ such that for any $t > T_0$,

$$\mathbf{E}\big(\big\|e(t)\big\|^2\big) \leqslant C \mathrm{e}^{-\epsilon(t-T_0)}$$

holds for any initial value ρ_k.

Theorem 1. *Suppose that Assumption* 1 *holds, and let* $\lambda_1 = 2(L + \lambda_{\max}(U \otimes B))$. *If the conditions*

(i) $\delta_\psi = (1+\vartheta_\psi)^2 \leqslant c$, *where c is an positive constant,* $\psi = 1, 2, \dots,$ *and*
(ii) $\lambda_1 - (1-c)\lambda < 0$

hold for the system, then network model (3) *can achieve global mean-square exponential synchronization.*

Proof. Construct a Lyapunov function $V(t) = (1/2)\sum_{k=1}^{N} e_k^{\mathrm{T}}(t) e_k(t)$. For any given τ_ψ's value t_ψ, we consider the corresponding pinning impulsive network.

When $t \neq T_\psi$, the derivative of V is

$$\begin{aligned}\dot V(t) &= \sum_{k=1}^N e_k^{\mathrm T}(t)\dot e_k(t) = \sum_{k=1}^N e_k^{\mathrm T}(t)\Biggl[\hat g\bigl(e_k(t)\bigr) + \sum_{j=1}^N \mu_{kj} B e_j(t)\Biggr] \\ &= \sum_{k=1}^N e_k^{\mathrm T}(t)\hat g\bigl(e_k(t)\bigr) + \sum_{k=1}^N e_k^{\mathrm T}(t)\sum_{j=1}^N \mu_{kj} B e_j(t) \\ &\leqslant 2LV(t) + e^{\mathrm T}(t)(U\otimes B)e(t) \leqslant \lambda_1 V(t).\end{aligned}$$

When $t = T_\psi$,

$$V\bigl(T_\psi^+\bigr) = (1+\vartheta_\psi)^2 V\bigl(T_\psi^-\bigr) = \delta_\psi V\bigl(T_\psi^-\bigr).$$

Therefore, the stochastic process $V(t)$ satisfies

$$\dot V(t) \leqslant \lambda_1 V(t), \quad t\neq \xi_\psi, \qquad V\bigl(\xi_\psi^+\bigr) = \delta_\psi V\bigl(\xi_\psi^-\bigr), \qquad V(T_0) = V_0.$$

Let stochastic process $u(t)$ be a solution of the system

$$\dot u(t) = \lambda_1 u(t), \quad t\neq \xi_\psi, \qquad u\bigl(\xi_\psi^+\bigr) = \delta_\psi u\bigl(\xi_\psi^-\bigr), \qquad u(T_0) = u_0 = V_0. \tag{4}$$

Obviously, we know that every sample track $u(t; T_0, u_0, \{t_\psi\}_{\psi=1}^\infty)$ of $u(t)$ is above $V(t)$'s corresponding sample track, and if we let $y(t) = u(t) - V(t)$, the state space of $y(t)$ is $[0,\infty)$. □

Definition 4. We say that the stochastic processes $x(t), y(t)$ satisfy $x(t) \leqslant y(t)$ if the state space of $z(t) = y(t) - x(t)$ is $[0,+\infty)$.

Next, we calculate the mean value of $u(t)$ and design controllers to achieve its synchronization.

Lemma 2. *Consider the random impulsive linear differential equation system* (4)*, where τ_ψ has exponential distribution. This equation is a first-order homogeneous linear ordinary differential equation with general solutions $C\mathrm{e}^{\lambda_1 t}$ in non-pulse time, where C is a constant. The solution of this initial value problem is*

$$u\bigl(t; T_0, u_0, \{\tau_\psi\}_1^\infty\bigr) = \begin{cases} u_0(\prod_{i=1}^\psi \delta_i)\mathrm{e}^{\lambda_1(t-T_0)}, & \xi_\psi \leqslant t < \xi_{\psi+1}, \\ u_0\mathrm{e}^{\lambda_1(t-T_0)}, & \xi_0 \leqslant t < \xi_1, \end{cases}$$

