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The nonlinear contraction in probabilistic cone b-metric spaces with application to integral equation

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Abstract. The probabilistic cone b-metric space is a novel concept that we describe in this study along with some of its fundamental topological properties and instances. We also established the fixed point theorem for the probabilistic nonlinear Banach contraction mapping on this kind of spaces. Many prior findings in the literature are generalized and unified by our findings. In order to illustrate the basic theorem in ordinary cone b-metric spaces, some related findings are also provided with an application to integral equation.

Keywords: probabilistic cone b-metric spaces, fixed point, φ-contraction, integral equation.

1 Introduction

Stefan Banach [4] was initiated the theory of metric fixed point, which is known as a major branch of modern mathematics through offering answers to numerous mathematical issues like differential equations, numerical treatment and network engineering [5, 13, 18]. Some mathematicians have developed this result by relaxing or modifying the contraction conditions [3, 7, 8, 20] or by introducing new metrics spaces [12, 14, 15, 23] with different structures. In this regard, by modifying the triangle inequality, Vulpe et al. [23] were the first to propose a generalized form of a metric space (b-metric space) in 1981 due to Berinde and Păcurar [6] according to the current bibliographical knowledge. In the same direction, Hussain and Shah [10] generalized the notion of a b-metric space by changing the set of real numbers by an ordered Banach space, and they proved some fixed point theorems on this type of spaces called the cone b-metric space, which is also a generalization of the idea of a cone metric space [2, 11, 14, 21].

Our objective is to extend the notion of the cone b-metric space to the probabilistic metric version as in [1, 8, 16], that will be a generalization of the probabilistic b-metric space introduced recently by Mbarki et al. [15]. Also, we inspect some of its topological proprieties and proved the fixed point theorem for the probabilistic nonlinear Banach contraction with some extensions to the ordinary cone b-metric spaces [9, 10].

This article is structured as follows. In Section 2, we discuss some fundamental ideas and findings in probabilistic and cone metric spaces. We introduce the concept of the probabilistic cone b-metric space in Section 3, while also going over some fundamental topological principles. Several fixed point theorems for nonlinear contraction in probabilistic cone b-metric spaces are proved in Section 4. Otherwise, in Section 5, we use several well-known preexisting results in ordinary cone b-metric spaces to demonstrate the validity of our findings. As an application, in Section 6, we substantiate the validity of our results by applying them to solve an integral equation.

2 Preliminaries

All over this paper, X represent a Banach space, and we denote by 0_X the zero of X.

Definition 1. Let $\mathcal{C}$ be a subset of a Banach space X, then $\mathcal{C}$ is a cone if:

(i) $\mathcal{C}$ is nonempty, closed and $\mathcal{C} \neq \{0_X\}$;
(ii) If $\theta, \vartheta \in [0, +\infty)$ and $x, y \in \mathcal{C}$, then $\theta x + \vartheta y \in \mathcal{C}$;
(iii) If both x and $-x$ are in $\mathcal{C}$, then $x = \{0_X\}$.

We note that for any given cone $\mathcal{C} \subset X$, $\preccurlyeq$ defines a partial ordering with respect to $\mathcal{C}$ by: $\theta \preccurlyeq \vartheta$ if and only if $\vartheta - \theta \in \mathcal{C}$; $\theta \prec \vartheta$ if $\theta \preccurlyeq \vartheta$; and $\theta \neq \vartheta$, while $\theta \prec\!\!\prec \vartheta$ will stand for $\vartheta - \theta \in \operatorname{Int} \mathcal{C}$ when $\operatorname{Int} \mathcal{C}$ is the interior of $\mathcal{C}$. In the following, we assume that all cones has nonempty interior.

A cone $\mathcal{C}$ is normal if there is a constant $M > 0$ such that $0_X \preccurlyeq \theta \preccurlyeq \vartheta$ implies that $\|\theta\| \preccurlyeq M\|\vartheta\|$ for all $\theta, \vartheta \in X$.

Definition 2. Let Γ be a nonempty set, $r \geqslant 1$ is a real number and $d_{\mathcal{C}} : \Gamma \times \Gamma \to X$ is a mapping that satisfies:

(i) $0_X \preccurlyeq d_{\mathcal{C}}(\theta, \vartheta)$ for all $\theta, \vartheta \in \Gamma$ with $\theta \neq \vartheta$ and $d_{\mathcal{C}}(\theta, \vartheta) = 0_X$ if and only if $\theta = \vartheta$;
(ii) $d_{\mathcal{C}}(\theta, \vartheta) = d_{\mathcal{C}}(\vartheta, \theta)$ for all $\theta, \vartheta \in \Gamma$;
(iii) $d_{\mathcal{C}}(\theta, \rho) \preccurlyeq r[d_{\mathcal{C}}(\theta, \vartheta) + d_{\mathcal{C}}(\vartheta, \rho)]$ for all $\theta, \vartheta, \rho \in \Gamma$.

