Nonlinear Analysis: Modelling and Control, Vol. 29, No. 4, 783–801
https://doi.org/10.15388/namc.2024.29.35460

A few generalizations of Kendall's tau. Part I: Construction*

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Received: June 14, 2023 / **Revised:** May 8, 2024 / **Published online:** June 25, 2024

Abstract. Complimenting our earlier work on generalizations of popular concordance measures in the sense of Scarsini for a pair of continuous random variables (X, Y) (such measures can be understood as functions of the bivariate copula C associated with (X, Y)), we focus on generalizations of Kendall's τ. In Part I, we give two forms of such measures and also provide general bounds for their values, which are sharp in certain cases and depend on the values of Spearman's ρ and the original Kendall's τ. Part II is devoted to the intrinsic meaning of presented Kendall's τ generalizations, their degree as polynomial-type concordance measures, and computational aspects.

Keywords: Kendall's tau, Scarsini axioms, bivariate copula, transformation, concordance measure, supermodular.

1 Introduction

Given a family of random variables, which often model important quantities in real life like risky positions in finance, contingent claims in insurance, losses due to natural disasters, etc., measuring and modeling various forms of dependence between pairs, triples, or larger subsets of random variables becomes important. The scientific literature contains many notions of dependence (e.g., positive/negative quadrant or orthant dependence, right/left tail increasingness or decreasingness, association, tail dependence, to name just a few; see, e.g., [18, Chap. 2]), as well as many measures (functionals) to quantify it (e.g., Pearson's correlation coefficient, Spearman's ρ, Kendall's τ, Gini's γ, tail dependence index, Schweizer and Wolff's σ, Hoeffding's dependence index, etc.; see [28, Chap. 5]) and to describe what and on which scale is measured. To put the theory and its applications on a solid ground, several axiom systems have been suggested to formalize the natural and useful properties of various measures of dependence. There are the axioms of Rényi [29] for measures of dependence of a pair of random variables, which in the

*This research was supported by the Research Council of Lithuania grant No. S-MIP-20-16.

copula setting (when the considered variables have continuous distributions) are provided, e.g., in [28, Definition 5.3.1].

For measures of concordance, which is the object of interest in this paper, there are the axioms of Scarsini [30] in the bivariate case and their generalizations to the multivariate setting, which are discussed in the works and references of Taylor et al. [10, 33], Joe [17], Nelsen [27], Dolati and Úbeda-Flores [8], Fuchs and Schmidt [11–13], Mesiar et al. [25, 26], Borroni [3], just to name a few. Slightly different sets of axioms can be found in [18] and [4], so one has to be careful about the set of axioms being used (as well as various statements that follow) and the precise meaning of the term "concordance measure". In this paper, we base our findings on the axioms of Scarsini (see Definition 3 below).

Theoretical studies on concordance measures, both bivariate and multivariate, have focused on their construction and properties (see, e.g., [3, 10, 12, 15, 16, 23, 33]), on the precise bounds for the values of such measures, in particular when some additional information is known, or for other objects given the value of some concordance measure (see, e.g., [1, 7, 20, 21, 36]).

In parallel, many of the above-mentioned theoretical notions and results have found applications in statistics and data science, where they are often used to test statistical hypotheses, e.g., independence vs. dependence, or are helpful in the estimation of quantities of interest. Rank correlation coefficients (Spearman's ρ, Kendall's τ, etc.) and their generalizations are often preferred in this context. Here we cite just a few recent papers in this direction; see [2, 6, 14] and the references therein.

Our contribution to the literature on concordance measures is several-fold: in this first part of a series of papers, we present new generalizations of Kendall's τ (see Theorems 2 and 3), employing a convex and properly normalized distortion function φ as well as a symmetrization procedure, which compliments our earlier paper [23] about similar generalizations of other popular concordance measures. We also provide bounds for the suggested concordance measures (see Proposition 1), which are sharp if a linear (trivial case) or a quadratic distortion function φ is used, and illustrate our findings using several examples. The second part of the series is devoted to a comparison of the classical Kendall's τ with the new measures, emphasizing the shift from a probability measure to weigh a random partition of the unit square to a convex (supermodular) capacity, opening up possibilities for applications in, e.g., the economic decision theory where such capacities have been successfully employed. Furthermore, there we also elaborate on the degree of such generalizations as polynomial-type concordance measures, providing additional examples to advance the line of research by Taylor et al. mentioned above.

The rest of this paper is structured as follows. In Section 2, we provide basic concepts and needed facts about copulas and concordance measures (in the sense of Scarsini) needed to state and prove our main results. Section 3 presents a few auxiliary results, which are useful when integrating with respect to the copula-induced measure and when proving monotonicity of functionals with respect to concordance order. Section 4 gives our main results (Theorems 2 and 3) with generalizations of Kendall's τ and their different forms. We also provide several examples and general bounds for the new measures, which depend on Spearman's ρ and classical Kendall's τ. Section 5 concludes.

2 Basic facts from copula theory

We begin by recalling the notion of a bivariate copula. Let $\mathbb{I} := [0,1]$.

Definition 1. A bivariate copula[1] (a copula for short) C is a function defined on $\mathbb{I}^2$ with values in $\mathbb{I}$ such that

- $C(x,0) = C(0,x) = 0$ and $C(x,1) = C(1,x) = x$ for any $x \in \mathbb{I}$,
- (2-increasingness) for all $x, x', y, y' \in \mathbb{I}$ with $x \leqslant x'$ and $y \leqslant y'$,

$$V_C\big([x,x'] \times [y,y']\big) = C(x',y') - C(x,y') - C(x',y) + C(x,y) \geqslant 0.$$

The first two conditions for C are also called the boundary conditions.

The set of bivariate copulas will be denoted by $\mathcal{C}$ (or, more precisely, $\mathcal{C}_2$ if we need explicit dependence on the dimension).

Many examples of copulas are known in the literature (see [9, 18, 28] and the references therein), among them, the most important are the comonotonicity copula $M(x,y) = \min\{x,y\} = x \wedge y$, independence copula $\Pi(x,y) = xy$, and countermonotonicity copula $W(x,y) = \max\{x+y-1, 0\} = (x+y-1)^+$ for $(x,y) \in \mathbb{I}^2$.

To each (bivariate) copula $C \in \mathcal{C}$, one can associate a Borel measure μ_C, which is doubly stochastic, i.e., $\mu_C(A \times \mathbb{I}) = \mu(\mathbb{I} \times A) = \lambda(A)$ for any Borel set $A \subset \mathcal{B}(\mathbb{I}^2)$, where λ denotes the Lebesgue measure, and such that $\mu_C((0,x] \times (0,y]) = C(x,y)$ for any $x, y \in \mathbb{I}$, and vice versa (see, e.g., [9, Thm. 3.1.2], where the result is stated for a general dimension $d \geqslant 2$). In what follows, integrals with respect to a copula $C \in \mathcal{C}$, e.g., $\int_{\mathbb{I}^2} f \,\mathrm{d}C$, will mean $\int_{\mathbb{I}^2} f \,\mathrm{d}\mu_C$.

On the set of bivariate copulas, one can consider a pointwise partial-order relation defined as follows:

Definition 2. (See [28, Def. 2.8.1].) For any $C_1, C_2 \in \mathcal{C}$, we say that C_1 is *smaller* (resp. *larger*) than C_2 and denote it by $C_1 \prec C_2$ (resp. $C_1 \succ C_2$) if $C_1(x,y) \leqslant C_2(x,y)$ (resp. $C_1(x,y) \geqslant C_2(x,y)$) for any $(x,y) \in \mathbb{I}^2$.

Concordance order, in the general d-dimensional setting (when $d \geqslant 2$), is defined as

$$\begin{aligned} C_1 \prec C_2 \quad \iff \quad & C_1(x_1,\dots,x_d) \leqslant C_2(x_1,\dots,x_d) \quad \text{and} \\ & \overline{C}_1(x_1,\dots,x_d) \leqslant \overline{C}_2(x_1,\dots,x_d) \quad \forall (x_1,\dots,x_d) \in \mathbb{I}^d, \end{aligned}$$

where $\overline{C}(x_1,\dots,x_d) = \mathbf{P}(U_1 > x_1, \dots, U_d > x_d)$, $U_1,\dots,U_d \sim U(\mathbb{I})$ are uniformly on $\mathbb{I}$ distributed random variables whose copula is $C \in \mathcal{C}_d$. In other words, $\overline{C}$ is the survival function associated with copula C. For $d = 2$, $\overline{C}(x,y) = 1 - x - y + C(x,y)$, so concordance order for bivariate copulas is equivalent to pointwise order.

