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Reckoning applications of $\mathcal{Z}$-iteration: Data dependence and solution to a delay Caputo fractional differential equation

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Abstract. In this study, we focus on demonstrating the stability of the three-step $\mathcal{Z}$-iterative scheme within the context of weak contraction mappings as defined by Berinde. Further, we attain results concerning stability, data dependence, and error accumulation of the $\mathcal{Z}$-iterative scheme. This article also includes a comparison of the convergence rates among various established iterative strategies. Several illustrative numerical examples are furnished to validate the accuracy and reliability of our findings. In the same spirit, we present an application that utilises the $\mathcal{Z}$-iterative technique on Banach spaces to attain the solution of a delay Caputo fractional differential equation, building upon our primary findings.

Keywords: weak contractions, stability, data dependency, error estimation, delay fractional differential equations.

1 Introduction and preliminaries

The successive approximation of fixed points or common fixed points plays an instrumental part in achieving solutions to many real-life problems arising in numerous research domains. Fixed point iterative methods, which are key to such approximation, have a substantial amount of development in the literature, which got going with the Picard iterative scheme. Owing to the fact that Picard iteration fails to converge to the fixed points of nonexpansive mappings, Mann [17] proposed a novel iteration technique, and thereafter, Ishikawa [15], Noor [18], Agarwal et al. [1], and many researchers have studied different features of many such techniques in the setting of various structures; see [2, 10, 12, 26, 28].

However, one can comprehensibly perceive that the efficacy of any iterative algorithm relies on certain aspects, namely, the stability and the rate of convergence. Quite obviously, the investigation of these attributes corresponding to an iterative scheme has attracted a lot of researchers as evidenced by various literary sources; see [2, 5, 11, 13, 14, 21]. In this paper, we look into the preceding features for $\mathcal{Z}$-iterative scheme proposed by Zaheer et al. [30] and defined as

$$\begin{aligned} &x_0 \in \mathscr{X}, \quad x_{n+1} = \mathcal{G} z_n, \\ &y_n = \mathcal{G}\big((1-\beta_n)x_n + \beta_n \mathcal{G} x_n\big), \\ &z_n = (1-\alpha_n)y_n + \alpha_n \mathcal{G} y_n. \end{aligned} \tag{1}$$

Besides, we compare the rate of convergence of $\mathcal{Z}$-iterative scheme (1) to that of other well-known iterative schemes, namely, Noor and Thakur iteration.

On the other hand, in recent years, many researchers have made contributions to data dependence results via any iteration method and also, approximated the error accumulation concerning the iteration. Readers are referred to [3, 4, 9, 14, 23–27] and references therein for a detailed study on the applicative viewpoint of various iterative techniques and related notions. Following the direction, in this sequel, we come by a data dependence result and an error estimation result pertaining to the newly introduced $\mathcal{Z}$-iterative scheme.

Berinde [6] introduced the notion of weak contractions, often known as almost-contraction maps, in 2003. In his paper, he demonstrated that the collection of weak contraction mappings is more general than the classes of contraction mappings and Zamfirescu mappings, and he presented existence and uniqueness results concerning fixed points of weak contraction mappings.

Definition 1. Let $\mathscr{X}$ be a Banach space and consider a self-mapping $\mathcal{G}$ on $\mathscr{X}$. Then $\mathcal{G}$ is called a weak contraction if there exists a constant $\mu \in (0,1)$ and a nonnegative constant L so that the following hold for all $x, y \in \mathscr{X}$:

$$\|\mathcal{G}x - \mathcal{G}y\| \leqslant \mu\|x-y\| + L\|y - \mathcal{G}x\|. \tag{2}$$

Further, the author also established the following result in the context of aforementioned maps.

Theorem 1. (*See* [6].) *Let $\mathscr{X}$ be a Banach space. Consider a self-mapping $\mathcal{G}$ on $\mathscr{X}$ satisfying* (2) *together with the inequality*

$$\|\mathcal{G}x - \mathcal{G}y\| \leqslant \mu\|x-y\| + L\|x - \mathcal{G}x\| \tag{3}$$

for all $x, y \in \mathscr{X}$. Then $\mathcal{G}$ owns a unique fixed point in $\mathscr{X}$.

The notion of stability related to a fixed point technique was originally brought forward by Harder and Hicks [13], which is as follows.

Definition 2. (See [20].) Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be a nonempty, closed, and convex subset of $\mathscr{X}$. Define a self-map $\mathcal{G} : \mathscr{C} \to \mathscr{C}$ and let (x_n) be a sequence formulated by an iterative scheme $x_{n+1} = \mathcal{F}(\mathcal{G}, x_n)$ converging to a fixed point ξ for any function $\mathcal{F}$. Suppose that (u_n) is any arbitrary sequence of (x_n) and also define a sequence (ϵ_n) given by $\epsilon_n = \|u_{n+1} - \mathcal{F}(\mathcal{G}, u_n)\|$. Then (x_n) will be called $\mathcal{G}$-stable if

$$\lim_{n\to\infty} \epsilon_n = 0 \iff \lim_{n\to\infty} u_n = \xi.$$

In 1990, Rhoades [22] extended Harder's [13] work and introduced the stability result for Picard and Mann iterative systems. Following that, Osilike [19] established the concept of weak stability, often known as almost-stability of iterative schemes, which is a weaker notion of stability than the one due to [20], and is discussed below.

Definition 3. (See [19].) Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be a nonempty, closed, and convex subset of $\mathscr{X}$. Define a self-map $\mathcal{G} : \mathscr{C} \to \mathscr{C}$ and let (x_n) be a sequence generated by an iterative scheme $x_{n+1} = \mathcal{F}(\mathcal{G}, x_n)$ converging to a fixed point ξ for any function $\mathcal{F}$. Suppose that (u_n) is an approximate sequence of (x_n) and also define a sequence (ϵ_n) given by $\epsilon_n = \|u_{n+1} - \mathcal{F}(\mathcal{G}, u_n)\|$. Then (x_n) will be called almost $\mathcal{G}$-stable if

$$\sum_{n=0}^{\infty} \epsilon_n < \infty \implies \lim_{n\to\infty} u_n = \xi.$$

Now we define an approximate operator corresponding to a nonexpansive mapping.

Definition 4. (See [6].) Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be a nonempty, closed, and convex subset of $\mathscr{X}$ and let $\mathcal{G}, \bar{\mathcal{G}} : \mathscr{C} \to \mathscr{C}$ be two mappings. Then $\bar{\mathcal{G}}$ is said to be an approximate operator for $\mathcal{G}$ if for some $\epsilon > 0$ and for each $x \in \mathscr{C}$, we have $\|\mathcal{G}x - \bar{\mathcal{G}}x\| \leqslant \epsilon$.

The succeeding lemma, originally conceived by Şoltuz and Grosan [27], is playing a vital role in this sequel.

Lemma 1. *Let (u_n), (v_n) be two nonnegative sequences of real numbers satisfying*

$$u_{n+1} \leqslant q u_n + v_n$$

for all $n \in \mathbb{N}$, $q \in [0, 1)$. If $\lim_{n\to\infty} v_n = 0$, then $\lim_{n\to\infty} u_n = 0$.

Now we note down the following definition, which is related to the comparison of two iterative process.

Definition 5. (See [21].) Let (τ_n) and (θ_n) be two sequences generated by two iterative algorithms converging to the same fixed point ξ. Then (τ_n) converges faster than (θ_n) if

$$\lim_{n\to\infty} \frac{\|\tau_n - \xi\|}{\|\theta_n - \xi\|} = 0.$$

Our paper is divided into six distinct sections. Section 2 demonstrates convergence and stability results. Then Section 3 provides comparison results of the $\mathcal{Z}$-iterative scheme with a few well-known iterative methods. The data dependence result is presented in Section 4. However, Section 5 provides a numerical illustration of the applicability of the results obtained in the previous sections. In Section 6, we bring out the error estimation of the $\mathcal{Z}$-iterative approach. Finally, in Section 7, we solve a particular type of delay Caputo fractional differential equation employing the $\mathcal{Z}$-iterative technique.

2 $\mathcal{Z}$-iterative scheme and convergence analysis

The purpose of this section is to affirm a few convergence and stability results involving Berinde weak contractions using $\mathcal{Z}$-iterative scheme (1). Also, we establish that the aforementioned iteration has at least first order of convergence.

Theorem 2. *Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be a nonempty, closed, and convex subset of the setting. Also, suppose that $\mathcal{G} : \mathscr{C} \to \mathscr{C}$ be a Berinde weak contraction satisfying* (3). *Let (x_n) be an iterative sequence generated by $\mathcal{Z}$-iterative scheme* (1)*, with two sequences of real numbers (α_n) and (β_n) in $(0,1)$ satisfying one of the subsequent assumptions, $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n = \infty$ and $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$. Then (x_n) converges strongly to a unique fixed point ξ of $\mathcal{G}$.*

Proof. Using $\mathcal{Z}$-iterative scheme (1), we have

$$\begin{aligned}
\|z_n - \xi\| &= \big\|(1-\alpha_n)y_n + \alpha_n\mathcal{G}y_n - \xi\big\| \\
&\leqslant (1-\alpha_n)\|y_n - \xi\| + \alpha_n\|\mathcal{G}y_n - \mathcal{G}\xi\| \\
&\leqslant (1-\alpha_n)\|y_n - \xi\| + \alpha_n\mu\|y_n - \xi\| \\
&\leqslant (1-\alpha_n+\alpha_n\mu)\|y_n - \xi\| = (1-\alpha_n(1-\mu))\|y_n - \xi\|.
\end{aligned}$$

Now,

$$\begin{aligned}
\|y_n - \xi\| &= \big\|\mathcal{G}\big((1-\beta_n)x_n + \beta_n\mathcal{G}x_n\big) - \xi\big\| \\
&\leqslant \mu\big\|(1-\beta_n)x_n + \beta_n\mathcal{G}x_n - \xi\big\| \\
&\leqslant \mu(1-\beta_n)\|x_n - \xi\| + \mu^2\beta_n\|x_n - \xi\| \\
&\leqslant \big(\mu - \mu\beta_n + \mu^2\beta_n\big)\|x_n - \xi\| = \mu\big(1-\beta_n(1-\mu)\big)\|x_n - \xi\|.
\end{aligned}$$

Then

$$\begin{aligned}
\|x_{n+1} - \xi\| &= \|\mathcal{G}z_n - \xi\| \leqslant \mu\|z_n - \xi\| \leqslant \mu\big(1-\alpha_n(1-\mu)\big)\|y_n - \xi\| \\
&\leqslant \mu\big(1-\alpha_n(1-\mu)\big)\mu\big(1-\beta_n(1-\mu)\big)\|x_n - \xi\| \\
&= \mu^2\big(1-\big(\alpha_n+\beta_n-\alpha_n\beta_n(1-\mu)\big)(1-\mu)\big)\|x_n - \xi\|.
\end{aligned}$$