In this case the triplet $(\Gamma, d_{\mathcal{C}}, r)$ is called a cone b-metric space, for short CbMS.

Now let us review some fundamental lemmas and concepts from probabilistic metric space.

Definition 3. Let $\mathcal{C}$ be a cone of a Banach X and $\sigma : \mathcal{C} \to [0, 1]$ is a function that satisfies:

(i) σ is continuous;
(ii) σ is nondecreasing;
(iii) $\sigma(0_X) = 0$ and $\sup_{t \in \mathcal{C}} \sigma(t) = 1$.

Then σ is a distance distribution function. We represent by Ω^+ the space of all distance distribution functions.

As an example, we state one of the element of Ω^+, which is the function μ_0 defined as

$$\mu_0(x) = \begin{cases} 0 & \text{if } x \notin \mathcal{C}, \\ 1 & \text{if } x \in \operatorname{Int} \mathcal{C}. \end{cases}$$

Definition 4. (See [22].) An operation τ on $[0,1]$ is a t-norm if for each $u, v, s \in [0,1]$, the following requirement are verified:

(i) $\tau(u,v) = \tau(v,u)$;
(ii) $\tau(u, \tau(v,s)) = \tau(\tau(u,v), s)$;
(iii) $\tau(u,v) < \tau(u,s)$ for $v < s$;
(iv) $\tau(u,1) = \tau(1,u) = u$.

Example 1. We cite here the most basic t-norms:

1. The minimum t-norm $\tau_M(u,v) = \operatorname{Min}(u,v)$.
2. The product t-norm $\tau_P(u,v) = u \cdot v$

Definition 5. (See [17].) We refer to a t-norm τ of H-type if the family $(\tau^n(x))_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$. It means that for all $\epsilon \in (0,1)$, there exist $\lambda \in (0,1)$ such that

$$t > 1 - \lambda \quad \Longrightarrow \quad \tau^n(t) > 1 - \epsilon \quad \text{for all } n \geqslant 1,$$

where for all $x \in [0,1]$ and $n \in \mathbb{N}$, we write

$$\tau^n(x) = \begin{cases} 1 & \text{if } n = 0, \\ \tau(\tau^{n-1}(x), x) & \text{otherwise.} \end{cases}$$

3 The probabilistic cone b-metric space

The notion of the probabilistic cone b-metric space, which is larger than the class of probabilistic metric spaces, is introduced in this section.

Definition 6. A quadruple $(\Gamma, \mathscr{F}, \tau, r)$ is called a probabilistic cone b-metric space (briefly PCbMS) if $\mathcal{C}$ is a cone of Banach space X, Γ is a nonempty set, $\mathscr{F}$ is a mapping from $\Gamma \times \Gamma$ into Ω^+, τ is a continuous t-norm, and $r \geqslant 1$ is a real number with the following requirement for all $s, v, u \in \Gamma$ and $x, y \in \operatorname{Int} \mathcal{C}$:

(i) $\mathscr{F}_{s,s} = \mu_0$;
(ii) $\mathscr{F}_{s,v} = \mu_0$ implies $s = v$;
(iii) $\mathscr{F}_{s,v} = \mathscr{F}_{v,s}$;
(iv) $\mathscr{F}_{s,v}(r(x+y)) \geqslant \tau(\mathscr{F}_{s,u}(x), \mathscr{F}_{u,v}(y))$.

It is obvious that if $r = 1$, then $(\Gamma, \mathscr{F}, \tau)$ is a probabilistic cone metric space.

Example 2. Let $X = \mathbb{R}^2$ with $\mathcal{C} = \{(x, y)\colon x, y \geqslant 0\} \subset X$ is a normal cone with a constant $M = 1$, $\Gamma = \mathbb{R}$, and $\mathscr{F} : \Gamma \times \Gamma \to \Omega^+$ is defined by

$$\mathscr{F}_{s,v}(t) = \frac{1}{\mathrm{e}^{|s-v|/\|t\|}} \quad \text{for all } t \in \operatorname{Int}\mathcal{C}.$$

We show that $(X, \mathscr{F}, \tau_p, 2)$ is a PCbMS space. Actually, we need to demonstrate the probabilistic triangle inequality since that (i), (ii) and (iii) of Definition 6 are trivially verified.