Then the famous Fréchet–Hoeffding bounds can be written succinctly as $W \prec C \prec M$ for any $C \in \mathcal{C}$. For any reasonable concordance measure $\kappa_{X,Y}$ in the sense of Scarsini

[1] One can also consider n-variate copulas for any $n \geqslant 2$ (see, e.g., [9, 18, 28]), but we will only be concerned with bivariate copulas in this paper.

(Kendall's τ, Spearman's ρ, Gini's γ, etc. are examples; see [28, Def. 5.1.7]), measuring the dependence between continuous random variables X and Y whose copula is C, an increase of C in concordance order means an increase in $\kappa_{X,Y}$, which justifies the name of the order.

2.1 Transformations of copulas generated by symmetries of their domain

In relation to the axioms of concordance measures, of particular importance are the transformations of bivariate (or, more generally, multivariate) copulas that are induced by the symmetries of their domain $\mathbb{I}^2$ (or $\mathbb{I}^d$ for $d > 2$ in higher dimensions). The group of symmetries of the unit square $\mathbb{I}^2$ can be generated by involutions $\pi : \mathbb{I}^2 \to \mathbb{I}^2$ (permutation, i.e., reflection with respect to the main diagonal) and $\sigma_1 : \mathbb{I}^2 \to \mathbb{I}^2$ (partial reflection with respect to the axis $x = 1/2$) given by

$$\pi(x, y) = (y, x) \quad \text{and} \quad \sigma_1(x, y) = (1 - x, y).$$

Involution means that $\pi^2 = \sigma_1^2 = \mathrm{e}$, the identity transformation. Also, one can get the partial reflection $\sigma_2(x, y) = (x, 1 - y)$ with respect to the axis $y = 1/2$ as $\sigma_2(x, y) = (\pi \circ \sigma_1 \circ \pi)(x, y)$. Combining the two reflections, we get the so-called total reflection

$$\varsigma(x, y) = (\sigma_1 \circ \sigma_2)(x, y) = (\sigma_2 \circ \sigma_1)(x, y) = (1 - x, 1 - y).$$

Altogether the group of symmetries of the unit square, also called the dihedral group D_4, has $8 = 2!\,2^2$ elements:

$$D_4 = \{\mathrm{e},\, \pi,\, \sigma_1,\, \sigma_2,\, \varsigma,\, \pi \circ \sigma_1,\, \pi \circ \sigma_2,\, \pi \circ \varsigma\}.$$

It is important to note that D_4 is not commutative since

$$\pi \circ \sigma_1 = \sigma_2 \circ \pi. \tag{1}$$

Given a symmetry $\xi \in D_4$, there is a corresponding transformation $\xi^* : \mathcal{C} \to \mathcal{C}$ given by

$$\xi^*(C)(x, y) := \mu_C\bigl(\xi\bigl([0, x] \times [0, y]\bigr)\bigr), \quad (x, y) \in \mathbb{I}^2. \tag{2}$$

For the partial reflections σ_1, σ_2 and total reflection ς, one easily gets

$$\begin{aligned} \sigma_1^*(C)(u, v) &= \mu_C\bigl([1 - u, 1] \times [0, v]\bigr) \\ &= \mu_C\bigl([0, 1] \times [0, v]\bigr) - \mu_C\bigl([0, 1 - u] \times [0, v]\bigr) \\ &= C(1, v) - C(1 - u, v) = v - C(1 - u, v), \\ \sigma_2^*(C)(u, v) &= C(u, 1) - C(u, 1 - v) = u - C(u, 1 - v), \\ \varsigma^*(C)(u, v) &= u + v - 1 + C(1 - u, 1 - v), \quad u, v \in \mathbb{I}, \end{aligned}$$

while the transpose of C is given by $C^{\mathrm{T}}(u, v) := \pi^*(C)(u, v) = C(v, u)$. Note that $\varsigma^*(C)(u, v) = \widehat{C}(u, v)$, the survival copula corresponding to C.

2.2 Scarsini's axioms of concordance measures

In this paper, we will be concerned with the family of functionals on the set of copulas $\mathcal{C}$, which measure the "degree of association" of continuous random variables having a given copula and preserve concordance order. This family was axiomatized by Scarsini in 1984 (see [30, 31]); for extensions to the multidimensional case, see [8, 32]).

Definition 3. (See [9, Def. 2.4.7].) A measure of concordance is a mapping $\kappa : \mathcal{C} \to \mathbb{R}$ such that

- (κ_1) κ is defined for every copula $C \in \mathcal{C}$,
- (κ_2) for every $C \in \mathcal{C}$, $\kappa(C) = \kappa(C^{\mathrm{T}})$,
- (κ_3) $\kappa(C_1) \leqslant \kappa(C_2)$ whenever $C_1 \prec C_2$,
- (κ_4) $\kappa(C) \in [-1, 1]$,
- (κ_5) $\kappa(\varPi) = 0$,
- (κ_6) $\kappa(\sigma_1^*(C)) = \kappa(\sigma_2^*(C)) = -\kappa(C)$ for the partial reflections σ_1 and σ_2 and any $C \in \mathcal{C}$,
- (κ_7) (continuity) if $C_n \to C$ uniformly[2] as $n \to \infty$, then $\lim_{n\to\infty} \kappa(C_n) = \kappa(C)$.

One can observe that some authors, e.g., Nelsen [28, Def. 5.1.17, property 2] and Fuchs [12, Sect. 3], also require

- (κ_5') $\kappa(M) = 1$,

which can be achieved by a simple normalization if the original concordance measure does not satisfy this condition.

The most commonly used concordance measures are Spearman's ρ, Kendall's τ, Gini's γ, Blomqvist's β; see [28, Chap. 5], [9, Sect. 2.4]. On the other hand, Spearman's foot-rule is not a concordance measure; see [28, Exercise 5.21].

These measures are succinctly defined in terms of the so-called *biconvex form*[3] given by

$$[C, D] := \int_{\mathbb{I}^2} C \,\mathrm{d}D, \quad C, D \in \mathcal{C}, \tag{3}$$

which is linear in each argument with respect to convex combinations of copulas, hence the terminology. In fact (see [28]), for a copula $C \in \mathcal{C}$,

- Spearman's ρ is given by $\rho_S(C) = 12[C, \varPi] - 3 = 12[C - \varPi, \varPi]$,
- Kendall's τ is defined as $\tau(C) = 4[C, C] - 1 = 4([C, C] - [\varPi, \varPi])$,
- Gini's γ is $\gamma(C) = 4([C, M] + [C, W]) - 2$.

In our earlier work [23], we have constructed several generalizations of Spearman's ρ, Gini's γ, etc., but Kendall's τ was not essentially considered in that paper due to a technical issue at that time. So below we mostly focus on this measure of concordance and its possible generalizations.

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[2] For copulas, pointwise convergence is enough.

[3] A more precise name, in our opinion, would be *convex-combinations-restricted bilinear form*, which is much longer, albeit clearer.

3 Auxiliary results

We begin with a straightforward extension of a theorem due to Li et al. [22], also presented in [9, Thm. 4.1.13]. The result is very useful when dealing with various integrals involving copulas.

Theorem 1. *Let C be a bivariate copula, and $f : \mathbb{I}^2 \to \mathbb{R}$ be an absolutely continuous function with respect to each argument and with essentially bounded partial derivatives. Then*

$$\int\limits_{\mathbb{I}^2} f \,\mathrm{d}C = \int\limits_0^1 f(x,1)\,\mathrm{d}x - \int\limits_{\mathbb{I}^2} \partial_2 f\, \partial_1 C \,\mathrm{d}\Pi = \int\limits_0^1 f(1,y)\,\mathrm{d}y - \int\limits_{\mathbb{I}^2} \partial_1 f\, \partial_2 C \,\mathrm{d}\Pi, \tag{4}$$

where $\partial_i g$ denotes the partial derivative of g with respect to the ith variable.