where $\psi = 1,2,\dots$. The mean value of the solution satisfies

$$\begin{aligned}&\mathbf{E}\bigl(\bigl|u\bigl(t; T_0, u_0, \{t_\psi\}_{\psi=1}^\infty\bigr)\bigr|\bigr) \\ &\quad\leqslant |u_0|\mathrm{e}^{(\lambda_1-\lambda)(t-T_0)}\Biggl(1 + \sum_{\psi=1}^\infty \prod_{i=1}^\psi |\delta_i| \frac{\lambda^\psi (t-T_0)^\psi}{\psi!}\Biggr).\end{aligned}$$

Proof. Take some value t_ψ of τ_ψ, and consider the corresponding pinning impulsive differential problem

$$\dot{u}(t) = \lambda_1 u(t), \quad t \neq T_\psi, \qquad u\bigl(T_\psi^+\bigr) = \delta_\psi u\bigl(T_\psi^-\bigr), \qquad u(T_0) = u_0 = V_0.$$

The solution of this system is obtained as

$$u\bigl(t; T_0, u_0, \{t_\psi\}_1^\infty\bigr) = \begin{cases} u_0 (\prod_{i=1}^{\psi} \delta_i) \mathrm{e}^{\lambda_1 (t-T_0)}, & T_\psi \leqslant t < T_{\psi+1}, \\ u_0 \mathrm{e}^{\lambda_1 (t-T_0)}, & T_0 \leqslant t < T_1, \end{cases}$$

where $\psi = 1, 2, \dots$. Taking all values of τ_ψ, the set of this term of solutions forms a stochastic process $u(t; T_0, u_0, \{\tau_\psi\}_1^\infty)$, which is the solution of system (4).

According to the stochastic process theory, the expectation of this solution is estimated as

$$\begin{aligned}
&\mathbf{E}\bigl(\bigl|u\bigl(t; T_0, u_0, \{\tau_\psi\}_{\psi=1}^\infty\bigr)\bigr|\bigr) \\
&= \sum_{\psi=0}^{\infty} \mathbf{E}\bigl(\bigl|u\bigl(t; T_0, u_0, \{\tau_\psi\}_{\psi=1}^\infty\bigr)\bigr| \bigm| A_\psi(t)\bigr) \mathbf{P}\bigl(A_\psi(t)\bigr) \\
&\leqslant |u_0| \mathrm{e}^{\lambda_1 (t-T_0)} \Biggl(\mathbf{P}\bigl(A_0(t)\bigr) + \sum_{\psi=1}^{\infty} \prod_{i=1}^{\psi} |\delta_i| \mathbf{P}\bigl(A_\psi(t)\bigr) \Biggr) \\
&= |u_0| \mathrm{e}^{\lambda_1 (t-T_0)} \mathrm{e}^{-\lambda (t-T_0)} \Biggl(1 + \sum_{\psi=1}^{\infty} \prod_{i=1}^{\psi} |\delta_i| \frac{\lambda^\psi (t-T_0)^\psi}{\psi!} \Biggr) \\
&= |u_0| \mathrm{e}^{(\lambda_1 - \lambda)(t-T_0)} \Biggl(1 + \sum_{\psi=1}^{\infty} \prod_{i=1}^{\psi} |\delta_i| \frac{\lambda^\psi (t-T_0)^\psi}{\psi!} \Biggr). \qquad \square
\end{aligned}$$

Remark 4. If δ_ψ satisfies $|\delta_\psi| \leqslant c$, then we have

$$\mathbf{E}\bigl(\bigl|u\bigl(t; T_0, u_0, \{\tau_\psi\}_{\psi=1}^\infty\bigr)\bigr|\bigr) \leqslant |u_0| \mathrm{e}^{(\lambda_1 - (1-c)\lambda)(t-T_0)}. \tag{5}$$

