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Analyzing crop production: Unraveling the impact of pests and pesticides through a fractional model

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Abstract. The continuous growth of the human population raises concerns about food, fiber, and agricultural insecurity. Meeting the escalating demand for agricultural products due to this population surge makes protecting crops from pests becomes imperative. While farmers use chemical pesticides as crop protectors, the extensive use of these chemicals adversely affects both human health and the environment. In this research work, we formulate a nonlinear mathematical model using the Caputo fractional (CF) operator to investigate the effects of pesticides on crop yield dynamics. We assume that pesticides are sprayed proportional to the density of pest density and pests not entirely reliant on crops. The feasibility of every possible nonnegative equilibrium and its stability characteristics are explored utilizing the stability theory of fractional differential equations. Our model analysis reveals that in a continuous spray approach, the roles of pesticide abatement rate and pesticide uptake rate can be interchanged. Furthermore, we have identified the optimal time profile for pesticide spraying rate. This profile proves effective in minimizing both the pest population and the associated costs. To provide a practical illustration of our analytical findings and to showcase the impact of key parameters on the system's dynamics, we conducted numerical simulations. These simulations are conducted employing the generalized Adams–Bashforth–Moulton method, which allowed us to vividly demonstrate the real-world implications of our research.

Keywords: mathematical model, crop production, Caputo fractional derivative, pesticides, stability.

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1 Introduction

Agricultural products play a vital role in society by meeting the basic needs of the human population and supplying raw materials to industries, thereby bolstering the country's economy. With the world's human population projected to reach about 10 billion by 2050, the demand for food, fiber, and agricultural products has surged. As per the 2020 report from the United Nations Food and Agricultural Organization, around 135 million individuals in 55 countries faced food insecurity [33]. The additional demand for agricultural products can be met by enhancing the productivity of existing agricultural fields or expanding into new areas, potentially converting forest or unused land into agricultural space. However, the conversion of forest land, which typically has poor soil fertility, necessitates a significant investment. Furthermore, forests significantly regulate biodiversity and ecosystems, and such conversion poses risks to these factors [15]. In light of these considerations, farmers are encouraged to adopt various irrigation techniques, high-yielding seed varieties, and mechanical technologies to augment productivity per unit area of agricultural fields and ensure food security. The strides made in the agricultural sector have proven helpful, yet they fall short of ensuring food security. This issue persists today due to the diverse array of pests, like the stalk borer affecting corn, the capitulum borer in sunflowers, the leaf folder in rice, and the codling moth targeting apples. These pests pose significant threats to growing crops by causing damage to their leaves, fruits, and roots, ultimately leading to reduced crop production and compromised quality [14]. Therefore, effective management of pest populations stands as a crucial factor in increasing crop production within agricultural fields.

Chemical insecticides offer a promising path towards sustainable agricultural yield. Farmers employ these insecticides to manage insect populations, and researches indicate substantial improvements in agricultural outcomes following their applications [31, 32]. Studies suggest that without insecticide application, there would be a considerable reduction in the yields of cereal crops by 32%, vegetables by 54%, and fruits by 78%, leading to significant losses [27]. Thus, insecticides play a pivotal role in enhancing global crop yields. The global use of pesticides has nearly doubled from 1990 to 2022 [34]. Numerous studies in the literature underscore the importance of pesticides in controlling pest populations in agricultural fields [4, 22]. While pesticides demonstrate effectiveness in minimizing crop losses with abundant use, their excessive application poses adverse effects on both human health and the environment [9, 19].

Mathematical models that capture the interplay among insect population, crop yield, and amount of insecticides have provided a foundation to mitigate these adverse effects on human health and agriculture crops (see [26, 30] and references therein). In a study by Wang et al. [29], a pest control model was introduced, and the impact of pesticide spraying in conjunction with the release of infected insects was investigated. The study concluded that this combination represents an effective strategy for pest control. Kar et al. [8] discussed the pest control by using pesticides along with introducing infected pests. To assess the impact of delayed responses to pesticides on a pest control approach and ascertain the lasting efficacy of pesticides, Liang et al. [12] designed an integrated model for pest management. Their research outcomes indicate the presence of an optimal number of pesticide spray frequency or an ideal release time frame for predators, which can optimize the economic threshold and control pest population. In a recent study, Misra and Yadav [16] presented a nonlinear model, taking into account that insecticide spraying is influenced by both insect density and crop yield losses. Their research discerns a specific range of insecticide spraying rate, showcasing bistability behavior within the model system. Furthermore, they identify the threshold value of the insecticide spraying rate beyond which the system stabilizes at equilibrium, resulting in higher crop yield.

All the pest control models discussed above are confined to traditional integer-order differential equations. As an integer-order derivative is local in nature, it lacks complete memory and therefore cannot accurately capture the physical dynamics of the model. In recent years, fractional calculus has gained significant recognition in the realm of mathematical modeling. Fractional models, due to their hereditary characteristics and memory effect, prove more relevant and beneficial in understanding natural phenomena [23, 28]. Caputo and Liouville–Caputo developed fractional-order operators, which have to be proven beneficial in constructing models with a broad range of real-life applications [21, 25]. Ameen et al. [1] introduced a fractional maize streak virus infection model using the Caputo fractional operator and employed optimal control to demonstrate the effects of chemical control, quarantine, and prevention in determining the most advantageous strategy for eradicating maize streak disease. From these studies it becomes evident that the Caputo fractional operator better expresses the dynamics of real-life phenomena due to its memory and hereditary properties. In this research endeavor, we develop a continuous spray pesticide model within a fractional environment using the Caputo fractional derivative.

The primary objective of this research work is to investigate the impact of continuous pesticide spraying on both crop yield and pest population, especially, when pests have an alternative food source. We aim to identify the key parameters that play a crucial role in pest control. Furthermore, in this study, we will analyze the effect of fractional-order derivative on the stability of equilibrium points and find the optimal strategy to reduce both the pest population and the expenses incurred in applying pesticides. To achieve these goals, the subsequent sections of this article are structured as follows. Section 2 covers the discussion on mathematical preliminaries. Following that, Section 3 elaborates on the modeling phenomenon, discussing the existence, uniqueness, positivity, and boundedness of solutions for the proposed mathematical model. Section 4 delves into equilibrium and stability analysis, while Section 5 extends the proposed model to an optimal control problem. Moving on to Section 6, a numerical scheme is developed, and further Section 7 presents the numerical simulations and the results of the optimal control strategy. Ultimately, the research paper concludes by drawing insights derived from the model analysis.