Proof. We follow the argument of the proof of [9, Thm. 4.1.13] with f in place of a transformed copula, $\phi(A)$: first, we prove the result for absolutely continuous copulas C, and second, we take advantage of the approximation of arbitrary copulas by sequences of absolutely continuous ones.

So to fix the ingredients, for any integer $n \geqslant 1$, consider Bernstein copulas $B^{(n)}$ that ∂-converge[4] to C; this is possible by [9, Thm. 4.5.8]. In other words, for any $v \in \mathbb{I}$,

$$\lim_{n\to\infty} \int\limits_{\mathbb{I}} \left|\partial_1 B^{(n)}(x,v) - \partial_1 C(x,v)\right| \mathrm{d}x = 0.$$

Then the bounded convergence theorem yields

$$\lim_{n\to\infty} \int\limits_{\mathbb{I}^2} \left|\partial_1 B^{(n)}(u,v) - \partial_1 C(u,v)\right| \mathrm{d}u\,\mathrm{d}v = 0.$$

As Bernstein copulas are absolutely continuous with continuous second-order partial derivatives, we can prove the first equality of (4) for them like is done in the proof of [9, Eq. (4.1.11)], that is, we have

$$\int\limits_{\mathbb{I}^2} f \,\mathrm{d}B^{(n)} = \int\limits_0^1 f(x,1)\,\mathrm{d}x - \int\limits_{\mathbb{I}^2} \partial_2 f \partial_1 B^{(n)} \,\mathrm{d}\Pi. \tag{5}$$

Then we only need to justify the passage to the limit as $n \to \infty$.

Since the partial derivative $\partial_2 f$ is assumed essentially bounded, we get

$$\left|\int\limits_{\mathbb{I}^2} \partial_2 f\big(\partial_1 B^{(n)} - \partial_1 C\big)\,\mathrm{d}\Pi\right| \leqslant \|\partial_2 f\|_\infty \int\limits_{\mathbb{I}^2} \left|\partial_1 B^{(n)} - \partial_1 C\right| \mathrm{d}\Pi \to 0 \tag{6}$$

[4] Note that ∂-convergence was not mentioned in the proof of [9, Thm. 4.1.13], but was implicitly used when passing to the limit.

as $n \to \infty$. Thus, taking the limit as $n \to \infty$ in (5) and using (6), we obtain the first equality in (4).

To get the second claimed equality in (4), one simply has to change the order of integration and repeat the same steps. $\square$

Example 1. To illustrate Theorem 1, consider several important choices for C, namely, for any f satisfying the conditions of the theorem:

(i) If $C = M$, then $\partial_1 M(u,v) = \mathbf{1}_{\{u<v\}}$, $\partial_2 M(u,v) = \mathbf{1}_{\{v<u\}}$, and so

$$\begin{aligned}\int\limits_{\mathbb{I}^2} f \,\mathrm{d}M &= \int\limits_0^1 f(x,1)\,\mathrm{d}x - \int\limits_0^1 \int\limits_u^1 \partial_2 f(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int\limits_0^1 f(x,1)\,\mathrm{d}x - \int\limits_0^1 \big(f(u,1) - f(u,u)\big)\,\mathrm{d}u \\ &= \int\limits_0^1 f(u,u)\,\mathrm{d}u.\end{aligned}$$

(ii) If $C = W$, then $\partial_1 W(u,v) = \partial_2 W(u,v) = \mathbf{1}_{\{u+v>1\}}$, and so

$$\begin{aligned}\int\limits_{\mathbb{I}^2} f \,\mathrm{d}W &= \int\limits_0^1 f(x,1)\,\mathrm{d}x - \int\limits_0^1 \int\limits_{1-u}^1 \partial_2 f(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int\limits_0^1 f(x,1)\,\mathrm{d}x - \int\limits_0^1 \big(f(u,1) - f(u,1-u)\big)\,\mathrm{d}u \\ &= \int\limits_0^1 f(u,1-u)\,\mathrm{d}u.\end{aligned}$$

Lemma 1. *Let $\varphi : [0,1] \to \mathbb{R}$ be a nondecreasing convex function. Then the mapping*

$$\mathcal{C}^2 \ni (C,D) \mapsto \int\limits_{\mathbb{I}^2} \varphi(C)\,\mathrm{d}D$$

is increasing in each place with respect to concordance order.

Proof. If $C_1 \prec C_2$, then $\varphi(C_1) \leqslant \varphi(C_2)$ on $\mathbb{I}^2$ as φ is assumed nondecreasing, and so

$$\int\limits_{\mathbb{I}^2} \phi(C_1)\,\mathrm{d}D \leqslant \int\limits_{\mathbb{I}^2} \phi(C_2)\,\mathrm{d}D.$$

On the other hand, if $D_1 \prec D_2$, then the result follows by combining the results of Topkis [35], Day [5] (see, e.g., [24, p. 219, 6.D.2.]), and Tchen [34, Thm. 2]. Indeed, since φ is nondecreasing and convex and any copula C is monotone and supermodular (monotone and L-superadditive in the terminology of [24]), by [24, 6.D.2.], $\varphi(C)$ is monotone supermodular. Now by [34, Thm. 2],

$$\int\limits_{\mathbb{I}^2} \phi(C)\,\mathrm{d}D_2 - \int\limits_{\mathbb{I}^2} \phi(C)\,\mathrm{d}D_1 \geqslant \int\limits_{\mathbb{I}^2} (D_2 - D_1)\,\mathrm{d}K \geqslant 0,$$

where K is the positive measure induced by the supermodular function $\varphi(C)$. $\square$

Remark 1. A similar statement is provided in Theorem 3 [4], where the nondecreasingness only with respect to D is considered.

4 Generalization of Kendall's τ

For the intended generalization of Kendall's τ and notational convenience, given a nonconstant, nondecreasing, and convex function $\varphi : [0,1] \to \mathbb{R}$, let

$$[C,D]_\varphi := \int\limits_{\mathbb{I}^2} \varphi(C)\,\mathrm{d}D = \bigl[\varphi(C), D\bigr], \quad C, D \in \mathcal{C}.$$

When $\varphi(x) = x$, $x \in \mathbb{I}$, $[C,D]_\varphi = [C,D]$, the usual biconvex form in (3) (for more details about its properties, see [11]), used to define various concordance measures for bivariate copulas. In fact, for Kendall's τ, we have

$$\tau(C) = 4[C,C] - 1 = \sum_{\xi \in R} (-1)^{|\xi|} \bigl[\xi^*(C), \xi^*(C)\bigr], \quad C \in \mathcal{C}, \tag{7}$$

where R denotes the commutative subgroup of D_4 generated by partial reflections of $\mathbb{I}^2$, that is, $R = \{\mathrm{e}, \sigma_1, \sigma_2, \varsigma = \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 \mid \sigma_1^2 = \sigma_2^2 = \mathrm{e}\}$, and $\xi^*(C)$ is defined in (2). It is also important to understand the meaning of these transformations at the level of random variables. Indeed, if U and V are random variables distributed uniformly on the interval $\mathbb{I}$ and joined by copula $C \in \mathcal{C}$, i.e., $(U,V) \sim C$, then

$$(1-U,\, V) \sim \sigma_1^*(C), \quad (U,\, 1-V) \sim \sigma_2^*(C), \quad \text{and} \quad (1-U,\, 1-V) \sim \varsigma^*(C).$$

We now generalize Kendall's τ, replacing $[C,D]$ in (7) by $[C,D]_\varphi$ and normalizing appropriately:

$$\tau_\varphi(C) := a_\varphi \sum_{\xi \in R} (-1)^{|\xi|} \bigl[\xi^*(C), \xi^*(C)\bigr]_\varphi, \quad C \in \mathcal{C}, \tag{8}$$

where

$$a_\varphi := \left(2 \int\limits_0^1 \bigl(\varphi(x) - \varphi(0)\bigr)\,\mathrm{d}x\right)^{-1},$$

which is positive as φ is assumed nonconstant, nondecreasing, and convex. In the case, $\varphi(x) = x$ on $\mathbb{I}$, $a_\varphi = 1$, and thus we recover in (8) the usual Kendall's τ in (7).

