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Optimal control of an infected prey–predator model with fear effect*

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Abstract. In this paper, we propose and analyze a prey–predator model with the functional response of Beddington–DeAngelis and the fear effect that have infection only in prey populations. We determine existence criteria of several equilibria, and the stability at different equilibria are presented. We exert pesticide control over prey and additional food control over predators, the optimal control is obtained by the Pontryagin maximum principle. We confirm that adding controls to the predator and prey yields better results. Further we enrich our analysis with the inclusion of the existence and uniqueness of the optimal control. Finally, some numerical results to illustrate our analysis are presented.

Keywords: prey–predator system, optimal control, fear effect, stability.

1 Introduction

The prey–predator system is an important concept in ecology that describes the interactions between prey and predators. The significance of this model lies in the in-depth understanding and interpretation of the food chain and food web structure in the ecosystem, as well as the interrelationship between prey and predators. Jana et al. [16], Mandal et al. [21] used mathematical models with the help of ordinary differential equations to describe the prey–predator system and made significant progress. When there are insects or other pests in the system that cause harm to crops, horticultural crops, livestock, human health, or other ecosystems, we call them pests. Pests cause direct harm to crops, horticultural crops and livestock, reduce yield and quality, may contribute to environmental pollution and have the potential to spread disease. Therefore, the selection and use of pest control methods have become more and more important, and the researches on pest control have been widely concerned by researchers.

Khatua et al. [18] proposed a new mathematical model using fuzzy inferences to investigate the impacts of global warming, water pollution, and harvesting of juvenile

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fishes on the production of mature Hilsa fishes. By applying optimal control theory, the most effective method for controlling pest populations can be found. This helps reduce the use of chemical pesticides, protect the ecosystem, reduce environmental pollution, and improve crop yield and quality. In the research of predator feeding systems with pests, the aim is to eliminate pests and obtain maximum benefits. In these studies, single pesticide control is generally applied to pest populations, and few studies have looked at simultaneous control of pests and predators to achieve better results. Therefore, in our current work, we apply pesticide control to pests, while applying alternative food control to predators and providing alternative food may increase predator density and provide better pest control [2]. Moreover, in the model, we also consider the fear effect, and the predation rate follows the Beddington–DeAngelis functional response term. Chottopadhyay et al. [11] studied a prey–predator models and became diseased in prey species.

Now to form the mathematical model, we hypothesize: prey groups are divided into susceptibility class $X(t)$ and infectious class $Y(t)$, these two groups differ in mortality and ability to escape predators $Z(t)$, $X(t)+Y(t)$ is the total biomass of the prey population, $Z(t)$ is the total biomass of the predator population. It is assumed that only susceptible prey groups $X(t)$ can reproduce according to logical laws, while infected prey $Y(t)$ cannot. It is assumed that the disease affects only the incidence of simple populations. λXY circulates in prey populations, and infected prey cannot recover from disease and cannot reproduce. The disease is only transmitted in prey populations, and the disease is not hereditary. Infected people do not recover or become immune. Susceptible prey is less likely to prey than infected prey, and we hypothesize the function of predator response to susceptible prey after Beddington–DeAngelis, $h(X,Z) = \alpha SZ/(1+e_1X+e_2Z)$, where α and e_1 are normal numbers that measure the effect of capture rate and processing time on feeding rate, respectively, while the normal number e_2 describes interference between predators.

However, the mere presence of a predator can alter the physiology of prey and even reduce its birth rate. Zanette et al. [26], Preisser et al. [23], Cresswell [13], Creel and Christianson [12], Pretelli et al. [24], Wang et al. [25] are the research works used mathematical models with fear effect. We use the $n(f,Z)$ fear item, including that f is the fear effect parameter and Z is the predator population density. It can be used as a fear item if the following conditions are met:

$$n(0,Z) = 1, \qquad \lim_{f\to\infty} n(f,Z) = 0, \qquad \frac{\partial n(f,Z)}{\partial f} < 0,$$
$$n(f,0) = 1, \qquad \lim_{Z\to\infty} n(f,Z) = 0, \qquad \frac{\partial n(f,Z)}{\partial Z} < 0.$$

In order to bring fear effect into the model, in the paper, the intrinsic growth rate is usually multiplied by $1 + fZ$. Please note that when $f = 0$, the model does not have the fear effect.

$$\mathrm{d}X = X\left(\frac{r}{1+fZ}\left(1 - \frac{X+Y}{K}\right) - \lambda Y - \frac{\alpha Z}{1+e_1X+e_2Z}\right)\mathrm{d}t, \tag{1a}$$

$$\mathrm{d}Y = Y(\lambda X - m - \beta Z)\,\mathrm{d}t, \tag{1b}$$

$$\mathrm{d}Z = Z\left(\frac{m_1 \alpha X}{1 + e_1 X + e_2 Z} + m_2 \beta Y - \mu\right)\mathrm{d}t. \tag{1c}$$

r is the intrinsic birth rate constant for the susceptible preys; K is the carrying capacity with which the susceptible preys obeys the logistic curve. m (μ) represents the natural mortality rate of infected prey (predator). β is the search rate, m_1 represents the conversion factor, and m_2 represents the mortality rate per unit consumed by the predator of infected prey.

The goal of this research is to minimize the number of pests at the least cost and with the least damage to the environment, and in addition, to study susceptible pests, infected pests, and their connections and interactions with predators. Unlike other studies, we used insecticides at regular intervals to achieve this goal. At the same time, other control factors are added to control the pest population. We modulate the system by providing additional food to the predator in the model. Considering the cost, environmental and ecological issues, we need to find an optimal solution between the cost and the use of pesticides and the provision of additional food. Therefore, we adopt the optimal control theory and apply the Pontryagin maximum principle to solve the problem. In ecological epidemiology, research on the optimal control theory of applying multivariate control to prey populations is still limited.

In recent years, many scholars at home and abroad have devoted themselves to the study of optimal control of population systems (Brauer and Soudack [7–10], Zhang [27], Lu et al. [19]). Bidhan et al. [4–6] was studied the optimal harvesting of several types of prey–predatory models with time delay in detail, Majee et al. [20] have constructed and solved an optimal monkeypox control strategy taking into account vaccination and treatment controls into consideration. System (1) has been modified to add pesticide control to the system. This control is represented by the variable u. Thus, the population difference equations for susceptible and infected pests are reduced by $\varepsilon_1 uX$ and $\varepsilon_2 uY$, respectively. Insecticides are known to have different effects on susceptible and infected prey. Therefore, we assume that infected preys are more susceptible to pesticides than susceptible prey, $\varepsilon_1 < \varepsilon_2$. It is clear that predators in the ecosystem are also affected by pesticides [8]. The predator population is decreasing at a rate of ε_3. In addition, the additional food supply we add to predators also affects predators and prey populations. Therefore, it is also a way to control the effects of pests in the model. Replace the initial model (1) with the following:

$$\begin{aligned}
\mathrm{d}X &= X\left(\frac{r}{1 + fZ}\left(1 - \frac{X+Y}{K}\right) - \lambda Y - \frac{\alpha Z}{1 + e_1 X + e_2 Z} - \varepsilon_1 u\right)\mathrm{d}t,\\
\mathrm{d}Y &= Y(\lambda X - m - \beta Z - \varepsilon_2 u)\,\mathrm{d}t,\\
\mathrm{d}Z &= Z\left(\frac{m_1 \alpha X}{1 + e_1 X + e_2 Z} + m_2 \beta Y - \mu - \varepsilon_3 u + g\left(1 - \frac{X+Y}{K}\right)\right)\mathrm{d}t.
\end{aligned} \tag{2}$$

In this paper, the dynamic behaviors and optimal control of differential system (1) are discussed. In Section 2, we find all the feasible equilibria of the system and their local stability analyses are given. In Section 3, we get the existence and uniqueness of solution for the optimal control problem. Finally, numerical simulations are given in Section 4.

2 Equilibrium and stability analysis

2.1 Equilibrium analysis

The prey–predator mathematical model with infection in prey has five equilibria, namely, the population-free equilibrium E_0. The population-free equilibrium represents an ecological situation where neither prey nor predator populations are present in the system. The equilibrium with susceptible prey only E_1, this equilibrium point illustrates an ecological situation where only prey population exists. This can happen in some ecological systems. The endemic equilibrium E_2, the disease-free equilibrium E_3, and the coexistence equilibrium E^* will be mentioned as follows.