2 Mathematical preliminaries

Definition 1. Let $g : [a, b] \to \mathbb{R}$ be a real valued function, then the (left) CF derivative of order γ is expressed as

$$ {}_{a}^{C}D_{t}^{\gamma} g(t) = \frac{1}{\Gamma(n-\gamma)} \int_{a}^{t} g^{n}(u)(t-u)^{n-\gamma-1}\,\mathrm{d}u, $$

where $n-1 < \gamma \leqslant n$, $n \in \mathbb{N}$. Similarly, the corresponding CF integral is given by

$$ {}_{a}^{C}I_{t}^{\gamma} g(t) = \frac{1}{\Gamma(\gamma)} \int_{a}^{t} g(u)(t-u)^{\gamma-1}\,\mathrm{d}u. $$

The Laplace transform of CF derivative of order γ such that $n-1 < \gamma \leqslant n$, $n \in \mathbb{N}$, can be written as

$$ \mathscr{L}\bigl\{{}_{0}^{C}D_{t}^{\gamma} g(t)\bigr\} = s^{\gamma}\mathscr{L}\bigl\{g(t)\bigr\} - \sum_{k=0}^{n-1} s^{\gamma-k-1} g^{(k)}(0), $$

where $g^{(k)}$ denotes the kth order derivative of the function $g(t)$, and for $n = 1$, we have

$$ \mathscr{L}\bigl\{{}_{0}^{C}D_{t}^{\gamma} g(t)\bigr\} = s^{\gamma}\mathscr{L}\bigl\{g(t)\bigr\} - g(0). $$

The Mittage-Leffler function with two parameters $\gamma_1 > 0$, $\gamma_2 > 0$ and $z \in \mathbb{C}$ is defined as

$$ E_{\gamma_1,\gamma_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma_1 k + \gamma_2)}. $$

3 Model formulation

Here, we formulate a nonlinear fractional model to enhance crop yield by controlling pest population using the pesticides. We consider three dynamical variables: crop yield $A(t)$, pest population $S(t)$, and pesticides $P(t)$. We postulate that the crop yield adheres to the logistic growth, characterized by an intrinsic growth rate denoted as r and a carrying capacity represented by K [30]. We further posit that the pest population is not entirely reliant on crop A, and in the absence of crop, the pest population follows logistic growth with an intrinsic growth rate denoted as u and carrying capacity L as they have limited availability of resources [3, 6]. Pests, however, attack crop, leading to a reduction in the yield of crop A at a rate αAS. Here, α is the agricultural crop consumption rate of pests, and this consumption of crops increases the growth rate of pests at rate $\theta\alpha AS$, where θ is the conversion efficiency of pests.

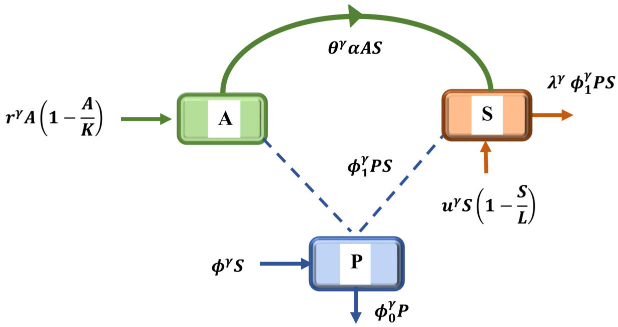

Figure 1. Schematic diagram for model (1).

We consider that farmers spray the pesticides proportional to pest population at a rate ϕS. The pesticides naturally deplete at the rate $\phi_0 P$, where ϕ_0 is the natural depletion rate of pesticides. Pests consume the pesticide, by which pesticides decrease at the rate $\phi_1 SP$, where ϕ_1 is the uptake rate of pesticide by pests. Due to this consumption of pesticides, pest population declines at the rate $\lambda\phi_1 SP$, where λ is the depletion rate of pests due to pesticides. Based on the above considerations, a schematic diagram is demonstrated in Fig. 1, and the following set of nonlinear fractional-order differential equations regulates the dynamics of the proposed model:

$$\begin{aligned} {}_0^C D_t^\gamma A &= r^\gamma A\left(1 - \frac{A}{K}\right) - \alpha^\gamma AS, \\ {}_0^C D_t^\gamma S &= u^\gamma S\left(1 - \frac{S}{L}\right) + \theta^\gamma\alpha^\gamma AS - \lambda^\gamma\phi_1^\gamma PS, \\ {}_0^C D_t^\gamma P &= \phi^\gamma S - \phi_0^\gamma P - \phi_1^\gamma PS. \end{aligned} \tag{1}$$

The initial conditions for the formulated model system (1) are

$$A(0) = A_0 \geqslant 0, \quad S(0) = S_0 \geqslant 0, \quad \text{and} \quad P(0) = P_0 \geqslant 0.$$

We correct the dimensions of both sides of the model system (1) by introducing γ in the required model parameters.

3.1 Properties of the model

Theorem 1. *Every nonnegative initial condition yields a distinct solution for the fractional-order system represented by* (1).

Proof. To establish both the existence and uniqueness of solutions for system (1) within the domain $\psi \times [0,T]$, where $\psi = \{(A,S,P) \in \mathbb{R}^3\colon \max\{|A|,|S|,|P|\} \leqslant \eta\}$ and $T < +\infty$. First, we denote $X = (A,S,P)$ and $\bar{X} = (\bar{A},\bar{S},\bar{P})$ and consider a mapping

$\mathcal{H}(X) = (\mathcal{H}_1(X), \mathcal{H}_2(X), \mathcal{H}_3(X))$, where

$$\mathcal{H}_1(X) = r^\gamma A\left(1 - \frac{A}{K}\right) - \alpha^\gamma AS,$$
$$\mathcal{H}_2(X) = u^\gamma S\left(1 - \frac{S}{L}\right) + \theta^\gamma\alpha^\gamma AS - \lambda^\gamma\phi_1^\gamma PS,$$
$$\mathcal{H}_3(X) = \phi^\gamma S - \phi_0^\gamma P - \phi_1^\gamma PS.$$