Let $s, v, u \in \Gamma$ and $t_1, t_2 \in \operatorname{Int}\mathcal{C}$. Since $\mathcal{C}$ is a normal cone, then $\|t_1 + t_2\|/\|t_2\| \geqslant 1$ and $\|t_1 + t_2\|/t_1 \geqslant 1$ for all $0_X \ll t_1, t_2$. Hence,

$$|s-u| \leqslant \frac{\|t_1+t_2\|}{\|t_1\|}|s-v| + \frac{\|t_1+t_2\|}{\|t_2\|}|v-u|,$$
$$\frac{|s-u|}{\|t_1+t_2\|} \leqslant \frac{|s-v|}{\|t_1\|} + \frac{|v-u|}{\|t_2\|}.$$

Therefore,

$$\frac{|s-u|}{2\|t_1+t_2\|} \leqslant \frac{|s-v|}{\|t_1\|} + \frac{|v-u|}{\|t_2\|},$$

which implies that

$$\mathrm{e}^{|s-u|/(2\|t_1+t_2\|)} \leqslant \mathrm{e}^{|s-v|/\|t_1\|}\mathrm{e}^{|v-u|/\|t_2\|}.$$

So,

$$\mathscr{F}_{s,u}\big(2(t_1+t_2)\big) \geqslant \tau_p\big(\mathscr{F}_{s,v}(t_1), \mathscr{F}_{v,u}(t_2)\big).$$

Thus, $(\Gamma, \mathscr{F}, \tau_p, 2)$ is PCbMS.

Definition 7. Let $(\Gamma, \mathscr{F}, \tau, r)$ be a PCbMS, then the neighborhoods family

$$\mathcal{R} = \big\{R_p(\epsilon, \lambda)\colon p \in \Gamma,\ 0_X \ll \epsilon \text{ and } \lambda > 0\big\},$$

where

$$R_p(\epsilon, \lambda) = \big\{q \in \Gamma\colon \mathscr{F}_{p,q}(\epsilon) > 1 - \lambda\big\}$$

is a topology in $(\Gamma, \mathscr{F})$ for all $p \in \Gamma$.

Lemma 1. *Let $(\Gamma, \mathscr{F}, \tau, r)$ be a PCbMS. If the t-norm τ is continuous, then the (ϵ, λ)-topology is a Hausdorff topology.*

Proof. Let $p, q \in \Gamma$ and $p \neq q$. Since $\mathscr{F} \in \Omega^+$, we put $\mathscr{F}_{p,q}(r\epsilon) = \lambda$ for some $\lambda \in (0, 1)$ and $0_X \ll \epsilon$. By the uniform continuity of τ, we have that for any $\lambda_0 \in (\lambda, 1)$, there exist $\lambda_1 \in (0, 1)$ such that

$$\tau(\lambda_1, \lambda_1) \geqslant \lambda_0.$$

Suppose that there exist $v \in R_p(r\epsilon, 1-\lambda_1) \cap R_q(r\epsilon, 1-\lambda_1)$. Then $\mathscr{F}_{p,v}(r\epsilon) > 1-(1-\lambda_1) = \lambda_1$ and $\mathscr{F}_{v,q}(r\epsilon) > 1-(1-\lambda_1) = \lambda_1$. Utilizing the triangle inequality, we get

$$\lambda = \mathscr{F}_{p,q}(r\epsilon) \geqslant \tau\left(\mathscr{F}_{p,v}\left(\frac{\epsilon}{2}\right), \mathscr{F}_{v,q}\left(\frac{\epsilon}{2}\right)\right) > \tau(\lambda_1, \lambda_1) \geqslant \lambda_0 > \lambda,$$

which is a contradiction, therefore, $R_p(r\epsilon, 1-\lambda_1) \cap R_q(r\epsilon, 1-\lambda_1) = \emptyset$. Hence, (ϵ, λ)-topology is a Hausdorff. $\square$

The notions of Cauchy sequence, completeness and convergence can all be applied to the case of a PCbMS as shown below.