1. The population-free equilibrium $E_0 = (0, 0, 0)$, which is always feasible. The population-free equilibrium represents an ecological situation where neither prey nor predator populations are present in the system. As for the population-free limit, this equilibrium point should be unstable.

2. The equilibrium with susceptible prey only $E_1 = (K, 0, 0)$, which is always feasible. This equilibrium point illustrates an ecological situation where only prey population exists. Under some specific conditions on system's parameters, this equilibrium point should be stable.

3. The endemic equilibrium $E_2 = (X_2, 0, Z_2)$, where $Z_2 = cX_2 + d$, $c = (m_1\alpha - \mu e_1)/\mu$, $d = -1/e_2$, X_2 is positive root of the following equation:

$$a_2X_2^2 + a_1X_2 + a_0 = 0, \tag{3}$$

where

$$\begin{aligned}
a_2 &= -\alpha f c^2 - \frac{re_1(1+c)}{K},\\
a_1 &= re_1(1+c) - \alpha c + 2\alpha f c d - \frac{r(1+e_1 d)}{K},\\
a_0 &= r(1+e_1 d) + \alpha d + \alpha f d^2.
\end{aligned}$$

Since $a_2 > 0$ and $a_0 < 0$, (3) has exactly one positive root. The equilibrium point E_2 illustrates an ecological situation where predator population does not exist.

4. The disease-free equilibrium $E_3 = (X_3, Y_3, 0)$, where $X_3 = m/\lambda$, $Y_3 = K - Km/r - m/\lambda$. The equilibrium E_3 is feasible if

$$K\lambda(r - m) - rm > 0. \tag{4}$$

The equilibrium point E_3 illustrates an ecological situation where infected prey does not exist in the system.

5. The coexistence equilibrium $E^* = (X^*, Y^*, Z^*)$, X^* is the solution of $AX^2 + BX + C = 0$, $a = 1 - e_2 m/\beta$, $b = e_1 + e_2\lambda/\beta$, where

$$\begin{aligned} Y^* &= -\frac{m_1 X^*}{(a + X^*) m_2 \beta} + \frac{\mu}{m_2\beta}, \qquad Z^* = \frac{\lambda X^* - m}{\beta}, \\ A &= \frac{m_1 f \lambda^2}{m_2 \beta^2} - \frac{b}{k} - \frac{\lambda^2 \mu b f}{m_2 \beta^2} - \frac{\alpha f \lambda^2}{\beta}, \\ B &= -\frac{a}{k} + br + \frac{m_1 r - r\mu b}{k m_2 \beta} + \frac{m_1 \lambda - \lambda \mu b}{m_2 \beta} \\ &\quad - \frac{m m_1 f \lambda + (a\lambda - bm)\lambda\mu f}{m_2 \beta^2} - \frac{\alpha\lambda - 2\alpha f m \lambda}{\beta}, \\ C &= ar - \frac{r\mu a}{k m_2 \beta} - \frac{r\beta\mu a - \lambda\mu a}{m_2 \beta^2} + \frac{\alpha m - \alpha f m^2}{\beta}. \end{aligned}$$

2.2 Stability analysis

Regarding local stability of equilibria of system (1), we have the following theorem.

Theorem 1. *System* (1) *has the following behavior at different equilibria*:

(i) *The equilibrium E_0 is unconditionally unstable.*

(ii) *The equilibrium E_1 is stable, provided that*

$$\lambda K - m < 0, \qquad \frac{K m_1 \alpha}{1 + e_1 K} - \mu < 0. \tag{5}$$

(iii) *The equilibrium E_2, if feasible, is stable, provided that*

$$\lambda X_2 - m - \beta Z_2 < 0.$$

(iv) *The equilibrium E_3, if feasible, is stable, provided that*

$$\frac{m_1 \alpha X_3 (1 + X_3)}{(1 + e_1 X_3)^2} + m_2 \beta Y_3 - \mu < 0. \tag{6}$$

(v) *The equilibrium E^*, if feasible, is locally asymptotically stable if and only if the following conditions hold*:

$$A_1 > 0, \qquad A_3 > 0, \qquad A_1 A_2 - A_3 > 0, \tag{7}$$

where A_i $(i = 1, 2, 3)$ are defined in the proof.

Proof. Jacobian matrix of system (1) is given by

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

where

$$\begin{aligned}
J_{11} &= \frac{r}{1+fY}(1-\frac{2X+Y}{K}) - \lambda Y - \frac{\alpha a Z}{(1+e_1X+e_2Z)^2},\\
J_{12} &= -\frac{rX}{K(1+fZ)} - \lambda X,\\
J_{13} &= -\frac{KrX}{(1+fZ)^2}\left(1-\frac{X+Y}{K}\right) - \frac{\alpha X(1+X)}{(1+e_1X+e_2Z)^2},\\
J_{21} &= \lambda Y, \qquad J_{22} = \lambda X - m - \beta Z, \qquad J_{23} = -\beta Y,\\
J_{31} &= \frac{m_1\alpha a Z(1+Z)}{(1+e_1X+e_2Z)^2}, \qquad J_{32} = m_2\beta Z,\\
J_{33} &= \frac{m_1\alpha a X(1+X)}{(1+e_1X+e_2Z)^2} + m_2\beta Y - \mu.
\end{aligned}$$

(i) Evaluating the Jacobian matrix at the equilibrium E_0 leads to the eigenvalues r, $-m$, and $-\mu$. Since one eigenvalue is always positive, the equilibrium E_0 is unconditionally unstable.

(ii) Evaluating the Jacobian matrix at the equilibrium E_1 leads to the eigenvalues $-r$, λKm, and $(m_1\alpha K)/(1+e_1K) - \mu$. One eigenvalue is always negative, while the remaining two are negative in view of conditions in (5). Note that the second condition in (5) is opposite to condition (4). Thus, the equilibrium E_1 is related to the equilibrium E_3 via transcritical bifurcation.

(iii) Evaluating the Jacobian matrix at the equilibrium E_2 immediately gives one eigenvalue $\lambda x_2 - m - \beta z_2$, while the other two are given by roots of the following quadratic equation:

$$\begin{aligned}
&\xi^2 - \left[\frac{r}{1+fZ_2}\left(1-\frac{2X_2}{K}\right) + \frac{m_1\alpha X_2(1+e_1X_2) - \alpha Z_2(1+e_2Z_2)}{(1+e_1X_2+e_2Z_2)^2} - \mu\right]\\
&\quad - \frac{m_1\alpha Z_2(1+e_2Z_2)}{(1+e_1X_2+e_2Z_2)^2}\left[\frac{KrX_2}{(1+fZ_2)^2}\left(1-\frac{X_2}{K}\right) + \frac{\alpha X_2(1+e_1X_2)}{(1+e_1X_2+e_2Z_2)^2}\right] = 0. \qquad (8)
\end{aligned}$$

Employing Routh–Hurwitz criterion, both roots of (8) are either negative or have negative real parts.

(iv) Evaluating the Jacobian matrix at the equilibrium E_3 gives one eigenvalue as $m_1\alpha X_3/(1+e_1X_3) + m_2\beta Y_3 - \mu$ and the other two are roots of the quadratic equation

$$\xi^2 - \left[r\left(1-\frac{2X_3+Y_3}{K}\right) - \lambda Y_3\right]\xi - \lambda Y_3\left(\frac{rX_3}{K} + \lambda X_3\right) = 0. \qquad (9)$$

Roots of (9) are either negative or have negative real parts if the coefficient of linear term is positive. In view of the second condition in (6), the roots of characteristic equation (9) are either negative or have negative real parts, while the first condition in (6) leads to negativity of the remaining eigenvalue $m_1\alpha X_3/(1+e_1X_3) + m_2\beta Y_3 - \mu$.