Now, for any $X, \bar{X} \in \psi$, we have

$$\begin{aligned}
&\left\|\mathcal{H}(X) - \mathcal{H}(\bar{X})\right\| \\
&= \left|\mathcal{H}_1(X) - \mathcal{H}_1(\bar{X})\right| + \left|\mathcal{H}_2(X) - \mathcal{H}_2(\bar{X})\right| + \left|\mathcal{H}_3(X) - \mathcal{H}_3(\bar{X})\right| \\
&= \left|r^\gamma A\left(1 - \frac{A}{K}\right) - \alpha^\gamma AS - r^\gamma \bar{A}\left(1 - \frac{\bar{A}}{K}\right) + \alpha^\gamma \bar{A}\bar{S}\right| \\
&\quad + \left|u^\gamma S\left(1 - \frac{S}{L}\right) + \theta^\gamma\alpha^\gamma AS - \lambda^\gamma\phi_1^\gamma PS \right. \\
&\quad \left. - u^\gamma \bar{S}\left(1 - \frac{\bar{S}}{L}\right) - \theta^\gamma\alpha^\gamma \bar{A}\bar{S} + \lambda^\gamma\phi_1^\gamma \bar{P}\bar{S}\right| \\
&\quad + \left|\phi^\gamma S - \phi_0^\gamma P - \phi_1^\gamma PS - \phi^\gamma \bar{S} + \phi_0^\gamma \bar{P} + \phi_1^\gamma \bar{P}\bar{S}\right| \\
&\leqslant \left(r^\gamma + \frac{2r^\gamma\eta}{K} + \alpha^\gamma\left(1 + \theta^\gamma\right)\eta\right)|A - \bar{A}| \\
&\quad + \left(u^\gamma + \frac{2u^\gamma\eta}{L} + \alpha^\gamma\left(1 + \theta^\gamma\right)\eta + \phi^\gamma + \left(1 + \lambda^\gamma\right)\phi_1^\gamma\eta\right)|S - \bar{S}| \\
&\quad + \left(\phi_0^\gamma + \left(1 + \lambda^\gamma\right)\phi_1^\gamma\eta\right)|P - \bar{P}| \\
&\leqslant \mathcal{G}\|X - \bar{X}\|,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G} = \max\biggl\{ & r^\gamma + \frac{2r^\gamma\eta}{K} + \alpha^\gamma\left(1 + \theta^\gamma\right)\eta, \\
& u^\gamma + \frac{2u^\gamma\eta}{L} + \alpha^\gamma\left(1 + \theta^\gamma\right)\eta + \phi^\gamma + \left(1 + \lambda^\gamma\right)\phi_1^\gamma\eta, \\
& \phi_0^\gamma + \left(1 + \lambda^\gamma\right)\phi_1^\gamma\eta\biggr\}.
\end{aligned}$$

Therefore, $\mathcal{H}(X)$ fulfils the Lipschitz condition concerning X, establishing the existence of a unique solution $X(t)$ for the formulated system (1) with the initial condition $X(0) = (A(0), S(0), P(0))$. □

The positivity of the model system (1) can be easily proved using the analysis akin to Li et al. [11]. We have formulated the following theorem to prove the boundedness of the formulated model system (1).

Theorem 2. *Every solution originating from* $\mathbb{R}^3_+$ *for system* (1) *remains bounded.*

Proof. Consider $W = A + S + P$, then

$$\begin{aligned} {}_0^C D_t^\gamma W &= {}_0^C D_t^\gamma A + {}_0^C D_t^\gamma S + {}_0^C D_t^\gamma P, \\ &\leqslant r^\gamma A - \frac{r^\gamma (A)^2}{K} + u^\gamma S - \frac{u^\gamma (S)^2}{L} - \alpha^\gamma \big(1-\theta^\gamma\big) AS + \phi^\gamma S - \phi_0 P, \\ &\leqslant -a(A+S+P) + 2r^\gamma A - \frac{r^\gamma (A)^2}{K} + 2u^\gamma S - \frac{u^\gamma (S)^2}{L}, \end{aligned}$$

where $a^\gamma = \min\{r^\gamma, (u^\gamma - \phi^\gamma), \phi_0{}^\gamma\} > 0$, provided $u^\gamma > \phi^\gamma$. Therefore,

$$ {}_0^C D_t^\gamma W \leqslant r^\gamma K + u^\gamma L - a^\gamma W. \tag{2}$$

Applying the Laplace transform on both sides of Eq. (2), we have

$$\mathscr{L}\big\{{}_0^C D_t^\gamma W(t)\big\} \leqslant \mathscr{L}\big\{r^\gamma K + u^\gamma L\big\} - a^\gamma \mathscr{L}\big\{W(t)\big\},$$

and then

$$\big(s^\gamma + a^\gamma\big)\mathscr{L}\big\{W(t)\big\} = W(0) + \frac{r^\gamma K + u^\gamma L}{s},$$

where $W(0)$ denotes the initial value of sum of all dynamical variables. Now by using inverse Laplace transform, we get

$$W(t) \leqslant \left(W(0) + \frac{r^\gamma K + u^\gamma L}{a^\gamma}\right) E_\gamma\big(-a^\gamma t^\gamma\big) + \frac{r^\gamma K + u^\gamma L}{a^\gamma},$$

where $E_\gamma(z)$ constitutes the Mittag-Leffler (ML) function with one parameter. Thus, $W(t) \leqslant (r^\gamma K + u^\gamma L)/a^\gamma$ as $t \to \infty$. □

4 Equilibrium and stability analysis

4.1 Equilibrium analysis

Here, we identify the attainable equilibria of the model system (1) by setting the derivatives of all dynamic variables with respect to time t to zero. As a result, we identify four nonnegative equilibria for model system (1), listed as follows:

(i) The *trivial equilibrium* $E_0(0,0,0)$ and the *pest free equilibrium* $E_1(K,0,0)$ always exist.

(ii) The *crop-free equilibrium* $E_2(0, S_2^*, P_2^*)$ always exists. The value of S_2^*, P_2^* satisfy the following relations:

$$u^\gamma\left(1 - \frac{S}{L}\right) - \lambda^\gamma \phi_1^\gamma P = 0, \tag{3}$$

$$\phi^\gamma S - \phi_0^\gamma P - \phi_1^\gamma PS = 0. \tag{4}$$

From Eq. (3), using the value of P in Eq. (4), we obtained a quadratic equation as

$$\frac{u^\gamma}{\lambda L}S^2 + \left(\phi^\gamma + \frac{u^\gamma \phi_0^\gamma}{\lambda^\gamma \phi_1^\gamma L} - \frac{u^\gamma}{\lambda^\gamma}\right)S - \frac{u^\gamma \phi_0^\gamma}{\lambda^\gamma \phi_1^\gamma} = 0. \tag{5}$$

Clearly, Eq. (5) has one positive root, and using this value of S_2^* in Eq. (4), we get the positive value of P_2^*.