Definition 8. A sequence $\{s_n\}$ in a PCbMS $(\Gamma, \mathscr{F}, \tau, r)$ is:

(i) Convergent to $s \in \Gamma$ if for any given $\lambda > 0$ and $0_X \lll \epsilon$, there exist $n_0(\epsilon, \lambda) \in \mathbb{N}$ that satisfies $\mathscr{F}_{s_n,s}(\epsilon) > 1-\lambda$ whenever $n \geqslant n_0(\epsilon, \lambda)$;

(ii) Strong Cauchy sequence if for any $\lambda > 0$ and $0_X \lll \epsilon$, there exist $n_0(\epsilon, \lambda) \in \mathbb{N}$ that satisfies $\mathscr{F}_{s_n,s_m}(\epsilon) > 1-\lambda$ whenever $n, m \geqslant n_0(\epsilon, \lambda)$.

A PCbMS $(\Gamma, \mathscr{F}, \tau, s)$ is complete if each Cauchy sequence in Γ is convergent in Γ.

Lemma 2. *Let $(\Gamma, \mathscr{F}, \tau, r)$ be a PCbMS with the continuous t-norm τ, then every convergence sequence is a Cauchy sequence.*

Proof. Take $\{s_n\}$ a sequence in Γ that converge to $s \in \Gamma$. Then for any $0_X \lll t$ and $\epsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$, which satisfies $\mathscr{F}_{s_n,s}(t) > 1-\epsilon$ for each $n \geqslant n_0$. Then, for all $n, m \geqslant n_0$ and $0_X \lll t$, we have

$$\mathscr{F}_{s_n,s_m}(rt) \geqslant \tau\left(\mathscr{F}_{s_n,s}\left(\frac{t}{2}\right), \mathscr{F}_{s,s_m}\left(\frac{t}{2}\right)\right) \geqslant \tau(1-\epsilon,\, 1-\epsilon).$$

Hence, by the continuity of τ, there us $\epsilon_0 \in (0,1)$ that satisfies

$$\tau(1-\epsilon,\, 1-\epsilon) \geqslant 1-\epsilon_0.$$

Therefore,

$$\mathscr{F}_{s_n,s_m}(rt) > 1-\epsilon_0,$$

which complete the proof. $\square$

The next lemma shows that every cone b-metric space is a PCbMS.

Lemma 3. *Let $(\Gamma, d_{\mathcal{C}}, r)$ be a cone b-metric space. We define $\mathscr{F} : \Gamma \times \Gamma \to \Omega^+$ by*

$$\mathscr{F}_{s,v}(t) = \mu_0\big(t - d_{\mathcal{C}}(s,v)\big) = \begin{cases} 0 & \textit{if}\, t \preccurlyeq d_{\mathcal{C}}(s,v), \\ 1 & \textit{if}\, d_{\mathcal{C}}(s,v) \prec t. \end{cases}$$

Then $(\Gamma, \mathscr{F}, \tau_M, r)$ is a PCbMS.

Proof. It is obvious that $F_{p,q}$ satisfies conditions (i), (ii) and (iii) of Definition 6. It left to prove the following inequality for every $s, v, u \in \Gamma$ and $0_X \prec\!\!\prec x, y$:

$$\mathscr{F}_{p,q}\big(r(x+y)\big) \geqslant \mathrm{Min}\big(\mathscr{F}_{s,u}(x), \mathscr{F}_{u,v}(y)\big). \tag{1}$$

Since $\mathscr{F}_{s,v}(r(x+y)) \in \{0,1\}$, then (1) holds if $\mathrm{Min}(\mathscr{F}_{s,u}(x), \mathscr{F}_{u,v}(y)) = 0$.

We suppose that $\mathrm{Min}(\mathscr{F}_{s,u}(x), \mathscr{F}_{u,v}(y)) = 1$, then $d_{\mathcal{C}}(s,u) \prec\!\!\prec x$ and $d_{\mathcal{C}}(u,v) \prec\!\!\prec y$. Since $(\Gamma, d_{\mathcal{C}}, r)$ is a cone b-metric space, we obtain

$$d_{\mathcal{C}}(s,v) \preccurlyeq r\big(d_{\mathcal{C}}(s,u) + d_{\mathcal{C}}(u,v)\big) \prec\!\!\prec r(x+y).$$

Hence, $\mathscr{F}_{s,v}(r(x+y)) = 1$, which proves that (1) also holds. Therefore, $(\Gamma, \mathscr{F}, \tau_M, r)$ is a PCbMS. □

However, it is simple to confirm that $(\Gamma, \mathscr{F}, \tau_M, r)$ is complete if and only if $(\Gamma, d_{\mathcal{C}}, r)$ is complete.