Therefore,

$$\|x_{n+1} - \xi\| \leqslant \mu^2\big(1-\big(\alpha_n+\beta_n-\alpha_n\beta_n(1-\mu))(1-\mu)\big)\|x_n - \xi\|.$$

Repeating the above steps, we obtain

$$\begin{aligned}
\|x_n-\xi\| &\leqslant \mu^2\bigl(1-\bigl(\alpha_{n-1}+\beta_{n-1}-\alpha_{n-1}\beta_{n-1}(1-\mu)\bigr)(1-\mu)\bigr) \\
&\quad \times \|x_{n-1}-\xi\| \\
&\implies \\
\|x_{n-1}-\xi\| &\leqslant \mu^2\bigl(1-\bigl(\alpha_{n-2}+\beta_{n-2}-\alpha_{n-2}\beta_{n-2}(1-\mu)\bigr)(1-\mu)\bigr) \\
&\quad \times \|x_{n-2}-\xi\| \\
&\implies \quad \cdots \quad \implies \\
\|x_1-\xi\| &\leqslant \mu^2\bigl(1-\bigl(\alpha_0+\beta_0-\alpha_0\beta_0(1-\mu)\bigr)(1-\mu)\bigr)\|x_0-\xi\|.
\end{aligned}$$

Therefore, inductively we obtain

$$\|x_{n+1}-\xi\| \leqslant \mu^{2(n+1)}\prod_{i=0}^{n}\bigl(1-\bigl(\alpha_i+\beta_i-\alpha_i\beta_i(1-\mu)\bigr)(1-\mu)\bigr)\|x_0-\xi\|. \tag{4}$$

We know that $1-x\leqslant \mathrm{e}^{-x}$, then the above equation can be written as

$$\|x_{n+1}-\xi\| \leqslant \mu^{2(n+1)}\mathrm{e}^{-(1-\mu)\sum_{i=0}^{\infty}(\alpha_i+\beta_i-\alpha_i\beta_i(1-\mu))}\|x_0-\xi\|.$$

Using one of the assumptions, $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\sum_{n=0}^{\infty}\beta_n=\infty$ and $\sum_{n=0}^{\infty}\alpha_n\beta_n=\infty$ and letting $n\to\infty$, we get

$$\lim_{n\to\infty}\|x_n-\xi\|=0.$$

As $\mu^2<\mu<1$, therefore (x_n) converges strongly to ξ. □

Theorem 3. *Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be a nonempty, closed, and convex subset of the setting. Also, assume that $\mathcal{G}:\mathscr{C}\to\mathscr{C}$ is a Berinde weak contraction mapping satisfying* (3) *with a fixed point ξ and (x_n) be the iterative sequence generated by $\mathcal{Z}$-iterative scheme* (1)*, where (α_n) and (β_n) are sequences of real numbers in $(0,1)$. Let $(q_n)\subseteq\mathscr{X}$ be any sequence and define a sequence (ϵ_n) in $\mathbb{R}$ as $\epsilon_n=\|q_{n+1}-\mathcal{F}(\mathcal{G},q_n)\|$, $u_n=(1-\alpha_n)v_n+\alpha_n\mathcal{G}v_n$, $v_n=\mathcal{G}((1-\beta_n)q_n+\beta_n\mathcal{G}q_n)$. Then $\mathcal{Z}$-iteration* (1)*, is almost $\mathcal{G}$-stable.*

Proof. Let (q_n) be any sequence, and we aim to affirm that (q_n) is almost stable with respect to $\mathcal{G}$. Suppose that $\sum_{n=0}^{\infty}\epsilon_n<\infty$. Then by using $\mathcal{Z}$-iterative scheme (1) we have

$$\begin{aligned}
\|q_{n+1}-\xi\| &\leqslant \bigl\|q_{n+1}-\mathcal{F}(\mathcal{G},q_n)\bigr\|+\bigl\|\mathcal{F}(\mathcal{G},q_n)-\xi\bigr\| \\
&= \epsilon_n+\|\mathcal{G}u_n-\xi\| \leqslant \epsilon_n+\mu\|u_n-\xi\|.
\end{aligned} \tag{5}$$

Again,

$$\begin{aligned}
\|u_n-\xi\| &= \bigl\|(1-\alpha_n)v_n+\alpha_n\mathcal{G}v_n-\xi\bigr\| \\
&\leqslant (1-\alpha_n)\|v_n-\xi\|+\alpha_n\|\mathcal{G}v_n-\xi\| \\
&\leqslant (1-\alpha_n)\|v_n-\xi\|+\alpha_n\mu\|v_n-\xi\| \\
&= \bigl(1-\alpha_n(1-\mu)\bigr)\|v_n-\xi\|.
\end{aligned} \tag{6}$$

Now,

$$\begin{aligned}\|v_n-\xi\| &= \big\|\mathcal{G}\big((1-\beta_n)q_n+\beta_n\mathcal{G}q_n\big)-\xi\big\| \\ &\leqslant \mu\big\|\big((1-\beta_n)q_n+\beta_n\mathcal{G}q_n\big)-\xi\big\| \\ &= \mu\big\|(1-\beta_n)(q_n-\xi)+\beta_n(\mathcal{G}q_n-\xi)\big\| \\ &\leqslant \mu(1-\beta_n)\|q_n-\xi\|+\beta_n\mu\|\mathcal{G}q_n-\xi\| \\ &\leqslant \mu(1-\beta_n)\|q_n-\xi\|+\beta_n\mu^2\|q_n-\xi\| \\ &= \mu\big(1-\beta_n(1-\mu)\big)\|q_n-\xi\|. \end{aligned} \tag{7}$$

Using (6), (7) in equation (5) we have

$$\begin{aligned}\|q_{n+1}-\xi\| &\leqslant \epsilon_n+\mu\big(1-\alpha_n(1-\mu)\big)\|v_n-\xi\| \\ &\leqslant \epsilon_n+\mu\big(1-\alpha_n(1-\mu)\big)\mu\big(1-\beta_n(1-\mu)\big)\|q_n-\xi\| \\ &= \epsilon_n+\mu^2\big(1-\alpha_n(1-\mu)\big)\big(1-\beta_n(1-\mu)\big)\|q_n-\xi\|. \end{aligned} \tag{8}$$

Now, from (8) we know, $0<(1-\alpha_n(1-\mu))\leqslant 1$ and $0<(1-\beta_n(1-\mu))\leqslant 1$, and using this fact, we have

$$\|q_{n+1}-\xi\|\leqslant \mu^2\|q_n-\xi\|+\epsilon_n.$$

Using Lemma 1, one can conclude that if $\lim_{n\to\infty}\epsilon_n=0$, then $\lim_{n\to\infty}\|q_n-\xi\|=0$, which shows that (q_n) converges to ξ, that is, $\lim_{n\to\infty}q_n=\xi$. Therefore, $\mathcal{Z}$-iterative scheme (1) is almost $\mathcal{G}$-stable. □

Theorem 4. *Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be a nonempty, closed, and convex subset of the setting. Assume that $\mathcal{G}:\mathscr{C}\to\mathscr{C}$ is a self-mapping on $\mathscr{X}$ with a fixed point ξ. Let (x_n) be defined by the $\mathcal{Z}$-iterative scheme* (1)*. Then the $\mathcal{Z}$-iterative scheme has at least first order of convergence.*

Proof. For any sequence (x_n), we denote $e_{x_n}=x_n-\xi$. By the Taylor series expansion about ξ, we obtain

$$\begin{aligned}\mathcal{G}x_n &= \mathcal{G}\xi+(\mathcal{G}'\xi)(x_n-\xi)+(\mathcal{G}''\xi)\frac{(x_n-\xi)^2}{2!}+O\big(e_{x_n}^3\big) \\ &= \xi+(\mathcal{G}'\xi)e_{x_n}+\frac{(\mathcal{G}''\xi)}{2!}e_{x_n}^2+O\big(e_{x_n}^3\big).\end{aligned}$$

Using this expansion, we have

$$\begin{aligned}e_{y_n} &= y_n-\xi=\mathcal{G}\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\xi \\ &= \mathcal{G}\xi+(\mathcal{G}'\xi)\big(\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\xi\big)+\frac{\mathcal{G}''\xi}{2!}\big(\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\xi\big)^2 \\ &\quad +O\big(e_{x_n}^3\big)-\xi\end{aligned}$$

$$
\begin{aligned}
&= (\mathcal{G}'\xi)\big[(1-\beta_n)(x_n-\xi)+\beta_n(\mathcal{G}x_n-\xi)\big] \\
&\quad + \frac{\mathcal{G}''\xi}{2!}\big[(1-\beta_n)(x_n-\xi)+\beta_n(\mathcal{G}x_n-\xi)\big]^2 + O\big(e_{x_n}^3\big) \\
&= (\mathcal{G}'\xi)(1-\beta_n)e_{x_n} + (\mathcal{G}'\xi)\beta_n(\mathcal{G}x_n-\xi) \\
&\quad + \frac{\mathcal{G}''\xi}{2!}\big[(1-\beta_n)^2e_{x_n}^2 + \beta_n^2(\mathcal{G}x_n-\xi)^2 + 2(1-\beta_n)\beta_n e_{x_n}(\mathcal{G}x_n-\xi)\big] \\
&\quad + O\big(e_{x_n}^3\big) \\
&= (\mathcal{G}'\xi)(1-\beta_n)e_{x_n} + (\mathcal{G}'\xi)\beta_n\bigg[(\mathcal{G}'\xi)e_{x_n} + \frac{(\mathcal{G}''\xi)}{2!}e_{x_n}^2 + O\big(e_{x_n}^3\big)\bigg] \\
&\quad + \frac{\mathcal{G}''\xi}{2!}\big[(1-\beta_n)^2e_{x_n}^2\big] + \frac{\mathcal{G}''\xi}{2!}\beta_n^2(\mathcal{G}x_n-\xi)^2 \\
&\quad + \frac{\mathcal{G}''\xi}{2!}\big[2(1-\beta_n)\beta_n e_{x_n}(\mathcal{G}x_n-\xi)\big] + O\big(e_{x_n}^3\big) \\
&= (\mathcal{G}'\xi)(1-\beta_n)e_{x_n} + (\mathcal{G}'\xi)\beta_n(\mathcal{G}'\xi)e_{x_n} + (\mathcal{G}'\xi)\beta_n\frac{(\mathcal{G}''\xi)}{2!}e_{x_n}^2 + O\big(e_{x_n}^3\big) \\
&\quad + \frac{\mathcal{G}''\xi}{2!}(1-\beta_n)^2e_{x_n}^2 + \frac{\mathcal{G}''\xi}{2!}\beta_n^2(\mathcal{G}x_n-\xi)^2 \\
&\quad + \frac{\mathcal{G}''\xi}{2!}\big(2(1-\beta_n)\beta_n e_{x_n}(\mathcal{G}x_n-\xi)\big) + O\big(e_{x_n}^3\big) \\
&= (\mathcal{G}'\xi)(1-\beta_n)e_{x_n} + (\mathcal{G}'\xi)^2\beta_n e_{x_n} + \frac{(\mathcal{G}'\xi)(\mathcal{G}''\xi)}{2!}\beta_n e_{x_n}^2 + \frac{\mathcal{G}''\xi}{2!}(1-\beta_n)^2e_{x_n}^2 \\
&\quad + \frac{\mathcal{G}''\xi}{2!}(\mathcal{G}'\xi)^2\beta_n^2e_{x_n}^2 + \frac{\mathcal{G}''\xi}{2!}\big(2(1-\beta_n)\beta_n e_{x_n}^2(\mathcal{G}'\xi)\big) + O\big(e_{x_n}^3\big) \\
&= \big[(\mathcal{G}'\xi)(1-\beta_n) + \beta_n(\mathcal{G}'\xi)^2\big]e_{x_n} + \bigg[\frac{(\mathcal{G}'\xi)(\mathcal{G}''\xi)}{2!}\beta_n + \frac{\mathcal{G}''\xi}{2!}(1-\beta_n)^2 \\
&\quad + \frac{\mathcal{G}''\xi}{2!}(\mathscr{G}'\xi)^2\beta_n^2 + \frac{\mathcal{G}''\xi}{2!}\cdot 2(1-\beta_n)\beta_n(\mathcal{G}'\xi)\bigg]e_{x_n}^2 + O\big(e_{x_n}^3\big).
\end{aligned}
$$