(v) The Jacobian matrix associated with the linearization of system (1) at the equilibrium E^* is

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= -\frac{X^* r}{(1+fZ^*)K} - \frac{\alpha Z^* X^* + \alpha (Z^*)^2}{(1+e_1X^*+e_2Z^*)^2}, \\
a_{12} &= -\frac{rX^*}{K(1+fZ^*)} - \lambda X^*, \\
a_{13} &= \frac{KrX^*}{(1+fZ^*)^2}\left(1-\frac{X^*}{K}\right) + \frac{\alpha X^*(1+X^*)}{(1+e_1X^*+e_2Z^*)^2}, \\
a_{21} &= \lambda Y^*, \qquad a_{22} = \lambda X^* - m - \beta Z^*, \qquad a_{23} = -\beta Y, \\
a_{31} &= \frac{m_1 \alpha Z^*(1+Z^*)}{(1+e_1X^*+e_2Z^*)^2}, \qquad a_{32} = m_2 \beta Z^*, \qquad a_{33} = 0.
\end{aligned}$$

Therefore, characteristic equation associated with the equilibrium E^* is

$$\xi^3 + A_1\xi^2 + A_2\xi + A_3 = 0, \tag{10}$$

where

$$\begin{aligned}
A_1 &= (Ar-\lambda)X^* + B\alpha Z^*(X^*+Z^*) + \beta Z^* + m, \\
A_2 &= -\left[ArX^* + \alpha BZ^*(X^*+Z^*)\right](-\lambda X^* + m + \beta Z^*) \\
&\quad + m_1 B\alpha Z^*(1+Z^*)\left[\frac{A^2 rX^*}{K}\left(1-\frac{X^*}{K}\right) + B\alpha X^*(1+X^*)\right], \\
A_3 &= A^2 m_2 \beta\lambda KrX^*Y^*Z^*\left(1-\frac{X^*}{K}\right) + Bm_2\lambda\alpha\beta X^*Y^*Z^*(1+X^*) \\
&\quad + Bm_1\alpha Z^*(1+Z^*)(-\lambda X^* + m + \beta Z^*) \\
&\quad \times \left[AKrX^*\left(1-\frac{X^*}{K}\right) + B\alpha X^*(1+X^*)\right] \\
&\quad + Am_2\beta^2 rX^*Y^*Z^* + Bm_2\alpha^2\beta^2 Y^*(Z^*)^2(X^*+Z^*).
\end{aligned}$$

Here

$$A = \frac{1}{1+fZ^*}K, \qquad B = \frac{1}{(1+e_1X^*+e_2Z^*)^2}.$$

Using Routh–Hurwitz criterion, roots of (10) are either negative or with negative real parts if and only if the conditions in (7) hold. Theorem has been proved. □

Remark 1. The characteristic values of the Jacobian matrix at equilibrium E_0 indicate that population-free equilibrium cannot be observed in natural ecosystems. Equilibrium E_1 can be observed if the mortality rate of infected prey is high, susceptible prey or/and infection rate is low, and predator mortality is high. For sufficiently large predator population mortality, the balance can be visualized by E_2. When the infection rate of infected prey is low and the mortality rate is high, a balance of E_3 can be achieved. In addition, the high growth rate of susceptible prey and the low consumption rate of predator populations enhance the stability of the equilibrium E_3. Furthermore, if the initial state of system (1) is close to the equilibrium point E^*, the solution trajectory not only remains near the equilibrium point E^* at all $t > 0$, but also close to the equilibrium component of E^* such as $t \to \infty$. Therefore, if the initial values of the state variables X, Y, and Z are close to X^*, Y^*, and Z^*, respectively, system (1) will eventually be stable with (7) holding the condition. That is, small perturbations in system variables do not affect the stability of the system under coexistence equilibrium.

3 The optimal control problem

In this section, we will analyze the effects on pest populations after applying pesticide control and providing additional food control for predators. In addition, we will study how to formulate an optimal control problem after the control is applied under the condition of controlling the cost and reducing the damage to the environment, and prove its existence and other properties. Our main goal is to reduce the population of susceptible and infected pests through the use of pesticides while exerting control over the predator's food. But given the cost of using pesticides, we need to use them as little as possible. In addition to the cost factor, the ecosystem does not exist in isolation, there are many relationships. When large amounts of pesticides are used, it is inevitable that there will be an impact on predators, which will have a negative impact on pest control. It can also cause harm to crops or other living things that we need. Therefore, we should minimize the square of the cost of applying pesticides and feeding predators, so that we can not only reduce the cost, but also minimize the side effects of pesticides.

The optimal control problem, we consider, aims the minimization of the objective function

$$\mathcal{J} = \int_0^{t_f} \left(k_1X + k_2Y + k_3u^2 + k_4g^2\right) \mathrm{d}t, \tag{11}$$

where k_1 and k_2 are constants, balancing the dimensions of variables X and Y, respectively. The normal numbers k_3 and k_4 balance the size of the secondary control, k_2Y represents the number of infected pests. The second term, k_1X, represents the number of susceptible pests. k_3u^2 is the cost of using pesticides. Finally, k_4g^2 is the cost of the food supply for the predator population. The cost of these controls is not linearly proportional to the increase in pesticide levels or the additional food supply. When pesticides are used in large quantities, they can make crops toxic and useless. Therefore, the quadratic functional (11) is most suitable for this model, and k_3u^2 reflects the severity of pesticide side effects [17]. We assume that the upper bounds for pest control and predator feeding are $u_{\max}$ and $g_{\max}$, respectively. Furthermore, we prove the existence and uniqueness of the optimal control solution.

3.1 Existence of solution for the optimal control problem

Problem (2), (11) is an optimal control problem in the Lagrange form

$$\begin{aligned} \mathcal{J}(x,u) &= \int_0^{t_f} \mathfrak{L}\bigl(t, X(t), u(t)\bigr)\,\mathrm{d}t \to \min, \\ x'(t) &= f\bigl(t, x(t), u(t)\bigr), \quad \text{a.e. } t \in [t_0, t_f]; \\ x(t_0) &= x_0, \end{aligned} \tag{12}$$

$x(t) \in AC([0,t_f];\mathbb{R}^n)$; $u(t) \in L^1((0,t_f];\mathbb{U}^n)$. Here $\mathbb{U} = [0, u_{1,\max}] \times \cdots \times [0, u_{m,\max}]$ ($u_{i,\max} \leqslant 1$, $i = 1,\dots,m$), AC stands for absolutely continuous, $x(t) = (x_1(t),\dots, x_n(t))$, and the control $u(t) = (u_1(t),\dots,u_m(t))$ for some natural numbers n and m.

In this context, a pair $(x,u) \in AC([0,t_f];\mathbb{R}^n) \times L^1([0,t_f];\mathbb{U})$ is feasible if it satisfies the control problem considered in (12). As usual, the set of all feasible pairs is denoted by F. The following theorem that we will use to prove existence of solution is contained in Theorem III.4.1 and Corollary III.4.1 in [14].

Lemma 1 [Existence of solutions for control problems]. *Suppose that f and $\mathfrak{L}$ are continuous and that there exist positive constants C_1 and C_2 such that, for $t \in \mathbb{R}$, $x, x_1, x_2 \in \mathbb{R}^N$, and $u \in \mathbb{R}^M$, we have*:

(i) $\|f(t,x,u)\| \leqslant C_1(1 + \|x\| + \|u\|)$;
(ii) $\|f(t,x_1,u) - f(t,x_2,u)\| \leqslant C_2\|x_1 - x_2\|(1 + \|u\|)$;
(iii) *F is nonempty*;
(iv) *$\mathbb{U}$ is closed*;
(v) *There is a compact set S such that $x(t_1) \in S$ for any state variable x*;
(vi) *$\mathbb{U}$ is convex, $f(t,x,u) = \alpha(t,x) + \beta(t,x)u$, and $\mathfrak{L}(t,x,\cdot)$ is convex on $\mathbb{U}$*;
(vii) *$\mathfrak{L}(t,x,u) > c_1|u|\beta - c_2$ for some $c_1 > 0$, $\beta > 1$.*

Consequently, an optimal solution (x^*,u^*) exists for minimizing $\mathcal{J}$ over the region F. Set $\mathbb{U} = [0,u_{\max}] \times [0, g_{\max}]$ with $u_{\max}, g_{\max} \leqslant 1$. Applying Lemma 1 to our problem, we obtain the following result.