Corollary 1. *We consider the generalized model of impulsive linear differential equation*

$$\dot{u}(t) = a(t) u(t), \quad t \neq \xi_\psi, \qquad u\bigl(\xi_\psi^+\bigr) = \delta_\psi u\bigl(\xi_\psi^-\bigr), \qquad u(T_0) = u_0,$$

where $a \in C(\mathbb{R}^+, \mathbb{R})$. The solution is obtained as

$$u\bigl(t; T_0, u_0, \{\tau_\psi\}_1^\infty\bigr) = \begin{cases} u_0 (\prod_{i=1}^{\psi} \delta_i) \mathrm{e}^{\int_{T_0}^{t} a(s)\,\mathrm{d}s}, & \xi_\psi \leqslant t < \xi_{\psi+1}, \\ u_0 \mathrm{e}^{\int_{T_0}^{t} a(s)\,\mathrm{d}s}, & \xi_0 \leqslant t < \xi_1, \end{cases}$$

where $\psi = 1, 2, \dots$, and the estimation of its expectation is

$$\begin{aligned}
&\mathbf{E}\bigl(\bigl|u\bigl(t; T_0, u_0, \{t_\psi\}_{\psi=1}^\infty\bigr)\bigr|\bigr) \\
&\quad \leqslant |u_0| \mathrm{e}^{\int_{T_0}^{t} a(s) - \lambda\,\mathrm{d}s} \Biggl(1 + \sum_{\psi=1}^{\infty} \prod_{i=1}^{\psi} |\delta_i| \frac{\lambda^\psi (t-T_0)^\psi}{\psi!} \Biggr).
\end{aligned}$$

Based on the above discussion, we complete the following proof.

Proof of Theorem 1. First, it follows from Definition 4 that $\mathbf{E}(V(t)) \leqslant \mathbf{E}(u(t))$. From (5) we get the estimation

$$\mathbf{E}\big(V(t)\big) \leqslant \mathbf{E}\big(u(t)\big) \leqslant |V_0|\mathrm{e}^{(\lambda_1-(1-c)\lambda)(t-T_0)}.$$

By condition (ii) there exists a constant $\epsilon > 0$ such that

$$\mathbf{E}\big(\|e(t)\|^2\big) = 2\mathbf{E}\big(V(t)\big) \leqslant 2|V_0|\mathrm{e}^{-\epsilon(t-T_0)},$$

which implies that the synchronization is achieved. □

Remark 5. Condition (ii) in Theorem 1 shows the relationship between system parameters and impulse parameters. If the system has a small Lipschitz coefficient L, which implies good properties, the impulse parameter λ can be reduced accordingly. The average impulse time interval $(1/\lambda)$ could be amplified, which implies a very weak impulse control, and the cost will be reduced. Otherwise, a strong impulse controller is needed for synchronization, and the required cost will be high.

In some networks, the signal transmission between nodes is unidirectional, which means that the outer coupling matrix U is not a symmetric matrix. Therefore, it is necessary to improve the network model.

Corollary 2. *Consider the dynamical network* (3) *in which* $U(t) = (\mu_{kj})$ *is not a symmetric matrix. Let the following condition hold for the system*:

(i) $\delta_\psi = (1 + \vartheta_\psi)^2 \leqslant c$, *where* c *is a positive constant,* $\psi = 1, 2, \dots,$
(ii) $\tilde{\lambda}_1 - (1-c)\lambda < 0$, *where* $\tilde{\lambda}_1 = 2L + \lambda_{\max}(U \otimes B + (U \otimes B)^{\mathrm{T}})$.