(iii) Equilibrium $E^*(A^*, S^*, P^*)$ exists, provided $S^* < r/\alpha$. Here, all the dynamical variables are present, and this satisfy the following relations:

$$r^\gamma\left(1 - \frac{A}{K}\right) - \alpha^\gamma S = 0, \tag{6}$$

$$u^\gamma\left(1 - \frac{S}{L}\right) + \theta^\gamma\alpha^\gamma A - \lambda^\gamma\phi_1^\gamma P = 0, \tag{7}$$

$$\phi^\gamma S - \phi_0^\gamma P - \phi_1^\gamma PS = 0. \tag{8}$$

Using the value of A from (6) and the value of P from Eq. (8) in Eq. (7), we obtain the following quadratic equation in S:

$$aS^2 + bS - c = 0, \tag{9}$$

where

$$\begin{aligned}
a &= \phi_1^\gamma\left(\frac{\theta^\gamma(\alpha^2)^\gamma K}{r} + \frac{u^\gamma}{L}\right),\\
b &= \phi_0^\gamma\left(\frac{\theta^\gamma(\alpha^2)^\gamma K}{r} + \frac{u^\gamma}{L}\right) + \lambda^\gamma\phi^\gamma\phi_1^\gamma - \phi_1^\gamma u^\gamma - \phi_1^\gamma\theta^\gamma\alpha^\gamma K,\\
c &= \phi_0^\gamma\left(u^\gamma + \theta^\gamma\alpha^\gamma K\right).
\end{aligned}$$

Clearly, Eq. (9) has one positive root, which will lie in the feasible region if $S^* < r^\gamma/\alpha^\gamma$, and using this value of S^*, we obtain positive values for A^* and P^*, respectively.

4.2 Stability analysis

Now, we perform the local stability analysis of the obtained feasible equilibria. Matignon [13], in 1996, discussed the local stability of an equilibrium for CF-order derivatives. According to this theory, an equilibrium is considered locally asymptotically stable if the arguments of all eigenvalues of the Jacobian matrix satisfy

$$|\mathrm{Arg}\,\lambda_i| > \frac{\gamma\pi}{2},$$

where $i = 1, 2, 3$ for system (1), and $0 < \gamma \leqslant 1$.

Theorem 3.

(i) *Equilibrium E_0 and E_1 are always unstable.*
(ii) *The crop-free equilibrium, E_2 is unstable if interior equilibrium E^* exists.*
(iii) *Equilibrium E^* is always locally asymptotically stable.*

Proof. For the formulated model system (1), matrix J is

$$J = \begin{bmatrix} r^\gamma(1 - \frac{2A}{K}) - \alpha^\gamma S & -\alpha^\gamma A & 0 \\ \theta^\gamma \alpha^\gamma S & u^\gamma(1 - \frac{2S}{L}) + \theta^\gamma \alpha^\gamma A - \lambda^\gamma \phi_1^\gamma P & -\lambda^\gamma \phi_1^\gamma S \\ 0 & \phi^\gamma - \phi_1^\gamma P & -(\phi_0^\gamma + \phi_1^\gamma S) \end{bmatrix}.$$

(i) Eigenvalues of the matrix J at E_0 are calculated as r^γ, u^γ, and $-\phi_0^\gamma$. Since r^γ and u^γ are always positive, thus $|\operatorname{Arg} r^\gamma| = 0 = |\operatorname{Arg} u^\gamma|$, and trivial equilibrium is always unstable. At equilibrium E_1, we get the three eigenvalues as $-r^\gamma$, $u^\gamma + \theta^\gamma \alpha^\gamma K$, and $-\phi_0^\gamma$. The positive sign of one eigenvalue indicates the instability of E_1.

(ii) At the equilibrium E_2, one eigenvalue of the Jacobian matrix is $(r^\gamma - \alpha^\gamma S_2^*)$, and the other two lie in the left half of complex plane. Hence, if $\alpha > (r^\gamma / S_2^*)^{(1/\gamma)} = \alpha^*$, then E_2 is stable; otherwise, unstable, i.e., if interior equilibrium E^* exists, then E_2 is unstable.

(iii) The characteristic polynomial for the matrix J_{E^*} is calculated as follows:

$$\chi^3 + A_1\chi^2 + A_2\chi + A_3 = 0, \tag{10}$$

where

$$\begin{aligned} A_1 &= \frac{r^\gamma A^*}{K} + \frac{u^\gamma S^*}{L} + \phi_0^\gamma + \phi_1^\gamma S^*, \\ A_2 &= \frac{r^\gamma A^*}{K}\frac{u^\gamma S^*}{L} + (\phi_0^\gamma + \phi_1^\gamma S^*)\left(\frac{r^\gamma A^*}{K} + \frac{u^\gamma S^*}{L}\right) + \lambda^\gamma \phi_0^\gamma \phi_1^\gamma P^* + \theta^\gamma (\alpha^2)^\gamma A^* S^*, \\ A_3 &= \frac{r\lambda^\gamma \phi_0^\gamma \phi_1^\gamma A^* P^*}{K} + \theta^\gamma (\alpha^2)^\gamma A^* S^* (\phi_0^\gamma + \phi_1^\gamma S^*) + \frac{r^\gamma A^*}{K}\frac{u^\gamma S^*}{L}(\phi_0^\gamma + \phi_1^\gamma S^*). \end{aligned}$$

It is evident that A_1 and A_3 are consistently positive. Additionally, after some algebraic manipulation, it is determined that $A_1 A_2 - A_3$ remains positive. Employing the *Routh–Hurwitz criterion*, it follows that the roots of the characteristic equation (10) lie in the left half of complex plane and $|\operatorname{Arg} \lambda_i| > \gamma\pi/2$. Consequently, we affirm the local asymptotic stability of E^*. □

5 The optimal control problem

Pesticide spraying is necessary to manage pest density effectively, but a successful intervention strategy aims to reduce pest density while minimizing associated costs. With this objective in mind, our focus is on determining the control function with respect to the formulated model system (1) to achieve the dual goal of minimizing pest density and reducing the overall cost of pesticide application. This study employs CF optimal control, leveraging an operator known for its efficiency and effectiveness, a concept aligned with the analysis by Baba and Bilgehan [2]. We examine the pesticide spraying rate as a Lebesgue measurable function, denoted by $v(t)$, operating within the finite interval $[0, t_f]$. Consequently, our main aim is to minimize the cost functional linked to this control function

$$J(v) = \int_0^{t_f} \left[w_1 S(t) + w_2 v^2(t)\right] \mathrm{d}t$$

with respect to

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= r^\gamma A\left(1 - \frac{A}{K}\right) - \alpha^\gamma AS, \\
\frac{\mathrm{d}S}{\mathrm{d}t} &= u^\gamma S\left(1 - \frac{S}{L}\right) - \theta^\gamma \alpha^\gamma AS - \lambda\phi_1 PS, \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= v^\gamma(t)S - \phi_0^\gamma P - \phi_1^\gamma PS.
\end{aligned} \tag{11}$$