Let

$$A = \big[(\mathcal{G}'\xi)(1-\beta_n) + \beta_n(\mathcal{G}'\xi)^2\big]$$

and

$$B = \bigg[\frac{(\mathcal{G}'\xi)(\mathcal{G}''\xi)}{2!}\beta_n + \frac{\mathcal{G}''\xi}{2!}(1-\beta_n)^2 + \frac{\mathcal{G}''\xi}{2!}(\mathcal{G}'\xi)^2\beta_n^2 + \frac{\mathcal{G}''\xi}{2!}2(1-\beta_n)\beta_n(\mathcal{G}'\xi)\bigg].$$

Then

$$e_{y_n} = Ae_{x_n} + Be_{x_n}^2 + O\big(e_{x_n}^3\big). \tag{9}$$

Also, we have

$$
\begin{aligned}
\mathcal{G}y_n &= \mathcal{G}\xi + (\mathcal{G}'\xi)(y_n-\xi) + (\mathcal{G}''\xi)\frac{(y_n-\xi)^2}{2!} + O\big(e_{y_n}^3\big) \\
&= \xi + (\mathcal{G}'\xi)e_{y_n} + \frac{(\mathcal{G}''\xi)}{2!}e_{y_n}^2 + O\big(e_{y_n}^3\big).
\end{aligned}
$$

Now,

$$
\begin{aligned}
e_{z_n} &= z_n - \xi = \big((1-\alpha_n)y_n + \alpha_n \mathcal{G} y_n\big) - \xi \\
&= (1-\alpha_n)(y_n - \xi) + \alpha_n(\mathcal{G} y_n - \xi) \\
&= (1-\alpha_n)(y_n - \xi) \\
&\quad + \alpha_n\left(\mathcal{G}\xi + (\mathcal{G}'\xi)(y_n - \xi) + \frac{(\mathcal{G}''\xi)}{2!}(y_n - \xi)^2 + O(e_{y_n}^3) - \xi\right) \\
&= (1-\alpha_n)e_{y_n} + \alpha_n\left((\mathcal{G}'\xi)e_{y_n} + \frac{(\mathcal{G}''\xi)}{2!}e_{y_n}^2 + O\big(e_{y_n}^3\big)\right) \\
&= (1-\alpha_n + \alpha_n(\mathcal{G}'\xi))e_{y_n} + \alpha_n \frac{(\mathcal{G}''\xi)}{2!}e_{y_n}^2 + O\big(e_{y_n}^3\big)
\end{aligned}
$$

and

$$
\begin{aligned}
e_{x_{n+1}} &= x_{n+1} - \xi = \mathcal{G} z_n - \xi \\
&= \mathcal{G}\xi + (\mathcal{G}'\xi)(z_n - \xi) + \frac{(\mathcal{G}''\xi)}{2!}(z_n - \xi)^2 + O\big(e_{z_n}^3\big) - \xi \\
&= (\mathcal{G}'\xi)e_{z_n} + \frac{(\mathcal{G}''\xi)}{2!}e_{z_n}^2 + O\big(e_{z_n}^3\big) \\
&= (\mathcal{G}'\xi)\left[\big(1-\alpha_n + \alpha_n(\mathcal{G}'\xi)\big)e_{y_n} + \alpha_n \frac{(\mathcal{G}''\xi)}{2!}e_{y_n}^2 + O\big(e_{y_n}^3\big)\right] \\
&\quad + \frac{(\mathcal{G}''\xi)}{2!}\left[\big(1-\alpha_n + \alpha_n(\mathcal{G}'\xi)\big)e_{y_n} + \alpha_n \frac{(\mathcal{G}''\xi)}{2!}e_{y_n}^2 + O\big(e_{y_n}^3\big)\right]^2 + O\big(e_{y_n}^3\big) \\
&= \big[(\mathcal{G}'\xi)\big(1-\alpha_n + \alpha_n(\mathcal{G}'\xi)\big)\big]e_{y_n} \\
&\quad + \left[\alpha_n(\mathcal{G}'\xi)\frac{(\mathcal{G}''\xi)}{2!} + \frac{(\mathcal{G}''\xi)}{2!}\big(1-\alpha_n + \alpha_n(\mathcal{G}'\xi)\big)^2\right]e_{y_n}^2 + O\big(e_{y_n}^3\big).
\end{aligned}
$$

Let

$$C = \big[(\mathcal{G}'\xi)\big(1-\alpha_n + \alpha_n(\mathcal{G}'\xi)\big)\big]$$

and

$$D = \left[\alpha_n(\mathcal{G}'\xi)\frac{(\mathcal{G}''\xi)}{2!} + \frac{(\mathcal{G}''\xi)}{2!}\big(1-\alpha_n + \alpha_n(\mathcal{G}'\xi)\big)^2\right].$$

Then

$$e_{x_{n+1}} = Ce_{y_n} + De_{y_n}^2 + O\big(e_{y_n}^3\big).$$

Now using (9) we have

$$
\begin{aligned}
e_{x_{n+1}} &= C\big[Ae_{x_n} + Be_{x_n}^2 + O\big(e_{x_n}^3\big)\big] + D\big[Ae_{x_n} + Be_{x_n}^2 + O\big(e_{x_n}^3\big)\big]^2 + O\big(e_{x_n}^3\big) \\
&= CAe_{x_n} + (BC + DA^2)e_{x_n}^2 + O\big(e_{x_n}^3\big) = A'e_{x_n} + B'e_{x_n}^2 + O\big(e_{x_n}^3\big).
\end{aligned}
$$

Here $A' = CA$ and $B' = (BC + DA^2)$. This implies that $\mathcal{Z}$-iterative scheme (1) has at least first order of convergence. □

3 Comparison with various iteration schemes

In this section, we compare the convergence rate of $\mathcal{Z}$-iterative scheme (1) with that of Noor iterative scheme [18] and Thakur iterative scheme [28]. Consider $\mathscr{X}$ be a Banach space and $\mathcal{G}$ be a self-map on a nonempty, closed, and convex subset $\mathscr{C}$ of the setting and (α_n), (β_n) and (γ_n) in $(0,1)$ be sequences satisfying certain conditions. For arbitrarily chosen $w_0 \in \mathscr{C}$, the Noor iterative scheme and Thakur iterative scheme are defined as

$$\begin{aligned} &w_0 \in \mathscr{X}, \quad w_{n+1} = (1-\alpha_n)w_n + \alpha_n \mathcal{G} v_n, \\ &v_n = (1-\beta_n)w_n + \beta_n \mathcal{G} u_n, \\ &u_n = (1-\gamma_n)w_n + \gamma_n \mathcal{G} w_n \end{aligned} \tag{10}$$

and

$$\begin{aligned} &w_0 \in \mathscr{X}, \quad w_{n+1} = (1-\alpha_n)\mathcal{G} w_n + \alpha_n \mathcal{G} v_n, \\ &v_n = (1-\beta_n)u_n + \beta_n \mathcal{G} u_n, \\ &u_n = (1-\gamma_n)w_n + \gamma_n \mathcal{G} w_n, \end{aligned} \tag{11}$$

respectively.

Theorem 5. *Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be any nonempty, closed, and convex subset of the setting. Also, suppose that $\mathcal{G}$ is a Berinde weak contraction on $\mathscr{C}$ satisfying* (3). *Let ξ be a fixed point of $\mathcal{G}$ and (x_n), (w_n), and (r_n) be the iterative sequences generated by $\mathcal{Z}$-iterative scheme* (1)*, Noor iterative scheme* (10)*, and Thakur iterative scheme* (11)*, respectively, and converge to a fixed point ξ with real sequences (α_n), (β_n), and (γ_n) in an interval $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, and $\lim_{n\to\infty} \gamma_n = 0$. Then the $\mathcal{Z}$-iterative method* (1) *converges faster than the Noor iteration* (10) *and Thakur iterative scheme* (11) *given that the initial point is same for all the schemes.*

Proof. From (4) we have

$$\|x_{n+1} - \xi\| \leqslant \mu^{2(n+1)} \prod_{i=0}^{n} \bigl[1 - \bigl(\alpha_i + \beta_i - \alpha_i\beta_i(1-\mu)\bigr)(1-\mu)\bigr] \|x_0 - \xi\|. \tag{12}$$