Then network (3) *can achieve globally mean-square exponential synchronization.*

The proof of Corollary 2 is similar to that of Theorem 1. The only difference is the derivative of $V(t)$, which is given by

$$\begin{aligned}\dot{V}(t) &= \sum_{k=1}^{N} e_k^{\mathrm{T}}(t)\dot{e}_k(t) = \sum_{k=1}^{N} e_k^{\mathrm{T}}(t)\Bigg[\hat{g}\big(e_k(t)\big) + \sum_{j=1}^{N}\mu_{kj}Be_j(t)\Bigg] \\ &= \sum_{k=1}^{N} e_k^{\mathrm{T}}(t)\hat{g}\big(e_k(t)\big) + \sum_{k=1}^{N} e_k^{\mathrm{T}}(t)\sum_{j=1}^{N}\mu_{kj}Be_j(t) \\ &\leqslant 2LV(t) + \frac{1}{2}e^{\mathrm{T}}(t)\big[(U\otimes B) + (U\otimes B)^{\mathrm{T}}\big]e(t) \leqslant \tilde{\lambda}_1 V(t).\end{aligned}$$

4 Numerical simulation

In this section, we provide a numerical example to verify the effectiveness of the theorem given in the previous section.

Consider the following chaotic nonlinear system as a single node in the network:

$$\dot{x}(t) = -Cx(t) + Af\bigl(x(t)\bigr) = g\bigl(x(t)\bigr), \tag{6}$$

where $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), f_3(x_3(t)))^{\mathrm{T}}$, $f_s(\phi) = (|\phi + 1| - |\phi - 1|)/2$, $s = 1, 2, 3$,

$$A = \begin{bmatrix} 1.16 & -1.5 & -1.5 \\ -1.5 & 1.16 & -2.0 \\ -1.2 & 2.0 & 1.16 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that the system satisfies the Lipschitz condition with Lipschitz constant 2.178. The trajectory of system (6) is shown in Fig. 1, and the time history of system (6) is shown in Fig. 2

This system forms a 5-node network in which the inner coupling matrix and the outer coupling matrix are

$$U = \begin{bmatrix} -3 & 1 & 1 & 1 & 0 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 0 & 1 & 1 & 1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

respectively. By simple calculation we find that $\lambda_1 = 2(L + \lambda_{\max}(U \otimes B)) < 4.5$. Suppose that the system has impulse gain $\vartheta_\psi = -0.5$, and as such $\delta_\psi = 0.25$. Choose the exponential distribution parameter $\lambda = 6$, i.e., $\mathbf{P}(\tau_\psi = k) = (6^k/k!)\mathrm{e}^{-6}$, $k = 1, 2, \ldots$. Thus, condition (ii) in Theorem 1 is satisfied. Initial values $x_k(0)$ are random numbers in the interval $[-1, 1]$. Each experiment is randomly generated with exponentially distributed impulse waiting time series. We ran the experiment for 10000 times. As shown in Figs. 3–5, with the proposed control, we realized the synchronization of the system.

We then calculate the mean value of the 10000 experiments and get the mean error of the system with the proposed control. Simulation results of the mean values of $e_k(t)$ are shown in Figs. 6–10. The line of each color represents the error of the component in the node.

Consider a more sparse matrix

$$U' = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}.$$

Letting $\vartheta_\psi = -1$ and $\lambda = 15$ to meet the conditions of Theorem 1, we then have simulation results shown in Figs. 11–18.

We can see that the results of the numerical simulation after replacing the sparser matrix are still in line with the theory.

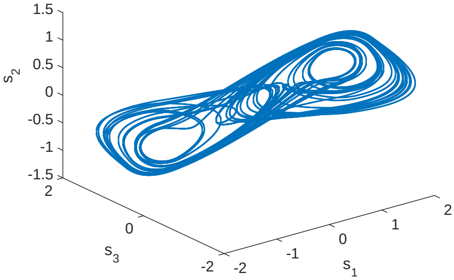

Figure 1. Simulated phase portrait of system (6) in the x, y, z-space.

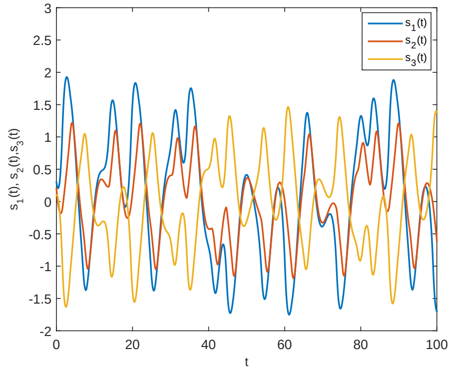

Figure 2. Time history of system (6).