Remark 2.

(i) Without loss of generality, we can assume $\varphi(0) = 0$. If this were not the case, defining $\varphi_0(x) = \varphi(x) - \varphi(0)$, we would still retain a nonconstant, nondecreasing, and convex function. Moreover, then we would also have $a_{\varphi_0} = a_\varphi$ and

$$[C, D]_{\varphi_0} = \big[\varphi_0(C), D\big] = \big[\varphi(C), D\big] - \varphi(0),$$

which yields $\tau_{\varphi_0}(C) = \tau_\varphi(C)$ for all $C \in \mathcal{C}$.

(ii) Furthermore, one can also normalize φ so that $\varphi(1) = 1$. Indeed, for a nonconstant, nondecreasing function φ with $\varphi(0) = 0$, we have $\varphi(1) > 0$, and so the function $\varphi_1(x) := \varphi(x)/\varphi(1)$ remains nonconstant, nondecreasing, and convex, giving $\tau_{\varphi_1}(C) = \tau_\varphi(C)$ for any $C \in \mathcal{C}$.

(iii) Finally, we can assume that φ is left-continuous at 1. Indeed, if $\varphi_2(x) := \varphi(x-)$, $x \in \mathbb{I}$, then, due to assumed convexity, φ is continuous on $(0, 1)$. Nondecreasingness and convexity imply that $\varphi(0) = \varphi(0+)$, so $\varphi_2(x)$ and $\varphi(x)$ can only differ at $x = 1$. But then due to continuity of any copula $D \in \mathcal{C}$, $[C, D]_{\varphi_2} = [C, D]_\varphi$ since

$$\int\limits_{\mathbb{I}^2} \big(\varphi(C) - \varphi_2(C)\big)\,\mathrm{d}D = \big(\varphi(1) - \varphi_2(1)\big)\mu_D\big(\big\{(1, 1)\big\}\big) = 0.$$

Theorem 2. *If $\varphi : [0, 1] \to \mathbb{R}$ is a nonconstant, nondecreasing, and convex function such that $\varphi(1) = \varphi(1-) = 1$, $\varphi(0) = 0$, then $\tau_\varphi : \mathcal{C} \to [-1, 1]$ is a measure of concordance, generalizing Kendall's τ.*

Proof. We need to check the axioms of Scarsini:

- Axiom (κ_1) clearly holds as $\tau_\varphi(C)$ is well defined for all $C \in \mathcal{C}$.
- Axiom (κ_2) holds since, on the one hand, $[\cdot, \cdot]_\varphi$ is symmetric with respect to the interchange of integration variables, that is, $[\varphi(C^{\mathrm{T}}), C^{\mathrm{T}}] = [\varphi(C), C]$ for any $C \in \mathcal{C}$. On the other hand, using (1), we have

$$\begin{aligned} \sigma^*_{2-i}\big(C^{\mathrm{T}}\big) &= \big(\sigma^*_i(C)\big)^{\mathrm{T}}, \quad i = 1, 2; \\ \varsigma^*\big(C^{\mathrm{T}}\big) &= \sigma^*_1\big(\sigma^*_2\big(C^{\mathrm{T}}\big)\big) = \sigma^*_1\big(\big(\sigma^*_1(C)\big)^{\mathrm{T}}\big) \\ &= \big(\sigma^*_2(\sigma^*_1(C)\big)^{\mathrm{T}} = \big(\varsigma^*(C)\big)^{\mathrm{T}}, \end{aligned}$$

  so that the positive terms in (8) remain unchanged, while the negative terms get interchanged when C is replaced by C^{T}, altogether keeping the value of $\tau_\varphi(C)$ unchanged.
- Axiom (κ_3) follows by Lemma 1.
- Axiom (κ_4) follows from Axioms (κ_3) and (κ_6) (still to be proved) and the fact that, due to normalization, $\tau_\varphi(M) = 1$.

- Axiom (κ_5) follows from the fact that $\xi^*(\Pi) = \Pi$ for any $\xi \in R$, and so, clearly, $\tau_\varphi(\Pi) = 0$.
- Axiom (κ_6) follows by observing that, e.g.,

$$\begin{aligned}
(-1)^{|\mathrm{e}|}\bigl[\sigma_1^*(C), \sigma_1^*(C)\bigr]_\varphi &= -(-1)^{|\sigma_1|}\bigl[\sigma_1^*(C), \sigma_1^*(C)\bigr]_\varphi;\\
(-1)^{|\sigma_1|}\bigl[\sigma_1^*\bigl(\sigma_1^*(C)\bigr), \sigma_1^*\bigl(\sigma_1^*(C)\bigr)\bigr]_\varphi &= -(-1)^{|\mathrm{e}|}[C, C]_\varphi;\\
(-1)^{|\sigma_2|}\bigl[\sigma_2^*\bigl(\sigma_1^*(C)\bigr), \sigma_2^*\bigl(\sigma_1^*(C)\bigr)\bigr]_\varphi &= -(-1)^{|\varsigma|}\bigl[\varsigma^*(C), \varsigma^* C\bigr]_\varphi;\\
(-1)^{|\varsigma|}\bigl[\varsigma^*\bigl(\sigma_1^*(C)\bigr), \varsigma^*\bigl(\sigma_1^*(C)\bigr)\bigr]_\varphi &= -(-1)^{|\sigma_2|}\bigl[\sigma_2^*(C), \sigma_2^*(C)\bigr]_\varphi,
\end{aligned}$$

  which produces $\tau_\varphi(\sigma_1^*(C)) = -\tau_\varphi(C)$. Similarly for $\xi \in \{\sigma_2, \varsigma\}$.
- Axiom (κ_7) follows by an application of [9, Lemma 2.4.8]. Indeed, let $C_n \to C$ uniformly as $n \to \infty$. As φ is assumed nondecreasing, convex, left-continuous at 1, and $\varphi(0) = 0$, it is nonnegative and continuous on $[0, 1]$. Thus $f_n := \varphi(C_n)$ is a nonnegative, uniformly continuous, and bounded function on $\mathbb{I}^2$. Taking $\nu_n := \mu_{C_n}$ as probability measures induced by copulas C_n, we see that condition (b) of Lemma 2.4.8 [9] is trivially satisfied for $\{f_n\}$, so that

$$\lim_{n\to\infty}[C_n, C_n]_\varphi = \lim_{n\to\infty}\int f_n \,\mathrm{d}\nu_n = \int f \,\mathrm{d}\nu = [C, C]_\varphi,$$

  where $f := \varphi(C)$ and $\nu = \mu_C$. We have also used the fact that uniform convergence of copulas C_n to C induces the weak (hence also vague) convergence of the corresponding doubly stochastic probability measures ν_n to ν (see, e.g., [9, Thm. 4.2.1]). □

Remark 3. Theorem 2 looks similar but differs essentially from [4, Cor. 1] in that, on the one hand, a different set of axioms is used to define a concordance measure, in particular, Cardin and Ferretti do not require the antisymmetry with respect to partial reflections σ_1 and σ_2 (Axiom (κ_6)). On the other hand, the symmetry with respect to π (Axiom (κ_2)) there follows from the assumptions on the integrand (function f, which in our notation is $f = \varphi \circ C$), while we get it from the form of τ_φ in (8). Also, the measure of Cardin and Ferretti extends beyond Kendall's τ because f need not depend on copula C.

The following result provides an alternative expression for τ_φ given in (8) whenever φ is a little smoother.