The quantities w_1 and w_2 denote positive weight functions. Our goal is to determine the optimal control $v^*(t)$ within the control set

$$V = \left\{v(t)\colon\ v(t) \text{ is measurable},\ 0 \leqslant v(t) \leqslant v^{\max}\right\}$$

within the interval $t \in [0, t_f]$, minimizing the functional $J(v)$ subject to the model system (11). After establishing the optimal control, we apply Pontryagin's principle [7,24] to derive the necessary conditions for optimal control. The Hamiltonian $\mathcal{M}$ corresponding to the control problem can be represented as

$$\begin{aligned}
&\mathcal{M}(A, S, P, v, \chi_1, \chi_2, \chi_3) \\
&\quad = w_1 S(t) + w_2 v^2(t) + \chi_1\left[r^\gamma A\left(1 - \frac{A}{K}\right) - \alpha^\gamma AS\right] \\
&\qquad + \chi_2\left[u^\gamma S\left(1 - \frac{S}{L}\right) + \theta^\gamma \alpha^\gamma AS - \lambda\phi_1 PS\right] + \chi_3\left[v^\gamma(t)S - \phi_0^\gamma P - \phi_1^\gamma PS\right].
\end{aligned}$$

Here, χ_j $(j = 1, \dots, 3)$ denotes the adjoint variables, and their corresponding differential equations are expressed as follows:

$$\begin{aligned}
{}_0^{RL}D_t^\gamma \chi_1 &= \frac{\partial \mathcal{M}}{\partial A} = \chi_1\left[r^\gamma\left(1 - \frac{2A}{K}\right) - \alpha^\gamma S\right] + \chi_2 \theta^\gamma \alpha^\gamma S, \\
{}_0^{RL}D_t^\gamma \chi_2 &= \frac{\partial \mathcal{M}}{\partial S} = w_1 - \chi_1 \alpha A + \chi_2\left[u^\gamma\left(1 - \frac{2A}{K}\right) - \theta^\gamma \alpha^\gamma S - \lambda^\gamma \phi_1^\gamma P\right] \\
&\qquad + \chi_3\left(v^\gamma(t) - \phi_1^\gamma P\right), \\
{}_0^{RL}D_t^\gamma \chi_3 &= \frac{\partial \mathcal{M}}{\partial P} = -\chi_2 \lambda^\gamma \phi_1^\gamma S - \chi_3\left(\phi_0^\gamma + \phi_1^\gamma S\right).
\end{aligned}$$

The transversality conditions, given by $\chi_j(t_f) = 0$ for $j = 1, \dots, 3$, must be satisfied. The optimal control v^* is determined through the optimality condition $\partial\mathcal{M}/\partial v = 0$ at $v = v^*$ within the set $t \in [0, t_f]$: $0 < v(t) < v^*(t)$. This condition yields $v^* = -\chi_3 S/(2w_2)$ within the interior of the set V. Through the imposition of constraints on the control variable, the optimal control $v^*(t)$ is derived as

$$v^*(t) = \max\left\{0,\ \min\left\{\frac{-\chi_3 S}{2w_2},\ v^{\max}\right\}\right\}.$$

6 Numerical scheme

Here, we give a numerical technique to assess and forecast the numerical stability of formulated fractional model (1) with initial conditions based on generalized Adams–Bashforth–Moulton method [10] for CF derivative. The Volterra integral equation for the nonlinear system

$$\begin{aligned} {}_0^C D_t^\gamma A &= f\big(t, z(t)\big), \quad 0 \leqslant t \leqslant T, \\ z^k(0) &= z_0^k, \quad k = 1, 2, 3, \dots, n,\ n = [\gamma], \end{aligned}$$

is

$$z(t) = \sum_{q=0}^{n-1} z_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\eta-1} f\big(s, z(s)\big)\,\mathrm{d}s. \tag{12}$$

In integration of (12), Diethelm et al. [5] applied the Adams–Bashforth–Moulton scheme. By defining $h = T/M$, $t_m = mh$, $m = 0, 1, 2, \dots, M \in \mathbb{Z}^+$, the model system (1) can be represented as follows:

$$\begin{aligned} A_{m+1} &= A_0 + \frac{h^\gamma}{\Gamma(\gamma+2)}\left[r^\gamma A_{m+1}^j\left(1 - \frac{A_{m+1}^j}{K}\right) - \alpha^\gamma A_{m+1}^j S_{m+1}^j\right] \\ &\quad + \frac{h^\gamma}{\Gamma(\gamma+2)} \sum_{i=0}^m a_{i,m+1}\left[r^\gamma A_i\left(1 - \frac{A_i}{K}\right) - \alpha^\gamma A_i S_i\right], \\ S_{m+1} &= S_0 + \frac{h^\gamma}{\Gamma(\gamma+2)}\left[u^\gamma S_{m+1}^j\left(1 - \frac{S_{m+1}^j}{L}\right) + \theta^\gamma \alpha^\gamma A_{m+1}^j S_{m+1}^j - \lambda^\gamma \phi_1^\gamma P_{m+1}^j S_{m+1}^j\right] \\ &\quad + \frac{h^\gamma}{\Gamma(\gamma+2)} \sum_{i=0}^m a_{i,m+1}\left[u^\gamma S_i\left(1 - \frac{S_i}{L}\right) + \theta^\gamma \alpha^\gamma A_i S_i - \lambda \phi_1 P_i S_i\right], \\ P_{m+1} &= P_0 + \frac{h^\gamma}{\Gamma(\gamma+2)}\left[\phi^\gamma S_{m+1}^j - \phi_0^\gamma P_{m+1}^j - \phi_1^\gamma P_{m+1}^j S_{m+1}^j\right] \\ &\quad + \frac{h^\gamma}{\Gamma(\gamma+2)} \sum_{i=0}^m a_{i,m+1}\left[\phi^\gamma S_i - \phi_0^\gamma P_i - \phi_1^\gamma P_i S_i\right], \end{aligned}$$

where

$$A_{m+1}^{j} = A_0 + + \frac{h^\gamma}{\Gamma(\gamma)} \sum_{i=0}^{m} \Theta_{i,m+1} \left[r^\gamma A_i \left(1 - \frac{A_i}{K} \right) - \alpha^\gamma A_i S_i \right],$$

