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Existence of solutions for a fractional Riemann–Stieltjes integral boundary value problem*

Yanfang Li[a], **Donal O'Regan**[b], **Jiafa Xu**[c,1]

[a]School of Arts and Sciences,
Suqian University,
Suqian, Jiangsu 223800, China
liyanfang998@163.com

[b]School of Mathematical and Statistical Sciences,
University of Galway,
Galway, Ireland
donal.oregan@universityofgalway.ie

[c]School of Mathematical Sciences,
Chongqing Normal University,
Chongqing 401331, China
xujiafa292@sina.com

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Abstract. In this paper, we study a Riemann–Liouville-type fractional Riemann–Stieltjes integral boundary value problem under some conditions regarding the spectral radius of the relevant linear operator. The existence of nontrivial solutions is obtained using topological degree, and our results improve and generalize some results in the literature.

Keywords: Riemann–Liouville-type fractional-order differential equations, integral boundary value problems, nontrivial solutions, topological degree.

1 Introduction

In this paper, we consider the existence of nontrivial solutions for the Riemann–Liouville-type fractional-order Riemann–Stieltjes integral boundary value problem

$$\begin{aligned} &-D_{0+}^{\alpha}\chi(t)+\tau\chi(t)=f\big(t,\chi(t)\big), \quad 0<t<1,\\ &\chi(0)=\chi'(0)=0, \qquad \chi(1)=\int_0^1 \chi(t)\,\mathrm{d}\gamma(t), \end{aligned} \tag{1}$$

where $2<\alpha<3$, D_{0+}^{α} is the Riemann–Liouville derivative, and f satisfies the following conditions.

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[1]Corresponding author.

(H1) $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$,
(H2) There exist $\sigma, \delta \in C([0,1], \mathbb{R}^+)$ and $\mathcal{K} \in C(\mathbb{R}, \mathbb{R}^+)$ with $\delta(t) \not\equiv 0$, $t \in [0,1]$, such that $f(t,x) \geqslant -\sigma(t) - \delta(t)\mathcal{K}(x)$ for all $x \in \mathbb{R}$, $t \in [0,1]$.
(H3) $\lim_{|x| \to +\infty} \mathcal{K}(x)/|x| = 0$.

We now consider conditions on τ and γ. Let

$$g(\tau) = \sum_{k=0}^{+\infty} \frac{(k\alpha + \alpha - 2)(k\alpha + \alpha - 3)\tau^k}{\Gamma(k\alpha + \alpha)}.$$

Then we have $g'(\tau) > 0$ on $(0, +\infty)$, $g(0) < 0$, and $\lim_{\tau \to +\infty} g(\tau) = +\infty$. Therefore, there exists a unique root $\tau^* > 0$ such that $g(\tau^*) = 0$.

Throughout our paper, we assume that τ and γ satisfy the conditions:

(H4) $\tau \in (0, \tau^*]$ is a constant.
(H5) γ is a function of bounded variation with $\gamma(t) \geqslant 0$, $t \in [0,1]$, and $\int_0^1 G(t)\,\mathrm{d}\gamma(t) \in [0, G(1))$, where

$$G(t) = t^{\alpha-1} \sum_{k=0}^{+\infty} \frac{\tau^k t^{\alpha k}}{\Gamma((k+1)\alpha)}.$$

Fractional calculus is used to describe problems in rheology, mechanics, material science, signal processing, and there is much research on fractional-order equations in the literature; see, for example, [1–6, 8–10, 12–24]. In [20] the authors investigated positive solutions for the resonant fractional multipoint boundary value problem

$$\begin{aligned}
&D_{0+}^{\alpha}\chi(t) + f\big(t, \chi(t), D_{0+}^{\beta}\chi(t)\big) = 0, \quad t \in (0,1),\\
&\chi(0) = \chi'(0) = 0, \qquad D_{0+}^{\beta}\chi(1) = \sum_{i=1}^{m} \eta_i D_{0+}^{\beta}\chi(\xi_i),
\end{aligned}$$

where D_{0+}^{α} is the Riemann–Liouville derivative, and in [22] the authors investigated multiple positive solutions for the higher-order fractional integral boundary value problem

$$\begin{aligned}
&-D_{0+}^{\eta-2}\big(\chi''(t)\big) + f\big(t, \chi(t)\big) = 0, \quad 0 < t < 1,\\
&\chi''(0) = \chi'''(0) = \cdots = \chi^{(n-2)}(0) = 0, \qquad D_{0+}^{\kappa-2}\chi''(1) = 0,\\
&\alpha\chi(0) - \beta\chi'(0) = \int_0^1 \chi(s)\,\mathrm{d}A(s), \qquad \gamma\chi(1) + \delta\chi'(1) = \int_0^1 \chi(s)\,\mathrm{d}B(s).
\end{aligned}$$