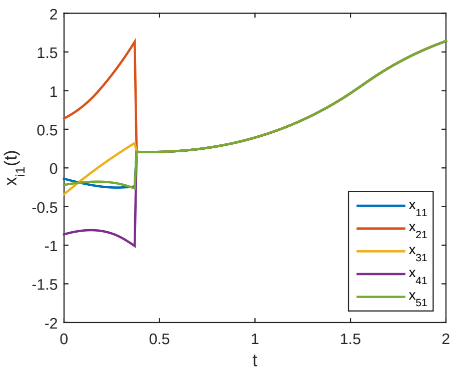

Figure 3. Time histories of $x_{i1}(t), i = 1, 2, 3, 4, 5,$ with inner coupling matrix U.

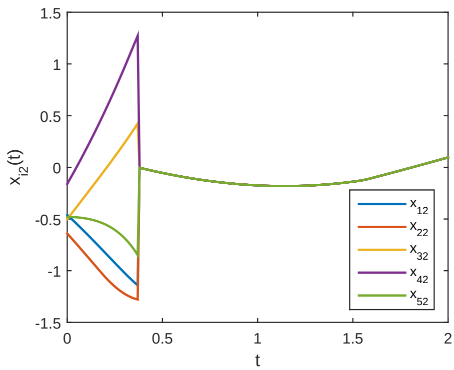

Figure 4. Time histories of $x_{i2}(t), i = 1, 2, 3, 4, 5,$ with inner coupling matrix U.

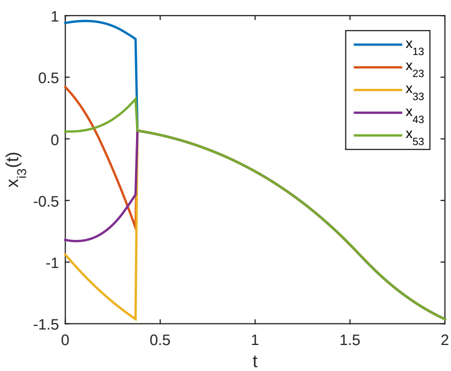

Figure 5. Time histories of $x_{i3}(t), i = 1, 2, 3, 4, 5,$ with inner coupling matrix U.

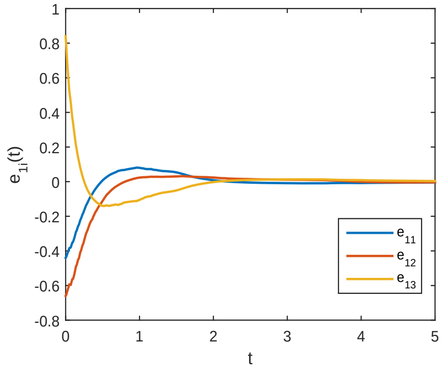

Figure 6. Time history of $e_{1i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U.

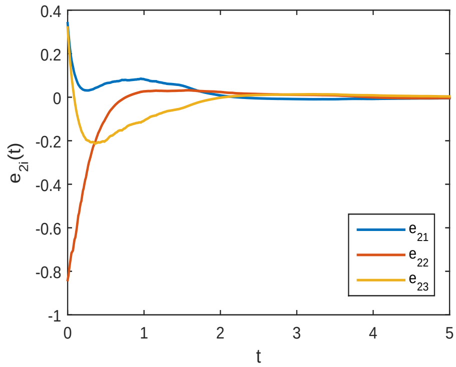

Figure 7. Time history of $e_{2i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U.

