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Positive solutions for a Hadamard-type fractional p-Laplacian integral boundary value problem*

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Abstract. In this paper, we study the existence of positive solutions to a Hadamard-type fractional integral boundary value problem using fixed point index. We construct a new linear operator and obtain our main results under some conditions concerning the spectral radius of this linear operator. Our method improves and generalizes some results in the literature.

Keywords: Hadamard-type fractional-order differential equations, integral boundary value problems, positive solutions, fixed point index.

1 Introduction

In this work, we study the existence of positive solutions to the Hadamard-type fractional p-Laplacian integral boundary value problem

$$\begin{aligned} &D^{\alpha}\big(\varphi_p\big(D^{\beta}\chi(t)\big)\big) = f\big(t,\chi(t)\big), \quad t \in (1,\mathrm{e}),\\ &D^{\beta}\chi(1) = D^{\beta}\chi(\mathrm{e}) = 0, \qquad \chi(1) = \delta\chi(1) = 0, \qquad \delta\chi(\mathrm{e}) = \int_1^{\mathrm{e}} \eta(t) g\big(t,\chi(t)\big)\frac{\mathrm{d}t}{t}, \end{aligned} \tag{1}$$

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where D^α, D^β are the Hadamard-type fractional derivatives with $\alpha \in (1,2]$, $\beta \in (2,3]$, $\delta\chi(t) = t\,\mathrm{d}\chi/\mathrm{d}t$, and $\varphi_p(s) = |s|^{p-2}s$ is the p-Laplacian, $p > 1$, $s \in \mathbb{R}$. The functions f, g, η satisfy:

(H1) $f, g \in C([1,\mathrm{e}] \times \mathbb{R}^+, \mathbb{R}^+)$,
(H2) η is a nonnegative continuous function on $[1,\mathrm{e}]$ with $\eta(t) \not\equiv 0$, $t \in [1,\mathrm{e}]$.

Fractional calculus and fractional boundary value problems arise in physical mechanics, anomalous diffusion, automatic control, biomedicine, etc.; we refer the reader to [1–6, 8, 10–16, 18, 20–35] and the references therein. For example, in [26] the author studied the following model in the fractional sense of the system for HIV-1 population dynamics:

$$\begin{aligned}
&D_t^\alpha u(t) + \lambda f\bigl(t, u(t), D_t^\beta u(t), v(t)\bigr) = 0, \quad 0 < t < 1,\\
&D_t^\gamma v(t) + \lambda g\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1,\\
&D_t^\beta u(0) = D_t^{\beta+1} u(0) = 0, \qquad D_t^\beta u(1) = \int_0^1 D_t^\beta u(s)\,\mathrm{d}A(s),\\
&v(0) = v'(0) = 0, \qquad v(1) = \int_0^1 v(s)\,\mathrm{d}B(s),
\end{aligned}$$

where $2 < \alpha, \gamma \leqslant 3$, $0 < \beta < 1$, u denotes the number of uninfected CD4$^+$T cells, and v denotes the number of infected cells, D_t^α, D_t^β, D_t^γ are the Riemann–Liouville derivatives.

In [1], the author studied the Riemann–Stieltjes integral boundary value problem of the Hadamard-type fractional differential equation

$$\begin{aligned}
&D^\alpha \xi(\varsigma) = \varOmega\bigl(\varsigma, \xi(\varsigma), D^\alpha \xi(\varsigma)\bigr), \quad \varsigma \in [1,T],\ T < \infty,\\
&\xi(1) = 0, \qquad \int_1^T \xi(s)\,\mathrm{d}A(s) = \frac{\mu}{\Gamma(\beta)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{\beta-1} \xi(s)\frac{\mathrm{d}s}{s} = \mu I^\beta \xi(\eta), \quad 1 < \eta < T,
\end{aligned}$$

where D^α is the Hadamard fractional derivative, I^β is the Hadamard fractional integral with $\alpha \in (1,2]$, $\beta \in (1,1)$, $\varOmega : [1,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition. Using Schauder's fixed point theorem and Banach's contraction principle, the author obtained the existence and uniqueness of nontrivial solutions. In [32], the author investigated the eigenvalue problem of the Hadamard-type fractional differential equations with a p-Laplacian

$$\begin{aligned}
&D^\beta\bigl(\varphi_p\bigl(D^\alpha x(t)\bigr)\bigr) = -\lambda a(t) f\bigl(x(t)\bigr), \quad \alpha \in (3,4],\ \beta \in (2,3],\ t \in [1,\mathrm{e}],\\
&x(1) = \delta x(1) = x(\mathrm{e}) = \delta x(\mathrm{e}) = 0,\\
&\varphi_p\bigl(D^\alpha x(1)\bigr) = \delta\varphi_p\bigl(D^\alpha x(1)\bigr) = 0, \qquad \varphi_p\bigl(D^\alpha x(\mathrm{e})\bigr) = \eta\varphi_p\bigl(D^\alpha x(\xi)\bigr),
\end{aligned}$$

and using the Guo–Krasnosel'skii fixed point theorem on cones, the author obtained some existence and nonexistence results for positive solutions under some $(p-1)$-superlinear and $(p-1)$-sublinear conditions. In [4], the authors used the five functional fixed point theorem to discuss multiple positive solutions for the p-Laplacian Hadamard fractional differential equation with multipoint boundary conditions

$$\begin{aligned}
&D^r\big(\varphi_p\big(D^\gamma v(t)\big)\big) = f\big(t, v(t), D^{\gamma-1}v(t)\big), \quad 2 < r,\ \gamma \leqslant 3,\ t \in (1, \mathrm{e}),\\
&v(1) = v'(1) = D^\gamma v(1) = \big(\varphi_p\big(D^\gamma v(1)\big)\big)' = 0,\\
&D^{r-1}\varphi_p\big(D^\gamma v(\mathrm{e})\big) = \sum_{i=1}^{m-2} a_i\varphi_p\big(D^\gamma v(\sigma_i)\big), \qquad D^{\gamma-1}v(\mathrm{e}) = \sum_{i=1}^{n-2} b_i v(\rho_i).
\end{aligned}$$

Note that the spectral theory of linear operators can be used to study differential equations, and in [29], the authors studied positive solutions to the Hadamard-type fractional integral boundary value problem

$$\begin{aligned}
&D^\mu\chi(t) + f\big(t, \chi(t)\big) = 0, \quad t \in (a, b),\\
&\chi(a) = \chi'(a) = 0, \qquad \chi(b) = \int_a^b h(t)\chi(t)\frac{\mathrm{d}t}{t},
\end{aligned}$$

where $f \in C([a, b] \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies the conditions

(HZ1) $\liminf_{\chi\to 0^+} f(t, \chi)/\chi > \lambda_1$, $\limsup_{\chi\to+\infty} f(t, \chi)/\chi < \lambda_1$ uniformly on $t \in [a, b]$,

(HZ2) $\limsup_{\chi\to 0^+} f(t, \chi)/\chi < \lambda_1$, $\liminf_{\chi\to+\infty} f(t, \chi)/\chi > \lambda_1$ uniformly on $t \in [a, b]$,

where λ_1 is the first eigenvalue of the operator

$$(L_{Z1}\chi)(t) = \int_a^b G(t, s)\chi(s)\frac{\mathrm{d}s}{s},$$

and G is the Green's function.

Motivated by the aforementioned works, in this paper, we use the fixed point index and the Leggett–Williams fixed point theorem to study the existence and multiplicity of positive solutions for (1). We first obtain two existence theorems when f, g grow superlinearly and sublinearly under some conditions concerning the spectral radius of a related linear operator. Then, when f, g satisfy some bounded conditions, we obtain a theorem concerning the multiplicity. We note here that our considered linear operator is totally different from L_{Z1} (see (10) in the following section). Finally, some examples are provided to verify our main results.

2 Preliminaries

In this section, we give the definition of the Hadamard-type fractional derivative; for more details, we refer the reader to [29–32].

Definition 1. The Hadamard derivative of fractional order q for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{\mathrm{d}}{\mathrm{d}t} \right)^n \int_1^t (\ln t - \ln s)^{n-q-1} g(s) \frac{\mathrm{d}s}{s}, \quad n-1 < q < n,$$

where $n = [q] + 1$, $[q]$ denotes the integer part of the real number q, and $\ln(\cdot) = \log_{\mathrm{e}}(\cdot)$.

Now, we calculate the Green's function associated with (1). Let $\varphi_p(D^\beta \chi(t)) = -\omega(t)$, $t \in [1, \mathrm{e}]$. Then from (1) we have

$$\begin{aligned} &-D^\alpha \omega(t) = f\bigl(t, \chi(t)\bigr), \quad t \in (1, \mathrm{e}), \\ &\omega(1) = \omega(\mathrm{e}) = 0, \end{aligned} \tag{2}$$

and we can obtain the following lemma.

Lemma 1. (*See* [13]). (2) *is equivalent to the Hammerstein integral equation*

$$\omega(t) = \int_1^{\mathrm{e}} G_\alpha(t, s) f\bigl(s, \chi(s)\bigr) \frac{\mathrm{d}s}{s},$$

where

$$G_\alpha(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\ln t)^{\alpha-1}(1 - \ln s)^{\alpha-1} - (\ln t - \ln s)^{\alpha-1}, & 1 \leqslant s \leqslant t \leqslant \mathrm{e}, \\ (\ln t)^{\alpha-1}(1 - \ln s)^{\alpha-1}, & 1 \leqslant t \leqslant s \leqslant \mathrm{e}. \end{cases}$$

By Lemma 1 and (1) we obtain the following boundary value problem:

$$\begin{aligned} &-D^\beta \chi(t) = \varphi_q\left(\int_1^{\mathrm{e}} G_\alpha(t, s) f\bigl(s, \chi(s)\bigr) \frac{\mathrm{d}s}{s} \right), \quad t \in (1, \mathrm{e}), \\ &\chi(1) = \delta\chi(1) = 0, \qquad \delta\chi(\mathrm{e}) = \int_1^{\mathrm{e}} \eta(t) g\bigl(t, \chi(t)\bigr) \frac{\mathrm{d}t}{t}, \end{aligned} \tag{3}$$

where q is the conjugate number of p, i.e., $1/p + 1/q = 1$.