The spectral theory of linear operators can be used to study differential equations; see [2, 13, 19, 21, 23]. In [23] the authors studied positive solutions for the fractional integral boundary value problem

$$\begin{aligned}
&D_{0+}^{\alpha}\chi(t) + h(t)f\big(t, \chi(t)\big) = 0, \quad 0 < t < 1,\\
&\chi(0) = \chi'(0) = \chi''(0) = 0, \qquad \chi(1) = \lambda \int_0^{\eta} \chi(s)\,\mathrm{d}s,
\end{aligned}$$

where $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies the conditions:

(HZ1) $\liminf_{\chi \to 0^+} f(t,\chi)/\chi > \lambda_1$, $\limsup_{\chi \to +\infty} f(t,\chi)/\chi < \lambda_1$ uniformly with respect to $t \in [0,1]$,

(HZ2) $\limsup_{\chi \to 0^+} f(t,\chi)/\chi < \lambda_1$, $\liminf_{\chi \to +\infty} f(t,\chi)/\chi > \lambda_1$ uniformly with respect to $t \in [0,1]$, where λ_1 is the first eigenvalue of the operator $(L_{Z1}\chi)(t) = \int_0^1 G(t,s)h(s)\chi(s)\,\mathrm{d}s$, and G is the Green's function.

Motivated by the aforementioned works, in this paper, we use topological degree to study nontrivial solutions for (1) under some conditions concerning the spectral radius of the relevant linear operator. Note the considered linear operator can include the Riemann–Stieltjes integral condition in (1), and the approach is quite different from previous works in the literature. Also, we note that our conditions are more general than (H1), (H2).

2 Preliminaries

We first provide some useful definitions and conclusions, which are used to obtain our main results.

Definition 1. (See [10, 14].) The α-order Riemann–Liouville fractional derivative of a function $\varphi : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha}\varphi(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1}\varphi(s)\,\mathrm{d}s,$$

where $n = [\alpha]+1$. $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0,+\infty)$.

Definition 2. (See [10,14].) The α-order Riemann–Liouville fractional integral of a function $\varphi : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s,$$

provided that the right-hand side is point-wise defined on $(0,+\infty)$.

In what follows, we will calculate the Green's function associated with (1).

Lemma 1. (*See* [17].) *Suppose that* (H1), (H2) *hold, and* $\psi \in L[0,1]$. *Then the boundary value problem*

$$-D_{0+}^{\alpha}\chi(t) + \tau\chi(t) = \psi(t), \quad 0 < t < 1,$$

$$\chi(0) = \chi'(0) = 0, \qquad \chi(1) = \int_0^1 \chi(t)\,\mathrm{d}\gamma(t)$$

has a unique solution

$$\chi(t) = \int_0^1 K(t,s)\psi(s)\,\mathrm{d}s + \frac{G(t)}{G(1)}\int_0^1 \chi(t)\,\mathrm{d}\gamma(t) \tag{2}$$

$$= \int_0^1 \Theta(t,s)\psi(s)\,\mathrm{d}s, \tag{3}$$

where

$$\Theta(t,s) = K(t,s) + \frac{G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}\int_0^1 K(t,s)\,\mathrm{d}\gamma(t),$$

$$K(t,s) = \frac{1}{G(1)}\begin{cases} G(t)G(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ G(t)G(1-s) - G(t-s)G(1), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Proof. We first consider the problem

$$-D_{0+}^{\alpha}\chi(t) + \tau\chi(t) = 0, \quad 0 < t < 1,$$

$$\chi(0) = \chi'(0) = 0, \qquad \chi(1) = \int_0^1 \chi(t)\,\mathrm{d}\gamma(t).$$

Using the method in [17, Lemma 2.1], we have

$$\chi(t) = c_1 G(t) + c_2 G'(t) + c_3 G''(t),$$

where $c_i \in \mathbb{R}$, $i = 1, 2, 3$. From $\chi(0) = \chi'(0) = 0$ we have $c_2 = c_3 = 0$. Hence, $\chi(1) = \int_0^1 \chi(t)\,\mathrm{d}\gamma(t)$ implies that

$$\chi(1) = c_1 G(1) = \int_0^1 \chi(t)\,\mathrm{d}\gamma(t) \quad \text{and} \quad c_1 = \frac{1}{G(1)}\int_0^1 \chi(t)\,\mathrm{d}\gamma(t).$$

Thus, we have

$$\chi(t) = \frac{G(t)}{G(1)}\int_0^1 \chi(t)\,\mathrm{d}\gamma(t). \tag{4}$$

Next, we consider the problem

$$-D_{0+}^{\alpha}\chi(t) + \tau\chi(t) = \psi(t), \quad 0 < t < 1,$$

$$\chi(0) = \chi'(0) = 0, \qquad \chi(1) = 0.$$

From [17, Lemma 2.1] we have

$$\chi(t) = \int_0^1 K(t,s)\psi(s)\,\mathrm{d}s. \tag{5}$$

Combining (4) and (5), we obtain (2).