Theorem 2. *There exist an optimal control pair (u^*,g^*) and a corresponding solution (X^*,Y^*,Z^*) of the initial value problem determined by* (2) *with initial condition $(X(0),Y(0),Z(0)) = (X_0,Y_0,Z_0)$ that minimizes the cost functional*

$$\mathcal{J} = \int_0^{t_f} \bigl(k_1X + k_2Y + k_3u^2 + k_4g^2\bigr)\,\mathrm{d}t$$

over $L^1([0,t_f];[0,u_{\max}] \times [0,g_{\max}])$.

Proof. To apply Lemma 1 to our problem, we set $\mathbb{U} = [0,u_{\max}] \times [0,g_{\max}]$ and $[t_0,t_1] = [0,t_f]$. To keep the expressions in the proofs short, we omit the dependency on time of the parameters. Adding the first two equations in (2), we get

$$\begin{aligned}\dot{X} + \dot{Y} &= \frac{r}{1+fZ}\left(1 - \frac{X+Y}{K}\right) - \frac{\alpha XZ}{1+e_1X+e_2Z} \\ &\quad - mY - \beta YZ - \varepsilon_1 uX - \varepsilon_2 uY \\ &\leqslant rX\left(1 - \frac{X+Y}{K}\right) - mY - \varepsilon_1 uX - \varepsilon_2 uY.\end{aligned}$$

We conclude that $X(t)+Y(t) \leqslant \max\{X_0+Y_0,\ K\}$ (since $X(t)+Y(t)$ is decreasing if $X(t)+Y(t) > K$). Thus we conclude that $X(t)+Y(t) \leqslant \max\{X_0+Y_0,\ K\} := M_1$.

$$\begin{aligned}\dot{z} &= P\left(\frac{m_1\alpha X}{1+e_1X+e_2Y} + m_2\beta Y - \mu\right) - \varepsilon_3 uZ + gZ\left(1 - \frac{X+Y}{K}\right) \\ &\leqslant Z\left(\frac{m_1\alpha}{e_1} + m_2\beta M_1 + g - \mu - \varepsilon_3 u\right),\end{aligned}$$

$z \leqslant \max\{Z_0, Z_0\mathrm{e}^{m_1\alpha+m_2\beta M_1+g}\} = M_2$.

By employing the aforementioned limits, we promptly ascertain conditions (i) and (ii). Conditions (iii) and (iv) are straightforward consequences of the definition of F and due to $\mathbb{U} = [0, u_{\max}] \times [0, g_{\max}]$. We consequently ascertain that all the state variables lie within the confined compact domain $\{(X,Y,Z) \in (\mathbb{R}^+)^3\colon\ 0 \leqslant X+Y+Z \leqslant M_1+M_2\}$, and condition (v) follows.

Given that the state equations are linearly reliant on the controls and $\mathfrak{L}$ is quadratic in the controls, we derive (vi). Finally,

$$\begin{aligned}\mathfrak{L} &= k_1Y + k_2X + k_3u^2 + k_4g^2 > \min\{k_3, k_4\}\left(u^2+g^2\right) \\ &> \min\{k_3 + k_4\}\left\|(u,g)\right\|^2,\end{aligned}$$

and we establish (vii) with $c_1 = \min\{k_3, k_4\}$.

After evaluating all assumptions, the result is emanated from Lemma 1. □

3.2 Characterization of the controls

First-order necessary conditions for optimality of a controlled trajectory are given by the Pontryagin maximum principle (cf. [1]; for a formulation adapted to a minimization problem, see [3]). Since we have a minimization problem, using the adjoint variable $p = (p_1, p_2, p_3) \in \mathbb{R}_3^+$, the Hamiltonian for the objective function (11) and the control system (2) is given by the following theorem.

Theorem 3. *The optimal control pair is given by*

$$\begin{aligned}u^* &= \min\left\{\max\left\{0, \frac{\varepsilon_1 X^* p_1^* + \varepsilon_2 Y^* p_2^* + \varepsilon_3 Z^* p_3^*}{2k_3}\right\}, u_{\max}\right\}, \qquad (13)\\ g^* &= \min\left\{\max\left\{0, \frac{p_3^* Z^*}{2k_4}\left(\frac{X^*+Y^*}{K} - 1\right)\right\}, g_{\max}\right\},\end{aligned}$$

where X^*, Y^*, Z^*, p_1^*, p_2^* and p_3^* are the optimal variables.

Proof.

$$\begin{aligned}
\mathcal{H} &= \sum_{i=1}^{3} p_i f_i(X,Y,Z) + k_1 Y + k_2 X + k_3 u^2 + k_4 g^2 \\
&= p_1\left[\frac{rX}{1+fZ}\left(1-\frac{X+Y}{K}\right) - \lambda XY - \frac{\alpha XZ}{1+e_1X+e_2Z} - \varepsilon_1 uX\right] \\
&\quad + p_2[\lambda XY - mY - \beta YZ - \varepsilon_2 uY] \\
&\quad + p_3\left[\frac{m_1\alpha XZ}{1+e_1X+e_2Z} + m_2\beta YZ - \mu Z - \varepsilon_3 uZ + gZ\left(1-\frac{X+Y}{K}\right)\right] \\
&\quad + k_1 Y + k_2 X + k_3 u^2 + k_4 g^2,
\end{aligned}$$

adjoint variables satisfying the following costate equations:

$$p_1(t) = -\frac{\partial\mathcal{H}}{\partial X}, \qquad p_2(t) = -\frac{\partial\mathcal{H}}{\partial Y}, \qquad p_3(t) = -\frac{\partial\mathcal{H}}{\partial Z},$$

so

$$\begin{aligned}
\dot{p_1} &= -k_2 - p_2\lambda Y - p_3\left[\frac{m_1\alpha XZ}{(1+e_1X+e_2Z)^2} - \frac{gZ}{K}\right] \\
&\quad - p_1\left[\frac{rX}{1+fz}\left(1-\frac{2X+Y}{K}\right) - \lambda Y \right. \\
&\quad \left. - \frac{\alpha Z(1+e_1X+e_2Z) - XZ}{(1+e_1X+e_2Z)^2} - \varepsilon_1 uX\right], \\
\dot{p_2} &= -k_1 - p_1\left[-\frac{rX}{K(1+fZ)}\left(1-\frac{X+Y}{K}\right) - \lambda X\right] \\
&\quad - p_2(\lambda X - m - \beta Z - \varepsilon_2 u) + p_3\left(m_2\beta Z - \frac{gZ}{K}\right), \\
\dot{p_3} &= p_1\left[\frac{\alpha X(1+X)}{(1+e_1X+e_2Z)^2} + \frac{rfX}{(1+fZ)^2}\left(1-\frac{X+Y}{K}\right)\right] + p_2\beta Y \\
&\quad - p_3\left[\frac{m_1\alpha X(1+X)}{(1+e_1X+e_2Z)^2} + m_2\beta Y - \mu - \varepsilon_3 u + g\left(1-\frac{X+Y}{K}\right)\right].
\end{aligned}$$

Since the terminal state $(X(t_f), Y(t_f), Z(t_f))$ is free, the transversality conditions are

$$p_1(t_f) = p_2(t_f) = p_3(t_f) = 0.$$

Using the Pontryagin maximum principle (PMP), we characterize the optimal controls u^* and g^*. The optimality conditions dictate that $\partial\mathcal{H}/\partial u = 0$ and $\partial\mathcal{H}/\partial g = 0$, that is,

$$u^* = \frac{\varepsilon_1 X^* p_1^* + \varepsilon_2 Y^* p_2^* + \varepsilon_3 Z^* p_3^*}{2k_3}, \qquad g^* = \frac{p_3^* Z^*}{2k_4}\left(\frac{X^*+Y^*}{K} - 1\right). \qquad \square$$

3.3 Uniqueness of solution for the optimal control problem

In this section, we shall demonstrate the uniqueness of the optimal control of (2). The local uniqueness result established in [22] is expanded to encompass the global uniqueness condition, that is, uniqueness over each time interval $[0, t_f]$. Consequently, it is postulated that we have two distinct variables $\xi = (X, Y, Z, p_1, p_2, p_3, p_4)$ and $\xi^* = (X^*, Y^*, Z^*, p_1, p_2, p_3)$, respectively pertaining to (u, g) and (u^*, g^*); subsequently, it is established that both minimization trajectories reside within the positive invariant region Γ embodying the contradiction $\xi = \xi^*$ in a small time interval $[0, T]$; if the interval $[0, T]$ overlaps with the optimal control problem's corresponding interval, the proof is accomplished. Otherwise, we deem T at the right limit of $[0, T]$ as the initial time, thus obtaining the interval $[T, 2T]$, because the estimate of T is only related to the maximum of the parameter on the invariant region Γ of the new control problem and the boundary of the state and costate variables, so it is the same as the obtained conclusion on the interval $[0, T]$. Finally, by repeating this process, we ascertain the global uniqueness for the entire interval $[0, t_f]$.