Lemma 2. (3) *is equivalent to the following integral equation*:

$$\chi(t) = \int_1^{\mathrm{e}} G_\beta(t, s) \varphi_q\left(\int_1^{\mathrm{e}} G_\alpha(s, \tau) f\bigl(\tau, \chi(\tau)\bigr) \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{d}s}{s} + \frac{(\ln t)^{\beta-1}}{\beta - 1} \int_1^{\mathrm{e}} \eta(t) g\bigl(t, \chi(t)\bigr) \frac{\mathrm{d}t}{t},$$

where

$$G_\beta(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} (\ln t)^{\beta-1}(1 - \ln s)^{\beta-2} - (\ln t - \ln s)^{\beta-1}, & 1 \leqslant s \leqslant t \leqslant \mathrm{e}, \\ (\ln t)^{\beta-1}(1 - \ln s)^{\beta-2}, & 1 \leqslant t \leqslant s \leqslant \mathrm{e}. \end{cases}$$

Proof. We first consider the boundary value problem

$$\begin{aligned} &-D^{\beta}\chi(t)=\varphi_q\left(\int_1^{\mathrm{e}} G_\alpha(t,s)f\big(s,\chi(s)\big)\frac{\mathrm{d}s}{s}\right), \quad t\in(1,\mathrm{e}),\\ &\chi(1)=\delta\chi(1)=\delta\chi(\mathrm{e})=0. \end{aligned} \tag{4}$$

From the methods in [31] we obtain

$$\chi(t)=c_1(\ln t)^{\beta-1}+c_2(\ln t)^{\beta-2}+c_3(\ln t)^{\beta-3}-\frac{1}{\Gamma(\beta)}\int_1^t(\ln t-\ln s)^{\beta-1}\varphi_q\big(\omega(s)\big)\frac{\mathrm{d}s}{s},$$

where $c_i\in\mathbb{R}$, $i=1,2,3$. Note that $\chi(1)=\delta\chi(1)=0$, and we have $c_2=c_3=0$. Moreover, $\delta\chi(\mathrm{e})=0$ implies that

$$\chi(\mathrm{e})=c_1(\beta-1)-\frac{\beta-1}{\Gamma(\beta)}\int_1^{\mathrm{e}}(1-\ln s)^{\beta-2}\varphi_q\left(\int_1^{\mathrm{e}}G_\alpha(s,\tau)f\big(\tau,\chi(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}=0,$$

and then

$$\begin{aligned} &\chi(t)\\ &=\frac{1}{\Gamma(\beta)}\int_1^{\mathrm{e}}(\ln t)^{\beta-1}(1-\ln s)^{\beta-2}\varphi_q\big(\omega(s)\big)\frac{\mathrm{d}s}{s}-\frac{1}{\Gamma(\beta)}\int_1^t(\ln t-\ln s)^{\beta-1}\varphi_q\big(\omega(s)\big)\frac{\mathrm{d}s}{s}\\ &=\int_1^{\mathrm{e}}G_\beta(t,s)\varphi_q\big(\omega(s)\big)\frac{\mathrm{d}s}{s}. \end{aligned} \tag{5}$$

Next, we study the boundary value problem

$$\begin{aligned} &-D^{\beta}\chi(t)=0, \quad t\in(1,\mathrm{e}),\\ &\chi(1)=\delta\chi(1)=0, \qquad \delta\chi(\mathrm{e})=\int_1^{\mathrm{e}}\eta(t)g\big(t,\chi(t)\big)\frac{\mathrm{d}t}{t}. \end{aligned} \tag{6}$$

From (6) we have

$$\chi(t)=\widetilde{c}_1(\ln t)^{\beta-1}+\widetilde{c}_2(\ln t)^{\beta-2}+\widetilde{c}_3(\ln t)^{\beta-3},$$

where $\widetilde{c}_i\in\mathbb{R}$, $i=1,2,3$. Similarly, we have $\widetilde{c}_2=\widetilde{c}_3=0$. Consequently, we obtain

$$\delta\chi(\mathrm{e})=\widetilde{c}_1(\beta-1)=\int_1^{\mathrm{e}}\eta(t)g\big(t,\chi(t)\big)\frac{\mathrm{d}t}{t},$$

$$\chi(t)=\frac{(\ln t)^{\beta-1}}{\beta-1}\int_1^{\mathrm{e}}\eta(t)g\big(t,\chi(t)\big)\frac{\mathrm{d}t}{t}. \tag{7}$$

Combining (4)–(7), we obtain the conclusion of this lemma. This completes the proof. □

Remark 1. If we use t, s to replace $\ln t$, $\ln s$, we can find that G_α, G_β are the Green's functions associated with the following two boundary value problems involving Riemann–Liouville-type fractional derivatives:

$$-D^{\alpha}_{0+}u(t) = \varrho_1(t), \quad t \in (0,1), \qquad u(0) = u(1) = 0 \tag{8}$$

and

$$-D^{\beta}_{0+}u(t) = \varrho_2(t), \quad t \in (0,1), \qquad u(0) = u'(0) = u'(1) = 0, \tag{9}$$

where D^{α}_{0+}, D^{β}_{0+} are the Riemann–Liouville-type fractional derivatives with $\alpha \in (1,2]$, $\beta \in (2,3]$. Therefore, in this paper, we do not construct new Green's functions, and we can study G_α, G_β from the properties of the Green's functions for (8)–(9).

Lemma 3. (*See* [8]). *The functions G_α, G_β have the following properties:*

(ii) $G_\alpha \in C([1,\mathrm{e}] \times [1,\mathrm{e}], \mathbb{R}^+)$ *and* $\Gamma(\alpha)G_\alpha(t,s) \leqslant 1$ *for* $t, s \in [1,\mathrm{e}]$,
(iii) $G_\beta \in C([1,\mathrm{e}] \times [1,\mathrm{e}], \mathbb{R}^+)$ *and* $\Gamma(\beta)G_\beta(t,s) \leqslant 1$ *for* $t, s \in [1,\mathrm{e}]$,
(iiii) $(\ln t)^{\beta-1}G_\beta(\mathrm{e},s) \leqslant G_\beta(t,s) \leqslant G_\beta(\mathrm{e},s)$ *for* $t, s \in [1,\mathrm{e}]$.

Let $E = C[1,\mathrm{e}]$ with the norm $\|\chi\| = \max_{1\leqslant t\leqslant \mathrm{e}} |\chi(t)|$. Define a cone P by $P = \{\chi \in E\colon \chi(t) \geqslant 0,\ t \in [1,\mathrm{e}]\}$. Then E is a Banach space, and P a closed cone on E. Moreover, E's conjugate space is $E^* = \{z\colon z$ has bounded variation on $[1,\mathrm{e}]\}$, and P's dual cone $P^* := \{z \in E^*\colon z$ is nondecreasing on $[1,\mathrm{e}]\}$. Let $\overline{\eta}(t) = \int_1^t \eta(s)\,\mathrm{d}s$, $t \in [1,\mathrm{e}]$. Then from (H2) we have that $\overline{\eta}$ is a nondecreasing continuous function on $[1,\mathrm{e}]$. Note that we can choose a sufficiently large positive constant Λ_η such that

$$\bigl|\overline{\eta}(t)\bigr| < \Lambda_\eta, \quad t \in [1,\mathrm{e}].$$

Hence, we can define a positive continuous function $\widetilde{\eta} = \overline{\eta} + \Lambda_\eta$ such that $\mathrm{d}\widetilde{\eta}/\mathrm{d}t = \eta(t)$, $t \in [1,\mathrm{e}]$, and $\widetilde{\eta} \in P^*$. Consequently, from Lemma 2 we can define an operator $T : E \to E$ as follows:

$$\begin{aligned}
&(T\chi)(t)\\
&\quad= \int_1^{\mathrm{e}} G_\beta(t,s)\varphi_q\biggl(\int_1^{\mathrm{e}} G_\alpha(s,\tau)f\bigl(\tau,\chi(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\biggr)\frac{\mathrm{d}s}{s} + \frac{(\ln t)^{\beta-1}}{\beta-1}\int_1^{\mathrm{e}} \eta(t)g\bigl(t,\chi(t)\bigr)\frac{\mathrm{d}t}{t}\\
&\quad= \int_1^{\mathrm{e}} G_\beta(t,s)\varphi_q\biggl(\int_1^{\mathrm{e}} G_\alpha(s,\tau)f\bigl(\tau,\chi(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\biggr)\frac{\mathrm{d}s}{s} + \frac{(\ln t)^{\beta-1}}{\beta-1}\int_1^{\mathrm{e}} g\bigl(t,\chi(t)\bigr)\frac{\mathrm{d}\widetilde{\eta}(t)}{t},
\end{aligned}$$

where $t \in [1,\mathrm{e}]$. Moreover, from Lemma 3(ii)–(iii) and (H1)–(H2) we have

$$T(P) \subset P,$$

and if there exists $\chi^* \in P \setminus \{0\}$ such that $T\chi^* = \chi^*$, then χ^* is the positive solution to (1).

Let μ, ν be two positive constants, and consider a linear operator as follows:

$$(L^*_{\mu,\nu}z)(\mathrm{e}^s) = \mu\int_1^{\mathrm{e}^s}\mathrm{d}\tau\int_1^{\mathrm{e}} G_{\beta,\alpha}(t,\tau)\frac{\mathrm{d}z(t)}{t} + \widetilde{\eta}(\mathrm{e}^s)\nu\int_1^{\mathrm{e}}(\ln t)^{\beta-1}\frac{\mathrm{d}z(t)}{t}, \tag{10}$$

where $z \in P^*$, $s \in [0,1]$, and

$$G_{\beta,\alpha}(t,s) = \int_1^{\mathrm{e}} G_\beta(t,\tau)G_\alpha(\tau,s)\frac{\mathrm{d}\tau}{\tau}, \quad t,s \in [1,\mathrm{e}].$$

Note that $L^*_{\mu,\nu}$ is a positive linear operator, i.e., $L^*_{\mu,\nu}(P^*) \subset P$. Next, we shall prove that the spectral radius of $L^*_{\mu,\nu}$, denoted by $r(L^*_{\mu,\nu})$, is positive.

Lemma 4. $r(L^*_{\mu,\nu}) > 0$.

Proof. Let $(L^*_\nu z)(s) = \widetilde{\eta}(s)\nu\int_1^{\mathrm{e}}(\ln t)^{\beta-1}\,\mathrm{d}z(t)/t$, $z \in P^*$, $s \in [1,\mathrm{e}]$. Then for all $n \in \mathbb{N}_+$, we find

$$\big((L^*_\nu)^n z\big)(s) = \widetilde{\eta}(s)\nu^n\left(\int_1^{\mathrm{e}}\eta(t)(\ln t)^{\beta-1}\frac{\mathrm{d}t}{t}\right)^{n-1}\int_1^{\mathrm{e}}(\ln t)^{\beta-1}\frac{\mathrm{d}z(t)}{t}, \quad s \in [1,\mathrm{e}].$$

Choose $z_0 = t/\mathrm{e}$, $t \in [1,\mathrm{e}]$, and let $\widetilde{\eta}_m = \min_{s\in[1,\mathrm{e}]}\widetilde{\eta}(s) > 0$. Then we have

$$\begin{aligned}\big\|(L^*_\nu)^n\big\| &= \sup_{\|z\|=1}\big\|\big((L^*_\nu)^n z\big)\big\| \geqslant \big\|\big((L^*_\nu)^n z_0\big)\big\| \\ &\geqslant \frac{\widetilde{\eta}_m}{\mathrm{e}}\nu^n\left(\int_1^{\mathrm{e}}\eta(t)(\ln t)^{\beta-1}\frac{\mathrm{d}t}{t}\right)^{n-1}\int_1^{\mathrm{e}}(\ln t)^{\beta-1}\frac{\mathrm{d}t}{t}.\end{aligned}$$

Consequently, Gelfand's theorem implies that

$$\begin{aligned}r(L^*_\nu) &= \liminf_{n\to+\infty}\sqrt[n]{\big\|(L^*_\nu)^n\big\|} \\ &\geqslant \liminf_{n\to+\infty}\sqrt[n]{\frac{\widetilde{\eta}_m}{\mathrm{e}}\nu^n\left(\int_1^{\mathrm{e}}\eta(t)(\ln t)^{\beta-1}\frac{\mathrm{d}t}{t}\right)^{n-1}\int_1^{\mathrm{e}}(\ln t)^{\beta-1}\frac{\mathrm{d}t}{t}} \\ &= \nu\int_1^{\mathrm{e}}\eta(t)(\ln t)^{\beta-1}\frac{\mathrm{d}t}{t} > 0.\end{aligned}$$

Therefore, we have $r(L^*_{\mu,\nu}) \geqslant r(L^*_\nu) > 0$. This completes the proof. □

Therefore, Lemma 4 and the Krein–Rutman theorem [17] imply that there exists $\vartheta_{\mu,\nu} \in P^* \setminus \{0\}$ such that

$$(L^*_{\mu,\nu}\vartheta_{\mu,\nu})(\mathrm{e}^s) = r(L^*_{\mu,\nu})\vartheta_{\mu,\nu}(\mathrm{e}^s), \quad s \in [0,1]. \tag{11}$$

Lemma 5. *Suppose that* (H1)–(H2) *hold, and let*

$$P_0 = \{\chi \in P\colon\ \chi(t) \geqslant (\ln t)^{\beta-1}\|\chi\|,\ t \in [1, \mathrm{e}]\}.$$

Then $T(P) \subset P_0$.