Theorem 3. *Let $\varphi : [0, 1] \to \mathbb{R}$ be a nondecreasing function such that $\varphi(1) = \varphi(1-) = 1$, $\varphi(0) = 0$. Assume, furthermore, that φ is nonconstant, differentiable, and convex on $(0, 1)$. Then the measure of concordance τ_φ given in* (8) *can be equivalently expressed as follows*:

$$\tau_\varphi(C) = a_\varphi\biggl\{\int\limits_{\mathbb{I}^2} \varphi'(\overline{C}) \,\mathrm{d}\Pi - \int\limits_{\mathbb{I}^2} G_\varphi(C)\,\partial_1 C\,\partial_2 C \,\mathrm{d}\Pi\biggr\}, \tag{9}$$

where for any $C \in \mathcal{C}$ *and* $u, v \in (0,1)$,

$$\begin{aligned} G_\varphi(C)(u,v) &:= \varphi'\big(C(u,v)\big) + \varphi'\big(\tilde\sigma_1^*(C)(u,v)\big) + \varphi'\big(\tilde\sigma_2^*(C)(u,v)\big) + \varphi'\big(\overline{C}(u,v)\big);\\ \tilde\sigma_1^*(C)(u,v) &= \sigma_1^*(C)(1-u,v) = v - C(u,v);\\ \tilde\sigma_2^*(C)(u,v) &= \sigma_2^*(C)(u,1-v) = u - C(u,v);\\ \overline{C}(u,v) &= 1-u-v+C(u,v) \quad (\textit{survival function corresponding to } C). \end{aligned}$$

Remark 4. Note that as φ is assumed convex, it is continuous on $(0,1)$ and hence absolutely continuous, so $\varphi'(x)$ exists for Lebesgue almost all $x \in (0,1)$ and is integrable on $(0,1)$, yet, due to level sets of copulas, which could have positive Lebesgue measure, $G_\varphi(C)(u,v)$ could become undefined on sets of positive Lebesgue measure on $\mathbb{I}^2$ without the requirement that φ is differentiable on $(0,1)$. Whether this poses a real problem, especially for the first integral in (9), is to be investigated.

Proof. First, observe that for any reflection $\xi \in R$ and any $C \in \mathcal{C}$, using Theorem 1 and the fact that $\xi^*(C) \in \mathcal{C}$, one has

$$\begin{aligned} \big[\xi^*(C), \xi^*(C)\big]_\varphi &= \int_{\mathbb{I}^2} \varphi\big(\xi^*(C)\big)\, \mathrm{d}\xi^*(C)\\ &= \int_0^1 \varphi\big(\xi^*(C)(1,y)\big)\, \mathrm{d}y - \int_{\mathbb{I}^2} \partial_1\varphi\big(\xi^*(C)\big)\, \partial_2\xi^*(C)\, \mathrm{d}\Pi\\ &= \int_0^1 \varphi(y)\, \mathrm{d}y - \int_{\mathbb{I}^2} \varphi'\big(\xi^*(C)\big)\, \partial_1\xi^*(C)\, \partial_2\xi^*(C)\, \mathrm{d}\Pi\\ &=: \int_0^1 \varphi(y)\, \mathrm{d}y - K_\varphi(\xi, C). \end{aligned}$$

Since

$$\sum_{\xi\in R} (-1)^{|\xi|} \int_0^1 \varphi(x)\, \mathrm{d}x = 0,$$

for the measure τ_φ in (8), one has

$$\tau_\varphi(C) = -a_\varphi \sum_{\xi \in R} (-1)^{|\xi|} K_\varphi(\xi, C). \tag{10}$$

It remains to investigate each term in (10).

• For $\xi = \mathrm{e}$, we obviously have

$$(-1)^{|\mathrm{e}|} K_\varphi(\mathrm{e}, C) = \int_{\mathbb{I}^2} \varphi'(C)\, \partial_1 C\, \partial_2 C\, \mathrm{d}\Pi. \tag{11}$$

• For $\xi = \sigma_1$, we get (using $\varphi(0) = 0$ and substitution $u = 1 - s$)

$$
\begin{aligned}
&(-1)^{|\sigma_1|} K_\varphi(\sigma_1, C) \\
&\quad = -\int_0^1 \int_0^1 \varphi'\bigl(v - C(1-u, v)\bigr)(\partial_1 C)(1-u, v)\bigl(1 - (\partial_2 C)(1-u, v)\bigr)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad = -\int_0^1 \left(\int_0^1 \varphi'\bigl(v - C(s, v)\bigr)\partial_1 C(s, v)\bigl(1 - \partial_2 C(s, v)\bigr)\,\mathrm{d}s\right)\mathrm{d}v \\
&\quad = \int_0^1 \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\bigl(\varphi\bigl(v - C(s, v)\bigr)\bigr)\,\mathrm{d}s + \int_0^1 \varphi'\bigl(v - C(s, v)\bigr)\partial_1 C(s, v)\partial_2 C(s, v)\,\mathrm{d}s\right)\mathrm{d}v \\
&\quad = -\int_0^1 \varphi(v)\,\mathrm{d}v + \int_{\mathbb{I}^2} \varphi'\bigl(\tilde\sigma_1^*(C)\bigr)\,\partial_1 C\,\partial_2 C\,\mathrm{d}\Pi.
\end{aligned} \tag{12}
$$

• For $\xi = \sigma_2$, we similarly obtain (using $\varphi(0) = 0$ and substitution $v = 1 - s$)

$$
\begin{aligned}
&(-1)^{|\sigma_2|} K_\varphi(\sigma_2, C) \\
&\quad = -\int_0^1 \int_0^1 \varphi'\bigl(u - C(u, 1-v)\bigr)\bigl(1 - (\partial_1 C)(u, 1-v)\bigr)(\partial_2 C)(u, 1-v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad = -\int_0^1 \left(\int_0^1 \varphi'\bigl(u - C(u, s)\bigr)\bigl(1 - \partial_1 C(u, s)\bigr)\partial_2 C(u, s)\,\mathrm{d}s\right)\mathrm{d}u \\
&\quad = \int_0^1 \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\bigl(\varphi\bigl(u - C(u, s)\bigr)\bigr)\,\mathrm{d}s + \int_0^1 \varphi'\bigl(u - C(u, s)\bigr)\partial_1 C(u, s)\partial_1 C(u, s)\,\mathrm{d}s\right)\mathrm{d}u \\
&\quad = -\int_0^1 \varphi(v)\,\mathrm{d}v + \int_{\mathbb{I}^2} \varphi'\bigl(\tilde\sigma_2^*(C)\bigr)\,\partial_1 C\,\partial_2 C\,\mathrm{d}\Pi.
\end{aligned} \tag{13}
$$

• For $\xi = \varsigma = \sigma_1 \circ \sigma_2$, taking advantage of (12), we finally have

$$
\begin{aligned}
&(-1)^{|\varsigma|} K_\varphi(\varsigma, C) \\
&\quad = K_\varphi\bigl(\sigma_1, \sigma_2^*(C)\bigr) = \int_0^1 \varphi(v)\,\mathrm{d}v - \int_{\mathbb{I}^2} \varphi'(\tilde\sigma_1^*\bigl(\sigma_2^*(C)\bigr)\,\partial_1(\sigma_2^*(C))\,\partial_2\bigl(\sigma_2^*(C)\bigr)\,\mathrm{d}\Pi \\
&\quad = \int_0^1 \varphi(v)\,\mathrm{d}v - \int_0^1\int_0^1 \varphi'\bigl(v - u + C(u, 1-v)\bigr)\bigl(1 - (\partial_1 C)(u, 1-v)\bigr)(\partial_2 C)(u, 1-v)\,\mathrm{d}\Pi
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 \varphi(v)\,\mathrm{d}v - \int_0^1\int_0^1 \varphi'\bigl(\overline{C}(u,s)\bigr)\bigl(1-\partial_1 C(u,s)\bigr)\,\partial_2 C(u,s)\,\mathrm{d}u\,\mathrm{d}s \\
&= \int_0^1 \varphi(v)\,\mathrm{d}v - \int_0^1 \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\bigl(\varphi\bigl(\overline{C}(u,s)\bigr)\bigr) + \varphi'\bigl(\overline{C}(u,s)\bigr)\,\mathrm{d}s\right)\mathrm{d}u \\
&\quad + \int_0^1\int_0^1 \varphi'\bigl(\overline{C}(u,s)\bigr)\,\partial_1 C(u,s)\,\partial_2 C(u,s)\,\mathrm{d}u\,\mathrm{d}s \\
&= 2\int_0^1 \varphi(v)\,\mathrm{d}v - \int_{\mathbb{I}^2} \varphi'\bigl(\overline{C}\bigr)\,\mathrm{d}\Pi + \int_{\mathbb{I}^2} \varphi'\bigl(\overline{C}\bigr)\,\partial_1 C\,\partial_2 C\,\mathrm{d}\Pi. \qquad (14)
\end{aligned}
$$