$$S_{m+1}^{j} = S_0 + \frac{h^\gamma}{\Gamma(\gamma)} \sum_{i=0}^{m} \Theta_{i,m+1} \left[u^\gamma S_i \left(1 - \frac{S_i}{L} \right) + \theta^\gamma \alpha^\gamma A_i S_i - \lambda \phi_1 P_i S_i \right],$$

$$P_{m+1}^{j} = P_0 + \frac{h^\gamma}{\Gamma(\gamma)} \sum_{i=0}^{m} \Theta_{i,m+1} \left[\phi^\gamma S_i - \phi_0^\gamma P_i - \phi_1^\gamma P_i S_i \right]$$

in which

$$a_{i,m+1} = \begin{cases} m^{\gamma_q+1} - (m - \gamma_q)(m+1)^{\gamma_q}, & i = 0, \\ (m-i+2)^{\gamma_q+1} - (n-i)^{\eta_q+1} - 2(n-i+1)^{\gamma_q+1}, & 1 \leqslant i \leqslant m, \\ 1, & i = m+1, \end{cases}$$

and

$$\Theta_{i,m+1} = \frac{h^{\gamma_q}}{\gamma_q} \big((m-i+1)^{\gamma_q} - (m-i)^{\gamma_q} \big), \quad q = 1, 2, 3.$$

7 Simulation results

In this section, we utilize the nonstandard finite difference scheme introduced in Section 6 to simulate the model system (1). The simulation involves a specific set of parameter values outlined in Table 1. Using these designated parameter values, we have calculated the equilibrium components of the formulated system (1) at $\gamma = 1$ given below:

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 \\ 0.04 \\ 0.33 \end{bmatrix}, \qquad E^* = \begin{bmatrix} 32.52 \\ 2.80 \\ 1.87 \end{bmatrix}.$$

The eigenvalues corresponding to the equilibrium E_0, E_1, E_2, and E^* are calculated as

$$\Lambda_0 = \begin{bmatrix} 0.2 \\ 0.1 \\ -0.01 \end{bmatrix}, \qquad \Lambda_2 = \begin{bmatrix} 0.199 \\ -0.006 + 0.031\mathrm{i} \\ -0.006 - 0.031\mathrm{i} \end{bmatrix},$$

$$\Lambda_1 = \begin{bmatrix} -0.2 \\ 0.85 \\ -0.01 \end{bmatrix}, \qquad \Lambda^* = \begin{bmatrix} -0.147 \\ -0.08 + 0.192\mathrm{i} \\ -0.08 - 0.192\mathrm{i} \end{bmatrix}.$$

Here, it is observed that the eigenvalue corresponding to the equilibrium E_0 is positive, implying its instability. The presence of a positive eigenvalue linked to E_1 designates it as an unstable equilibrium point. Similarly, the existence of a positive eigenvalue related

Table 1. Biological interpretations of the parameters utilized in the model system (1) and their respective values along with units.

Parameters	Descriptions	Values	Units
r	Intrinsic growth rate of crop	0.2	day^{-1}
α	Crop consumption rate by pests	0.025	$\text{kg}^{-1}\text{day}^{-1}$
K	Carrying capacity of crop	50	kg
u	Intrinsic growth rate of pest population	0.1	day^{-1}
L	Carrying capacity of pest population	10	pest
θ	Conversion efficiency	0.6	pest kg^{-1}
ϕ	Rate of spraying pesticides	0.1	$\text{ml pest}^{-1}\ \text{kg}^{-1}$
ϕ_0	Natural depletion rate of pesticides	0.01	day^{-1}
ϕ_1	Uptake rate of pesticide by pests	0.05	$\text{pest}^{-1}\ \text{day}^{-1}$
λ	Depletion rate of pests due to pesticides	6	pest ml^{-1}

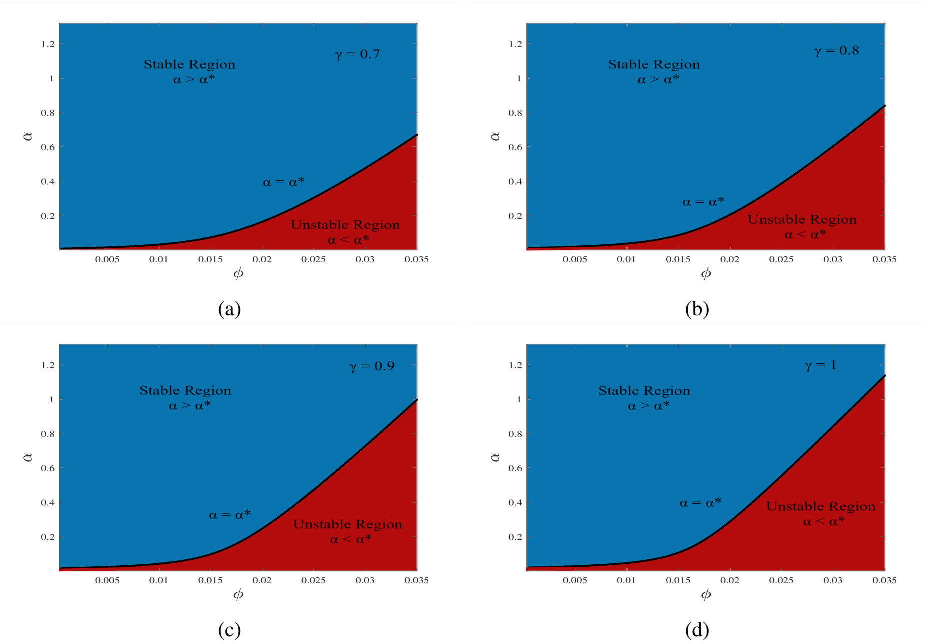

Figure 2. Effect of γ on the stability region of E_2. The remaining parameter values are consistent with those listed in Table 1.

to E_2 also characterizes it as an unstable equilibrium point. It is important to note that for these parameter values, an interior equilibrium exists, confirming the instability of E_2, thereby affirming Theorem 1. In contrast, for the equilibrium E^*, two eigenvalues exhibit negative real parts, while one eigenvalue remains negative. This configuration indicates the local asymptotic stability of E^*. Numerically, Theorem 3 is corroborated for these parameter values.