4 Fixed point theorem in probabilistic cone b-metric space

For this section, we assume that $(\Gamma, \mathscr{F}, \tau, r)$ is a PCbMS with a continuous t-norm τ, and we represent by χ the class of all function $\varphi : \mathcal{C} \to \mathcal{C}$ verifying

- If $\theta, \vartheta \in \mathcal{C}$ and $\theta \preccurlyeq \vartheta$, then $\varphi(\theta) \preccurlyeq \varphi(\vartheta)$;
- $\varphi(\theta) \preccurlyeq \theta$ for all $\theta \in \mathcal{C}$;
- $\lim_{n\to+\infty} \|\varphi^n(\theta)\| = 0$ for all $\theta \in \mathcal{C}$.

We shall now present the definition of φ-contraction in PCbMS.

Definition 9. Let $(\Gamma, \mathscr{F}, \tau, r)$ be a PCbMS. A mapping $f : \Gamma \to \Gamma$ is called a probabilistic φ-contraction if there exist $\varphi \in \chi$ such that for every $u, v \in \Gamma$ and $t \in \mathcal{C}$, we get

$$\mathscr{F}_{fu,fv}\big(\varphi(t)\big) \geqslant \mathscr{F}_{u,v}(rt). \tag{2}$$

Theorem 1. *Let $(\Gamma, \mathscr{F}, \tau, r)$ be a complete PCbMS with $\mathrm{Ran}\,\mathscr{F} \subset \mathcal{D}^+$ and a continuous t-norm of H-type τ. Let $f : \Gamma \to \Gamma$ be φ-probabilistic contraction, where $\varphi \in \chi$. Then f admits a unique fixed point.*

Proof. Let $s_0 \in \Gamma$, and we define a sequence $\{s_n\}$ by

$$s_n = f(s_{n-1}) = f^n(s_0) \quad \text{for all } n \geqslant 1.$$

From the contractivity condition we obtain for all $n, m \in \mathbb{N}$ such that $m > n$ and $t \in \mathcal{C}$, the following:

$$\begin{aligned}
\mathscr{F}_{s_n,s_{n+1}}\big(\varphi^n(t)\big) &\geqslant \mathscr{F}_{s_{n-1},s_n}\big(r\varphi^{n-1}(t)\big) \geqslant \mathscr{F}_{s_{n-1},s_n}\big(\varphi^{n-1}(t)\big) \\
&\geqslant \mathscr{F}_{s_{n-2},s_{n-1}}\big(r\varphi^{n-2}(t)\big) \geqslant \mathscr{F}_{s_{n-2},s_{n-1}}\big(\varphi^{n-2}(t)\big) \\
&\geqslant \cdots \geqslant \mathscr{F}_{s_0,s_1}(rt) \geqslant \mathscr{F}_{s_0,s_1}(t).
\end{aligned}$$

Next, we prove that $\lim_{n\to+\infty} \mathscr{F}_{s_n,s_{n+1}}(t) = 1$ for all $t \in \mathcal{C}$.

Since $\operatorname{Ran}\mathscr{F} \subset \Omega^+$, then there exist a t_0 for $0_X \lll t_0 \lll t$ such that for any $\lambda \in (0,1]$, we have $\mathscr{F}_{s_0,s_1}(t_0) > 1-\lambda$. As $\varphi \in \chi$, then there is $n_0 \in \mathbb{N}$ such that $\|\varphi^n(t_0)\| < \lambda$ for each $n \geqslant n_0$.

From the monotonicity of $\mathscr{F}$ we get

$$\mathscr{F}_{s_n,s_{n+1}}(t) \geqslant \mathscr{F}_{s_n,s_{n+1}}\bigl(\varphi^n(t_0)\bigr) \geqslant \mathscr{F}_{s_0,s_1}(t_0) > 1-\lambda \quad \text{for all } n \geqslant n_0,$$

which is implied that

$$\lim_{n\to+\infty} \mathscr{F}_{s_n,s_{n+1}}(t) = 1 \quad \text{for all } t \in \mathcal{C}. \tag{3}$$

Since $t-\varphi(t) \in \mathcal{C}$, we demonstrate by induction that for any $k > 2$,

$$\mathscr{F}_{s_n,s_{n+k}}(rt) \geqslant \tau^{k-1}\bigl(\mathscr{F}_{s_n,s_{n+1}}(t-\varphi(t))\bigr). \tag{4}$$

Inequality (4) is satisfied for $k = 3$.

Now, suppose that (4) holds for $k > 2$.