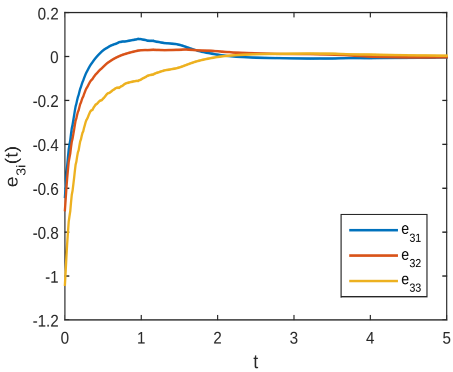

Figure 8. Time history of $e_{3i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U.

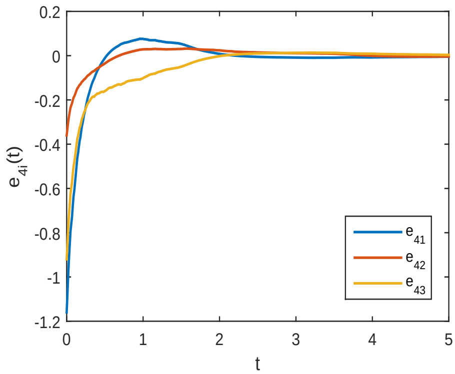

Figure 9. Time history of $e_{4i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U.

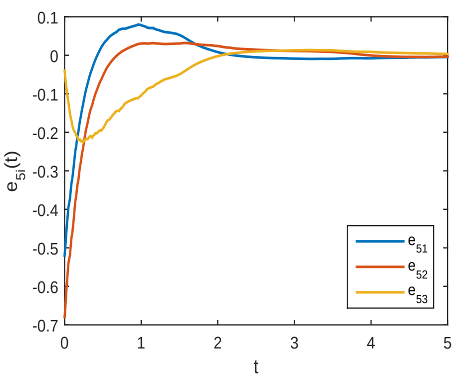

Figure 10. Time history of $e_{5i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U.

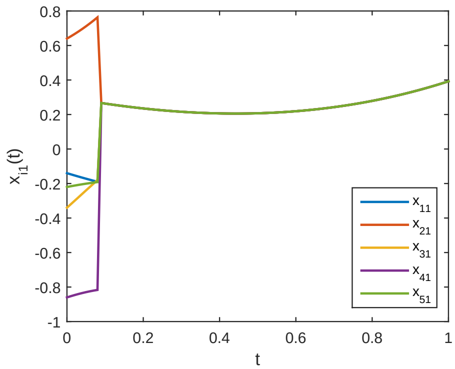

Figure 11. Time histories of $x_{i1}(t)$, $i = 1, 2, 3, 4, 5$, with inner coupling matrix U_2.

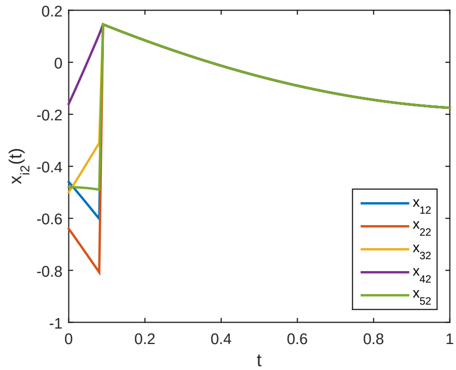

Figure 12. Time histories of $x_{i2}(t)$, $i = 1, 2, 3, 4, 5$, with inner coupling matrix U_2.

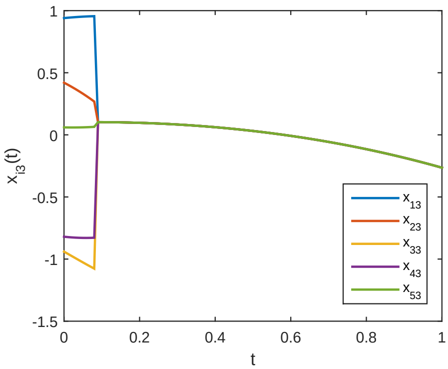

Figure 13. Time histories of $x_{i3}(t)$, $i = 1, 2, 3, 4, 5$, with inner coupling matrix U_2.