By some similar calculations as in Theorem 2, we obtain the following estimates for Noor iterative scheme (10):

$$\begin{aligned} &\|u_n - \xi\| \\ &\quad = \bigl\|(1-\gamma_n)w_n + \gamma_n \mathcal{G} w_n - \xi\bigr\| \leqslant (1-\gamma_n)\|w_n - \xi\| + \gamma_n \|\mathcal{G} w_n - \xi\| \\ &\quad \leqslant (1-\gamma_n)\|w_n - \xi\| + \gamma_n \mu \|w_n - \xi\| = \bigl[1 - \gamma_n(1-\mu)\bigr]\|w_n - \xi\|, \\ &\|v_n - \xi\| \\ &\quad = \bigl\|(1-\beta_n)w_n + \beta_n \mathcal{G} u_n - \xi\bigr\| \leqslant (1-\beta_n)\|w_n - \xi\| + \beta_n \|\mathcal{G} u_n - \xi\| \\ &\quad \leqslant (1-\beta_n)\|w_n - \xi\| + \beta_n \mu \|u_n - \xi\| \\ &\quad \leqslant (1-\beta_n)\|w_n - \xi\| + \beta_n \mu\bigl[1 - \gamma_n(1-\mu)\bigr]\|w_n - \xi\| \\ &\quad = \bigl[1 - \beta_n\bigl(1 - \mu\bigl(1 - \gamma_n(1-\mu)\bigr)\bigr)\bigr]\|w_n - \xi\|, \end{aligned}$$

and

$$\begin{aligned}
&\|w_{n+1}-\xi\| \\
&\quad = \big\|(1-\alpha_n)w_n+\alpha_n\mathcal{G}v_n-\xi\big\| = \big\|(1-\alpha_n)w_n+\alpha_n\xi-\alpha_n\xi+\alpha_n\mathcal{G}v_n-\xi\big\| \\
&\quad = \big\|(1-\alpha_n)(w_n-\xi)+\alpha_n(\mathcal{G}v_n-\xi)\big\| \geqslant \big\|(1-\alpha_n)(w_n-\xi)-\alpha_n(\mathcal{G}v_n-\xi)\big\| \\
&\quad \geqslant \big|(1-\alpha_n)\|w_n-\xi\|-\alpha_n\mu\|v_n-\xi\|\big| \\
&\quad \geqslant \big|(1-\alpha_n)\|w_n-\xi\|-\alpha_n\mu\big(1-\beta_n\big(1-\mu\big(1-\gamma_n(1-\mu)\big)\big)\big)\|w_n-\xi\|\big| \\
&\quad = \big|\big(1-\alpha_n\big(1+\mu\big(1-\beta_n\big(1-\mu\big(1-\gamma_n(1-\mu)\big)\big)\big)\big)\big)\big|\|w_n-\xi\| \\
&\quad = \prod_{i=0}^{n}\big[1-\alpha_i\big(1+\mu\big(1-\beta_i\big(1-\mu\big(1-\gamma_i(1-\mu)\big)\big)\big)\big)\big]\|w_0-\xi\|.
\end{aligned}$$

Therefore, for Noor iterative scheme, we get the following estimate:

$$\|w_{n+1}-\xi\| \geqslant \prod_{i=0}^{n}\big[1-\alpha_i\big(1+\mu\big(1-\beta_i\big(1-\mu\big(1-\gamma_i(1-\mu)\big)\big)\big)\big)\big]\|w_0-\xi\|. \tag{13}$$

Following similar steps to that of $\mathcal{Z}$-iterative scheme (1) and Noor iterative scheme (10), one can find out an estimate for Thakur iterative scheme (11) as below:

$$\begin{aligned}
\|r_{n+1}-\xi\| \geqslant \mu^{(n+1)}\prod_{i=0}^{n}&\big[1-\alpha_i\big(1+\alpha_i\big(1-\beta_i(1-\mu)\big)\big(1-\gamma_i(1-\mu)\big)\big)\big] \\
&\times\|r_0-\xi\|.
\end{aligned} \tag{14}$$

Now using (12), (13), (14), and the assumption $x_0=w_0=r_0$, we have

$$0 \leqslant \frac{\|x_{n+1}-\xi\|}{\|w_{n+1}-\xi\|} \leqslant \frac{\mu^{2(n+1)}\prod_{i=0}^{n}[1-(\alpha_i+\beta_i-\alpha_i\beta_i(1-\mu))(1-\mu)]}{\prod_{i=0}^{n}[1-\alpha_i(1+\mu(1-\beta_i(1-\mu(1-\gamma_i(1-\mu)))))]}$$

and

$$0 \leqslant \frac{\|x_{n+1}-\xi\|}{\|r_{n+1}-\xi\|} \leqslant \frac{\mu^{2(n+1)}\prod_{i=0}^{n}[1-(\alpha_i+\beta_i-\alpha_i\beta_i(1-\mu))(1-\mu)]}{\mu^{(n+1)}\prod_{i=0}^{n}[1-\alpha_i(1+\alpha_i(1-\beta_i(1-\mu))(1-\gamma_i(1-\mu)))]}.$$

Now, we define

$$\phi_n = \frac{\|x_{n+1}-\xi\|}{\|w_{n+1}-\xi\|} = \frac{\mu^{2(n+1)}\prod_{i=0}^{n}[1-(\alpha_i+\beta_i-\alpha_i\beta_i(1-\mu))(1-\mu)]}{\prod_{i=0}^{n}[1-\alpha_i(1+\mu(1-\beta_i(1-\mu(1-\gamma_i(1-\mu)))))]}$$

and

$$\theta_n = \frac{\|x_{n+1}-\xi\|}{\|r_{n+1}-\xi\|} = \frac{\mu^{2(n+1)}\prod_{i=0}^{n}[1-(\alpha_i+\beta_i-\alpha_i\beta_i(1-\mu))(1-\mu)]}{\mu^{(n+1)}\prod_{i=0}^{n}[1-\alpha_i(1+\alpha_i(1-\beta_i(1-\mu))(1-\gamma_i(1-\mu)))]}.$$

Then, using the ratio test, we have

$$\frac{\phi_{n+1}}{\phi_n} = \frac{\mu^2[1-(\alpha_{n+1}+\beta_{n+1}-\alpha_{n+1}\beta_{n+1}(1-\mu))(1-\mu)]}{[1-\alpha_{n+1}(1+\mu(1-\beta_{n+1}(1-\mu(1-\gamma_{n+1}(1-\mu)))))]}$$

and

$$\frac{\theta_{n+1}}{\theta_n} = \frac{\mu[1-(\alpha_{n+1}+\beta_{n+1}-\alpha_{n+1}\beta_{n+1}(1-\mu))(1-\mu)]}{[1-\alpha_{n+1}(1+\alpha_{n+1}(1-\beta_{n+1}(1-\mu))(1-\gamma_{n+1}(1-\mu)))]}.$$

Employing the assumptions $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, and $\lim_{n\to\infty}\gamma_n = 0$, we obtain $l_1 = \lim_{n\to\infty}\phi_{n+1}/\phi_n = \mu^2 < \mu < 1$ and $l_2 = \lim_{n\to\infty}\theta_{n+1}/\theta_n = \mu < 1$. Since $l_1, l_2 < 1$, the ratio test deduces that the series $\sum_{n=0}^{\infty}\phi_n$ and $\sum_{n=0}^{\infty}\theta_n$ converge. Then we conclude that $\lim_{n\to\infty}\phi_n = 0$ and $\lim_{n\to\infty}\theta_n = 0$. Therefore,

$$\lim_{n\to\infty}\frac{\|x_{n+1}-\xi\|}{\|w_{n+1}-\xi\|} = \lim_{n\to\infty}\frac{\|x_{n+1}-\xi\|}{\|r_{n+1}-\xi\|} = 0.$$

Hence, from Definition 5 we infer that (x_n) converges faster than (w_n) and (r_n) to a fixed point ξ. □

4 Data dependence result

This section consists of a data dependence result using the $\mathcal{Z}$-iterative scheme (1).

Theorem 6. *Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be a nonempty, closed, and convex subset of the setting. Also, suppose that $\mathcal{G}:\mathscr{C}\to\mathscr{C}$ be a weak contraction satisfying* (3) *with a fixed point ξ, $\bar{\mathcal{G}}$ be an approximate operator of $\mathcal{G}$ with a fixed point $\bar{x}$, and (x_n) be a sequence defined by $\mathcal{Z}$-iteration* (1) *for $\mathcal{G}$. Assume that $(\bar{x}_n)$ is the sequence for $\bar{\mathcal{G}}$ defined by*

$$\begin{aligned} \bar{x}_{n+1} &= \bar{\mathcal{G}}\bar{z}_n, \\ \bar{y}_n &= \bar{\mathcal{G}}\big((1-\beta_n)\bar{x}_n + \beta_n\bar{\mathcal{G}}\bar{x}_n\big), \\ \bar{z}_n &= (1-\alpha_n)\bar{y}_n + \alpha_n\bar{\mathcal{G}}\bar{y}_n, \end{aligned} \tag{15}$$

where (α_n) and (β_n) are real sequences in $(0,1)$ for all $n\in\mathbb{N}$. If $\mathcal{G}\xi=\xi$ and $\bar{\mathcal{G}}\bar{x}=\bar{x}$ as $\bar{x}_n\to\bar{x}$ when $n\to\infty$, then we have

$$\|\xi-\bar{x}\| \leqslant \left(\frac{1+3\mu}{1-\mu}\right)\epsilon$$

for a given fixed number $\epsilon>0$.

Proof. Let us consider

$$\begin{aligned} \|x_{n+1}-\bar{x}_{n+1}\| &= \|\mathcal{G}z_n - \bar{\mathcal{G}}\bar{z}_n\| \\ &\leqslant \|\mathcal{G}z_n - \mathcal{G}\bar{z}_n\| + \|\mathcal{G}\bar{z}_n - \bar{\mathcal{G}}\bar{z}_n\| \\ &\leqslant \mu\|z_n-\bar{z}_n\| + L\|z_n-\mathcal{G}z_n\| + \|\mathcal{G}\bar{z}_n - \bar{\mathcal{G}}\bar{z}_n\| \\ &\leqslant \mu\|z_n-\bar{z}_n\| + L\|z_n-\mathcal{G}z_n\| + \epsilon. \end{aligned}$$

Now,

$$\begin{aligned}\|z_n-\bar z_n\| &= \big\|(1-\alpha_n)y_n+\alpha_n\mathcal{G}y_n-\big((1-\alpha_n)\bar y_n+\alpha_n\bar{\mathcal{G}}\bar y_n\big)\big\|\\
&= \big\|(1-\alpha_n)(y_n-\bar y_n)+\alpha_n(\mathcal{G}y_n-\bar{\mathcal{G}}\bar y_n)\big\|\\
&\leqslant (1-\alpha_n)\|y_n-\bar y_n\|+\alpha_n\big\|\big[(\mathcal{G}y_n-\mathcal{G}\bar y_n)+(\mathcal{G}\bar y_n-\bar{\mathcal{G}}\bar y_n)\big]\big\|\\
&\leqslant (1-\alpha_n)\|y_n-\bar y_n\|+\alpha_n\|\mathcal{G}y_n-\mathcal{G}\bar y_n\|+\alpha_n\|\mathcal{G}\bar y_n-\bar{\mathcal{G}}\bar y_n\|\\
&\leqslant (1-\alpha_n)\|y_n-\bar y_n\|+\alpha_n\big[\mu\|y_n-\bar y_n\|+L\|y_n-\mathcal{G}y_n\|\big]\\
&\quad+\alpha_n\|\mathcal{G}\bar y_n-\bar{\mathcal{G}}\bar y_n\|\\
&\leqslant (1-\alpha_n)\|y_n-\bar y_n\|+\alpha_n\mu\|y_n-\bar y_n\|+\alpha_n L\|y_n-\mathcal{G}y_n\|+\alpha_n\epsilon\\
&= \big(1-\alpha_n(1-\mu)\big)\|y_n-\bar y_n\|+\alpha_n L\|y_n-\mathcal{G}y_n\|+\alpha_n\epsilon. \end{aligned}\tag{16}$$