Using (2) and the monotonicity of τ, we have

$$\begin{aligned}
\mathscr{F}_{s_n,s_{n+k+1}}(rt) &= \mathscr{F}_{s_n,s_{n+k+1}}\bigl(r\bigl(t-\varphi(t)\bigr) + r\varphi(t)\bigr) \\
&\geqslant \tau\bigl(\mathscr{F}_{s_n,s_{n+1}}\bigl(t-\varphi(t)\bigr), \mathscr{F}_{s_{n+1},s_{n+1+k}}\bigl(\varphi(t)\bigr) \\
&\geqslant \tau\bigl(\mathscr{F}_{s_n,s_{n+1}}\bigl(t-\varphi(t)\bigr), \mathscr{F}_{s_n,s_{n+k}}(rt)\bigr) \\
&\geqslant \tau\bigl(\mathscr{F}_{s_n,s_{n+1}}\bigl(t-\varphi(t)\bigr), \tau^{k-1}\bigl(\mathscr{F}_{s_n,s_{n+1}}\bigl(t-\varphi(t)\bigr)\bigr)\bigr) \\
&= \tau^k\bigl(\mathscr{F}_{s_n,s_{n+1}}\bigl(t-\varphi(t)\bigr)\bigr).
\end{aligned}$$

Hence, (4) is proved for all $k > 2$.

Finally, we prove that $\{s_n\}$ is a Cauchy.

Let $\lambda \in (0,1)$, since τ is of H-type, we can find $\delta > 0$ that satisfies

$$\tau^n(v) > 1-\lambda \quad \text{for all } v \in (1-\delta,\, 1] \text{ and } n \in \mathbb{N}. \tag{5}$$

Since $(t-\varphi(t))/r \in \mathcal{C}$, then by (3) and the hypothesis of the theorem we get

$$\lim_{n\to+\infty} \mathscr{F}_{s_n,s_{n+1}}\left(\frac{1}{r}\bigl(t-\varphi(t)\bigr)\right) = 1.$$

So, there is $N \in \mathbb{N}$ that satisfies $\mathscr{F}_{s_n,s_{n+1}}((t-\varphi(t))/r) > 1-\delta$ for all $n \geqslant N$.

From (4) and (5) we obtain

$$\begin{aligned}
\mathscr{F}_{s_n,s_{n+k}}(t) &\geqslant \tau^{k-1}\left(\mathscr{F}_{s_n,s_{n+1}}\left(\frac{1}{r}\bigl(t-\varphi(t)\bigr)\right)\right) \\
&> 1-\lambda \quad \text{for all } n \geqslant N, \text{ and } k > 1.
\end{aligned}$$

Hence, $\{s_n\}$ is a Cauchy. From the completeness of Γ we suppose that $\{s_n\}$ converges to some $s \in \Gamma$. We will demonstrate that s is a fixed point of f.

By the b-triangular inequality and (2) we get

$$\begin{aligned}\mathscr{F}_{s,fs}(rt) &\geqslant \tau\left(\mathscr{F}_{s,s_n}\left(\frac{t}{2}\right), \mathscr{F}_{s_n,fs}\left(\frac{t}{2}\right)\right)\\ &\geqslant \tau\left(\mathscr{F}_{s,s_n}\left(\frac{t}{2}\right), \mathscr{F}_{fs_{n-1},fs}\left(\varphi\left(\frac{t}{2}\right)\right)\right)\\ &\geqslant \tau\left(\mathscr{F}_{s,s_n}\left(\frac{t}{2}\right), \mathscr{F}_{s_{n-1},s}\left(\frac{rt}{2}\right)\right).\end{aligned}$$

Letting $n \to +\infty$, we get

$$\mathscr{F}_{s,fs}(rt) \geqslant 1 \quad \text{for all } t \in \mathcal{C}.$$

Thus implies that s is a fixed point of f. Finally, we should demonstrate the uniqueness of s. Indeed, we suppose that there exists $y \in \Gamma$ with $y \neq s$ such that $fy = y$.

For all $t \in \mathcal{C}$, we have

$$\mathscr{F}_{s,y}\big(\varphi(t)\big) = \mathscr{F}_{fs,fy}\big(\varphi(t)\big) \geqslant \mathscr{F}_{s,y}(rt) \geqslant \mathscr{F}_{s,y}(t).$$

From that $\varphi \in \chi$ we get

$$\mathscr{F}_{s,y}\big(\varphi(t)\big) = \mathscr{F}_{s,y}(t). \tag{6}$$

By (6) we can easily show by induction that

$$\mathscr{F}_{s,y}\big(\varphi^n(t)\big) = \mathscr{F}_{s,y}(t) \quad \text{for all } n \geqslant 1.$$

Since $\lim_{n\to+\infty} \|\varphi^n(t)\| = 0$, we conclude that $s = y$. □

5 Related results in cone b-metric spaces

We demonstrate the equivalent fixed point theorem in ordinary CbMS as an application of the main result.