This is a direct result of Lemma 3(iiii), so we omit its proof.

Lemma 6. (*See* [9]). *Suppose* $\Omega \subset E$ *is a bounded open set and* $A : \overline{\Omega} \cap P \to P$ *is a completely continuous operator. If there exists* $\chi_0 \in P \setminus \{0\}$ *such that* $\chi - A\chi \neq \lambda\chi_0$ *for all* $\lambda \geqslant 0$, $\chi \in \partial\Omega \cap P$, *then* $i(A, \Omega \cap P, P) = 0$.

Lemma 7. (*See* [9]). *Let* $\Omega \subset E$ *be a bounded open set with* $0 \in \Omega$. *Suppose* $A : \overline{\Omega} \cap P \to P$ *is a completely continuous operator. If* $\chi \neq \lambda A\chi$ *for all* $\chi \in \partial\Omega \cap P$, $0 \leqslant \lambda \leqslant 1$, *then* $i(A, \Omega \cap P, P) = 1$.

Lemma 8. (*See* [7]). *Let* $\theta > 0$, $n \geqslant 1$, $a_i \geqslant 0$ $(i = 1, 2, \dots, n)$, *and* $\psi \in C([a, b], \mathbb{R}^+)$. *Then*

$$\left(\int_a^b \psi(t)\,\mathrm{d}t\right)^{\theta} \leqslant (b-a)^{\theta-1}\int_a^b \big(\psi(t)\big)^{\theta}\,\mathrm{d}t \quad \textit{and} \quad \left(\sum_{i=1}^n a_i\right)^{\theta} \leqslant n^{\theta-1}\sum_{i=1}^n a_i^{\theta}$$

for all $\theta \geqslant 1$;

$$\left(\int_a^b \psi(t)\,\mathrm{d}t\right)^{\theta} \geqslant (b-a)^{\theta-1}\int_a^b \big(\psi(t)\big)^{\theta}\,\mathrm{d}t \quad \textit{and} \quad \left(\sum_{i=1}^n a_i\right)^{\theta} \geqslant n^{\theta-1}\sum_{i=1}^n a_i^{\theta}$$

for all $0 < \theta \leqslant 1$.

Let E be a real Banach space with a cone P. A map $\widetilde{\beta} : P \to \mathbb{R}^+$ is said to be a nonnegative continuous concave functional on P if $\widetilde{\beta}$ is continuous and

$$\widetilde{\beta}\big(t\chi + (1-t)\mathcal{Y}\big) \geqslant t\widetilde{\beta}(\chi) + (1-t)\widetilde{\beta}(\mathcal{Y}) \quad \text{for all } \chi, \mathcal{Y} \in P,\ t \in [0, 1].$$

Let $\widetilde{a}$, $\widetilde{b}$ be two numbers with $0 < \widetilde{a} < \widetilde{b}$, and $\widetilde{\beta}$ be a nonnegative continuous concave functional on P. We define the following convex sets:

$$P_{\widetilde{a}} = \{\chi \in P\colon\ \|\chi\| < \widetilde{a}\}, \quad \partial P_{\widetilde{a}} = \{\chi \in P\colon\ \|\chi\| = \widetilde{a}\},$$
$$\overline{P}_{\widetilde{a}} = \{\chi \in P\colon\ \|\chi\| \leqslant \widetilde{a}\}, \quad P(\widetilde{\beta}, \widetilde{a}, \widetilde{b}) = \{\chi \in P\colon\ \widetilde{a} \leqslant \widetilde{\beta}(\chi),\ \|\chi\| \leqslant \widetilde{b}\}.$$

Lemma 9. (*See* [19]). *Let* $T : \overline{P}_{\widetilde{c}} \to \overline{P}_{\widetilde{c}}$ *be completely continuous, and let* $\widetilde{\beta}$ *be a nonnegative continuous concave functional on* P *such that* $\widetilde{\beta}(\chi) \leqslant \|\chi\|$ *for* $\chi \in \overline{P}_{\widetilde{c}}$. *Suppose that there exist* $0 < \widetilde{d} < \widetilde{a} < \widetilde{b} \leqslant \widetilde{c}$ *such that*

(i) $\{\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{b})\colon\ \widetilde{\beta}(\chi) > \widetilde{a}\} \neq \emptyset$ *and* $\widetilde{\beta}(T\chi) > \widetilde{a}$ *for* $\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{b})$,
(ii) $\|T\chi\| < \widetilde{d}$ *for* $\|\chi\| \leqslant \widetilde{d}$,
(iii) $\widetilde{\beta}(T\chi) > \widetilde{a}$ *for* $\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{c})$ *with* $\|T\chi\| > \widetilde{b}$.

Then T *has at least three fixed points* χ_1, χ_2, χ_3 *in* $\overline{P}_{\widetilde{c}}$ *such that*

$$\|\chi_1\| < \widetilde{d}, \quad \widetilde{a} < \widetilde{\beta}(\chi_2) \quad \textit{and} \quad \|\chi_3\| > \widetilde{d}, \quad \widetilde{\beta}(\chi_3) < \widetilde{a}.$$

3 Main results

Note that the operator T can be expressed as

$$(T\chi)(t) = \int_0^1 G_\beta\big(t, \mathrm{e}^s\big)\left(\int_0^1 G_\alpha\big(\mathrm{e}^s, \mathrm{e}^\tau\big) f\big(\mathrm{e}^\tau, \chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau\right)^{1/(p-1)} \mathrm{d}s$$

$$+ \frac{(\ln t)^{\beta-1}}{\beta-1}\int_0^1 \eta\big(\mathrm{e}^\tau\big) g\big(\mathrm{e}^\tau, \chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau, \quad t \in [1, \mathrm{e}].$$

Now let $p^* = \max\{1,\, p-1\}$, $p_* = \min\{1,\, p-1\}$, $\eta_m = \max_{t\in[1,\mathrm{e}]} \eta(t)$, and

$$\widetilde{\omega}_{\alpha,\beta} = \left[\frac{2}{\Gamma(\beta)}\right]^{p_*-1}\big[\Gamma(\alpha)\big]^{1-p_*/(p-1)}, \qquad \overline{\omega}_{\alpha,\beta} = \left[\frac{2}{\Gamma(\beta)}\right]^{p^*-1}\big[\Gamma(\alpha)\big]^{1-p^*/(p-1)}.$$

Now, we list our assumptions for our nonlinearities f, g.

(H3) There exist $\mu_1, \nu_1 > 0$ such that $r(L^*_{\mu_1,\nu_1}) > 1$ and

$$\liminf_{\chi\to+\infty} \frac{f(t,\chi)}{\chi^{p-1}} \geqslant \left(\frac{\mu_1}{\widetilde{\omega}_{\alpha,\beta}}\right)^{(p-1)/p_*},$$
$$\liminf_{\chi\to+\infty} \frac{g(t,\chi)}{\chi} \geqslant (\beta-1)(2\eta_m)^{(1-p_*)/p_*}\nu_1^{1/p_*}$$

uniformly on $t \in [1, \mathrm{e}]$.

(H4) There exist $\mu_2, \nu_2 > 0$ such that $r(L^*_{\mu_2,\nu_2}) < 1$ and

$$\limsup_{\chi\to 0^+} \frac{f(t,\chi)}{\chi^{p-1}} \leqslant \left(\frac{\mu_2}{\overline{\omega}_{\alpha,\beta}}\right)^{(p-1)/p^*},$$
$$\limsup_{\chi\to 0^+} \frac{g(t,\chi)}{\chi} \leqslant (\beta-1)(2\eta_m)^{(1-p^*)/p^*}\nu_2^{1/p^*}$$

uniformly on $t \in [1, \mathrm{e}]$.

(H5) There exist $\mu_3, \nu_3 > 0$ such that $r(L^*_{\mu_3,\nu_3}) > 1$ and

$$\liminf_{\chi\to 0^+} \frac{f(t,\chi)}{\chi^{p-1}} \geqslant \left(\frac{\mu_3}{\widetilde{\omega}_{\alpha,\beta}}\right)^{(p-1)/p_*},$$
$$\liminf_{\chi\to 0^+} \frac{g(t,\chi)}{\chi} \geqslant (\beta-1)(2\eta_m)^{(1-p_*)/p_*}\nu_3^{1/p_*}$$

uniformly on $t \in [1, \mathrm{e}]$.

(H6) There exist $\mu_4, \nu_4 > 0$ such that $r(L^*_{\mu_4,\nu_4}) < 1$ and

$$\limsup_{\chi\to+\infty} \frac{f(t,\chi)}{\chi^{p-1}} \leqslant \left(\frac{2^{1-p^*/(p-1)}\mu_4}{\overline{\omega}_{\alpha,\beta}}\right)^{(p-1)/p^*},$$
$$\limsup_{\chi\to+\infty} \frac{g(t,\chi)}{\chi} \leqslant (\beta-1)(4\eta_m)^{(1-p^*)/p^*}\nu_4^{1/p^*}$$

uniformly on $t \in [1, \mathrm{e}]$.