Similarly,

$$\begin{aligned}\|y_n-\bar y_n\| &= \big\|\mathcal{G}\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\bar{\mathcal{G}}\big((1-\beta_n)\bar x_n+\beta_n\bar{\mathcal{G}}\bar x_n\big)\big\|\\
&\leqslant \big\|\mathcal{G}\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\mathcal{G}\big((1-\beta_n)\bar x_n+\beta_n\bar{\mathcal{G}}\bar x_n\big)\big\|\\
&\quad+\big\|\mathcal{G}\big((1-\beta_n)\bar x_n+\beta_n\bar{\mathcal{G}}\bar x_n\big)-\bar{\mathcal{G}}\big((1-\beta_n)\bar x_n+\beta_n\bar{\mathcal{G}}\bar x_n\big)\big\|\\
&\leqslant \mu\big\|\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\big((1-\beta_n)\bar x_n+\beta_n\bar{\mathcal{G}}\bar x_n\big)\big\|\\
&\quad+L\big\|\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\mathcal{G}\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)\big\|+\epsilon\\
&= \mu\big\|\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\big((1-\beta_n)\bar x_n+\beta_n\bar{\mathcal{G}}\bar x_n\big)\big\|\\
&\quad+L\big\|\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n-\xi\big)-\big(\mathcal{G}\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\xi\big)\big\|+\epsilon\\
&\leqslant \mu\big\|\big((1-\beta_n)x_n+\beta_n\mathcal{G}x_n\big)-\big((1-\beta_n)\bar x_n+\beta_n\bar{\mathcal{G}}\bar x_n\big)\big\|\\
&\quad+L\big(1-\beta_n+\beta_n\mu+\mu(1-\beta_n)+\mu^2\beta_n\big)\|x_n-\xi\|+\epsilon\\
&\leqslant \mu(1-\beta_n)\|x_n-\bar x_n\|+\mu\beta_n\|\mathcal{G}x_n-\bar{\mathcal{G}}\bar x_n\|\\
&\quad+L\big[1-\beta_n(1-\mu)\big](1+\mu)\|x_n-\xi\|+\epsilon\\
&\leqslant \mu(1-\beta_n)\|x_n-\bar x_n\|+\mu\beta_n\big[\mu\|x_n-\bar x_n\|+L\|x_n-\mathcal{G}x_n\|\big]+\mu\beta_n\epsilon\\
&\quad+L\big[1-\beta_n(1-\mu)\big](1+\mu)\|x_n-\xi\|+\epsilon\\
&= \mu\big[1-\beta_n(1-\mu)\big]\|x_n-\bar x_n\|+\mu\beta_n L\|x_n-\mathcal{G}x_n\|+\mu\beta_n\epsilon\\
&\quad+L\big[1-\beta_n(1-\mu)\big](1+\mu)\|x_n-\xi\|+\epsilon. \end{aligned}\tag{17}$$

Therefore, from (16) and (17) we have

$$\begin{aligned}&\|x_{n+1}-\bar x_{n+1}\|\\
&\quad\leqslant \mu\|z_n-\bar z_n\|+L\|z_n-\mathcal{G}z_n\|+\epsilon\\
&\quad\leqslant \mu\big[\big(1-\alpha_n(1-\mu)\big)\|y_n-\bar y_n\|+\alpha_n L\|y_n-\mathcal{G}y_n\|+\alpha_n\epsilon\big]+L\|z_n-\mathcal{G}z_n\|+\epsilon\\
&\quad= \mu\big[1-\alpha_n(1-\mu)\big]\|y_n-\bar y_n\|+\mu\alpha_n L\|y_n-\mathcal{G}y_n\|+\mu\alpha_n\epsilon+L\|z_n-\mathcal{G}z_n\|+\epsilon\\
&\quad\leqslant \mu\big[1-\alpha_n(1-\mu)\big]\big[\mu\big[1-\beta_n(1-\mu)\big]\|x_n-\bar x_n\|+\mu\beta_n L\|x_n-\mathcal{G}x_n\|\\
&\qquad+\mu\beta_n\epsilon+L\big[1-\beta_n(1-\mu)\big](1+\mu)\|x_n-\xi\|+\epsilon\big]\\
&\qquad+\mu\alpha_n L\|y_n-\mathcal{G}y_n\|+\mu\alpha_n\epsilon+L\|z_n-\mathcal{G}z_n\|+\epsilon\end{aligned}$$

$$
\begin{aligned}
&= \mu\big[1-\alpha_n(1-\mu)\big]\mu\big[1-\beta_n(1-\mu)\big]\|x_n-\bar{x}_n\| \\
&\quad + \mu\big[1-\alpha_n(1-\mu)\big]\mu\beta_n L\|x_n-\mathcal{G}x_n\| + \mu\big[1-\alpha_n(1-\mu)\big]\mu\beta_n\epsilon \\
&\quad + \mu\big[1-\alpha_n(1-\mu)\big]L\big[1-\beta_n(1-\mu)\big](1+\mu)\|x_n-\xi\| \\
&\quad + \mu\big[1-\alpha_n(1-\mu)\big]\epsilon + \mu\alpha_n L\|y_n-\mathcal{G}y_n\| + \mu\alpha_n\epsilon + L\|z_n-\mathcal{G}z_n\| + \epsilon.
\end{aligned}
$$

Now, using the fact $\alpha_n < 1$, $\beta_n < 1$, also $\mu^2 < \mu$ and $[1-\alpha_n(1-\mu)] < 1$, $[1-\beta_n(1-\mu)] < 1$ for all $n \in \mathbb{N}$ in above inequality, we obtain

$$
\begin{aligned}
\|x_{n+1}-\bar{x}_{n+1}\| &\leqslant \mu\big[1-\alpha_n(1-\mu)\big]\|x_n-\bar{x}_n\| + \mu\beta_n L\|x_n-\mathcal{G}x_n\| \\
&\quad + \mu\beta_n\epsilon + 2\mu L\|x_n-\xi\| + \mu\epsilon + \mu\alpha_n L\|y_n-\mathcal{G}y_n\| \\
&\quad + \mu\alpha_n\epsilon + L\|z_n-\mathcal{G}z_n\| + \epsilon \\
&\leqslant \mu\|x_n-\bar{x}_n\| + \mu L\|x_n-\mathcal{G}x_n\| + \mu\epsilon + 2\mu L\|x_n-\xi\| \\
&\quad + \mu\epsilon + \mu L\|y_n-\mathcal{G}y_n\| + \mu\epsilon + L\|z_n-\mathcal{G}z_n\| + \epsilon \\
&= \mu\|x_n-\bar{x}_n\| + \mu L\|x_n-\mathcal{G}x_n\| + 2\mu L\|x_n-\xi\| \\
&\quad + \mu L\|y_n-\mathcal{G}y_n\| + L\|z_n-\mathcal{G}z_n\| + (1+3\mu)\epsilon. \qquad (18)
\end{aligned}
$$

Using Theorem 2, we know that $x_n \to \xi$ as $n \to \infty$. Therefore, taking limit on both sides of (18), we have

$$
\|\xi-\bar{x}\| \leqslant \mu\|\xi-\bar{x}\| + (1+3\mu)\epsilon = \frac{1+3\mu}{1-\mu}\epsilon.
$$

Therefore, we have

$$
\|\xi-\bar{x}\| \leqslant \frac{1+3\mu}{1-\mu}\epsilon. \qquad \square
$$

Remark 1. If we consider an additional assumption that $\lim_{n\to\infty}\alpha_n = 0$ on the sequence of real numbers (α_n) in Theorem 6, then we have a finer estimate for the upper bound of the error in approximating $\bar{x}$ by ξ, and we come by the following:

$$
\|\xi-\bar{x}\| \leqslant \frac{1+2\mu}{1-\mu}\epsilon.
$$

The applicability of Theorem 6 can be realised from the following example given below.

5 Numerical illustration

In this section, we construct a few nontrivial numerical examples to ascertain the applicability of our obtained findings.

Example 1. Let $\mathscr{C} = [-1,1]$ be a nonempty subset of a Banach space $\mathscr{X} = \mathbb{R}$ equipped with the usual norm and consider a self-map $\mathcal{G} : \mathscr{C} \to \mathscr{C}$ as

$$
\mathcal{G}x = \begin{cases} \frac{11}{30}\sin\frac{11x}{30}, & -1 \leqslant x < 0, \\ -\frac{11}{30}\sin\frac{11x}{30}, & 0 \leqslant x \leqslant 1. \end{cases}
$$

Table 1. Impact of different initial points on iteration techniques.

Initial Values	Noor Iteration	Thakur Iteration	$\mathcal{Z}$-Iteration
-1	17	10	7
-0.7	17	10	6
-0.4	17	10	6
-0.2	17	10	6
0.2	16	10	6
0.4	16	10	6
0.6	16	10	6
0.8	16	10	6

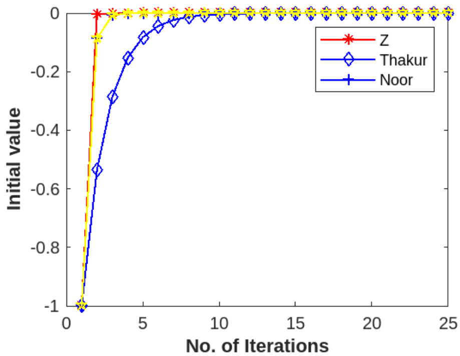

Figure 1. Convergence behaviour of iteration techniques for $x_0 = -1$.

One can easily observe that $\mathcal{G}$ is a weak contraction mapping satisfying (3) with $\mu \in [121/900, 1)$. Using this mapping $\mathcal{G}$, we verify that the $\mathcal{Z}$-iteration (1) converges faster than Noor and Thakur iterative schemes; see Table 1. For this, we choose the control sequences $(\alpha_n) = (n+2)/(n+3)$, $(\beta_n) = (n+3)/(n+4)$, $(\gamma_n) = (n+4)/(n+5)$. Further, for fixed initial value $x_0 = -1$ and control sequences $(\alpha_n) = 0.5$, $(\beta_n) = 0.6$, $(\gamma_n) = 0.7$, we plot Fig. 1, which shows that the convergence rate of $\mathcal{Z}$-iteration (1) is faster than the Noor iteration (10) and Thakur iteration (11).