Theorem 4. *Assuming that there is a solution of the optimal control problem in Ω_1, the optimal control is unique in the interval $[0, t_f]$.*

Proof. We initially demonstrate the uniqueness of the solution to the OCP problem (X, Y, Z, p_1, p_2, p_3) on some interval $[0, T]$ for some $T \in \mathbb{R}^+$, eventually less than t_f. We assume that we have two optimal controls corresponding to trajectories and state equations (X, Y, Z), (p_1, p_2, p_3) and $(\overline{X}, \overline{Y}, \overline{Z})$, $(\overline{p}_1, \overline{p}_2, \overline{p}_3)$, and we will show that the two coincide in some small interval. Consider the change of variables

$$X(t) = \mathrm{e}^{\theta t} x(t), \qquad Y(t) = \mathrm{e}^{\theta t} y(t), \qquad Z(t) = \mathrm{e}^{\theta t} z(t)$$

and

$$p_1(t) = \mathrm{e}^{-\theta t} \phi_1(t), \qquad p_2(t) = \mathrm{e}^{-\theta t} \phi_2(t), \qquad p_3(t) = \mathrm{e}^{-\theta t} \phi_3(t).$$

By the first equation in (12), we get

$$\begin{aligned} \theta \mathrm{e}^{\theta t} x + \mathrm{e}^{\theta t} \dot{x} = {} & \frac{r}{1 + fz} \mathrm{e}^{\theta t} x \left(1 - \frac{\mathrm{e}^{\theta t} x + \mathrm{e}^{\theta t} y}{K}\right) \\ & - \lambda \mathrm{e}^{2\theta t} xy - \frac{\alpha \mathrm{e}^{2\theta t} xz}{1 + e_1 \mathrm{e}^{\theta t} x + e_2 \mathrm{e}^{\theta t} z} - \varepsilon_1 u \mathrm{e}^{\theta t} x, \end{aligned}$$

thus

$$\begin{aligned} \theta x + \dot{x} &= \frac{rx}{1 + fz\mathrm{e}^{\theta t}} \left[1 - \frac{\mathrm{e}^{\theta t}(x + y)}{K}\right] - \lambda \mathrm{e}^{\theta t} xy - \frac{\alpha xz}{\mathrm{e}^{-\theta t} + e_1 x + e_2 z} - \varepsilon_1 u x, \\ \theta \overline{x} + \dot{\overline{x}} &= \frac{r\overline{x}}{1 + f\overline{z}\mathrm{e}^{\theta t}} \left[1 - \frac{\mathrm{e}^{\theta t}(\overline{x} + \overline{y})}{K}\right] - \lambda \mathrm{e}^{\theta t} \overline{xy} - \frac{\alpha \overline{xz}}{\mathrm{e}^{-\theta t} + e_1 \overline{x} + e_2 \overline{z}} - \varepsilon_1 u \overline{x}. \end{aligned}$$

Multiplying by $(s-\overline{s})$, integrating from 0 to T, and noting that $s(0)=\overline{s}(0)$, we have

$$\begin{aligned}
&\theta\int_0^T\left[(x-\overline{x})^2\,\mathrm{d}t+\frac{1}{2}\big(x(T)-\overline{x}(T)\big)^2\right]\\
&\quad= r\int_0^T\left[\frac{x(x-\overline{x})}{1+fz\mathrm{e}^{\theta t}}-\frac{\overline{x}(x-\overline{x})}{1+f\overline{z}\mathrm{e}^{\theta t}}-r\int_0^T\left[\frac{x+y}{K(\mathrm{e}^{-\theta t}+fz)}-\frac{\overline{x}+\overline{y}}{K(\mathrm{e}^{-\theta t}+f\overline{z})}\right](x-\overline{x})\right]\mathrm{d}t\\
&\qquad-\lambda\int_0^T \mathrm{e}^{\theta t}(xy-\overline{xy})(x-\overline{x})\,\mathrm{d}t\\
&\qquad-\alpha\int_0^T\left(\frac{xz}{\mathrm{e}^{-\theta t}+e_1x+e_2z}-\frac{\overline{xz}}{\mathrm{e}^{-\theta t}+e_1\overline{x}+e_2\overline{z}}\right)(x-\overline{x})\,\mathrm{d}t\\
&\qquad-\varepsilon_1\int_0^T(ux-\overline{ux})(x-\overline{x})\,\mathrm{d}t.
\end{aligned}$$

Recall that

$$(x-\overline{x})(y-\overline{y})\leqslant\frac{1}{2}\big[(x-\overline{x})^2+(y-\overline{y})^2\big] \tag{14}$$

and that, for each $x,y,z,\overline{x},\overline{y},\overline{z}>0$, there is $C>0$ (depending on $x,y,z,\overline{x},\overline{y},\overline{z}$) such that

$$(xy-\overline{xy})(z-\overline{z})=C[(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2]. \tag{15}$$

Using (14), (15), we can get

$$\begin{aligned}
&\int_0^T\left[\frac{xz}{\mathrm{e}^{-\theta t}+e_1x+e_2z}-\frac{\overline{xz}}{\mathrm{e}^{-\theta t}+e_1\overline{x}+e_2\overline{z}}\right]\\
&\quad=\int_0^T\frac{\mathrm{e}^{-\theta t}xz+\overline{x}xz+xz\overline{z}-\mathrm{e}^{-\theta t}\overline{xz}-x\overline{xz}-\overline{xz}z}{(\mathrm{e}^{-\theta t}+e_1x+e_2z)(\mathrm{e}^{-\theta t}+e_1\overline{x}+e_2\overline{z})}\,\mathrm{d}t\\
&\quad=\int_0^T\frac{[\mathrm{e}^{-\theta t}(xz-\overline{xz})+x\overline{x}(z-\overline{z})+\overline{z}z(x-\overline{x})](x-\overline{x})}{(\mathrm{e}^{-\theta t}+x+z)(\mathrm{e}^{-\theta t}+e_1\overline{x}+e_2\overline{z})}\,\mathrm{d}t\\
&\quad\leqslant C\mathrm{e}^{\theta T}\int_0^T[2(x-\overline{x})^2+(z-\overline{z})^2]\,\mathrm{d}t\\
&\qquad+\frac{x\overline{x}}{2}\int_0^T[(x-\overline{x})^2+(z-\overline{z})^2+z\overline{z}(x-\overline{x})^2]\,\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \left(\frac{rx}{1+fz\mathrm{e}^{\theta t}} - \frac{r\overline{x}}{1+f\overline{z}\mathrm{e}^{\theta t}}\right)(x-\overline{x})\,\mathrm{d}t \\
&\quad= \int_0^T \frac{(x-\overline{x}+fx\overline{z}\mathrm{e}^{\theta t} - fz\overline{x}\mathrm{e}^{\theta t})(x-\overline{x})}{(1+fz\mathrm{e}^{\theta t})(1+f\overline{z}\mathrm{e}^{\theta t})}\,\mathrm{d}t \\
&\quad= \int_0^T \frac{(x-\overline{x})^2 + f\mathrm{e}^{\theta t}(xz-\overline{xz})(x-\overline{x})}{(1+fz\mathrm{e}^{\theta t})(1+f\overline{z}\mathrm{e}^{\theta t})}\,\mathrm{d}t \\
&\quad\leqslant \frac{C\mathrm{e}^{\theta T}}{f}\int_0^T \left[2(x-\overline{x})^2 + (z-\overline{z})^2\right]\mathrm{d}t,
\end{aligned}$$

so

$$\begin{aligned}
&\theta\int_0^T (x-\overline{x})^2\,\mathrm{d}t + \frac{1}{2}\big(x(T)-\overline{x}(T)\big)^2 \\
&\quad= \big(C_1 + C_1\mathrm{e}^{\theta T}\big)\int_0^T \left[(x-\overline{x})^2 + (y-\overline{y})^2 + (z-\overline{z})^2(u-\overline{u})^2\right]\mathrm{d}t. \qquad (16)
\end{aligned}$$