Theorem 1. *Suppose that* (H1)–(H4) *hold. Then* (1) *has at least one positive solution.*

Proof. From (H3) there exist $d_1, d_2 > 0$ such that

$$f(t,\chi) \geqslant \left(\frac{\mu_1}{\widetilde{\omega}_{\alpha,\beta}}\right)^{(p-1)/p_*}\chi^{p-1} - d_1,$$
$$g(t,\chi) \geqslant (\beta-1)(2\eta_m)^{(1-p_*)/p_*}\nu_1^{1/p_*}\chi - d_2, \quad t \in [1,\mathrm{e}],\ \chi \in \mathbb{R}^+.$$

Noting that $p_*, p_*/(p-1) \in (0,1]$, we have

$$\left[f(t,\chi)+d_1\right]^{p_*/(p-1)} \leqslant f^{p_*/(p-1)}(t,\chi) + d_1^{p_*/(p-1)},$$

and

$$\left[g(t,\chi)+d_2\right]^{p_*} \leqslant g^{p_*}(t,\chi) + d_2^{p_*}.$$

Consequently, we find

$$\begin{aligned} f^{p_*/(p-1)}(t,\chi) &\geqslant \left[f(t,\chi)+d_1\right]^{p_*/(p-1)} - d_1^{p_*/(p-1)} \\ &\geqslant \left[\left(\frac{\mu_1}{\widetilde{\omega}_{\alpha,\beta}}\right)^{(p-1)/p_*}\chi^{p-1}\right]^{p_*/(p-1)} - d_1^{p_*/(p-1)} \\ &= \frac{\mu_1}{\widetilde{\omega}_{\alpha,\beta}}\chi^{p_*} - d_1^{p_*/(p-1)} \end{aligned} \tag{12}$$

and

$$\begin{aligned} g^{p_*}(t,\chi) &\geqslant \left[g(t,\chi)+d_2\right]^{p_*} - d_2^{p_*} \\ &\geqslant \left[(\beta-1)(2\eta_m)^{(1-p_*)/p_*}\nu_1^{1/p_*}\chi\right]^{p_*} - d_2^{p_*} \\ &= (\beta-1)^{p_*}(2\eta_m)^{1-p_*}\nu_1\chi^{p_*} - d_2^{p_*}, \quad t \in [1,\mathrm{e}],\ \chi \in \mathbb{R}^+. \end{aligned} \tag{13}$$

Now, we prove that the set

$$W_1 = \left\{\chi \in P\colon\ \chi = T\chi + \lambda\overline{\phi},\ \lambda \geqslant 0\right\}$$

is bounded in P, where $\overline{\phi} \in P_0$ is a given element. Note that if there exists $\chi \in W_1$, from Lemma 5 we have

$$\chi \in P_0. \tag{14}$$

Moreover, $\lambda \geqslant 0$ and $\overline{\phi} \in P_0$ enable us to find

$$\chi(t) \geqslant (T\chi)(t), \quad t \in [1,\mathrm{e}].$$

Therefore, we have

$$
\begin{aligned}
\chi^{p_*}(t) \geqslant{} & \Biggl[\int_0^1 G_\beta\bigl(t,\mathrm{e}^s\bigr)\Biggl(\int_0^1 G_\alpha\bigl(\mathrm{e}^s,\mathrm{e}^\tau\bigr) f\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr)^{1/(p-1)}\mathrm{d}s \\
& + \frac{(\ln t)^{\beta-1}}{\beta-1}\int_0^1 \eta\bigl(\mathrm{e}^\tau\bigr) g\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{p_*} \\
\geqslant{} & 2^{p_*-1}\Biggl[\int_0^1 \frac{\Gamma(\beta)G_\beta\bigl(t,\mathrm{e}^s\bigr)}{\Gamma(\beta)}\Biggl(\int_0^1 \frac{\Gamma(\alpha)G_\alpha\bigl(\mathrm{e}^s,\mathrm{e}^\tau\bigr)}{\Gamma(\alpha)} f\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr)^{1/(p-1)}\mathrm{d}s\Biggr]^{p_*} \\
& + 2^{p_*-1}\Biggl[\frac{(\ln t)^{\beta-1}}{\beta-1}\int_0^1 \frac{\eta(\mathrm{e}^\tau)}{\eta_m}\eta_m g\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{p_*} \\
\geqslant{} & 2^{p_*-1}\int_0^1 \frac{[\Gamma(\beta)G_\beta\bigl(t,\mathrm{e}^s\bigr)]^{p_*}}{[\Gamma(\beta)]^{p_*}}\Biggl(\int_0^1 \frac{\Gamma(\alpha)G_\alpha(\mathrm{e}^s,\mathrm{e}^\tau)}{\Gamma(\alpha)} f\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr)^{p_*/(p-1)}\mathrm{d}s \\
& + 2^{p_*-1}\biggl[\frac{(\ln t)^{(\beta-1)}}{\beta-1}\biggr]^{p_*}\int_0^1 \biggl[\frac{\eta(\mathrm{e}^\tau)}{\eta_m}\biggr]^{p_*}\eta_m^{p_*} g^{p_*}\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau \\
\geqslant{} & 2^{p_*-1}\int_0^1 \frac{\Gamma(\beta)G_\beta(t,\mathrm{e}^s)}{[\Gamma(\beta)]^{p_*}}\int_0^1 \frac{\Gamma(\alpha)G_\alpha(\mathrm{e}^s,\mathrm{e}^\tau)}{[\Gamma(\alpha)]^{p_*/(p-1)}} f^{p_*/(p-1)}\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s \\
& + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\ln t)^{\beta-1}\int_0^1 \frac{\eta(\mathrm{e}^\tau)}{\eta_m}\eta_m^{p_*} g^{p_*}\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau \\
={} & \biggl[\frac{2}{\Gamma(\beta)}\biggr]^{p_*-1}[\Gamma(\alpha)]^{1-p_*/(p-1)}\int_1^{\mathrm{e}}\int_1^{\mathrm{e}} G_\beta(t,s)G_\alpha(s,\tau) f^{p_*/(p-1)}\bigl(\tau,\chi(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}s}{s} \\
& + \frac{(2\eta_m)^{p_*-1}}{(\beta-1)^{p_*}}(\ln t)^{\beta-1}\int_1^{\mathrm{e}} \eta(\tau) g^{p_*}\bigl(\tau,\chi(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}. \qquad (15)
\end{aligned}
$$

This, together with (12)–(13), implies that

$$
\begin{aligned}
\chi^{p_*}(t) \geqslant{} & \widetilde{\omega}_{\alpha,\beta}\int_1^{\mathrm{e}} G_{\beta,\alpha}(t,\tau)\biggl[\frac{\mu_1}{\widetilde{\omega}_{\alpha,\beta}}\chi^{p_*}(\tau) - d_1^{p_*/(p-1)}\biggr]\frac{\mathrm{d}\tau}{\tau} \\
& + \frac{(2\eta_m)^{p_*-1}}{(\beta-1)^{p_*}}(\ln t)^{\beta-1}\int_1^{\mathrm{e}} \eta(t)\bigl[(\beta-1)^{p_*}(2\eta_m)^{1-p_*}\nu_1\chi^{p_*}(t) - d_2^{p_*}\bigr]\frac{\mathrm{d}t}{t}
\end{aligned}
$$

$$= \mu_1 \int\limits_1^{\mathrm{e}} G_{\beta,\alpha}(t,s)\chi^{p_*}(s)\frac{\mathrm{d}s}{s} + (\ln t)^{\beta-1}\nu_1 \int\limits_1^{\mathrm{e}} \chi^{p_*}(s)\frac{\mathrm{d}\widetilde{\eta}(s)}{s}$$
$$- \frac{d_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} - \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}. \tag{16}$$

Therefore, multiplying by $\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)/t$ on both sides of (16) and integrating over $[1,\mathrm{e}]$, from (11) we obtain

$$\begin{aligned}
&\int\limits_1^{\mathrm{e}} \chi^{p_*}(t)\frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} \\
&\geqslant \int\limits_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t}\left(\mu_1 \int\limits_1^{\mathrm{e}} G_{\beta,\alpha}(t,s)\chi^{p_*}(s)\frac{\mathrm{d}s}{s} + (\ln t)^{\beta-1}\nu_1 \int\limits_1^{\mathrm{e}} \chi^{p_*}(s)\frac{\mathrm{d}\widetilde{\eta}(s)}{s}\right) \\
&\quad - \left[\frac{\mathrm{d}_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}\right]\int\limits_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} \\
&= \int\limits_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t}\left(\mu_1 \int\limits_0^{1} G_{\beta,\alpha}\big(t,\mathrm{e}^s\big)\chi^{p_*}(\mathrm{e}^s)\,\mathrm{d}s + (\ln t)^{\beta-1}\nu_1 \int\limits_0^{1} \chi^{p_*}\big(\mathrm{e}^s\big)\eta\big(\mathrm{e}^s\big)\,\mathrm{d}s\right) \\
&\quad - \left[\frac{\mathrm{d}_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}\right]\int\limits_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} \\
&= \int\limits_0^{1} \frac{\chi^{p_*}(\mathrm{e}^s)}{\mathrm{e}^s}\,\mathrm{d}\left(\mu_1\int\limits_1^{\mathrm{e}^s}\mathrm{d}\tau\int\limits_1^{\mathrm{e}} G_{\beta,\alpha}(t,\tau)\frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} + \widetilde{\eta}(\mathrm{e}^s)\nu_1\int\limits_1^{\mathrm{e}}(\ln t)^{\beta-1}\frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t}\right) \\
&\quad - \left[\frac{\mathrm{d}_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}\right]\int\limits_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} \\
&= \int\limits_0^{1} \frac{\chi^{p_*}(\mathrm{e}^s)}{\mathrm{e}^s}\,\mathrm{d}\big(L^*_{\mu_1,\nu_1}\vartheta_{\mu_1,\nu_1}\big)\big(\mathrm{e}^s\big) \\
&\quad - \left[\frac{\mathrm{d}_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}\right]\int\limits_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} \\
&= r\big(L^*_{\mu_1,\nu_1}\big)\int\limits_0^{1} \frac{\chi^{p_*}(\mathrm{e}^s)}{\mathrm{e}^s}\,\mathrm{d}\vartheta_{\mu_1,\nu_1}\big(\mathrm{e}^s\big) \\
&\quad - \left[\frac{\mathrm{d}_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}\right]\int\limits_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t}
\end{aligned}$$

$$= r(L^*_{\mu_1,\nu_1}) \int_1^{\mathrm{e}} \chi^{p_*}(t)\frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} - \left[\frac{\mathrm{d}_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}\right] \int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t}. \tag{17}$$

Note that $\vartheta_{\mu_1,\nu_1} \in P^* \setminus \{0\}$, and from the definition of the Riemann–Stieltjes integral we have

$$\int_1^{\mathrm{e}} \chi^{p_*}(t)\frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} = \lim_{\lambda\to 0}\sum_{i=1}^n \frac{\chi^{p_*}(\xi_i)}{\xi_i}\left[\vartheta_{\mu_1,\nu_1}(t_i) - \vartheta_{\mu_1,\nu_1}(t_{i-1})\right] \geqslant 0 \tag{18}$$

and

$$\begin{aligned}\int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} &= \lim_{\lambda\to 0}\sum_{i=1}^n \frac{1}{\xi_i}\left[\vartheta_{\mu_1,\nu_1}(t_i) - \vartheta_{\mu_1,\nu_1}(t_{i-1})\right] \\ &\geqslant \frac{1}{\mathrm{e}}\left[\vartheta_{\mu_1,\nu_1}(\mathrm{e}) - \vartheta_{\mu_1,\nu_1}(1)\right] > 0\end{aligned}$$

for all divisions t_i: $1 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \mathrm{e}$, $\lambda = \max_{1\leqslant i\leqslant n}(t_i - t_{i-1})$, $\xi_i \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

Note that $r(L^*_{\mu_1,\nu_1}) > 1$, and from (17) we obtain

$$\int_1^{\mathrm{e}} \chi^{p_*}(t)\frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t} \leqslant \frac{\left[\frac{\mathrm{d}_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}\right]\int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t}}{r(L^*_{\mu_1,\nu_1}) - 1}.$$

From (14) we have

$$\|\chi\| \leqslant \sqrt[p_*]{\frac{\left[\frac{\mathrm{d}_1^{p_*/(p-1)}\widetilde{\omega}_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{2^{p_*-1}}{(\beta-1)^{p_*}}(\eta_m d_2)^{p_*}\right]\int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t}}{\left[r(L^*_{\mu_1,\nu_1}) - 1\right]\int_1^{\mathrm{e}} (\ln t)^{(\beta-1)p_*}\frac{\mathrm{d}\vartheta_{\mu_1,\nu_1}(t)}{t}}}.$$