Substituting Eqs. (11)–(14) into (10) gives (9), finishing the proof. □

Remark 5. Whenever φ' is convex, letting

$$
\begin{aligned}
x_1 &= C(u,v), & x_3 &= v - C(u,v),\\
x_2 &= u - C(u,v), & x_4 &= 1 - u - v + C(u,v),
\end{aligned}
$$

we have $x_i \in \mathbb{I}$, $i = 1,\dots,4$, $\sum_{i=1}^4 x_i = 1$ for any $u, v \in \mathbb{I}$ and $C \in \mathcal{C}$. Also, if $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ and $g(\boldsymbol{x}) = \sum_{i=1}^4 \varphi'(x_i)$, then it follows that $g : \mathbb{I}^4 \to \mathbb{R}$ is Schur-convex (cf. [24, p. 92, Prop. C.1]), and so

$$
4\varphi'\left(\frac{1}{4}\right) \leqslant g(\boldsymbol{x}) \leqslant \varphi'(1) + 3\varphi'(0) \qquad (15)
$$

since

$$
\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \prec (x_1, x_2, x_3, x_4) \prec (1, 0, 0, 0),
$$

where "$\prec$" denotes *majorization* relation for vectors (as defined by Hardy, Littlewood and Pólya; see, e.g., [24, p. 80, Def. A.1,]), i.e., for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$
\boldsymbol{x} \prec \boldsymbol{y} \iff \sum_{i=1}^k x_{[i]} \leqslant \sum_{i=1}^k y_{[i]} \quad \forall k = 1, 2, \dots, n-1, \qquad \sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}
$$

with $x_{[i]}$, $y_{[i]}$ denoting the ith largest entries of $\boldsymbol{x}$, $\boldsymbol{y}$, respectively. For more information about majorization and its applications, see the book [24]. Furthermore, the bounds in (15) are sharp since for $\varphi(t) = t^2$, we get equalities. Also, if φ' is concave, then g is Schur-concave, and the inequalities in (15) are reversed.

Example 2. As a sanity check, choose $\varphi(x) = x$, $x \in \mathbb{I}$. Then, clearly, $a_\varphi = 1$, $\int_{\mathbb{I}^2} \varphi'(\overline{C})\,\mathrm{d}\Pi = 1$, $G_\varphi(C) = 4$, and as expected,

$$
\tau_\varphi(C) = 1 - 4\int_{\mathbb{I}^2} \partial_1 C\,\partial_2 C\,\mathrm{d}\Pi = \tau(C).
$$

As a nontrivial example, we have

Example 3. Consider $\varphi(x) = x^2$, $x \in \mathbb{I}$. Then $a_\varphi = 3/2$,

$$\int_{\mathbb{I}^2} \varphi'(\overline{C})\,\mathrm{d}\Pi = 2\int_{\mathbb{I}^2} \overline{C}\,\mathrm{d}\Pi = 2\int_{\mathbb{I}^2} C\,\mathrm{d}\Pi = \frac{\rho_S(C)+3}{6},$$

where ρ_S denotes Spearman's rho. Also,

$$\begin{aligned} G_\varphi(C)(u,v) &= 2\bigl\{C(u,v) + \bigl[v - C(u,v)\bigr] + [u - C(u,v)] \\ &\quad + \bigl[1 - u - v + C(u,v)\bigr]\bigr\} \\ &= 2, \end{aligned}$$

and so

$$\tau_\varphi(C) = \frac{3}{2}\biggl(\frac{\rho_S(C)+3}{6} - 2\int_{\mathbb{I}^2} \partial_1 C\,\partial_2 C\,\mathrm{d}\Pi\biggr) = \frac{1}{4}\rho_S(C) + \frac{3}{4}\tau(C),$$

a much simpler expression compared to (8) or (9). We also note a curious similarity with a concordance measure considered by Borroni [3, Eq. (27)], where the author obtained $\gamma_{\varphi_\Delta}(C) = (1/4)\rho_S(C) + (3/4)\gamma(C)$ with $\gamma(C)$ being the Gini's gamma of copula C.

For more general power functions, we have

Proposition 1. *Consider* $\varphi(x) = x^p$, $x \in \mathbb{I}$, $p \geqslant 1$. *Then*

(i) *for* $p \geqslant 2$,

$$\begin{aligned} &\frac{p(p+1)}{2}\biggl[\biggl(\frac{\rho_S(C)+3}{12}\biggr)^{p-1} + \frac{\tau(C)-1}{4}\biggr] \\ &\quad\leqslant \tau_\varphi(C) \leqslant \frac{p(p+1)}{2}\biggl[\frac{\rho_S(C)+3}{12} + \frac{\tau(C)-1}{4^{p-1}}\biggr]; \end{aligned} \tag{16}$$

(ii) *for* $p \in [1,2)$, *the inequalities in* (16) *are reversed.*

Proof. First notice that, for $\varphi(x) = x^p$, we have $a_\varphi = (p+1)/2$, and for all $x \in \mathbb{I}$,

$$\varphi'(x) = px^{p-1}\begin{cases} \leqslant px & \text{if } p \geqslant 2, \\ \geqslant px & \text{if } p \in [1,2). \end{cases}$$

Thus, using Jensen's inequality, for $p \geqslant 2$, we obtain

$$\begin{aligned} \varphi'\biggl(\frac{\rho_S(C)+3}{12}\biggr) &= \varphi'\biggl(\int_{\mathbb{I}^2} \overline{C}\,\mathrm{d}\Pi\biggr) \leqslant \int_{\mathbb{I}^2} \varphi'(\overline{C})\,\mathrm{d}\Pi \\ &\leqslant p\int_{\mathbb{I}^2} \overline{C}\,\mathrm{d}\Pi = p\frac{\rho_S(C)+3}{12}. \end{aligned}$$

If $p \in [1,2)$, the inequalities above are reversed.

Secondly, if $p \geqslant 2$, then using (15), we have

$$p4^{2-p} \leqslant G_\varphi(C)(u,v) \leqslant p \quad \forall (u,v) \in \mathbb{I}^2,\ C \in \mathcal{C}. \tag{17}$$

Thus

$$\begin{aligned}\frac{p}{4}\bigl(\tau(C)-1\bigr) &= -p\int\limits_{\mathbb{I}^2} \partial_1 C\,\partial_2 C\,\mathrm{d}\Pi \leqslant -\int\limits_{\mathbb{I}^2} G_\varphi(C)\partial_1 C\,\partial_2 C\,\mathrm{d}\Pi \\ &\leqslant -p4^{2-p}\int\limits_{\mathbb{I}^2} \partial_1 C\,\partial_2 C\,\mathrm{d}\Pi = p4^{2-p}\bigl(\tau(C)-1\bigr).\end{aligned}$$

Plugging in the obtained estimates into (9) completes the proof in the case $p \geqslant 2$.