The effect of γ on the stability of equilibrium E_2 is depicted in Fig. 2 for the parameter values detailed in Table 1. The figure clearly shows that as the order of the derivative

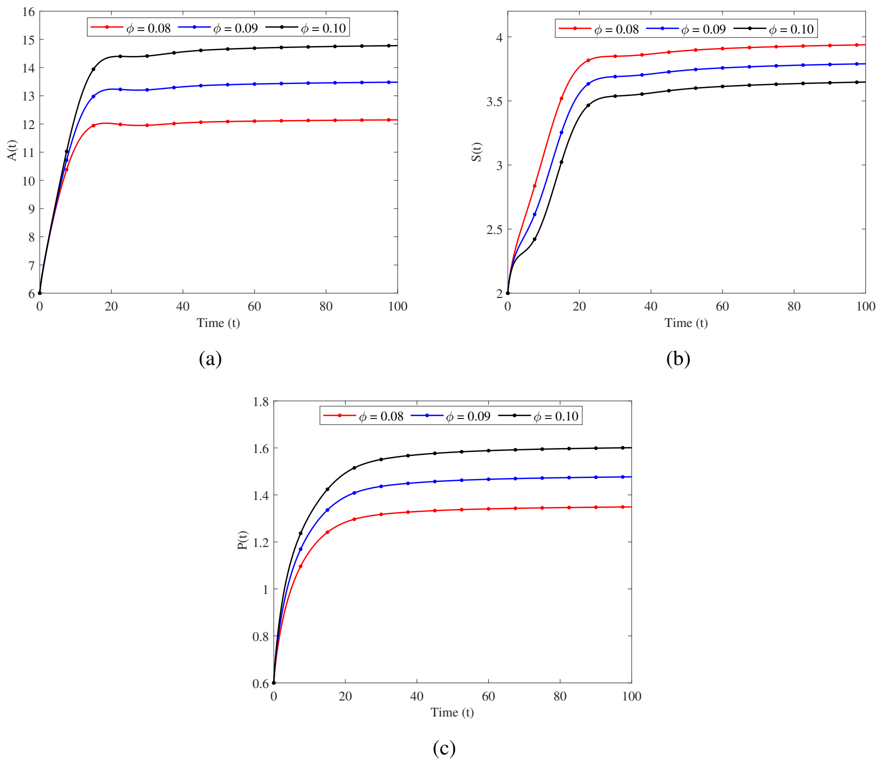

Figure 3. Effect of spraying rate of pesticide on A, S, and P at $\gamma = 0.8$. The remaining parameter values are consistent with those listed in Table 1.

increases, the stability region of equilibrium E_2 decreases. This phenomenon occurs because, as the order of the fractional derivative increases, the system's memory fades as discussed in Naik et al. [20]. We present a variation plot in Fig. 3 to observe the impact of pesticide spraying on agricultural crop yield and pest density across different values of ϕ. In this plot, three distinct values of the parameter ϕ are considered, revealing a trend, where an increase in ϕ results in a reduction in pest density, subsequently leading to increased crop yield. Examining the effects of the pesticide depletion rate, ϕ_0, on both crop yield and pest population density in Fig. 4, a contrasting trend emerges. This visualization illustrates that an increase in the pesticide depletion rate prompts a rise in pest density, ultimately leading to a decrease in crop yield.

Additionally, we present a surface plot illustrating the combined impact of α and λ at $\gamma = 0.8$ on the equilibrium level of crop yield (A^*). As depicted in Fig. 5(a), it is observed that when the parameter α is at its maximum level and λ is at its minimum level, the crop yield reaches its minimum level. However, as the value of λ increases, while the parameter α remains constant, the crop yield shows an upward trend. Conversely, if the parameter λ is held constant and the value of α is increased, the crop yield decreases.

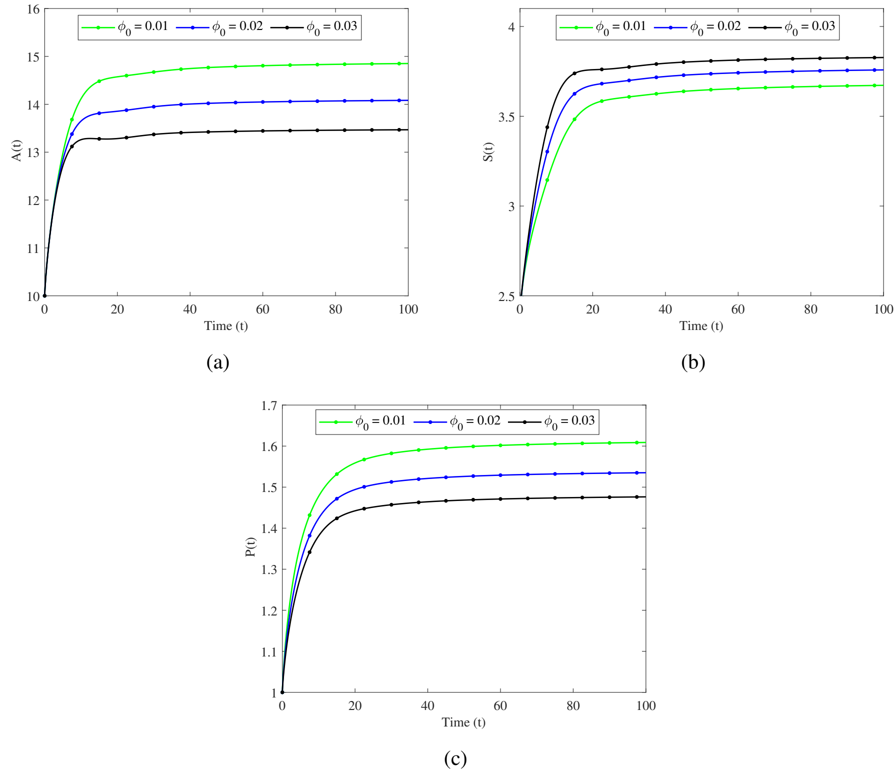

Figure 4. Effect of depletion rate of pesticide on A, S, and P at $\gamma = 0.8$. The remaining parameter values are consistent with those listed in Table 1.

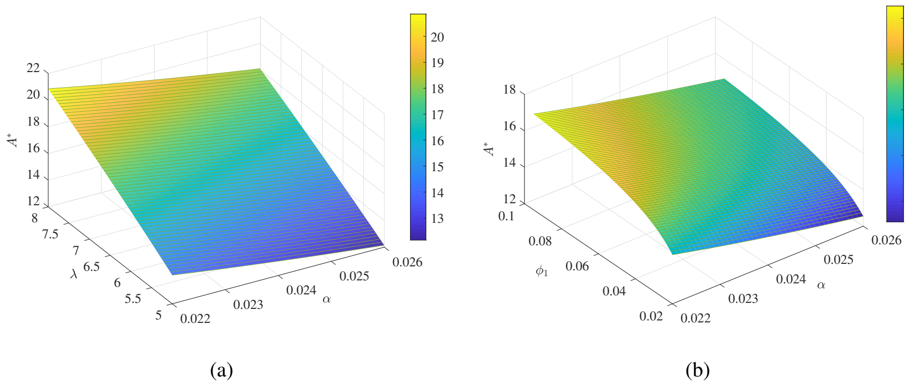

Figure 5. Surface plot illustrating the variation in the equilibrium level of crop yield (A^*) with changes in the values of (a) λ and α and (b) ϕ_1 and α at $\gamma = 0.8$. The remaining parameter values are consistent with those listed in Table 1.