This implies that W_1 is a bounded set as required. Taking $R_1 > \sup W_1$, when $\chi \in \partial B_{R_1} \cap P$ and $\lambda \geqslant 0$, we obtain

$$\chi \neq T\chi + \lambda\overline{\phi},$$

where $B_{R_1} = \{\chi \in E\colon \|\chi\| < R_1\}$. Hence, Lemma 6 implies that

$$i(T,\, B_{R_1} \cap P, P) = 0. \tag{19}$$

From (H4) there exists $r_1 \in (0, R_1)$ such that

$$\begin{aligned} &f(t,\chi) \leqslant \left(\frac{\mu_2}{\overline{\omega}_{\alpha,\beta}}\right)^{(p-1)/p^*}\chi^{p-1}, \\ &g(t,\chi) \leqslant (\beta-1)(2\eta_m)^{(1-p^*)/p^*}\nu_2^{1/p^*}\chi, \quad t \in [1,\mathrm{e}],\ \chi \in [0, r_1]. \end{aligned} \tag{20}$$

For this r_1, we prove that

$$\chi \neq \lambda T\chi, \quad \chi \in \partial B_{r_1} \cap P,\ \lambda \in [0,1], \tag{21}$$

where $B_{r_1} = \{\chi \in E\colon \|\chi\| < r_1\}$. Suppose the contrary. Then there exist $\chi \in \partial B_{r_1} \cap P$ and $\lambda \in [0,1]$ such that

$$\chi = \lambda T\chi.$$

This, together with (20), implies that

$$\begin{aligned}
\chi^{p^*}(t) &\leqslant \Biggl[\int_0^1 G_\beta\bigl(t,\mathrm{e}^s\bigr)\Biggl(\int_0^1 G_\alpha\bigl(\mathrm{e}^s,\mathrm{e}^\tau\bigr) f\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr)^{1/(p-1)}\,\mathrm{d}s \\
&\quad + \frac{(\ln t)^{\beta-1}}{\beta-1}\int_0^1 \eta\bigl(\mathrm{e}^\tau\bigr) g\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{p^*} \\
&\leqslant 2^{p^*-1}\Biggl[\int_0^1 \frac{\Gamma(\beta)G_\beta(t,\mathrm{e}^s)}{\Gamma(\beta)} \\
&\qquad \times \Biggl(\int_0^1 \frac{\Gamma(\alpha)G_\alpha(\mathrm{e}^s,\mathrm{e}^\tau)}{\Gamma(\alpha)}\biggl(\frac{\mu_2}{\overline{\omega}_{\alpha,\beta}}\biggr)^{(p-1)/p^*}\chi^{p-1}\bigl(\mathrm{e}^\tau\bigr)\,\mathrm{d}\tau\Biggr)^{1/(p-1)}\,\mathrm{d}s\Biggr]^{p^*} \\
&\quad + 2^{p^*-1}\Biggl[\frac{(\ln t)^{\beta-1}}{\beta-1}\int_0^1 \frac{\eta(\mathrm{e}^\tau)}{\eta_m}\eta_m(\beta-1)(2\eta_m)^{(1-p^*)/p^*}\nu_2^{1/p^*}\chi\bigl(\mathrm{e}^\tau\bigr)\,\mathrm{d}\tau\Biggr]^{p^*} \\
&\leqslant 2^{p^*-1}\int_0^1 \frac{\Gamma(\beta)G_\beta(t,\mathrm{e}^s)}{[\Gamma(\beta)]^{p^*}}\int_0^1 \frac{\Gamma(\alpha)G_\alpha(\mathrm{e}^s,\mathrm{e}^\tau)}{[\Gamma(\alpha)]^{p^*/(p-1)}}\frac{\mu_2}{\overline{\omega}_{\alpha,\beta}}\chi^{p^*}\bigl(\mathrm{e}^\tau\bigr)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + 2^{p^*-1}\frac{(\ln t)^{\beta-1}}{(\beta-1)^{p^*}}\int_0^1 \frac{\eta(\mathrm{e}^\tau)}{\eta_m}\bigl[\eta_m(\beta-1)\bigr]^{p^*}(2\eta_m)^{1-p^*}\nu_2\chi^{p^*}\bigl(\mathrm{e}^\tau\bigr)\,\mathrm{d}\tau \\
&= \mu_2\int_1^{\mathrm{e}} G_{\beta,\alpha}(t,s)\chi^{p^*}(s)\frac{\mathrm{d}s}{s} + (\ln t)^{\beta-1}\nu_2\int_1^{\mathrm{e}} \eta(s)\chi^{p^*}(s)\frac{\mathrm{d}s}{s}.
\end{aligned} \tag{22}$$

Consequently, multiplying by $\mathrm{d}\vartheta_{\mu_2,\nu_2}(t)/t$ on both sides of (22) and integrating over $[1,\mathrm{e}]$, from (11) we have

$$\begin{aligned}
&\int_1^{\mathrm{e}} \chi^{p^*}(t)\frac{\mathrm{d}\vartheta_{\mu_2,\nu_2}(t)}{t} \\
&\quad \leqslant \int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_2,\nu_2}(t)}{t}\Biggl(\mu_2\int_1^{\mathrm{e}} G_{\beta,\alpha}(t,s)\chi^{p^*}(s)\frac{\mathrm{d}s}{s} + (\ln t)^{\beta-1}\nu_2\int_1^{\mathrm{e}} \eta(s)\chi^{p^*}(s)\frac{\mathrm{d}s}{s}\Biggr)
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^{e} \frac{d\vartheta_{\mu_2,\nu_2}(t)}{t}\left(\mu_2\int_0^1 G_{\beta,\alpha}(t,e^s)\chi^{p^*}(e^s)\,ds + (\ln t)^{\beta-1}\nu_2\int_0^1 \eta(e^s)\chi^{p^*}(e^s)\,ds\right)\\
&= \int_0^1 \frac{\chi^{p^*}(e^s)}{e^s}\,d\left(\mu_2\int_1^{e^s} d\tau\int_1^{e} G_{\beta,\alpha}(t,\tau)\frac{d\vartheta_{\mu_2,\nu_2}(t)}{t} + \widetilde{\eta}(e^s)\nu_2\int_1^{e}(\ln t)^{\beta-1}\frac{d\vartheta_{\mu_2,\nu_2}(t)}{t}\right)\\
&= \int_0^1 \frac{\chi^{p^*}(e^s)}{e^s}\,d(L^*_{\mu_2,\nu_2}\vartheta_{\mu_2,\nu_2})(e^s) = r(L^*_{\mu_2,\nu_2})\int_0^1 \frac{\chi^{p^*}(e^s)}{e^s}\,d\vartheta_{\mu_2,\nu_2}(e^s)\\
&= r(L^*_{\mu_2,\nu_2})\int_1^{e} \chi^{p^*}(t)\frac{d\vartheta_{\mu_2,\nu_2}(t)}{t}.
\end{aligned}
\tag{23}
$$

Using a similar method as in (18), we have

$$
\int_1^{e} \chi^{p^*}(t)\frac{d\vartheta_{\mu_2,\nu_2}(t)}{t} \geqslant 0.
$$

Note that $r(L^*_{\mu_2,\nu_2}) < 1$, and by (23) we have

$$
\lim_{\lambda\to 0}\sum_{i=1}^{n}\frac{\chi^{p^*}(\xi_i)}{\xi_i}\left[\vartheta_{\mu_2,\nu_2}(t_i) - \vartheta_{\mu_2,\nu_2}(t_{i-1})\right] =: \int_1^{e}\chi^{p^*}(t)\frac{d\vartheta_{\mu_2,\nu_2}(t)}{t} = 0. \tag{24}
$$

Note that $\vartheta_{\mu_2,\nu_2} \in P^* \setminus \{0\}$, (24) holds for all divisions t_i, so we obtain $\chi^{p^*}(t) \equiv 0$, $t \in [1, e]$. Hence, this contradicts $\chi \in \partial B_{r_1} \cap P$, and (21) holds. Therefore, Lemma 7 implies that

$$
i(T,\, B_{r_1}\cap P,\, P) = 1. \tag{25}
$$

From (19) and (25) we have

$$
i\big(T,\, (B_{R_1}\setminus \overline{B}_{r_1})\cap P,\, P\big) = i(T,\, B_{R_1}\cap P,\, P) - i(T,\, B_{r_1}\cap P,\, P) = -1.
$$

Therefore, the operator T has at least one fixed point in $(B_{R_1}\setminus \overline{B}_{r_1})\cap P$. Thus, (1) has at least one positive solution. This completes the proof. □

Theorem 2. *Suppose that* (H1)–(H2) *and* (H5)–(H6) *hold. Then* (1) *has at least one positive solution.*

Proof. From (H5) there exists a sufficiently small $r_2 > 0$ such that

$$
\begin{aligned}
&f(t,\chi) \geqslant \left(\frac{\mu_3}{\widetilde{\omega}_{\alpha,\beta}}\right)^{(p-1)/p_*}\chi^{p-1},\\
&g(t,\chi) \geqslant (\beta-1)(2\eta_m)^{(1-p_*)/p_*}\nu_3^{1/p_*}\chi, \quad t\in[1,e],\ \chi\in[0,r_2].
\end{aligned}
\tag{26}
$$

For this r_2, we prove that

$$\chi \neq T\chi + \lambda\widetilde{\phi}, \quad \chi \in \partial B_{r_2} \cap P,\ \lambda \geqslant 0, \tag{27}$$

where $\widetilde{\phi}$ is a fixed element in P, and $B_{r_2} = \{\chi \in E\colon \|\chi\| < r_2\}$. Suppose the contrary. Then there exist $\chi \in \partial B_{r_2} \cap P$, $\lambda \geqslant 0$ such that

$$\chi = T\chi + \lambda\widetilde{\phi},$$

and from (15), (26) we have

$$\begin{aligned}\chi^{p_*}(t) &\geqslant \widetilde{\omega}_{\alpha,\beta}\int_1^{\mathrm{e}}\int_1^{\mathrm{e}} G_\beta(t,s)G_\alpha(s,\tau)\frac{\mu_3}{\widetilde{\omega}_{\alpha,\beta}}\chi^{p_*}(\tau)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}s}{s} \\ &\quad + \frac{(2\eta_m)^{p_*-1}}{(\beta-1)^{p_*}}(\ln t)^{\beta-1}\int_1^{\mathrm{e}} \eta(\tau)(\beta-1)^{p_*}(2\eta_m)^{1-p_*}\nu_3\chi^{p_*}(\tau)\frac{\mathrm{d}\tau}{\tau} \\ &= \mu_3\int_1^{\mathrm{e}} G_{\beta,\alpha}(t,s)\chi^{p_*}(s)\frac{\mathrm{d}s}{s} + (\ln t)^{\beta-1}\nu_3\int_1^{\mathrm{e}}\eta(s)\chi^{p_*}(s)\frac{\mathrm{d}s}{s}.\end{aligned} \tag{28}$$