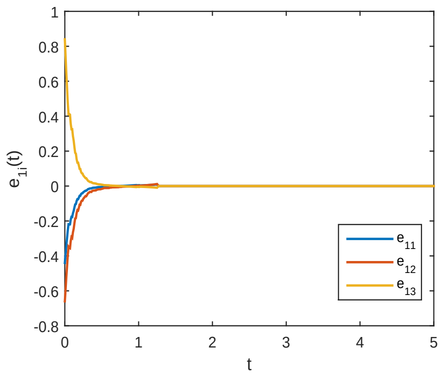

Figure 14. Time history of $e_{1i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U'.

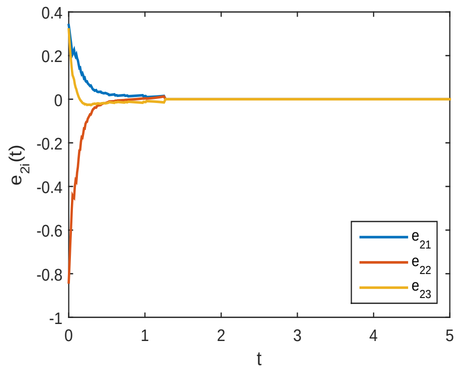

Figure 15. Time history of $e_{2i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U'.

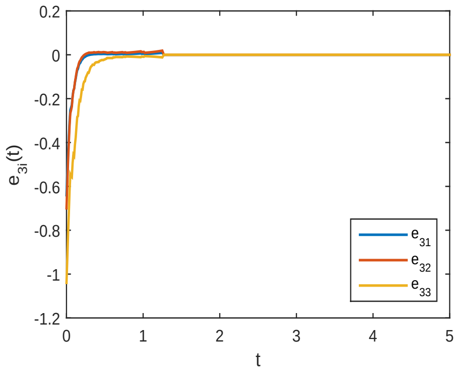

Figure 16. Time history of $e_{3i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U'.

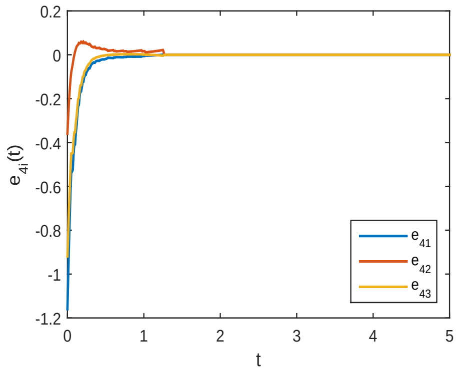

Figure 17. Time history of $e_{4i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U'.

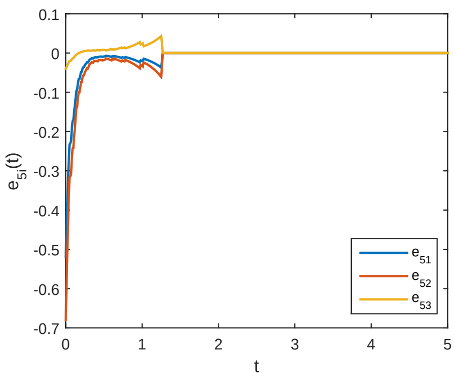

Figure 18. Time history of $e_{5i}(t)$, $i = 1, 2, 3$, with inner coupling matrix U'.

5 Conclusion

In this paper, we investigate the global mean-square exponential synchronization of the random impulsive dynamical network. Using Lyapunov theory, linear matrix inequality, and stochastic process theory, we have designed control mechanisms to achieve the synchronization of the network. We obtain synchronization criteria and consider the relationship between network parameters and impulsive parameters. A numerical example is provided to illustrate the validity of our analytical results.

In our future work, we will combine random impulse with stochastic differential equations to study the synchronization control problem for a variety of systems. For example, we will consider a mean-square exponential synchronization problem for the stochastic dynamical network via random impulse occurring at random times.

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