Example 2. Let $\mathscr{C} = [0, 5]$ be a nonempty subset of a Banach space $\mathscr{X} = \mathbb{R}$ equipped with the usual norm and consider a self-map $\mathcal{G} : \mathscr{C} \to \mathscr{C}$ as

$$\mathcal{G}(x) = \frac{(x+1)}{3} - \log(x+1), \quad x \in [0, 5].$$

Here we inspect that $\mathcal{G}$ is a weak contraction as it satisfies (3) for $\mu = 1/3$ for all $x, y \in [0, 5]$, and a fixed point of mapping $\mathcal{G}$ is $\xi = 0.31942396$. We can define a self map $\mathcal{S} : \mathscr{C} \to \mathscr{C}$ such that

$$\mathcal{S}(x) = \frac{x}{4.87} - \frac{(x-0.4)^3}{112.98} - \frac{(x-0.7)^5}{7835.50} + \frac{(x+0.9)^7}{125430.92}. \tag{19}$$

Using MATLAB 2023a software, we have

$$\max_{x\in[0,5]} \|\mathcal{G}x - \mathcal{S}x\| = 0.7398.$$

Hence, for all $x \in \mathscr{C}$, there is a fixed positive $\epsilon = 0.7398$ satisfying $\|\mathcal{G}x - \mathcal{S}x\| \leqslant 0.7398 = \epsilon$. Thus, by Definition 4 we can conclude that $\mathcal{S}$ is an approximate operator of $\mathcal{G}$, and also, from (19) we infer that $\bar{x} = 0.00074057$ is the unique fixed point of $\mathcal{S}$ in $\mathscr{C}$. The distance between the two fixed points ξ and $\bar{x}$ is $\|\xi - \bar{x}\| = 0.31868339$. Let us consider (19) and choose control sequences $\alpha_n = (n+2)/(n+3) = \beta_n$ in (15). We have

$$\begin{aligned}
\bar{x}_{n+1} &= \frac{\bar{z}_n}{4.87} - \frac{(\bar{z}_n - 0.4)^3}{112.98} - \frac{(\bar{z}_n - 0.7)^5}{7835.50} + \frac{(\bar{z}_n + 0.9)^7}{125430.92},\\
\bar{y}_n &= \mathcal{S}\Bigg((1-\beta_n)\bar{x}_n + \beta_n\bigg(\frac{\bar{x}_n}{4.87} - \frac{(\bar{x}_n - 0.4)^3}{112.98} - \frac{(\bar{x}_n - 0.7)^5}{7835.50}\\
&\quad + \frac{(\bar{x}_n + 0.9)^7}{125430.92}\bigg)\Bigg),\\
\bar{z}_n &= (1-\alpha_n)\bar{y}_n + \alpha_n\bigg(\frac{\bar{y}_n}{4.87} - \frac{(\bar{y}_n - 0.4)^3}{112.98} - \frac{(\bar{y}_n - 0.7)^5}{7835.50} + \frac{(\bar{y}_n + 0.9)^7}{125430.92}\bigg).
\end{aligned} \tag{20}$$

The sequence $(\bar{x}_n)$ is formulated using (20), which converges to a fixed point $\bar{x} = 0.00074057$. Also, from Theorem 6 we compute the succeeding estimate given by

$$\|\xi - \bar{x}\| \leqslant \frac{(1 + \frac{3}{3})}{(1 - \frac{1}{3})} \cdot 0.7398 = 2.2194.$$

Clearly, from above we have

$$\|\xi - \bar{x}\| \leqslant \left(\frac{1 + 3\mu}{1 - \mu}\right)\epsilon.$$

6 Error estimation of $\mathcal{Z}$-iterative scheme

In this section, we derive an error estimation result concerning the $\mathcal{Z}$-iterative scheme. Assume that $\mathscr{X}$ is a Banach space and $\mathscr{C}$ is any nonempty subset of the setting. Let $\mathcal{G} : \mathscr{C} \to \mathscr{C}$ be a contraction mapping. Define the errors of $\mathcal{G}x_n$, $\mathcal{G}y_n$, and $\mathcal{G}z_n$ by $p_n = \mathcal{G}x_n - \bar{\mathcal{G}}\bar{x}_n$, $q_n = \mathcal{G}y_n - \bar{\mathcal{G}}\bar{y}_n$, and $r_n = \mathcal{G}z_n - \bar{\mathcal{G}}\bar{z}_n$, where $\bar{\mathcal{G}}\bar{x}_n$, $\bar{\mathcal{G}}\bar{y}_n$, and $\bar{\mathcal{G}}\bar{z}_n$ are exact values, and $\mathcal{G}x_n$, $\mathcal{G}y_n$, and $\mathcal{G}z_n$ are approximated values. The theory of errors implies that (p_n), (q_n), and (r_n) are bounded. Let $\mathfrak{B} = \max\{B_p, B_q, B_r\}$, where $B_p = \sup\|p_n\|$, $B_q = \sup\|q_n\|$, and $B_r = \sup\|r_n\|$ are the absolute error boundaries of $\mathcal{G}x_n$, $\mathcal{G}y_n$, and $\mathcal{G}z_n$, respectively, and accumulated the error in (1). Hence, we can set (1) as (15). Let

$$\begin{gathered}
\|x_0\| = \|\bar{x}_0\|, \qquad \|x_1 - \bar{x}_1\| = \|\mathcal{G}z_0 - \bar{\mathcal{G}}\bar{z}_0\| = \|r_0\|,\\
\|x_2 - \bar{x}_2\| = \|r_1\|, \qquad \|x_3 - \bar{x}_3\| = \|r_2\|.
\end{gathered}$$

Following as above, we obtain

$$\|x_{n+1} - \bar{x}_{n+1}\| = \|r_n\|.$$

Now,

$$\begin{aligned}
\|y_0 - \bar{y}_0\| &= \big\|\mathcal{G}\big((1-\beta_0)x_0 + \beta_0\mathcal{G}x_0\big) - \bar{\mathcal{G}}\big((1-\beta_0)\bar{x}_0 + \beta_0\bar{\mathcal{G}}\bar{x}_0\big)\big\| \\
&\leqslant \big\|\mathcal{G}(1-\beta_0)x_0 + \beta_0\mathcal{G}x_0\big) - \mathcal{G}(1-\beta_0)\bar{x}_0 + \beta_0\bar{\mathcal{G}}\bar{x}_0)\big\| \\
&\quad + \big\|\mathcal{G}(1-\beta_0)\bar{x}_0 + \beta_0\bar{\mathcal{G}}\bar{x}_0) - \bar{\mathcal{G}}(1-\beta_0)\bar{x}_0 + \beta_0\bar{\mathcal{G}}\bar{x}_0\big)\big\| \\
&\leqslant \mu(1-\beta_0)\|x_0 - \bar{x}_0\| + \mu\beta_0\|\mathcal{G}x_0 - \bar{\mathcal{G}}\bar{x}_0\| + \epsilon \\
&= \mu(1-\beta_0)\|x_0 - \bar{x}_0\| + \mu\beta_0\|p_0\| + \epsilon.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|y_1 - \bar{y}_1\| &\leqslant \mu(1-\beta_1)\|x_1 - \bar{x}_1\| + \mu\beta_1\|p_1\| + \epsilon, \\
\|y_2 - \bar{y}_2\| &\leqslant \mu(1-\beta_2)\|x_2 - \bar{x}_2\| + \mu\beta_2\|p_2\| + \epsilon.
\end{aligned}$$

Following as above, we obtain

$$\|y_n - \bar{y}_n\| \leqslant \mu(1-\beta_n)\|x_n - \bar{x}_n\| + \mu\beta_n\|p_n\| + \epsilon$$

and

$$\begin{aligned}
\|z_0 - \bar{z}_0\| &= \big\|\big((1-\alpha_0)y_0 + \alpha_0\mathcal{G}y_0\big) - \big((1-\alpha_0)\bar{y}_0 + \alpha_0\bar{\mathcal{G}}\bar{y}_0\big)\big\| \\
&\leqslant (1-\alpha_0)\|y_0 - \bar{y}_0\| + \alpha_0\|\mathcal{G}y_0 - \bar{\mathcal{G}}\bar{y}_0\| \\
&= (1-\alpha_0)\|y_0 - \bar{y}_0\| + \alpha_0\|q_0\|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|z_1 - \bar{z}_1\| &\leqslant (1-\alpha_1)\|y_1 - \bar{y}_1\| + \alpha_1\|q_1\|, \\
\|z_2 - \bar{z}_2\| &\leqslant (1-\alpha_2)\|y_2 - \bar{y}_2\| + \alpha_2\|q_2\|.
\end{aligned}$$

Following as above, we obtain

$$\|z_n - \bar{z}_n\| \leqslant (1-\alpha_n)\|y_n - \bar{y}_n\| + \alpha_n\|q_n\|.$$

Now,

$$\begin{aligned}
\|z_n - \bar{z}_n\| &\leqslant (1-\alpha_n)\|y_n - \bar{y}_n\| + \alpha_n\|q_n\| \\
&\leqslant (1-\alpha_n)\big[\mu(1-\beta_n)\|x_n - \bar{x}_n\| + \mu\beta_n\|p_n\| + \epsilon\big] + \alpha_n\|q_n\| \\
&= \mu(1-\alpha_n)(1-\beta_n)\|x_n - \bar{x}_n\| + \mu(1-\alpha_n)\beta_n\|p_n\| \\
&\quad + (1-\alpha_n)\epsilon + \alpha_n\|q_n\|.
\end{aligned}$$

Now, we define

$$\begin{aligned}
\varepsilon_n^{(1)} &= \|x_{n+1} - \bar{x}_{n+1}\| = \|r_n\|, \\
\varepsilon_n^{(2)} &= \|y_n - \bar{y}_n\| \leqslant \mu(1-\beta_n)\|x_n - \bar{x}_n\| + \mu\beta_n\|p_n\| + \epsilon
\end{aligned}$$

and

$$\begin{aligned}\varepsilon_n^{(3)} &= \|z_n - \bar{z}_n\| \\ &\leqslant \mu(1-\alpha_n)(1-\beta_n)\|x_n - \bar{x}_n\| + \mu(1-\alpha_n)\beta_n\|p_n\| \\ &\quad + (1-\alpha_n)\epsilon + \alpha_n\|q_n\|.\end{aligned}$$

Therefore, at the end of $(n+1)$ iterations, the error of $\mathcal{Z}$-iteration (1) is accumulated to $\varepsilon_n^{(1)}$, $\varepsilon_n^{(2)}$, and $\varepsilon_n^{(3)}$.