Hence, multiplying by $\mathrm{d}\vartheta_{\mu_3,\nu_3}(t)/t$ on both sides of (28) and integrating over $[1,\mathrm{e}]$, from (11) we have

$$\begin{aligned}&\int_1^{\mathrm{e}}\chi^{p_*}(t)\frac{\mathrm{d}\vartheta_{\mu_3,\nu_3}(t)}{t} \\ &\quad \geqslant \int_1^{\mathrm{e}}\frac{\mathrm{d}\vartheta_{\mu_3,\nu_3}(t)}{t}\left(\mu_3\int_1^{\mathrm{e}} G_{\beta,\alpha}(t,s)\chi^{p_*}(s)\frac{\mathrm{d}s}{s} + (\ln t)^{\beta-1}\nu_3\int_1^{\mathrm{e}}\eta(s)\chi^{p_*}(s)\frac{\mathrm{d}s}{s}\right) \\ &\quad = r(L^*_{\mu_3,\nu_3})\int_1^{\mathrm{e}}\chi^{p_*}(t)\frac{\mathrm{d}\vartheta_{\mu_3,\nu_3}(t)}{t}.\end{aligned}$$

Using a similar method as in the proof of Theorem 1, note that $r(L^*_{\mu_3,\nu_3}) > 1$, and we have

$$\chi^{p_*}(t) \equiv 0, \quad t \in [1,\mathrm{e}].$$

This contradicts $\chi \in \partial B_{r_2} \cap P$, and (27) holds. Therefore, Lemma 6 implies that

$$i(T,\ B_{r_2} \cap P,\ P) = 0. \tag{29}$$

From (H6) there exist $d_3, d_4 > 0$ such that

$$\begin{aligned}f(t,\chi) &\leqslant \left(\frac{2^{1-p^*/(p-1)}\mu_4}{\overline{\omega}_{\alpha,\beta}}\right)^{(p-1)/p^*}\chi^{p-1} + d_3, \\ g(t,\chi) &\leqslant (\beta-1)(4\eta_m)^{(1-p^*)/p^*}\nu_4^{1/p^*}\chi + d_4, \quad t \in [1,\mathrm{e}],\ \chi \in \mathbb{R}^+.\end{aligned}$$

Now, we prove that the set $W_2 = \{\chi \in P\colon\ \chi = \lambda T\chi,\ \lambda \in [0,1]\}$ is bounded in P. Note that if there exists $\chi \in W_2$, from Lemma 5 we have

$$\chi \in P_0. \tag{30}$$

This implies that

$$\begin{aligned}
\chi^{p^*}(t) &\leqslant \Bigg[\int_0^1 G_\beta\big(t,\mathrm{e}^s\big)\bigg(\int_0^1 G_\alpha\big(\mathrm{e}^s,\mathrm{e}^\tau\big)f\big(\mathrm{e}^\tau,\chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau\bigg)^{1/(p-1)}\mathrm{d}s \\
&\quad + \frac{(\ln t)^{\beta-1}}{\beta-1}\int_0^1 \eta\big(\mathrm{e}^\tau\big)g\big(\mathrm{e}^\tau,\chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau\Bigg]^{p^*} \\
&\leqslant 2^{p^*-1}\int_0^1 \frac{\Gamma(\beta)G_\beta(t,\mathrm{e}^s)}{[\Gamma(\beta)]^{p^*}}\int_0^1 \frac{\Gamma(\alpha)G_\alpha(\mathrm{e}^s,\mathrm{e}^\tau)}{[\Gamma(\alpha)]^{p^*/(p-1)}} f^{p^*/(p-1)}\big(\mathrm{e}^\tau,\chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + 2^{p^*-1}\frac{(\ln t)^{\beta-1}}{(\beta-1)^{p^*}}\int_0^1 \frac{\eta(\mathrm{e}^\tau)}{\eta_m}\eta_m^{p^*}g^{p^*}\big(\mathrm{e}^\tau,\chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau \\
&\leqslant \overline{\omega}_{\alpha,\beta}\int_0^1 G_\beta\big(t,\mathrm{e}^s\big)\int_0^1 G_\alpha\big(\mathrm{e}^s,\mathrm{e}^\tau\big)\bigg(\bigg(\frac{2^{1-p^*/p-1}\mu_4}{\overline{\omega}_{\alpha,\beta}}\bigg)^{(p-1)/p^*}\chi^{p-1}\big(\mathrm{e}^\tau\big) \\
&\qquad + d_3\bigg)^{p^*/(p-1)}\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + 2^{p^*-1}\frac{(\ln t)^{\beta-1}}{(\beta-1)^{p^*}}\int_0^1 \frac{\eta(\mathrm{e}^\tau)}{\eta_m}\eta_m^{p^*}\big[(\beta-1)(4\eta_m)^{(1-p^*)/p^*}\nu_4^{1/p^*}\chi\big(\mathrm{e}^\tau\big)+d_4\big]^{p^*}\mathrm{d}\tau \\
&\leqslant 2^{p^*/(p-1)-1}\overline{\omega}_{\alpha,\beta}\int_0^1 G_\beta\big(t,\mathrm{e}^s\big)\int_0^1 G_\alpha\big(\mathrm{e}^s,\mathrm{e}^\tau\big)\frac{2^{1-p^*/(p-1)}\mu_4}{\overline{\omega}_{\alpha,\beta}}\chi^{p^*}\big(\mathrm{e}^\tau\big)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \frac{2^{p^*/(p-1)-1}\overline{\omega}_{\alpha,\beta}d_3^{p^*/(p-1)}}{\Gamma(\alpha)\Gamma(\beta)} \\
&\quad + 4^{p^*-1}\frac{(\ln t)^{\beta-1}}{(\beta-1)^{p^*}}\int_0^1 \frac{\eta(\mathrm{e}^\tau)}{\eta_m}\eta_m^{p^*}(\beta-1)^{p^*}(4\eta_m)^{1-p^*}\nu_4\chi^{p^*}\big(\mathrm{e}^\tau\big)\,\mathrm{d}\tau \\
&\quad + \frac{4^{p^*-1}}{(\beta-1)^{p^*}}(\eta_m d_4)^{p^*} \\
&= \mu_4\int_1^{\mathrm{e}} G_{\beta,\alpha}(t,s)\chi^{p^*}(s)\frac{\mathrm{d}s}{s} + (\ln t)^{\beta-1}\nu_4\int_1^{\mathrm{e}} \eta(s)\chi^{p^*}(s)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{2^{p^*/(p-1)-1}\overline{\omega}_{\alpha,\beta}d_3^{p^*/(p-1)}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{4^{p^*-1}}{(\beta-1)^{p^*}}(\eta_m d_4)^{p^*}.
\end{aligned} \tag{31}$$

Hence, multiplying by $\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)/t$ on both sides of (31) and integrating over $[1, \mathrm{e}]$, from (11) and the method used in the proof of Theorem 1 we have

$$\begin{aligned}
&\int_1^{\mathrm{e}} \chi^{p^*}(t)\frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t} \\
&\quad\leqslant \int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t}\left(\mu_4 \int_1^{\mathrm{e}} G_{\beta,\alpha}(t,s)\chi^{p^*}(s)\frac{\mathrm{d}s}{s} + (\ln t)^{\beta-1}\nu_4 \int_1^{\mathrm{e}} \eta(s)\chi^{p^*}(s)\frac{\mathrm{d}s}{s}\right) \\
&\qquad + \left(\frac{2^{p^*/(p-1)-1}\overline{\omega}_{\alpha,\beta}d_3^{p^*/(p-1)}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{4^{p^*-1}}{(\beta-1)^{p^*}}(\eta_m d_4)^{p^*}\right)\int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t} \\
&\quad\leqslant r(L^*_{\mu_4,\nu_4})\int_1^{\mathrm{e}} \chi^{p^*}(t)\frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t} \\
&\qquad + \left(\frac{2^{p^*/(p-1)-1}\overline{\omega}_{\alpha,\beta}d_3^{p^*/(p-1)}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{4^{p^*-1}}{(\beta-1)^{p^*}}(\eta_m d_4)^{p^*}\right)\int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t}.
\end{aligned}$$

Note that $r(L^*_{\mu_4,\nu_4}) < 1$ and (30), we have

$$\begin{aligned}
\int_1^{\mathrm{e}} (\ln t)^{(\beta-1)p^*}\|\chi\|^{p^*}\frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t} &\leqslant \int_1^{\mathrm{e}} \chi^{p^*}(t)\frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t} \\
&\leqslant \frac{(\frac{2^{\frac{p^*}{p-1}-1}\overline{\omega}_{\alpha,\beta}d_3^{p^*/(p-1)}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{4^{p^*-1}}{(\beta-1)^{p^*}}(\eta_m d_4)^{p^*})\int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t}}{1 - r(L^*_{\mu_4,\nu_4})},
\end{aligned}$$

and thus

$$\|\chi\| \leqslant \sqrt[p^*]{\frac{(\frac{2^{\frac{p^*}{p-1}-1}\overline{\omega}_{\alpha,\beta}d_3^{p^*/(p-1)}}{\Gamma(\alpha)\Gamma(\beta)} + \frac{4^{p^*-1}}{(\beta-1)^{p^*}}(\eta_m d_4)^{p^*})\int_1^{\mathrm{e}} \frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t}}{[1 - r(L^*_{\mu_4,\nu_4})]\int_1^{\mathrm{e}} (\ln t)^{(\beta-1)p^*}\frac{\mathrm{d}\vartheta_{\mu_4,\nu_4}(t)}{t}}}.$$

This implies that W_2 is a bounded set as required. Taking $R_2 > \sup W_2$ and $R_2 > r_2$, when $\chi \in \partial B_{R_2} \cap P$ and $\lambda \in [0, 1]$, we obtain

$$\chi \neq \lambda T\chi,$$

where $B_{R_2} = \{\chi \in E\colon \|\chi\| < R_2\}$. Hence, Lemma 7 implies that

$$i(T,\, B_{R_2} \cap P,\, P) = 1. \tag{32}$$

From (29) and (32) we have

$$i\bigl(T,\, (B_{R_2} \setminus \overline{B}_{r_2}) \cap P,\, P\bigr) = i(T,\, B_{R_2} \cap P,\, P) - i(T,\, B_{r_2} \cap P,\, P) = 1.$$

Therefore, the operator T has at least one fixed point in $(B_{R_2} \setminus \overline{B}_{r_2}) \cap P$. This means that (1) has at least one positive solution. This completes the proof. $\square$

Now, we study the multiplicity of positive solutions to (1). Note that by Lemma 5, if $t_0 \in (1, \mathrm{e})$, then we have

$$T(P) \subset P_1,$$

where

$$P_1 = \Big\{\chi \in P\colon \min_{t\in[t_0,\mathrm{e}]} \chi(t) \geqslant (\ln t_0)^{\beta-1}\|\chi\|\Big\}.$$

Theorem 3. *Suppose that* (H1)–(H2) *and the following conditions hold*:

(H7) *There exist* $\zeta_1, \zeta_2 > 0$ *with*

$$\delta_{\zeta_1,\zeta_2} := \zeta_1 \frac{2^{p^*+p^*/(p-1)-2}}{[\Gamma(\beta)]^{p^*}[\Gamma(\alpha)]^{p^*/(p-1)}} + \zeta_2 4^{p^*-1} \frac{\eta_m^{p^*}}{(\beta-1)^{p^*}} < 1$$

such that

$$\limsup_{\chi\to+\infty} \frac{f(t,\chi)}{\chi^{p-1}} \leqslant \zeta_1^{(p-1)/p^*}, \quad \limsup_{\chi\to+\infty} \frac{g(t,\chi)}{\chi} \leqslant \zeta_2^{1/p^*} \textit{ uniformly on } t \in [1,\mathrm{e}];$$