Corollary 1. *Let $(\Gamma, d_{\mathcal{C}}, r)$ be a complete cone b-metric space with $\varphi \in \chi$ and f, a self-mapping of $(\Gamma, d_{\mathcal{C}}, r)$, that satisfies for all $u, v \in \Gamma$,*

$$d_{\mathcal{C}}(fu, fv) \preccurlyeq \varphi\left(\frac{d_{\mathcal{C}}(u,v)}{r}\right). \tag{7}$$

Then f admits a unique fixed point s. Additionally, $f^n(u) \to s$ for all $u \in \Gamma$.

Proof. Define a mapping $\mathscr{F} : \Gamma \times \Gamma \to D^+$ that satisfies for all $t \in \mathcal{C}$,

$$\mathscr{F}_{u,v}(t) = \mu_{d_{\mathcal{C}}(u,v)}(t) = \begin{cases} 0 & \text{if } t \preccurlyeq d_{\mathcal{C}}(u,v),\\ 1 & \text{if } d_{\mathcal{C}}(u,v) \prec t.\end{cases}$$

Since $(\Gamma, d_{\mathcal{C}}, r)$ is complete, then $(\Gamma, \mathscr{F}, \tau_M, r)$ is a complete PCbMS.

We need to prove that condition (7) implies that f is a probabilistic φ-contraction mapping in $(\Gamma, \mathscr{F}, \tau_M, r)$. We have two cases for that.

Case 1. If $\mathscr{F}_{u,v}(rt) = 0$, then $\mathscr{F}_{fu,fv}(\varphi(t)) \geqslant \mathscr{F}_{u,v}(rt)$.

Case 2. If $\mathscr{F}_{u,v}(rt) = 1$, then $d_{\mathcal{C}}(u,v)/r \prec t$. Since φ is nondecreasing, we obtain

$$d_{\mathcal{C}}(fu,fv) \preccurlyeq \varphi\left(\frac{d_{\mathcal{C}}(u,v)}{r}\right) \prec \varphi(t).$$

It follows from (7) that $\mathscr{F}_{fu,fv}(\varphi(t)) = 1$, which suggests that

$$\mathscr{F}_{fu,fv}\big(\varphi(t)\big) \geqslant \mathscr{F}_{u,v}(rt).$$

So, from Cases 1 and 2 we conclude that f is a probabilistic φ-contraction. Therefore, f admits a fixed point. □

In the above corollary, if we take $r = 1$, then we have the result obtained in [19].

Corollary 2. (*See* [19].) *Let $(\Gamma, d_{\mathcal{C}})$ be a complete cone metric space, where $\mathcal{C}$ is a normal cone with normal constant, and f is a self-mapping of $(\Gamma, d_{\mathcal{C}})$ that satisfies for all $u, v \in \Gamma$,*

$$d_{\mathcal{C}}(fu,fv) \preccurlyeq \varphi\big(d_{\mathcal{C}}(u,v)\big)$$

with $\varphi \in \chi$. Then f admits a unique fixed point s.

By taking $\varphi(t) = rkt$ for all $t \in \mathcal{C}$ and $k \in [0,1)$ we get the result obtained by Huang and Xu in [9].

Corollary 3. (*See* [9].) *Let $(\Gamma, d_{\mathcal{C}})$ be a complete CMS, and let f be self-mapping of $(\Gamma, d_{\mathcal{C}})$ that satisfies for all $u, v \in \Gamma$,*

$$d_{\mathcal{C}}(fu,fv) \preccurlyeq k d_{\mathcal{C}}(u,v),$$

where $k \in [0,1)$. Then f admits a unique fixed point s. Furthermore, the sequence $\{f^n u\}$ converges to s for each $n \geqslant 1$.