Similar patterns are observed in the surface plot shown in Fig. 5(b), which depicts the combined effect of parameters α and ϕ_1 on crop yield. It is apparent from Figs. 5(a) and 5(b) that to increase crop yield, either pesticides with a high depletion rate or pesticide with high uptake rate must be used.

7.1 Optimal control results

Figures 6(a) and 6(b) display the simulations of pest density when subjected to time-dependent optimal control $v(t)$ as well as when no control strategies are implemented. Here, we have solved the fractional-order Caputo problem using the Adams–Bashforth numerical scheme developed in Section 6. The numerical scheme is applied to these equations in the same way as applied to model system (1). The weight factor selected for minimizing pesticide spraying is taken as $w_1 = 1$ and $w_2 = 1$ when a maximum value of v is set at 1. All other parameter values remain consistent with those presented in Table 1. It is evident from Figs. 6(a) and 6(b) that optimal control is effective to control pest

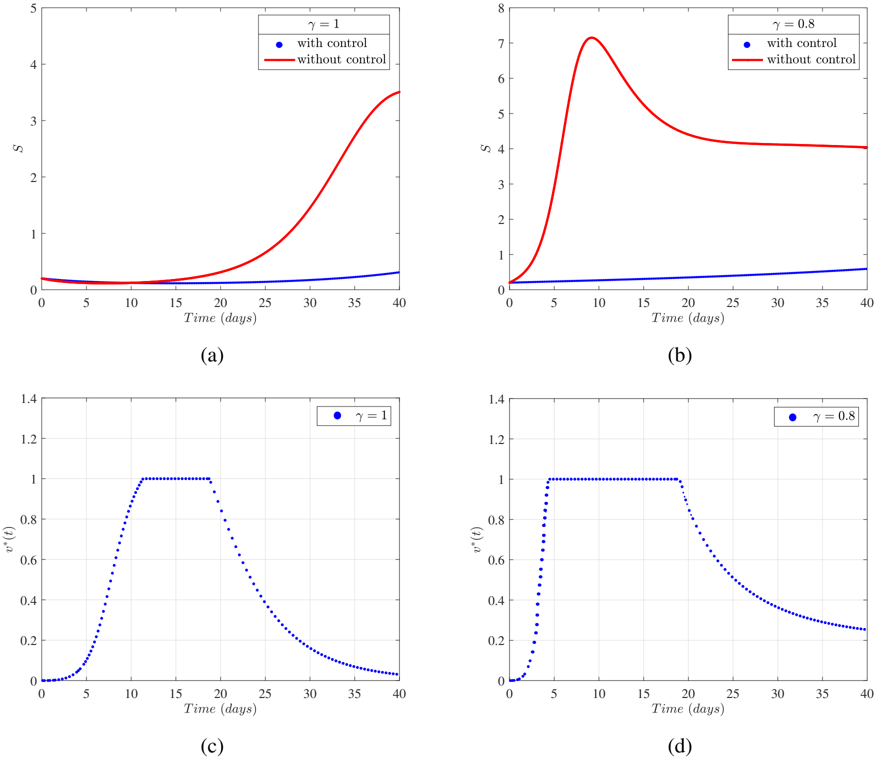

Figure 6. Density of pests with and without control along with the control profile $v(t)$ for $\gamma = 1$ and $\gamma = 0.8$, respectively.

population. Figures 6(c) and 6(d) demonstrate that the most effective strategy for pesticide application entails spraying at maximum rates for 11 to 19 days for $\gamma = 1$, and for 4 to 19 days when $\gamma = 0.8$ after which the application rates decrease. This phenomenon's biological significance is that fractional systems exhibit a memory property. Due to this property, pests might refrain from consuming parts of the crop plant sprayed with insecticides, necessitating long-term and extensive spraying to manage the pest problems effectively.

8 Conclusion

A major concern for farming communities has always been insect outbreaks, which threats crop yield and its quality. Chemical pesticides are the most used insect control treatment because they are cost-effective, time-efficient, and simple to use. This article introduces a three-dimensional nonlinear mathematical model aimed at investigating the impact of pesticides on pest control in agricultural crop fields. The model assumes that farmers employ chemical control methods recognized as potent agents for pest control. The analysis reveals that both the trivial equilibrium and the pest-free equilibrium consistently prove unstable. The crop-free equilibrium, however, is unstable if an interior equilibrium exists. Furthermore, the model analysis underscores the vital role of pesticides in eradicating pests from agricultural fields and enhancing crop yield. Numerical results, supporting the analytical findings, are presented utilizing the Adams–Bashforth criterion. Both the analytical and numerical analysis highlight the significance of ecological parameters in the system, particularly, the crop consumption rate, pesticide abatement rate, and pesticide spraying rate. As in previous studies [16, 30], we have also found that when the crop consumption rate surpasses a threshold, the pest-free equilibrium becomes unstable. This indicates that the crop consumption rate serves as a destabilizing factor even when insects do not entirely depend on crops. To illustrate the long-term effects of these crucial parameters on crop yield and other considered dynamic variables, time series plots for various parameter values are depicted.

Our results from this research complement several recent studies suggesting that pesticide abatement and pesticide uptake rates are crucial parameters to control pest population [17, 18]. In a continuous spray approach, the functions of pesticide abatement and pesticide uptake rates are interchangeable. This means that in order to minimize crop loss, either employ a pesticide with a high abatement rate or use a pesticide with a high uptake rate. Additionally, as farmers desire to adopt a tactic that reduces both the pest population and the expenses incurred in applying pesticides, the constructed model is expanded into an optimal control predicament by incorporating the variable nature of the pesticide spraying rate over time. The optimal time profile for pesticide spraying rate is identified, which effectively minimizes both pest population and the associated costs. This type of study in agricultural modeling, utilizing the CF operator, wherein insects are not entirely dependent on crops, has not been attempted yet. This novelty underscores the significance of our obtained results, offering valuable insights into effective strategies for pesticide management and carrying significant implications for policymakers.

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