(H8) *There exists* $\widetilde{d} > 0$ *and* $\zeta_3, \zeta_4 > 0$ *with*

$$\zeta_3 \frac{2^{p^*-1}}{[\Gamma(\beta)]^{p^*}[\Gamma(\alpha)]^{p^*/(p-1)}} + \zeta_4 2^{p^*-1} \frac{\eta_m^{p^*}}{(\beta-1)^{p^*}} < 1$$

such that

$$f(t,\chi) \leqslant \zeta_3^{(p-1)/p^*} \widetilde{d}^{p-1}, \quad g(t,\chi) \leqslant \zeta_4^{1/p^*} \widetilde{d}, \quad \chi \in [0,\widetilde{d}],\ t \in [1,\mathrm{e}];$$

(H9) *There exist* $\widetilde{a}$ *with* $\widetilde{a} > \widetilde{d}$ *and* $\zeta_5, \zeta_6 > 0$ *with*

$$\zeta_5 \frac{(\ln t_0)^{(\beta-1)p_*} 2^{p_*-1}}{[\Gamma(\beta)]^{p_*}[\Gamma(\alpha)]^{p_*/(p-1)}} + \zeta_6 (\ln t_0)^{(\beta-1)p_*} 2^{p_*-1} \frac{\eta_m^{p_*}}{(\beta-1)^{p_*}} > 1$$

such that

$$f(t,\chi) \geqslant \zeta_5^{(p-1)/p_*} \widetilde{a}^{p-1}, \quad g(t,\chi) \geqslant \zeta_6^{1/p_*} \widetilde{a}, \quad \chi \in \left[\widetilde{a}, \frac{\widetilde{a}}{(\ln t_0)^{\beta-1}}\right],\ t \in [1,\mathrm{e}].$$

Then (1) *has at least three positive solutions.*

Proof. From (H7) there exist $c_1, c_2 > 0$ such that

$$f(t,\chi) \leqslant \zeta_1^{(p-1)/p^*} \chi^{p-1} + c_1, \quad g(t,\chi) \leqslant \zeta_2^{1/p^*} \chi + c_2, \quad t \in [1,\mathrm{e}],\ \chi \in \mathbb{R}^+.$$

Taking

$$\widetilde{c} \geqslant \sqrt[p^*]{\frac{\frac{c_1^{p^*/(p-1)}2^{p^*+\frac{p^*}{p-1}-2}}{[\Gamma(\beta)]^{p^*}[\Gamma(\alpha)]^{p^*/(p-1)}} + c_2^{p^*}4^{p^*-1}(\frac{\eta_m}{\beta-1})^{p^*}}{1-\delta_{\zeta_1,\zeta_2}}},$$

for $\|\chi\| \leqslant \widetilde{c}$, we have

$$\begin{aligned}
&(T\chi)^{p^*}(t) \\
&\quad \leqslant \Bigg[\int_0^1 G_\beta\big(t,\mathrm{e}^s\big)\bigg(\int_0^1 G_\alpha\big(\mathrm{e}^s,\mathrm{e}^\tau\big)f\big(\mathrm{e}^\tau,\chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau\bigg)^{1/(p-1)}\mathrm{d}s \\
&\qquad + \frac{(\ln t)^{\beta-1}}{\beta-1}\int_0^1 \eta\big(\mathrm{e}^\tau\big)g\big(\mathrm{e}^\tau,\chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau\Bigg]^{p^*} \\
&\quad \leqslant 2^{p^*-1}\int_0^1 G_\beta^{p^*}\big(t,\mathrm{e}^s\big)\int_0^1 G_\alpha^{p^*/(p-1)}\big(\mathrm{e}^s,\mathrm{e}^\tau\big)f^{p^*/(p-1)}\big(\mathrm{e}^\tau,\chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad + 2^{p^*-1}\bigg[\frac{(\ln t)^{\beta-1}}{\beta-1}\bigg]^{p^*}\int_0^1 \eta^{p^*}\big(\mathrm{e}^\tau\big)g^{p^*}\big(\mathrm{e}^\tau,\chi\big(\mathrm{e}^\tau\big)\big)\,\mathrm{d}\tau \\
&\quad = 2^{p^*-1}\int_1^{\mathrm{e}} \frac{[\Gamma(\beta)G_\beta(t,s)]^{p^*}}{[\Gamma(\beta)]^{p^*}}\int_1^{\mathrm{e}} \frac{[\Gamma(\alpha)G_\alpha(s,\tau)]^{p^*/(p-1)}}{[\Gamma(\alpha)]^{p^*/(p-1)}}f^{p^*/(p-1)}\big(\tau,\chi(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}s}{s} \\
&\qquad + 2^{p^*-1}\bigg[\frac{(\ln t)^{\beta-1}}{\beta-1}\bigg]^{p^*}\int_1^{\mathrm{e}} \eta^{p^*}(\tau)g^{p^*}\big(\tau,\chi(\tau)\big)\frac{\mathrm{d}\tau}{\tau} \\
&\quad \leqslant \frac{2^{p^*+\frac{p^*}{p-1}-2}}{[\Gamma(\beta)]^{p^*}\big[\Gamma(\alpha)\big]^{p^*/(p-1)}}\int_1^{\mathrm{e}}\int_1^{\mathrm{e}} \big(\zeta_1\chi^{p^*}(\tau) + c_1^{p^*/(p-1)}\big)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}s}{s} \\
&\qquad + 4^{p^*-1}\bigg(\frac{\eta_m}{\beta-1}\bigg)^{p^*}\int_1^{\mathrm{e}} \big(\zeta_2\chi^{p^*}(\tau) + c_2^{p^*}\big)\frac{\mathrm{d}\tau}{\tau} \\
&\quad \leqslant \delta_{\zeta_1,\zeta_2}\widetilde{c}^{p^*} + \frac{c_1^{p^*/(p-1)}2^{p^*+p^*/(p-1)-2}}{[\Gamma(\beta)]^{p^*}[\Gamma(\alpha)]^{p^*/(p-1)}} + c_2^{p^*}4^{p^*-1}\bigg(\frac{\eta_m}{\beta-1}\bigg)^{p^*} \\
&\quad \leqslant \widetilde{c}^{p^*}.
\end{aligned} \tag{33}$$

This implies that $T: \overline{P}_{\widetilde{c}} \to \overline{P}_{\widetilde{c}}$.

For $\chi \in P_1$, define $\widetilde{\beta}(\chi) = \min_{t\in[t_0,\mathrm{e}]}\chi(t)$. Then $\widetilde{\beta}$ is a nonnegative continuous concave functional on P_1, and the following inequality holds:

$$\widetilde{\beta}(\chi) \leqslant \max_{t\in[1,\mathrm{e}]}\chi(t) = \|\chi\|, \quad \chi \in P_1.$$

If we take $\chi(t) \equiv (\widetilde{a} + \widetilde{a}/(\ln t_0)^{\beta-1})/2 > \widetilde{a}$, then this $\chi \in \{\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{a}/(\ln t_0)^{\beta-1})\colon \widetilde{\beta}(\chi) > \widetilde{a}\}$. Consequently, we see that $\{\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{a}/(\ln t_0)^{\beta-1})\colon \widetilde{\beta}(\chi) > \widetilde{a}\} \neq \emptyset$ and $\widetilde{\beta}(T\chi) > \widetilde{a}$ for $\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{a}/(\ln t_0)^{\beta-1})$. Moreover, for $\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{a}/(\ln t_0)^{\beta-1})$, $\widetilde{\beta}(\chi) > \widetilde{a}$, and we have

$$\frac{\widetilde{a}}{(\ln t_0)^{\beta-1}} \geqslant \|\chi\| \geqslant \widetilde{\beta}(\chi) > \widetilde{a}.$$

Therefore, by (H9) we obtain

$$\begin{aligned}
&\bigl[\widetilde{\beta}(T\chi)(t)\bigr]^{p_*} \\
&\quad= \Biggl[\min_{t\in[t_0,\mathrm{e}]} \Biggl\{ \int_0^1 G_\beta\bigl(t,\mathrm{e}^s\bigr)\Biggl(\int_0^1 G_\alpha\bigl(\mathrm{e}^s,\mathrm{e}^\tau\bigr) f\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr)^{1/(p-1)}\mathrm{d}s \\
&\qquad+ \frac{(\ln t)^{\beta-1}}{\beta-1}\int_0^1 \eta\bigl(\mathrm{e}^\tau\bigr) g\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr\}\Biggr]^{p_*} \\
&\quad\geqslant (\ln t_0)^{(\beta-1)p_*}\Biggl[\int_0^1 G_\beta\bigl(\mathrm{e},\mathrm{e}^s\bigr)\Biggl(\int_0^1 G_\alpha\bigl(\mathrm{e}^s,\mathrm{e}^\tau\bigr) f\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr)^{1/(p-1)}\mathrm{d}s \\
&\qquad+ \frac{1}{\beta-1}\int_0^1 \eta\bigl(\mathrm{e}^\tau\bigr) g\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{p_*} \\
&\quad\geqslant (\ln t_0)^{(\beta-1)p_*}2^{p_*-1} \\
&\qquad\times \int_0^1 \frac{[\Gamma(\beta)G_\beta(\mathrm{e},\mathrm{e}^s)]^{p_*}}{[\Gamma(\beta)]^{p_*}} \int_0^1 \frac{[\Gamma(\alpha)G_\alpha(\mathrm{e}^s,\mathrm{e}^\tau)]^{p_*/(p-1)}}{[\Gamma(\alpha)]^{p_*/(p-1)}} f^{p_*/(p-1)}\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad+ (\ln t_0)^{(\beta-1)p_*}2^{p_*-1}\frac{1}{(\beta-1)^{p_*}}\int_0^1 \frac{\eta^{p_*}(\mathrm{e}^\tau)}{\eta_m^{p_*}}\eta_m^{p_*} g^{p_*}\bigl(\mathrm{e}^\tau,\chi\bigl(\mathrm{e}^\tau\bigr)\bigr)\,\mathrm{d}\tau \\
&\quad\geqslant \frac{(\ln t_0)^{(\beta-1)p_*}2^{p_*-1}}{[\Gamma(\beta)]^{p_*}[\Gamma(\alpha)]^{p_*/(p-1)}}\int_1^{\mathrm{e}}\int_1^{\mathrm{e}} \zeta_5\widetilde{a}^{p_*}\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}s}{s} \\
&\qquad+ (\ln t_0)^{(\beta-1)p_*}2^{p_*-1}\frac{\eta_m^{p_*}}{(\beta-1)^{p_*}}\int_1^{\mathrm{e}} \zeta_6\widetilde{a}^{p_*}\frac{\mathrm{d}\tau}{\tau} \\
&\quad> \widetilde{a}^{p_*}.
\end{aligned}$$