Theorem 7. *We consider $\mathcal{G}$, $\mathfrak{B}$, $\varepsilon_n^{(1)}$, $\varepsilon_n^{(2)}$, and $\varepsilon_n^{(3)}$ same as above, and let ϵ be a fixed value. If $\sum_{i=0}^{\infty}\alpha_i = \infty$ or $\sum_{i=0}^{\infty}\beta_i = \infty$, then the error accumulation of $\mathcal{Z}$-iteration scheme* (1) *is bounded and does not exceed a number $\mathfrak{K}$.*

Proof. From above we have

$$\left\|\varepsilon_n^{(1)}\right\| = \|r_n\| = \mathfrak{B} \leqslant \mathfrak{B} + \epsilon \leqslant \mathfrak{K}. \tag{21}$$

Also,

$$\begin{aligned}\left\|\varepsilon_n^{(2)}\right\| &= \mu(1-\beta_n)\|x_n - \bar{x}_n\| + \mu\beta_n\|p_n\| + \epsilon \\ &= \mu(1-\beta_n)\|\varepsilon_{n-1}^{(1)}\| + \mu\beta_n\|p_n\| + \epsilon \\ &\leqslant \mu(1-\beta_n)\mathfrak{B} + \mu\beta_n\mathfrak{B} + \epsilon \leqslant \mathfrak{B} + \epsilon \leqslant \mathfrak{K}\end{aligned} \tag{22}$$

and

$$\begin{aligned}\left\|\varepsilon_n^{(3)}\right\| &= \mu(1-\alpha_n)(1-\beta_n)\|x_n - \bar{x}_n\| \\ &\quad + \mu(1-\alpha_n)\beta_n\|p_n\| + (1-\alpha_n)\epsilon + \alpha_n\|q_n\| \\ &= \mu(1-\alpha_n)(1-\beta_n)\|\varepsilon_{n-1}^{(1)}\| + \mu(1-\alpha_n)\beta_n\|p_n\| \\ &\quad + (1-\alpha_n)\epsilon + \alpha_n\|q_n\| \\ &\leqslant \mu(1-\alpha_n)(1-\beta_n)\mathfrak{B} + \mu(1-\alpha_n)\beta_n\mathfrak{B} \\ &\quad + (1-\alpha_n)\epsilon + \alpha_n\mathfrak{B} \\ &\leqslant (1-\alpha_n)\mathfrak{B} + (1-\alpha_n)\epsilon + \alpha_n\mathfrak{B} \\ &= \mathfrak{B} + (1-\alpha_n)\epsilon \leqslant \mathfrak{B} + \epsilon \leqslant \mathfrak{K}.\end{aligned} \tag{23}$$

Therefore, from (21), (22), and (23) we have

$$\max\left\{\left\|\varepsilon_n^{(1)}\right\|, \left\|\varepsilon_n^{(2)}\right\|, \left\|\varepsilon_n^{(3)}\right\|\right\} \leqslant \mathfrak{K}. \qquad \square$$

7 Application to delay Caputo fractional differential equations

The key objective of this section is to exhibit the applicability of $\mathcal{Z}$-iterative scheme (1) to find out the solution of a certain kind of delay fractional differential equation. One can note that, in 1967, mathematician Caputo invented a novel kind of fractional differentiation known as Caputo fractional derivative and defined as

$${}^{c}D_s^{\alpha}\mathcal{F}(s) = \frac{1}{\Gamma(\alpha - n)}\int_b^s \mathcal{F}^n(\tau)(s-\tau)^{n-\alpha-1}\,\mathrm{d}\tau, \quad (n-1 < \alpha < n).$$

Many researchers enquired for the solution of various types of delay fractional-order differential equations employing several different methodologies as documented in the literature [8, 16, 29]. In this sequel, we utilize the $\mathcal{Z}$-iterative scheme (1) with $\alpha \in (0,1)$ to estimate the solution of a certain kind of delay Caputo fractional differential equation. Let $\delta > 0$ and $\psi \in C([r-\delta,\, r]; \mathbb{R}^n)$ be any continuous mapping. Here we consider the ensuing delay Caputo fractional differential equation

$$^{c}D^{\alpha} v(s) = g\bigl(s,\, v(s),\, v(s-\delta)\bigr), \quad s \in [r, \kappa], \tag{24}$$

with initial condition

$$v(s) = \psi(s), \quad s \in [r-\tau,\, r], \tag{25}$$

where $v \in \mathbb{R}^n$, $g : [r,\kappa] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, where $s > 0$ and $\kappa > 0$. Assume that the subsequent assumptions are satisfied:

(P1) g satisfies the Lipschitz condition, then there is a constant $\mathscr{L}_g > 0$ such that

$$\bigl\|g(s,v,q) - g(s,w,m)\bigr\| \leqslant \mathscr{L}_g\bigl(\|v-w\| + \|q-m\|\bigr)$$

for all $s \in \mathbb{R}^+$ and $v, w, q, m \in \mathbb{R}^n$.

(P2) There is a real number $\lambda_{\mathscr{L}} > 0$ depended on $\mathscr{L}_g$ satisfying $\lambda_{\mathscr{L}} > 2\mathscr{L}_g$, which implies $2\mathscr{L}_g/\lambda_{\mathscr{L}} < 1$.

A map $\varrho \in C([r-\delta,\, \kappa]; \mathbb{R}^n) \cap C^1([r,\kappa]; \mathbb{R}^n)$ will be a solution of the initial value problem if it satisfies (24) with condition (25). From [16] one can verify that finding the solution of (24) with condition (25) is equivalent to obtaining the solution of the ensuing integral equation

$$v(s) = \psi(s) + \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g\bigl(\tau,\, v(\tau),\, v(\tau-\delta)\bigr)\, \mathrm{d}\tau$$

for all $s \in [r,\kappa]$ with $v(s) = \psi(s)$ for all $s \in [r-\delta,\, \kappa]$. Define a norm $\|\cdot\|_{\lambda_{\mathscr{L}}}$ on $C([r-\delta,\, r]; \mathbb{R}^n)$ by

$$\|\psi\|_{\lambda_{\mathscr{L}}} = \frac{\sup \|\psi(s)\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)}$$

for all $\psi \in C([r-\delta,\, r]; \mathbb{R}^n)$, where $\mathcal{M}_\alpha : \mathbb{R} \to \mathbb{R}$ is Mittag-Leffler function such that

$$\mathcal{M}_\alpha(s) = \sum_{n=0}^{\infty} \frac{s^n}{\Gamma(\alpha n + 1)}$$

for all $s \in \mathbb{R}$. It is clear that $C([r-\delta,\, r]; \mathbb{R}^n,\, \|\cdot\|_{\lambda_{\mathscr{L}}})$ is a Banach space. In their research article, Wang et al. [29] attested the existence and uniqueness of a solution of delay differential equations (24) with condition (25) given that hypothesis (P1) holds. The subsequent result brings forth an approximation of the solution using $\mathcal{Z}$-iterative algorithm (1).

Theorem 8. *Suppose that ψ and g are two functions as discussed above. If hypotheses* (P1) *and* (P2) *hold, then the differential equation* (24) *with condition* (25) *possesses a unique solution $\varrho \in C([r-\delta,\, \kappa];\mathbb{R}^n) \cap C^1([r,\kappa];\mathbb{R}^n)$, and the sequence (x_n) generated by $\mathcal{Z}$-iterative scheme* (1) *converges to ϱ.*

Proof. From [7] one can assert the existence of a unique solution ϱ. Suppose that (x_n) is a sequence generated by $\mathcal{Z}$-iterative scheme (1) and define $\mathcal{G}$ on $C([r-\delta,\, \kappa];\mathbb{R}^n) \cap C^1([r,\kappa];\mathbb{R}^n)$ as

$$(\mathcal{G}v)(s) = \begin{cases} \psi(r) + \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g(\tau, v(\tau), v(\tau-\delta))\, \mathrm{d}\tau, & s \in [r,\kappa], \\ \psi(s), & s \in [r-\tau,\, r]. \end{cases}$$

Here we prove that using $\mathcal{Z}$-iterative scheme (1) together with conditions (P1) and (P2), (x_n) converges to a unique solution, i.e., $x_n \to \varrho$ as $n \to \infty$ of problem (24) with condition (25). For this, let us consider

$$t_n = (1-\beta_n)x_n + \beta_n \mathcal{G}x_n.$$

Then we obtain

$$\begin{aligned}
&\|t_n - \varrho\| \\
&\quad = \big\|\big((1-\beta_n)x_n + \beta_n \mathcal{G}x_n\big) - \varrho\big\| \\
&\quad \leqslant (1-\beta_n)\|x_n - \varrho\| + \beta_n\|\mathcal{G}x_n - \varrho\| \\
&\quad = (1-\beta_n)\|x_n - \varrho\| + \beta_n\|\mathcal{G}x_n - \mathcal{G}\varrho\| \\
&\quad = (1-\beta_n)\|x_n - \varrho\| \\
&\qquad + \beta_n \Bigg\| \psi(r) + \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g\big(\tau,\, x_n(\tau),\, x_n(\tau-\delta)\big)\, \mathrm{d}\tau \\
&\qquad - \psi(r) - \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g\big(\tau,\, \varrho(\tau),\, \varrho(\tau-\delta)\big)\, \mathrm{d}\tau \Bigg\| \\
&\quad = (1-\beta_n)\|x_n - \varrho\| \\
&\qquad + \beta_n \Bigg\| \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g\big(\tau,\, x_n(\tau),\, x_n(\tau-\delta)\big)\, \mathrm{d}\tau \\
&\qquad - \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g\big(\tau,\, \varrho(\tau),\, \varrho(\tau-\delta)\big)\, \mathrm{d}\tau \Bigg\| \\
&= (1-\beta_n)\|x_n - \varrho\| \\
&\qquad + \frac{\beta_n}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} \big\| g\big(\tau,\, x_n(\tau),\, x_n(\tau-\delta)\big) - g\big(\tau,\, \varrho(\tau),\, \varrho(\tau-\delta)\big) \big\|\, \mathrm{d}\tau.
\end{aligned}$$

Taking supremum over the interval $[r-\delta,\,\kappa]$ on both sides of the above inequality, we derive

$$\begin{aligned}
&\sup\|t_n-\varrho\| \\
&\quad\leqslant (1-\beta_n)\sup\|x_n-\varrho\| \\
&\qquad+\sup\frac{\beta_n}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\big\|g\big(\tau,\,x_n(\tau),\,x_n(\tau-\delta)\big)-g\big(\tau,\,\varrho(\tau),\,\varrho(\tau-\delta)\big)\big\|\,\mathrm{d}\tau \\
&\quad\leqslant (1-\beta_n)\sup\|x_n-\varrho\| \\
&\qquad+\sup\frac{\beta_n}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\mathscr{L}_g\big(\big\|x_n(\tau)-\varrho(\tau)\big\|+\big\|x_n(\tau-\delta)-\varrho(\tau-\delta)\big\|\big)\,\mathrm{d}\tau \\
&\quad\leqslant (1-\beta_n)\sup\|x_n-\varrho\| \\
&\qquad+\frac{\beta_n\mathscr{L}_g}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\Big(\sup\big\|x_n(\tau)-\varrho(\tau)\big\|+\sup\big\|x_n(\tau-\delta)-\varrho(\tau-\delta)\big\|\Big)\,\mathrm{d}\tau.
\end{aligned}$$