If $p \in [1,2)$, φ' is concave, and the bounds in (17) are reversed, so that again we obtain the needed claim. □

Example 4. To illustrate Theorem 3, we consider Farlie–Gumbel–Morgenstern (FGM) family of copulas

$$C_\theta(u,v) = uv\bigl(1+\theta(1-u)(1-v)\bigr), \quad \theta \in [-1,1].$$

It is well known that for this family, $\rho_S(C_\theta) = \theta/3$ and $\tau(C_\theta) = 2\theta/9$ (see, e.g., [19, p. 213]). Using Maple 2018.1 software, we have computed the expressions of $\tau_\varphi(C_\theta)$ for $\varphi(t) = t^p$ with $p = 1,2,3,4,5,10$:

$$\tau_\varphi(C_\theta) = \begin{cases} \tau(C_\theta) = \frac{\theta}{3} & \text{for } p = 1, \\ \frac{1}{4}\rho_S(C_\theta) + \frac{3}{4}\tau(C_\theta) = \frac{\theta}{4} & \text{for } p = 2, \\ \frac{6\theta}{25} + \frac{2\theta^3}{3675} & \text{for } p = 3, \\ \frac{2\theta}{9} + \frac{\theta^3}{1176} & \text{for } p = 4, \\ \frac{10\theta}{49} + \frac{5\theta^3}{5292} + \frac{\theta^5}{533610} & \text{for } p = 5, \\ \frac{5\theta}{36} + \frac{5\theta^3}{8281} + \frac{\theta^5}{189280} + \frac{5\theta^7}{147695184} + \frac{\theta^9}{12694752720} & \text{for } p = 10. \end{cases}$$

One can observe (see Fig. 1) that the range of $\tau_\varphi(C_\theta)$ first widens ($p = 2$) and then narrows as p changes from 3 to 10 when θ ranges in the interval $[-1,1]$. Notice the dominating influence of linear terms in all graphs. Also, computations become considerably longer for larger p.

As another graphical illustration, we provide Figs. 2 and 3, where we plot the function $G_\varphi(C)$ for $\varphi(x) = x^p$, $p \in \{1.1, 1.5, 2.1\}$ and $p \in \{5, 10, 20\}$, respectively, and several copulas $C \in \{W, C_{\mathrm{FGM}}, C_{\mathrm{Clayton}}, W\}$ with $C_{\mathrm{FGM}}(u,v) = uv(1+\alpha(1-u)(1-v))$, $\alpha = -0.6$, being one of Farlie–Gumbel–Morgenstern copulas, and $C_{\mathrm{Clayton}}(u,v) = (\max(u^{-\theta}+v^{-\theta}-1,0))^{-1/\theta}$, $\theta = 0.8$, being a particular Clayton copula; $(u,v) \in \mathbb{I}^2$. Notice the change from concavity (for $1 \leqslant p < 2$) to convexity (for $p > 2$) and the respective lower and upper bounds (both being p). Also, observe that the higher the p, the more the corners of the unit square become relevant, putting a larger weight on the extreme dependence in the expression of $\tau_\varphi(C)$ (see Eq. (9)).

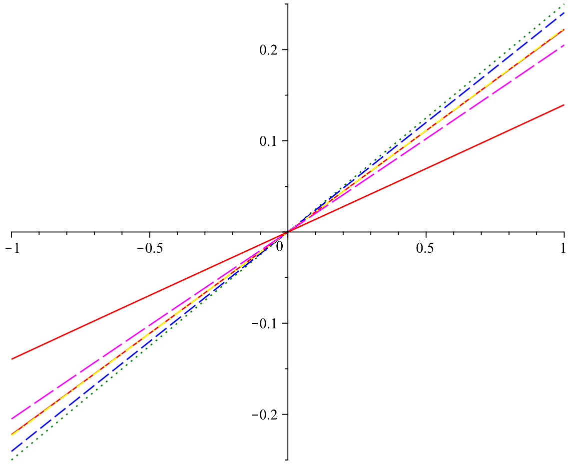

Figure 1. Generalized Kendall's τ_φ for FGM copula C_θ, $\theta \in [-1,1]$, and $\varphi(t) = t^p$ with $p = 1$ (red, solid), $p = 2$ (green, dotted), $p = 3$ (blue, dashed), $p = 4$ (yellow, dashed-dotted), and $p = 10$ (magenta, long-dashed).

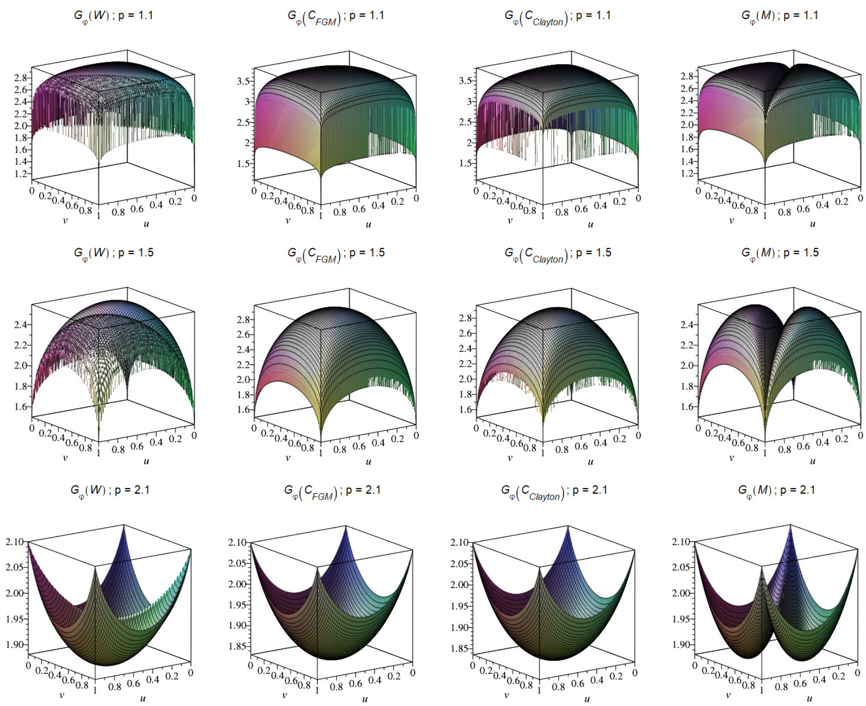

Figure 2. Plots of $G_\varphi(C)$ with $\varphi(x) = x^p$ for $p \in \{1.1, 1.5, 2.1\}$ and $C \in \{W, C_{\mathrm{FGM}}, C_{\mathrm{Clayton}}, M\}$. Plots produced using Maple 2018.1 software.

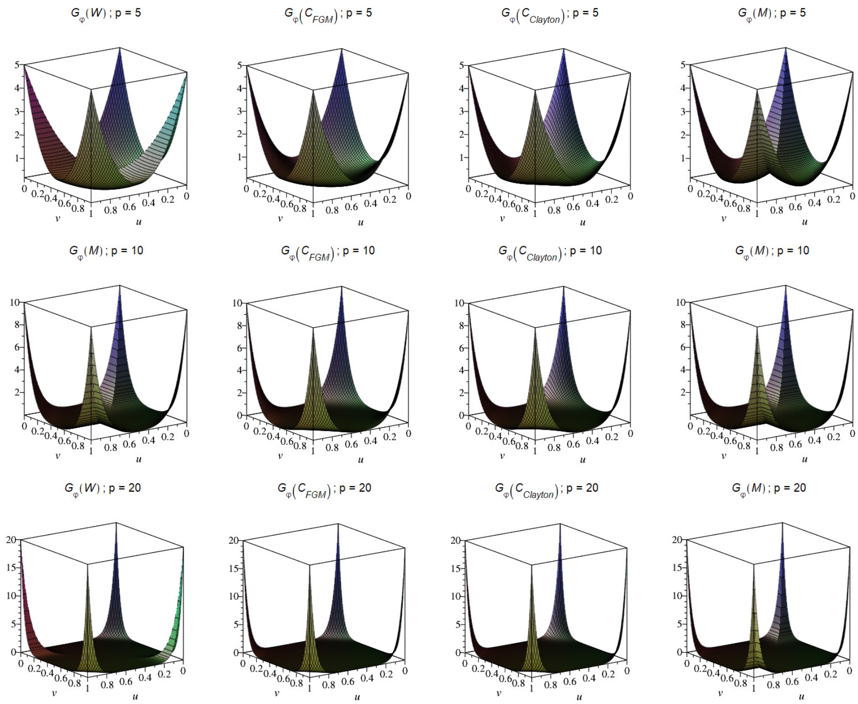

Figure 3. Plots of $G_\varphi(C)$ with $\varphi(x) = x^p$ for $p \in \{5, 10, 20\}$ and $C \in \{W, C_{\mathrm{FGM}}, C_{\mathrm{Clayton}}, M\}$. Plots produced using Maple 2018.1 software.