By the second equation in (1), we get

$$\theta\mathrm{e}^{\theta t}y + \mathrm{e}^{\theta t}\dot{y} = \lambda\mathrm{e}^{2\theta t}xy - m\mathrm{e}^{\theta t}y - \beta\mathrm{e}^{2\theta t}yz - \varepsilon_2 u\mathrm{e}^{\theta t}y,$$

thus

$$\theta y + \dot{y} = \lambda xy\mathrm{e}^{\theta t} - my - \beta yz\mathrm{e}^{\theta t} - \varepsilon_2 uy.$$

Use the same method as above, we can get

$$\begin{aligned}
&\theta\int_0^T (y-\overline{y})^2\,\mathrm{d}t + \frac{1}{2}\big(y(T)-\overline{y}(T)\big)^2 \\
&\quad= \int_0^T \big[\mathrm{e}^{-\theta t}\lambda(xy-\overline{xy})(y-\overline{y})\,\mathrm{d}t - m(y-\overline{y})^2 \\
&\qquad - \mathrm{e}^{-\theta t}\beta(yz-\overline{yz})(y-\overline{y}) - \varepsilon_2(uy-\overline{uy})(y-\overline{y})\big]\,\mathrm{d}t \\
&\quad\leqslant \big(\widetilde{C}_2\mathrm{e}^{\theta T} + C_2\big)\int_0^T \left[(x-\overline{x})^2 + (y-\overline{y})^2 + (z-\overline{z})^2 + (u-\overline{u})^2\right]\mathrm{d}t \qquad (17)
\end{aligned}$$

and

$$\theta\int_0^T (z-\overline{z})^2\,\mathrm{d}t + \frac{1}{2}\big(z(T)-\overline{z}(T)\big)^2$$
$$= \int_0^T \left[\frac{m_1\alpha xz}{\mathrm{e}^{-\theta t}+e_1x+e_2z} - \frac{m_1\alpha\overline{xz}}{\mathrm{e}^{-\theta t}+e_1\overline{x}+e_2\overline{z}}\right](z-\overline{z})\,\mathrm{d}t$$
$$+ \int_0^T m_2\beta(yz-\overline{yz})(z-\overline{z})\,\mathrm{d}t$$
$$- \mu\int_0^T (z-\overline{z})^2\,\mathrm{d}t - \varepsilon_3\int_0^T (uz-\overline{uz})(z-\overline{z})\,\mathrm{d}t + g\int_0^T (z-\overline{z})^2\,\mathrm{d}t$$
$$- \int_0^T \frac{\mathrm{e}^{\theta t}(x^2+xy-\overline{x}^2-\overline{xy})(z-\overline{z})}{K}\,\mathrm{d}t$$
$$\leqslant \big(\widetilde{C}_3\mathrm{e}^{\theta T}+C_3\big)\int_0^T \big[(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2+(u-\overline{u})^2\big]\,\mathrm{d}t. \tag{18}$$

Considering the equations for the adjoint equations and reasoning similarly, we can conclude that there are constants $A_1, \widetilde{A}_1, A_2, \widetilde{A}_2, A_3, \widetilde{A}_3 > 0$ (depending on the state variables and adjoint equations) such that

$$\theta\int_0^T (\phi_i-\overline{\phi}_i)^2\,\mathrm{d}t + \frac{1}{2}\big(\phi_i(0)-\overline{\phi}_i(0)\big)^2$$
$$\leqslant \big(\widetilde{A}_i\mathrm{e}^{\theta T}+A_i\big)\int_0^T \big[(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2\big]\,\mathrm{d}t$$
$$+ \big(\widetilde{A}_i\mathrm{e}^{\theta T}+A_i\big)\int_0^T \big[(\phi_1-\overline{\phi}_1)^2+(\phi_2-\overline{\phi}_2)^2+(\phi_3-\overline{\phi}_3)^2+(u-\overline{u})^2\big]\,\mathrm{d}t, \tag{19}$$

$i=1,2,3$. By (13), we obtain

$$u-\overline{u} = \frac{\varepsilon_1}{2k_3}(x\phi_1-\overline{x}\overline{\phi}_1) + \frac{\varepsilon_2}{2k_3}(y\phi_2-\overline{y}\overline{\phi}_2) + \frac{\varepsilon_3}{2k_3}(z\phi_3-\overline{z}\overline{\phi}_3),$$

and we conclude that $C_4 > 0$ (depending on the state variables and adjoint equations) makes

$$(u-\overline{u})^2 = \left[\frac{\varepsilon_1}{2k_3}(x\phi_1-\overline{x}\overline{\phi}_1) + \frac{\varepsilon_2}{2k_3}(y\phi_2-\overline{y}\overline{\phi}_2) + \frac{\varepsilon_3}{2k_3}(z\phi_3-\overline{z}\overline{\phi}_3)\right]^2$$

$$
\begin{aligned}
&= \bigg[\frac{\varepsilon_1}{2k_3}\big(x(\phi_1-\overline{\phi}_1)+(x-\overline{x})\overline{\phi}_1\big)+\frac{\varepsilon_2}{2k_3}\big(y(\phi_2-\overline{\phi}_2)+(y-\overline{y})\overline{\phi}_2\big)\\
&\quad+\frac{\varepsilon_3}{2k_3}\big(z(\phi_3-\overline{\phi}_3)+(z-\overline{z})\overline{\phi}_3\big)\bigg]^2\\
&\leqslant C_4\big[(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2\\
&\quad+(\phi_1-\overline{\phi}_1)^2+(\phi_2-\overline{\phi}_2)^2+(\phi_3-\overline{\phi}_3)^2\big]\,\mathrm{d}t,
\end{aligned}
\tag{20}
$$

and for the control g, we have

$$
\begin{aligned}
(g-\overline{g})^2 &= \frac{1}{4k_4^2}\bigg[\frac{\mathrm{e}^{\theta t}[(\phi_3xz-\overline{\phi}_3\overline{xz})+(\phi_3yz-\overline{\phi}_3\overline{yz})]}{K}+\phi_3z-\overline{\phi}_3\overline{z}\bigg]^2\\
&\leqslant C_8\big[(\phi_3-\overline{\phi}_3)^2+(z-\overline{z})^2\big]\\
&\quad+\frac{C_9\mathrm{e}^{\theta T}}{K}\big[(x-\overline{x})^2+(y-\overline{y})^2+2(z-\overline{z})^2+2(\phi_3-\overline{\phi}_3)^2\big]\,\mathrm{d}t.
\end{aligned}
\tag{21}
$$

Finally, we have all we need to prove our result. Define

$$\varPsi(t)=(x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2$$

and

$$\varPhi(t)=(\phi_1-\overline{\phi}_1)^2+(\phi_2-\overline{\phi}_2)^2+(\phi_3-\overline{\phi}_3)^2,$$

and observe that $\varPsi(t)>0$ and $\varPhi(t)>0$ for all t.