Now, dividing both sides of the above inequality by $\mathcal{M}_\alpha(\lambda_{\mathscr{L}}s^\alpha)$, we get

$$\begin{aligned}
&\frac{\sup\|t_n-\varrho\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}}s^\alpha)} \\
&\quad\leqslant\frac{(1-\beta_n)\sup\|x_n-\varrho\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}}s^\alpha)} \\
&\qquad+\frac{\beta_n\mathscr{L}_g}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\left(\frac{\sup\|x_n(\tau)-\varrho(\tau)\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}}s^\alpha)}+\frac{\sup\|x_n(\tau-\delta)-\varrho(\tau-\delta)\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}}s^\alpha)}\right)\mathrm{d}\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|t_n-\varrho\|_{\lambda_{\mathscr{L}}} \\
&\quad\leqslant(1-\beta_n)\|x_n-\varrho\|_{\lambda_{\mathscr{L}}} \\
&\qquad+\frac{\beta_n\mathscr{L}_g}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\big(\big\|x_n(\tau)-\varrho(\tau)\big\|_{\lambda_{\mathscr{L}}}+\big\|x_n(\tau-\delta)-\varrho(\tau-\delta)\big\|_{\lambda_{\mathscr{L}}}\big)\,\mathrm{d}\tau \\
&\quad=(1-\beta_n)\|x_n-\varrho\|_{\lambda_{\mathscr{L}}}+\beta_n(2\mathscr{L}_g)\frac{\|x_n-\varrho\|_{\lambda_{\mathscr{L}}}}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\,\mathrm{d}\tau \\
&\quad=(1-\beta_n)\|x_n-\varrho\|_{\lambda_{\mathscr{L}}}+\frac{\beta_n(2\mathscr{L}_g)}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}}s^\alpha)}\frac{\|x_n-\varrho\|_{\lambda_{\mathscr{L}}}}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\mathcal{M}_\alpha\big(\lambda_{\mathscr{L}}s^\alpha\big)\,\mathrm{d}\tau \\
&\quad=(1-\beta_n)\|x_n-\varrho\|_{\lambda_{\mathscr{L}}}+\frac{\beta_n(2\mathscr{L}_g)}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}}s^\alpha)}\|x_n-\varrho\|_{\lambda_{\mathscr{L}}}\left({}^cI^\alpha\left({}^cD^\alpha\left(\frac{\mathcal{M}_\alpha(\lambda_{\mathscr{L}}s^\alpha)}{\lambda_{\mathscr{L}}}\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= (1-\beta_n)\|x_n-\varrho\|_{\lambda_{\mathscr{L}}} + \frac{\beta_n(2\mathscr{L}_g)}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)}\|x_n-\varrho\|_{\lambda_{\mathscr{L}}}\left(\frac{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)}{\lambda_{\mathscr{L}}}\right) \\
&= (1-\beta_n)\|x_n-\varrho\|_{\lambda_{\mathscr{L}}} + \beta_n\frac{(2\mathscr{L}_g)}{\lambda_{\mathscr{L}}}\|x_n-\varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|x_n-\varrho\|_{\lambda_{\mathscr{L}}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\sup\|y_n-\varrho\| \\
&\quad= \sup\|\mathcal{G}t_n-\varrho\| = \sup\|\mathcal{G}t_n-\mathcal{G}\varrho\| \\
&\quad= \sup\left\|\psi(r)+\frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}g\bigl(\tau,\,t_n(\tau),\,t_n(\tau-\delta)\bigr)\,\mathrm{d}\tau\right. \\
&\qquad\left. -\psi(r)-\frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}g\bigl(\tau,\,\varrho(\tau),\,\varrho(\tau-\delta)\bigr)\,\mathrm{d}\tau\right\| \\
&\quad= \sup\left\|\frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}g\bigl(\tau,\,t_n(\tau),\,t_n(\tau-\delta)\bigr)\,\mathrm{d}\tau\right. \\
&\qquad\left. -\frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}g\bigl(\tau,\,\varrho(\tau),\,\varrho(\tau-\delta)\bigr)\,\mathrm{d}\tau\right\| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\sup\bigl\|g\bigl(\tau,\,t_n(\tau),\,t_n(\tau-\delta)\bigr)-g\bigl(\tau,\,\varrho(\tau),\,\varrho(\tau-\delta)\bigr)\bigr\|\,\mathrm{d}\tau \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\mathscr{L}_g\Bigl(\sup\bigl\|t_n(\tau)-\varrho(\tau)\bigr\|+\sup\bigl\|t_n(\tau-\delta)-\varrho(\tau-\delta)\bigr\|\Bigr)\,\mathrm{d}\tau.
\end{aligned}
$$

Now, dividing both sides of the above inequality by $\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)$,

$$
\begin{aligned}
&\frac{\sup\|y_n-\varrho\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)} \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\mathscr{L}_g\left(\frac{\sup\|t_n(\tau)-\varrho(\tau)\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)}+\frac{\sup\|t_n(\tau-h)-\varrho(\tau-\delta)\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)}\right)\mathrm{d}\tau,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\|y_n-\varrho\|_{\lambda_{\mathscr{L}}} &\leqslant \frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\mathscr{L}_g\bigl(\bigl\|t_n(\tau)-\varrho(\tau)\bigr\|_{\lambda_{\mathscr{L}}}+\bigl\|t_n(\tau-\delta)-\varrho(\tau-\delta)\bigr\|_{\lambda_{\mathscr{L}}}\bigr)\,\mathrm{d}\tau \\
&= (2\mathscr{L}_g)\|t_n-\varrho\|_{\lambda_{\mathscr{L}}}\frac{1}{\Gamma(\alpha)}\int_r^s(s-\tau)^{\alpha-1}\,\mathrm{d}\tau
\end{aligned}
$$

$$= \frac{(2\mathscr{L}_g)\|t_n - \varrho\|_{\lambda_{\mathscr{L}}}}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)} \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} \mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)\,\mathrm{d}\tau$$
$$= \frac{(2\mathscr{L}_g)\|t_n - \varrho\|_{\lambda_{\mathscr{L}}}}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)} \left({}^c I^\alpha \left({}^c D^\alpha \left(\frac{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)}{\lambda_{\mathscr{L}}}\right)\right)\right)$$
$$= \frac{2\mathscr{L}_g}{\lambda_{\mathscr{L}}} \|t_n - \varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|t_n - \varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|x_n - \varrho\|_{\lambda_{\mathscr{L}}}.$$

Similarly, we can show that $\|z_n - \varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|y_n - \varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|x_n - \varrho\|_{\lambda_{\mathscr{L}}}$. Also, we have

$$\begin{aligned}
&\sup\|x_{n+1} - \varrho\| \\
&\quad= \sup\|\mathcal{G} z_n - \varrho\| = \sup\|\mathcal{G} z_n - \mathcal{G}\varrho\| \\
&\quad= \sup\Bigg\|\psi(r) + \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g(\tau, z_n(\tau), z_n(\tau-\delta))\,\mathrm{d}\tau \\
&\qquad - \psi(r) - \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g(\tau, \varrho(\tau), \varrho(\tau-\delta))\,\mathrm{d}\tau\Bigg\| \\
&\quad= \sup\Bigg\|\frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g(\tau, z_n(\tau), z_n(\tau-\delta))\,\mathrm{d}\tau \\
&\qquad - \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} g(\tau, \varrho(\tau), \varrho(\tau-\delta))\,\mathrm{d}\tau\Bigg\| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} \mathscr{L}_g\Big(\sup\big\|z_n(\tau) - \varrho(\tau)\big\| + \sup\big\|z_n(\tau-\delta) - \varrho(\tau-\delta)\big\|\Big)\,\mathrm{d}\tau.
\end{aligned}$$

Now, dividing both sides of the above inequality by $\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)$,

$$\begin{aligned}
&\frac{\sup\|x_{n+1} - \varrho\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)} \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} \mathscr{L}_g\left(\frac{\sup\|z_n(\tau) - \varrho(\tau)\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)} + \frac{\sup\|z_n(\tau-\delta) - \varrho(\tau-\delta)\|}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)}\right)\mathrm{d}\tau
\end{aligned}$$

which implies the following:

$$\begin{aligned}
&\|x_{n+1} - \varrho\|_{\lambda_{\mathscr{L}}} \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1} \mathscr{L}_g\big(\big\|z_n(\tau) - \varrho(\tau)\big\|_{\lambda_{\mathscr{L}}} + \big\|z_n(\tau-\delta) - \varrho(\tau-\delta)\big\|_{\lambda_{\mathscr{L}}}\big)\,\mathrm{d}\tau \\
&\quad= (2\mathscr{L}_g)\|z_n - \varrho\|_{\lambda_{\mathscr{L}}} \frac{1}{\Gamma(\alpha)} \int_r^s (s-\tau)^{\alpha-1}\,\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(2\mathscr{L}_g)\|z_n - \varrho\|_{\lambda_{\mathscr{L}}}}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)} \frac{1}{\Gamma(\alpha)} \int\limits_r^s (s-\tau)^{\alpha-1} \mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)\, \mathrm{d}\tau \\
&= \frac{(2\mathscr{L}_g)\|z_n - \varrho\|_{\lambda_{\mathscr{L}}}}{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)} \left({}^c I^\alpha \left({}^c D^\alpha \left(\frac{\mathcal{M}_\alpha(\lambda_{\mathscr{L}} s^\alpha)}{\lambda_{\mathscr{L}}} \right)\right)\right) \\
&= \frac{2\mathscr{L}_g}{\lambda_{\mathscr{L}}} \|z_n - \varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|z_n - \varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|y_n - \varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|x_n - \varrho\|_{\lambda_{\mathscr{L}}}.
\end{aligned}
$$

Hence,

$$\|x_{n+1} - \varrho\|_{\lambda_{\mathscr{L}}} \leqslant \|x_n - \varrho\|_{\lambda_{\mathscr{L}}}.$$

Let us consider $a_n = \|x_n - \varrho\|_{\lambda_{\mathscr{L}}}$, then we have

$$a_{n+1} \leqslant a_n, \quad n \in \mathbb{N}.$$

Clearly, (a_n) is a monotone decreasing sequence of positive reals, and further, is also bounded. Therefore, we have

$$\lim_{n\to\infty} a_n = \inf(a_n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|x_n - \varrho\|_{\lambda_{\mathscr{L}}} = 0,$$

and this shows that the sequence (x_n) converges to ϱ. □

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