5 Conclusions and future directions

In this Part I, we have investigated generalizations of the popular Kendall's τ. We have showed two forms for such generalizations and have illustrated how to compute them for particular cases of the distortion function φ. In the general case, we have provided two-sided bounds for the values of generalized Kendall's τ, τ_φ, which are sharp if either linear (trivial case) or quadratic distortion function φ is used. In Part II, we will look into the intrinsic meaning of our generalizations and establish that they are achieved by replacing a probability measure μ_C (induced by a given copula C) with a nonadditive measure (in our case, convex (supermodular) capacity, $\nu = \varphi \circ \mu_C$). Such measures have found their place in the economic decision theory, so we hope that our generalizations could be of use there, too.

References

1. J.Y. Ahn, S. Fuchs, On minimal copulas under the concordance order, *J. Optim. Theory Appl.*, **184**(3):762–780, 2020, `https://doi.org/10.1007/s10957-019-01618-4`.

2. R.F. Barber, M. Kolar, ROCKET: Robust confidence intervals via Kendall's tau for transelliptical graphical models, *Ann. Stat.*, **46**(6B):3422–3450, 2018, `https://doi.org/10.48550/arXiv.1502.07641`.
3. C.G. Borroni, Mutual association measures, *Stat. Methods Appl.*, **28**(4):571–591, 2019, `https://doi.org/10.1007/s10260-018-00448-4`.
4. M. Cardin, P. Ferretti, Comparing bivariate risks: Measures of dependence, concordance, diversification, *Rendiconti per gli Studi Economici Quantitativi*, pp. 61–74, 01 2002.
5. P.W. Day, Rearrangement inequalities, *Can. J. Math.*, **24**(5):930–943, 1972, `https://doi.org/10.4153/CJM-1972-093-x`.
6. W. Deng, J. Wang, R. Zhang, Measures of concordance and testing of independence in multivariate structure, *J. Multivariate Anal.*, **191**:105035, 2022, `https://doi.org/10.1016/j.jmva.2022.105035`.
7. A. Derumigny, J.-D. Fermanian, On Kendall's regression, *J. Multivariate Anal.*, **178**:104610, 2020, `https://doi.org/10.1016/j.jmva.2020.104610`.
8. A. Dolati, M. Úbeda-Flores, On measures of multivariate concordance, *J. Probab. Stat. Sci.*, **4**(2):147–164, 2006.
9. F. Durante, C. Sempi, *Principles of Copula Theory*, CRC Press, Boca Raton, FL, 2016.
10. H.H. Edwards, M.D. Taylor, Characterizations of degree one bivariate measures of concordance, *J. Multivariate Anal.*, **100**(8):1777–1791, 2009, `https://doi.org/10.1016/j.jmva.2009.02.003`.
11. S. Fuchs, A biconvex form for copulas, *Depend. Model.*, **4**:63–75, 2016, `https://doi.org/10.1515/demo-2016-0003`.
12. S. Fuchs, Copula-induced measures of concordance, *Depend. Model.*, **4**:205–214, 2016, `https://doi.org/10.1515/demo-2016-0011`.
13. S. Fuchs, K.D. Schmidt, Bivariate copulas: Transformations, asymmetry and measures of concordance, *Kybernetika*, **50**(1):109–125, 2014, `https://doi.org/10.14736/kyb-2014-1-0109`.
14. S. Fuchs, K.D. Schmidt, On order statistics and Kendall's tau, *Stat. Probab. Lett.*, **169**(2021):108972, 2021, `https://doi.org/10.1016/j.spl.2020.108972`.
15. I. Gijbels, V. Kika, M. Omelka, On the specification of multivariate association measures and their behaviour with increasing dimension, *J. Multivariate Anal.*, **182**:104704, 2021, `https://doi.org/10.1016/j.jmva.2020.104704`.
16. I. Gijbels, M. Matterne, Study of partial and average conditional Kendall's tau, *Depend. Model.*, **9**(1):82–120, 2021, `https://doi.org/10.1515/demo-2021-0104`.
17. H. Joe, Multivariate concordance, *J. Multivariate Anal.*, **35**(1):12–30, 1990, `https://doi.org/10.1016/0047-259X(90)90013-8`.
18. H. Joe, *Multivariate Models and Dependence Concepts*, Monogr. Stat. Appl. Probab., Vol. 73, Chapman & Hall, London, 1997.
19. H. Joe, *Dependence modeling with copulas*, Monogr. Stat. Appl. Probab., Vol. 134, CRC Press, Boca Raton, FL, 2015.

20. N. Kamnitui, C. Genest, P. Jaworski, W. Trutschnig, On the size of the class of bivariate extreme-value copulas with a fixed value of Spearman's rho or Kendall's tau, *J. Math. Anal. Appl.*, **472**(1):920–936, 2019, https://doi.org/10.1016/j.jmaa.2018.11.057.
21. D. Kokol Bukovšek, N. Stopar, On the exact regions determined by Kendall's tau and other concordance measures, *Mediterr. J. Math.*, **20**(3):147, 2023, https://doi.org/10.1007/s00009-023-02350-0.
22. X. Li, P. Mikusiński, M.D. Taylor, Some integration-by-parts formulas involving 2-copulas, in C.M. Cuadras, J. Fortiana, J. Rodríguez-Lallena (Eds.), *Distributions with given marginals and statistical modelling*, pp. 153–159, Kluwer, Dordrecht, 2003, https://doi.org/10.1007/978-94-017-0061-0_16.
23. M. Manstavičius, Diversity of bivariate concordance measures, *Mathematics*, **10**(1103), 2022, https://doi.org/10.3390/math10071103.
24. A.W. Marshall, I. Olkin, B.C. Arnold, *Inequalities: Theory of Majorization and Its Applications,* 2nd ed., Springer Ser. Stat., Springer, New York, 2011, https://doi.org/10.1007/978-0-387-68276-1.
25. R. Mesiar, A. Kolesárová, A. Sheikhi, Convex concordance measures, *Fuzzy Sets Syst.*, **441**: 366–377, 2022, https://doi.org/10.1016/j.fss.2022.01.001.
26. R. Mesiar, A. Kolesárová, A. Sheikhi, S. Shvydka, Convex weak concordance measures and their constructions, *Fuzzy Sets Syst.*, **478**, 2024, https://doi.org/10.1016/j.fss.2023.108841.
27. R.B. Nelsen, Concordance and copulas: A survey, in *Distributions with Given Marginals and Statistical Modelling*, Kluwer, Dordrecht, 2002, pp. 169–178, https://doi.org/10.1007/978-94-017-0061-0_18.
28. R.B. Nelsen, *An Introduction to Copulas,* 2nd ed., Springer, New York, 2006, https://doi.org/10.1007/0-387-28678-0.
29. A. Rényi, On measures of dependence, *Acta Math. Hung.*, **10**:441–451, 1959, https://doi.org/10.1007/BF02024507.
30. M. Scarsini, On measures of concordance, *Stochastica*, **8**(3):201–218, 1984, http://eudml.org/doc/38916.
31. M. Scarsini, Strong measures of concordance and convergence in probability, *Riv. Mat. Sci. Econ. Soc.*, **7**:39–44, 1984, https://doi.org/10.1007/BF02106373.
32. M.D. Taylor, Multivariate measures of concordance, *Ann. Inst. Stat. Math.*, **59**(4):789–806, 2007, https://doi.org/10.1007/s10463-006-0076-2.
33. M.D. Taylor, Multivariate measures of concordance for copulas and their marginals, *Depend. Model.*, **4**:224–236, 2016, https://doi.org/10.1515/demo-2016-0013.
34. A.H. Tchen, Inequalities for distributions with given marginals, *Ann. Probab.*, **8**(4):814–827, 1980, https://doi.org/10.1214/aop/1176994668.
35. D.M. Topkis, *Ordered Optimal Solutions*, PhD thesis, Stanford University, Stanford, CA, 1968.
36. M. Úbeda Flores, Best-possible bounds on the set of copulas with a given value of Gini's gamma, *Iran. J. Fuzzy Syst.*, **19**(5):35–45, 2022, https://doi.org/10.22111/ijfs.2022.7155.