This implies that

$$\widetilde{\beta}(T\chi) > \widetilde{a}.$$

Next, we assert that $\|T\chi\| < \widetilde{d}$ for $\|\chi\| \leqslant \widetilde{d}$. In fact, if $\chi \in \overline{P}_{\widetilde{d}}$, from (H8) and (33) we have

$$\begin{aligned}(T\chi)^{p^*}(t) &\leqslant \frac{2^{p^*-1}}{[\Gamma(\beta)]^{p^*}[\Gamma(\alpha)]^{p^*/(p-1)}} \int_1^{\mathrm{e}}\int_1^{\mathrm{e}} f^{p^*/(p-1)}\big(\tau, \chi(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}s}{s} \\ &\quad + 2^{p^*-1}\left(\frac{\eta_m}{\beta-1}\right)^{p^*}\int_1^{\mathrm{e}} g^{p^*}(\tau, \chi(\tau))\frac{\mathrm{d}\tau}{\tau} \\ &\leqslant \frac{2^{p^*-1}}{[\Gamma(\beta)]^{p^*}[\Gamma(\alpha)]^{p^*/(p-1)}} \int_1^{\mathrm{e}}\int_1^{\mathrm{e}} \zeta_3 \widetilde{d}^{p^*}\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}s}{s} + 2^{p^*-1}\left(\frac{\eta_m}{\beta-1}\right)^{p^*}\int_1^{\mathrm{e}} \zeta_4 \widetilde{d}^{p^*}\frac{\mathrm{d}\tau}{\tau} \\ &< \widetilde{d}^{p^*}.\end{aligned}$$

This shows that $\|T\chi\| < \widetilde{d}$, and this implies that

$$T : \overline{P}_{\widetilde{d}} \to P_{\widetilde{d}} \quad \text{for } \chi \in \overline{P}_{\widetilde{d}}.$$

Finally, we assert that if $\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{c})$ and $\|T\chi\| > \widetilde{a}/(\ln t_0)^{\beta-1}$, then $\widetilde{\beta}(T\chi) > \widetilde{a}$. To see this, if $\chi \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{c})$ and $\|T\chi\| > \widetilde{a}/(\ln t_0)^{\beta-1}$, then from Lemma 3(iiii) we have

$$\begin{aligned}\widetilde{\beta}(T\chi)(t) &= \min_{t\in[t_0,\mathrm{e}]}\left\{\int_1^{\mathrm{e}} G_\beta(t,s)\varphi_q\left(\int_1^{\mathrm{e}} G_\alpha(s,\tau) f\big(\tau, \chi(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\right. \\ &\quad \left. + \frac{(\ln t)^{\beta-1}}{\beta-1}\int_1^{\mathrm{e}} \eta(t) g\big(t, \chi(t)\big)\frac{\mathrm{d}t}{t}\right\} \\ &\geqslant (\ln t_0)^{\beta-1}\left(\int_1^{\mathrm{e}} G_\beta(\mathrm{e},s)\varphi_q\left(\int_1^{\mathrm{e}} G_\alpha(s,\tau) f\big(\tau, \chi(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\right. \\ &\quad \left. + \frac{1}{\beta-1}\int_1^{\mathrm{e}} \eta(t) g\big(t, \chi(t)\big)\frac{\mathrm{d}t}{t}\right) \\ &\geqslant (\ln t_0)^{\beta-1}\|T\chi\|.\end{aligned}$$

Consequently, we have

$$\widetilde{\beta}(T\chi) \geqslant (\ln t_0)^{\beta-1}\|T\chi\| > (\ln t_0)^{\beta-1}\frac{\widetilde{a}}{(\ln t_0)^{\beta-1}} = \widetilde{a}.$$

As a result, all the conditions of Lemma 9 are satisfied by taking $\widetilde{b} = \widetilde{a}/(\ln t_0)^{\beta-1}$. Hence, T has at least three fixed points, i.e., (1) has at least three positive solutions χ_i $(i = 1, 2, 3)$ such that

$$\|\chi_1\| < \widetilde{d}, \quad \widetilde{a} < \widetilde{\beta}(\chi_2), \quad \text{and} \quad \|\chi_3\| > \widetilde{d} \quad \text{with } \widetilde{\beta}(\chi_3) < \widetilde{a}.$$

This completes the proof. □

Now, we will provide some examples to verify our main results. Let $\alpha = 1.5$, $\beta = 2.5$, $p = 1.5$, $t_0 = \sqrt{\mathrm{e}}$, $\eta(t) = \ln t$, $\widetilde{\eta}(t) = t\ln t - t + 2$, $t \in [1, \mathrm{e}]$. Then (H2) holds, and we can calculate the following values: $\Gamma(\alpha) = 0.89$, $\Gamma(\beta) = 1.33$, $p_* = 0.5$, $p^* = 1$, $\eta_m = 1$, $\widetilde{\omega}_{\alpha,\beta} = 0.82$, $\overline{\omega}_{\alpha,\beta} = 1.12$. From Lemma 4 we have

$$r(L^*_{\mu,\nu}) \geqslant \nu \int_1^{\mathrm{e}} \ln t (\ln t)^{\beta-1} \frac{\mathrm{d}t}{t} = \frac{\nu}{\beta+1}. \tag{34}$$

Moreover, note that from Lemma 3(ii)–(iii) we obtain

$$\begin{aligned} r(L^*_{\mu,\nu}) &\leqslant \max_{s\in[0,1]} \left\{ \mu \int_1^{\mathrm{e}^s} \mathrm{d}\tau \int_1^{\mathrm{e}} G_{\beta,\alpha}(t,\tau) \frac{\mathrm{d}t}{t} + \widetilde{\eta}(\mathrm{e}^s)\nu \int_1^{\mathrm{e}} (\ln t)^{\beta-1} \frac{\mathrm{d}t}{t} \right\} \\ &\leqslant \frac{2\nu}{\beta} + \frac{\mu(\mathrm{e}-1)}{\Gamma(\alpha)\Gamma(\beta)}. \end{aligned} \tag{35}$$

Example 1. Let $\mu_1 = 0.82$, $\nu_1 = 4$, $\mu_2 = 0.3$, $\nu_2 = 0.6$, and

$$f(t,\chi) = \frac{\ln t + 1}{4} \chi^{0.6}, \quad g(t,\chi) = \frac{\ln t + 2}{3} \chi^2, \quad t \in [1, \mathrm{e}],\ \chi \in \mathbb{R}^+.$$

Then from (34)–(35) we have $r(L^*_{\mu_1,\nu_1}) > 1$, $r(L^*_{\mu_2,\nu_2}) < 1$, and

$$\begin{aligned} \liminf_{\chi\to+\infty} \frac{f(t,\chi)}{\chi^{p-1}} &= \liminf_{\chi\to+\infty} \frac{\frac{\ln t+1}{4}\chi^{0.6}}{\chi^{0.5}} = +\infty, \\ \liminf_{\chi\to+\infty} \frac{g(t,\chi)}{\chi} &= \liminf_{\chi\to+\infty} \frac{\frac{\ln t+2}{3}\chi^{2}}{\chi} = +\infty, \\ \limsup_{\chi\to0^+} \frac{f(t,\chi)}{\chi^{p-1}} &= \limsup_{\chi\to0^+} \frac{\frac{\ln t+1}{4}\chi^{0.6}}{\chi^{0.5}} = 0, \\ \limsup_{\chi\to0^+} \frac{g(t,\chi)}{\chi} &= \limsup_{\chi\to0^+} \frac{\frac{\ln t+2}{3}\chi^{2}}{\chi} = 0 \end{aligned}$$

uniformly on $t \in [1, \mathrm{e}]$. These imply that (H1), (H3)–(H4) are satisfied.

Example 2. Let $\mu_3 = 0.9$, $\nu_1 = 7$, $\mu_3 = 0.2$, $\nu_4 = 0.5$, and

$$f(t,\chi) = \frac{2\ln t + 1}{7} \chi^{0.4}, \quad g(t,\chi) = \frac{3\ln t + 2}{9} \chi^{1/2}, \quad t \in [1, \mathrm{e}],\ \chi \in \mathbb{R}^+.$$

Then from (34)–(35) we have $r(L^*_{\mu_3,\nu_3}) > 1$, $r(L^*_{\mu_4,\nu_4}) < 1$, and

$$\begin{aligned} \liminf_{\chi\to0^+} \frac{f(t,\chi)}{\chi^{p-1}} &= \liminf_{\chi\to0^+} \frac{\frac{2\ln t+1}{7}\chi^{0.4}}{\chi^{0.5}} = +\infty, \\ \liminf_{\chi\to0^+} \frac{g(t,\chi)}{\chi} &= \liminf_{\chi\to0^+} \frac{\frac{3\ln t+2}{9}\chi^{1/2}}{\chi} = +\infty, \end{aligned}$$

$$\limsup_{\chi\to+\infty}\frac{f(t,\chi)}{\chi^{p-1}}=\limsup_{\chi\to+\infty}\frac{\frac{2\ln t+1}{7}\chi^{0.4}}{\chi^{0.5}}=0,$$
$$\limsup_{\chi\to+\infty}\frac{g(t,\chi)}{\chi}=\limsup_{\chi\to+\infty}\frac{\frac{3\ln t+2}{9}\chi^{1/2}}{\chi}=0$$

uniformly on $t\in[1,\mathrm{e}]$. These imply that (H1), (H5)–(H6) are satisfied.

Example 3. Let $\widetilde{a}=100$, $\widetilde{d}=1$, $\zeta_1=0.2$, $\zeta_2=0.7$, $\zeta_3=0.5$, $\zeta_4=0.7$, $\zeta_5=1.22$, $\zeta_6=1.48$, and

$$f(t,\chi)=\begin{cases}\frac{\ln t+3\chi}{8}, & \chi\in[0,1],\ t\in[1,\mathrm{e}],\\ \frac{\ln t}{8}+\frac{397}{792}\chi-\frac{25}{198}, & \chi\in[1,100],\ t\in[1,\mathrm{e}],\\ \frac{\ln t}{8}+50, & \chi\in[100,10000],\ t\in[1,\mathrm{e}],\\ \frac{\ln t}{8}+5\sqrt[4]{\chi}, & \chi\geqslant 10000,\ t\in[1,\mathrm{e}],\end{cases}$$
$$g(t,\chi)=\begin{cases}\frac{4\ln t+3\chi}{10}, & \chi\in[0,1],\ t\in[1,\mathrm{e}],\\ \frac{2\ln t}{5}+\frac{2197}{990}\chi-\frac{190}{99}, & \chi\in[1,100],\ t\in[1,\mathrm{e}],\\ \frac{2\ln t}{5}+220, & \chi\in[100,1000],\ t\in[1,\mathrm{e}],\\ \frac{2\ln t}{5}+22\sqrt[3]{\chi}, & \chi\geqslant 1000,\ t\in[1,\mathrm{e}].\end{cases}$$

Then

(i) $$\limsup_{\chi\to+\infty}\frac{f(t,\chi)}{\chi^{p-1}}=\limsup_{\chi\to+\infty}\frac{\frac{\ln t}{8}+5\sqrt[4]{\chi}}{\sqrt{\chi}}=0,$$
$$\limsup_{\chi\to+\infty}\frac{\frac{2\ln t}{5}+22\sqrt[3]{\chi}}{\chi}=0\quad\text{uniformly on } t\in[1,\mathrm{e}],$$

(ii) $$f(t,\chi)\leqslant 0.5\leqslant\zeta_3^{(p-1)/p^*}\widetilde{d}^{p-1}=0.71,$$
$$g(t,\chi)\leqslant 0.7=\zeta_4^{1/p^*}\widetilde{d},\quad \chi\in[0,1],\ t\in[1,\mathrm{e}],$$

(iii) $$f(t,\chi)\geqslant 50\geqslant\zeta_5^{(p-1)/p_*}\widetilde{a}^{p-1}=12.2,$$
$$g(t,\chi)\geqslant 220\geqslant\zeta_6^{1/p_*}\widetilde{a}=219,\quad \chi\in[100,282.85],\ t\in[1,\mathrm{e}].$$

Consequently, (H1) and (H7)–(H9) hold.

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