Adding Eqs. (16)–(21), we obtain for the sum of left-hand sides

$$\frac{1}{2}\big[\varPsi(T)+\varPhi(0)\big]+\theta\int_0^T\big[\varPsi(t)+\varPhi(t)\big]\,\mathrm{d}t\leqslant\big(D+\widetilde{D}\mathrm{e}^{2\theta T}\big)\int_0^T\big[\varPsi(t)+\varPhi(t)\big]\,\mathrm{d}t,$$

where $D=\sum_{i=1}^4 C_i+\sum_{i=1}^3 A_i$ and $\widetilde{D}=\sum_{i=1}^4\widetilde{C}_i+\sum_{i=1}^3\widetilde{A}_i$. Thus

$$\frac{1}{2}\big[\varPsi(T)+\varPhi(0)\big]+\big(\theta-D-\widetilde{D}\mathrm{e}^{2\theta T}\big)\int_0^T\big[\varPsi(t)+\varPhi(t)\big]\,\mathrm{d}t\leqslant 0. \tag{22}$$

We now choose θ so that

$$\theta>D+\widetilde{D}.$$

Subsequently, we choose T such that

$$T<\frac{1}{2\theta}\ln\frac{\theta-D}{\widetilde{D}},$$

then

$$\mathrm{e}^{2\theta T}<\frac{\theta-D}{\widetilde{D}}.$$

It follows that $\theta - D - \widetilde{D}e^{2\theta T} > 0$, so inequality (22) can hold if and only if, for all $t \in [0, T]$, we have $x(t) = \overline{x}(t)$, $y(t) = \overline{y}(t)$, $z(t) = \overline{z}(t)$, $\phi_1(t) = \overline{\phi}_1(t)$, $\phi_2(t) = \overline{\phi}_2(t)$, and $\phi_3(t) = \overline{\phi}_3(t)$. But this is equivalent to $X(t) = \overline{X}(t)$, $Y(t) = \overline{Y}(t)$, $Z(t) = \overline{Z}(t)$, $p_1(t) = \overline{P}_1(t)$, $p_2(t) = \overline{p}_2(t)$ and $p_3(t) = \overline{p}_3(t)$. Therefore, the uniqueness of the optimal control is established in a small interval $[0, T]$.

To culminate this proof, we suggest that if $T > t_f$, the conclusion naturally follows. Conversely, we can attain uniqueness on the interval $[T, 2T]$ for the optimal control problem whose initial conditions at time T align with the values of X, Y, and Z at T (note that we still acquire constants α, D, $\overline{D}$, and thus we retain the same T). Continuing in an analogous manner, we ascertain, after a finite number of iterations, that we possess uniqueness on the interval $[0, t_f]$, and thus, the proof is complete. □

4 Examples and numerical simulations

On the basis of complying with infectious diseases and ecology, we refer to [15] and other articles to ensure the scientificity and validity of parameter values. More importantly, these values are taken in strict accordance with the range of values in our proof condition. We focus on the qualitative behavior of the proposed system (1) and (2). In the following concrete example, we can see the value of our assumed parameter. The goal of our study is to minimize the objective function, so as to minimize the total number of susceptible and infected pests under the premise of considering the cost, and we set the corresponding weight to 1. In a time interval of 20 units, this unit may be equivalent to days, weeks, or months, here we take the year as the unit. We set the initial values of the susceptible pest, infected pest, and the predator to 1, 0.6, and 0.1, respectively. Then we will solve the state variable equation and the adjoint variable equation in the given time interval. Before that, the fear function for different fear levels is graphically represented in Fig. 1.

From Fig. 1 we can clearly see the relationship between the magnitude of the fear factor and the birth rate of susceptible prey populations. When the fear coefficient is 0, the birth rate remains unchanged, and the fear coefficient is inversely proportional to the birth rate, and the bigger the fear coefficient, the closer the birth rate is to 0.

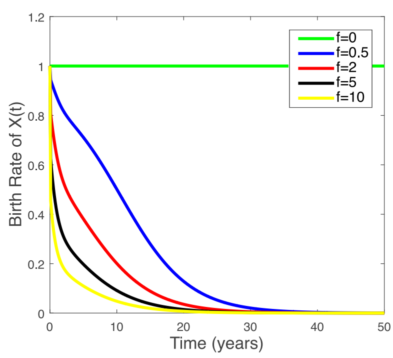


Figure 1. The relationship between fear level and natural growth rate of prey.

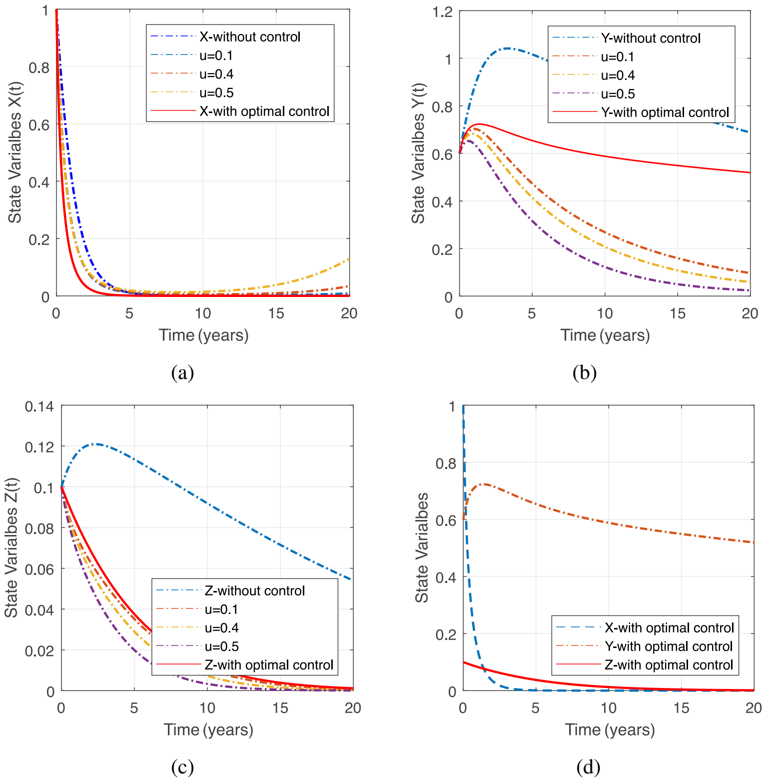


Figure 2. Control u of infected prey–predator model with the fear.

We solve numerical simulation of the optimal control problem using Runge–Kutta fourth-order iterative method. Let us consider the following example:

$$\begin{aligned}
\mathrm{d}X &= X\left(\frac{0.5}{1+0.5Z}\left(1-\frac{X+Y}{0.49}\right)-0.6Y-\frac{0.7Z}{1+0.1X+0.1Z}-0.1u\right)\mathrm{d}t,\\
\mathrm{d}Y &= Y(0.6S-0.5-0.7Z-0.1u)\,\mathrm{d}t,\\
\mathrm{d}Z &= Z\left(\frac{0.01*0.7X}{1+0.1X+0.1Z}+0.6*0.7Y-0.4+0.1u+0.1\left(1-\frac{X+Y}{0.49}\right)\right)\mathrm{d}t
\end{aligned}$$

with the initial value $(1, 0.6, 0.1)$. Choose and keep the initial value, different values of control u were selected, and it was obviously observed that the population density of susceptible prey $X(t)$ and infected prey $Y(t)$ decreased significantly after different levels of control u were added compared with those without control as shown in Fig. 2.

From Fig. 2(a) the population density of susceptible prey $X(t)$ is the largest without control, and the difference is not large when $u = 0.1, 0.4, 0.5$, and the population density is the smallest when the optimal control is taken. From Fig. 2(b) the population density of infected prey decreased significantly with the strengthening of control u, and the optimal

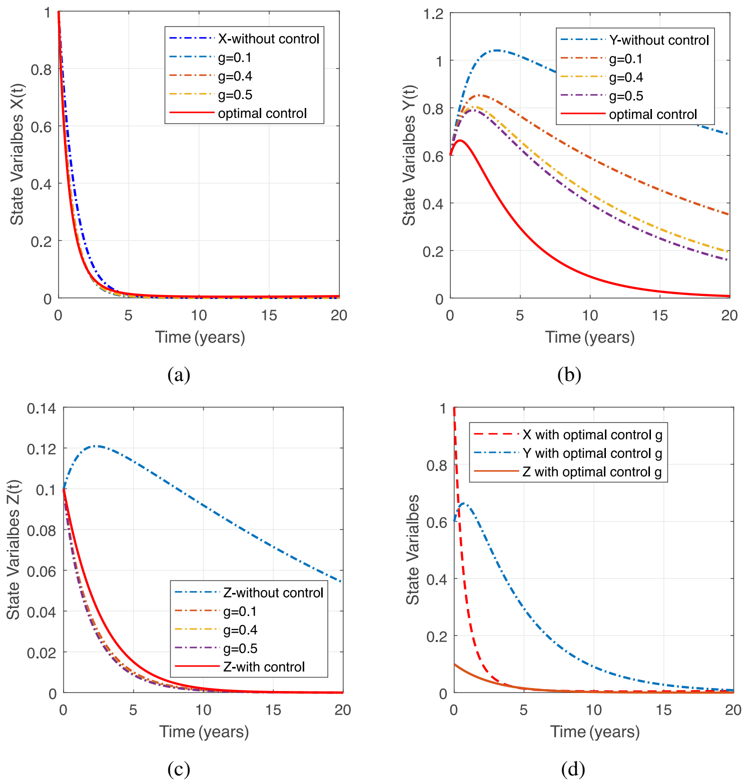


Figure 3. Control g of infected prey–predator model with the fear.