6 Application

In this section, we consider $X = C([0,p],\mathbb{R})$, the space of all real valued continuous functions on $[0,p]$ endowed with the norm $\|u\|_\infty = \sup_{t\in[0,p]} |u(t)|$, where $p > 0$, $u \in C([0,p],\mathbb{R})$ and $\mathcal{C} = \{u \in X\colon u \geqslant 0\}$. By using our previous findings we provide an example of how fixed point methods are typically used to integral equations. Specifically, we look at the following integral equation:

$$u(t) = \int_0^p \Lambda\big(t,w,u(s)\big)\,\mathrm{d}w + h(t) \quad \text{for all } t \in [0,p], \tag{8}$$

where $\Lambda\colon [0,p]\times[0,p]\times\mathbb{R} \to \mathbb{R}$ and $h\colon [0,p] \to \mathbb{R}$. The induced metric $d_{\mathcal{C}} : C([0,p],\mathbb{R}) \times C([0,p],\mathbb{R}) \to \mathbb{R}$ is defined as

$$d_{\mathcal{C}}(u,v) = \sup_{t\in[0,p]} \big|u(t) - v(t)\big|. \tag{9}$$

Then it is clear that $(C([0,p],\mathbb{R}),d_{\mathcal{C}},1)$ is a complete CbMS with $r=1$. Define $\mathscr{F}: C([0,p],\mathbb{R})\times C([0,p],\mathbb{R})\to\Omega^{+}$ as

$$\mathscr{F}_{u,v}(t)=\mu_0\bigl(t-d_{\mathcal{C}}(u,v)\bigr).$$

Then by Lemma 3 we have that $(C([0,p],\mathbb{R}),\mathscr{F},\tau_M,1)$ is a complete PCbMS.

Theorem 2. *Let* $(C([0,p],\mathbb{R}),\mathscr{F},\tau_M,1)$ *be a complete PCbMS as defined above, and let* $\Lambda\in C([0,p]\times[0,p]\times\mathbb{R},\mathbb{R})$ *be an operator that verify the next conditions*:

(i) $\Lambda(t,w,.):\mathbb{R}\to\mathbb{R}$ *is increasing for all* $t,w\in[0,p]$.

(ii) *There are a continuous function* $\rho:[0,p]\times[0,p]\to\mathbb{R}^{+}$ *and a control function* $\varphi\in\chi$ *satisfying for all* $t,w\in[0,p]$ *and* $u,v\in\mathbb{R}$,

$$\bigl|\Delta(t,w,u)-\Delta(t,w,v)\bigr|<\rho(t,w)\varphi\bigl(d_{\mathcal{C}}(u,v)\bigr); \tag{10}$$

(iii) $\sup_{t\in[0,p]}\int_0^p\rho(t,w)\,\mathrm{d}w=1$.

Then the integral equation (8) *has a unique solution* $u^{*}\in(C([0,p],\mathbb{R})$.

Proof. Take $f:(C([0,p],\mathbb{R})\to(C([0,p],\mathbb{R})$ by

$$fu(t)=\int\limits_0^p\Lambda\bigl(t,w,u(w)\bigr)\,\mathrm{d}w+h(t)\quad\text{for all } t\in[0,p].$$

Then, for each $u,v\in(C([0,p],\mathbb{R})$, we have from (10) and (9)

$$\begin{aligned}\bigl|fu(t)-fv(t)\bigr|&=\left|\left(\int\limits_0^p\Lambda\bigl(t,w,u(w)\bigr)-\Lambda\bigl(t,w,v(w)\bigr)\right)\mathrm{d}w\right|\\&\leqslant\int\limits_0^p\rho(t,w)\varphi\bigl(\bigl|u(w)-v(w)\bigr|\bigr)\\&\leqslant\varphi\bigl(\|u-v\|_\infty\bigr)\int\limits_0^p\rho(t,w)\,\mathrm{d}w\\&=\varphi\bigl(\|u-v\|_\infty\bigr).\end{aligned}$$

Therefore,

$$d_{\mathcal{C}}(fu,fv)\leqslant\varphi\bigl(d_{\mathcal{C}}(u,v)\bigr)\quad\text{for each } u,v\in C\bigl([0,p],\mathbb{R}\bigr).$$

Since the space $(C([0,p],\mathbb{R}),\mathscr{F},\tau_M,1)$ is a complete PCbMS, then the induced space $(C([0,p],\mathbb{R}),d_{\mathcal{C}},1)$ is also a complete CbMS. So, from Corollary 1 f admits a unique fixed point $u^{*}\in(C([0,p],\mathbb{R})$, which is also the unique solution of the integral equation (8). □

7 Conclusion

As a generalization of probabilistic b-metric space, we developed a new idea in this study that we named probabilistic cone b-metric space. We also obtained certain fixed point results in these kinds of spaces. As a consequence of our main result, we found the corresponding fixed point result on usual CbMS with an application to integral equation for support the results thus obtained.

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