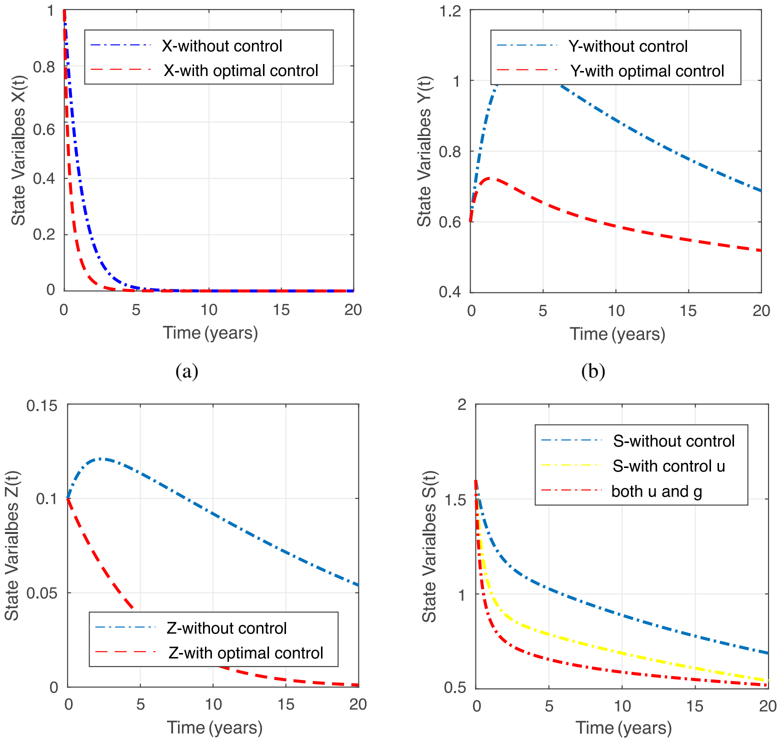
control was between 0 and 0.1. Fig. 2(c) is similar to Fig. 2(b), but it can be seen that pesticide control has a large impact on predators with optimal control having a relatively small effect on predator populations.

Different values of control g were selected, and it was obviously observed that the population density of susceptible prey $X(t)$ and infected prey $Y(t)$ decreased significantly after different levels of control g was added compared with those without control as shown in Fig. 3.

From Fig. 3(a) we can get susceptible prey $X(t)$ has the largest population density without control, and it can be seen that control g has less effect on susceptible prey. From Fig. 3(b) the population density of infected prey decreased significantly with the strengthening of control u, and the optimal control was greater than 0.5. It can be seen that the optimal control is between 0 and 0.1 in Fig. 3(c).

We choose $a = 0.3$; $m = 0.01$; $\beta = 0.5$; $r = 0.9$; $k_1 = 0.1$; $k_2 = 0.1$; $k_3 = 0.05$; $m_1 = 0.1$; $m_2 = 0.6$; $\mu = 0.4$; $\lambda = 0.6$; $K = 0.4$; $\varepsilon_1 = 0.9$; $\varepsilon_2 = 0.9$; $\varepsilon_3 = 0.9$.

Using the above methods, we study the optimal control of model (2). Let $S(t) = X(t)+Y(t)$. It can be observed in Fig. 4 that when the control u and g are both optimized, the pest control effect is the best.



(a) (b) (c) (d)

Figure 4. Control both u and g of infected prey–predator model.

5 Conclusions

Research on pest control is of great importance to both agriculture and ecological environment. First of all, pests will cause serious damage to crops and plants, resulting in reduced yield or even crop failure, causing major economic losses to agricultural production. Second, conventional chemical pesticide control can negatively impact the environment and human health, so we need to develop more environmentally friendly and sustainable pest control methods. Research on pest control can help us better understand the ecology and behavior patterns of pests, provide a basis for formulating more scientific and efficient control strategies, and promote the protection of biodiversity and the maintenance of ecological balance. With the above factors, we take a reasonable approach to study the dynamic behavior of pests and use pesticides and the food supply of predators for optimal control.

This paper systematically studies the prey–predator model with fear effect and draws some conclusions. The interaction between predators and susceptible prey is thought to be a functional response of Beddington–DeAngelis, and the birth rate of prey species is affected by predator populations. We certify that the birth rate of prey populations is inversely proportional and gradually decreases as the density of predator populations increases. In addition, the equilibrium point stability of the proposed model is analyzed, so we can better understand and manage the operation and evolution of biological systems. The equilibrium E_0 is unconditionally unstable. Note that the second condition in (5) is opposite to condition (4). Thus, the equilibrium E_1 is related to the equilibrium E_3 via transcritical bifurcation. Furthermore, if the initial state of system (1) is close to the equilibrium point E^*, the solution trajectory not only remains near the equilibrium point E^* at all $t > 0$, but also close to the equilibrium component of E^* such as $t \to \infty$. Therefore, if the initial values of the state variables X, Y, and Z are close to X^*, Y^*, and Z^*, respectively, the system (1) will eventually be stable with (7) holding the condition. That is, small perturbations in system variables do not affect the stability of the system under coexistence equilibrium.

Furthermore, the optimal control of the model is explored, an optimal strategy is proposed by considering the maximization of returns. Employing the Pontryagin maximum principle, the optimal control strategy for reaching the maximum value of the indicator is derived. It is proved that when the two controls are applied at the same time, there is an optimal control (u^*, g^*). We assume that there are two different variables $\xi = (X, Y, Z, p_1, p_2, p_3, p_4)$ and $\xi^* = (X^*, Y^*, Z^*, p_1, p_2, p_3)$, respectively corresponding to (u, g) and (u^*, g^*). Then we make a change of variable, proving that the contradiction $\xi = \xi^*$ is in a small time interval $[0, T]$. Finally, by iterating the process, we get the uniqueness of the entire interval $[0, t_f]$ of optimal control. On the basis of previous studies, more complex situations were considered, and numerical simulations confirmed that the control effect was better than that of chemical control alone when control was applied together with ecological control. This has catalyzed advancements in eco-epidemiology for multidimensional and environmentally friendly pest control. Guided by these research findings, we can devise more effective biological control strategies, reduce reliance on chemical pesticides, and improve the sustainable use of crop and fishery resources.

Pesticides have an important role in pest control, there are also some limitations, and the continuous evolution of pest resistance may make pest control more and more difficult. In the process of application, pesticides may pollute the soil, water, and air, causing harm to the ecological environment. In addition, residues on agricultural products may also pose a potential threat to human health. And some pesticides will remain in agricultural products for a long time, which may cause food safety hazards. The abuse of pesticides can also have long-term effects on ecosystems such as on soil microbes and other organisms. Therefore, although pesticides have a certain role in pest control, their limitations are also obvious. In order to solve these problems, it is necessary to explore and promote more sustainable, environmentally friendly, and selective pest control methods such as biological control, sex pheromone control, natural enemy release. Therefore, for future research purposes, we sought to refine the form of the optimal control question by introducing differences between harmful but less costly chemical pesticides and environmentally friendly but more risky and costly biological controls.

The results of numerical analysis do not represent the likelihood that we will reach the same conclusion in real situations, the content of the study can serve as an objective reference for real cases in which prey–predator interactions are indeed important in systems. In limited cases, we should use numerical analysis to analyze the data in real cases so as to provide more information and countermeasures for similar cases in the future. The environment is complex and variable, and there are many random factors that are difficult to predict, which can lead to bias in results. Therefore, the numerical simulation in this paper is considered under the assumed ideal environment. However, our simulation results in multiple scenarios with multiple sets of parameters provide a theoretical basis for further research that considers more factors and is closer to reality.

On the basis of this thesis, we can also study nonlinear cases. In addition, the time-delay parameter may affect the stable configuration of the eco-epidemiological system, and we can extend the eco-epidemiology model by considering the delay factor in model development.

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