We multiply both sides of (2) by $\mathrm{d}\gamma(t)$ and integrate over $[0,1]$, and then we have

$$\int_0^1 \chi(t)\,\mathrm{d}\gamma(t) = \int_0^1\int_0^1 K(t,s)\psi(s)\,\mathrm{d}s\,\mathrm{d}\gamma(t) + \int_0^1 \frac{G(t)}{G(1)}\,\mathrm{d}\gamma(t)\int_0^1 \chi(t)\,\mathrm{d}\gamma(t)$$

and

$$\int_0^1 \chi(t)\,\mathrm{d}\gamma(t) = \frac{G(1)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}\int_0^1\int_0^1 K(t,s)\psi(s)\,\mathrm{d}s\,\mathrm{d}\gamma(t).$$

Consequently, we have

$$\begin{aligned}\chi(t) &= \int_0^1 K(t,s)\psi(s)\,\mathrm{d}s + \frac{G(t)}{G(1)}\frac{G(1)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}\int_0^1\int_0^1 K(t,s)\psi(s)\,\mathrm{d}s\,\mathrm{d}\gamma(t)\\ &= \int_0^1 \Theta(t,s)\psi(s)\,\mathrm{d}s.\end{aligned}$$

This is (3). This completes the proof. □

Lemma 2. *Suppose that* $s^* \in (0,1)$ *such that* $s^* = (1-s^*)^{\alpha-2}$. *Then* $\Theta(t,s)$ *has the following properties*:

(i) $\Theta(t,s) \geqslant \kappa_1 s(1-s)^{\alpha-1}t^{\alpha-1}$ *for all* $t, s \in [0,1]$;
(ii) $\Theta(t,s) \leqslant \kappa_2 s(1-s)^{\alpha-1}$ *for all* $t, s \in [0,1]$, *where*

$$\kappa_1 = \frac{\int_0^1 (1-t)t^{\alpha-1}\,\mathrm{d}\gamma(t)}{[G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)]G(1)[\Gamma(\alpha)]^3},$$
$$\kappa_2 = \frac{[G'(1)]^2}{G(1)s^*}\left[1 + \frac{G(1)\gamma(1)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}\right],$$

(iii) $\Theta(t,s) \leqslant G^2(1)/(G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t))\,t^{\alpha-1}(1-s)^{\alpha-1}$ *for all* $t, s \in [0,1]$.

Proof. We first use [17, Thm. 3.1] and note that for all $t, s \in [0,1]$,

$$\frac{1}{G(1)[\Gamma(\alpha)]^2} s(1-s)^{\alpha-1}(1-t)t^{\alpha-1} \leqslant K(t,s) \leqslant \frac{[G'(1)]^2}{G(1)s^*} s(1-s)^{\alpha-1}. \tag{6}$$

Also, we note that

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \leqslant G(t) \leqslant t^{\alpha-1}G(1), \quad t \in [0,1]. \tag{7}$$

Therefore, from (6), (7) we have

$$\begin{aligned}
\Theta(t,s) &= K(t,s) + \frac{G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 K(t,s)\,\mathrm{d}\gamma(t) \\
&\geqslant \frac{t^{\alpha-1}}{[G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)]\Gamma(\alpha)} \int_0^1 \frac{1}{G(1)[\Gamma(\alpha)]^2} s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}\,\mathrm{d}\gamma(t) \\
&= \kappa_1 s(1-s)^{\alpha-1}t^{\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
\Theta(t,s) &= K(t,s) + \frac{G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 K(t,s)\,\mathrm{d}\gamma(t) \\
&\leqslant \frac{[G'(1)]^2}{G(1)s^*} s(1-s)^{\alpha-1} + \frac{G(1)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 \frac{[G'(1)]^2}{G(1)s^*} s(1-s)^{\alpha-1}\,\mathrm{d}\gamma(t) \\
&= \frac{[G'(1)]^2}{G(1)s^*} s(1-s)^{\alpha-1}\left[1 + \frac{G(1)\gamma(1)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}\right] \\
&= \kappa_2 s(1-s)^{\alpha-1}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\Theta(t,s) &= K(t,s) + \frac{G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 K(t,s)\,\mathrm{d}\gamma(t) \\
&\leqslant \frac{G(t)G(1-s)}{G(1)} + \frac{G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 \frac{G(t)G(1-s)}{G(1)}\,\mathrm{d}\gamma(t) \\
&= \frac{G^2(1)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} t^{\alpha-1}(1-s)^{\alpha-1}, \quad t \in [0,1].
\end{aligned}$$

This completes the proof. □

Let $E := C[0,1]$, $\|\vartheta\| := \max_{t\in[0,1]} |\vartheta(t)|$, $P := \{\vartheta \in E\colon\ \vartheta(t) \geqslant 0\ \forall t \in [0,1]\}$. Then $(E, \|\cdot\|)$ is a real Banach space, and P is a cone on E. Define an operator $\Psi : E \to E$

as follows:

$$(\Psi\vartheta)(t) = \int_0^1 K(t,s) f\big(s, \vartheta(s)\big)\,\mathrm{d}s + \frac{G(t)}{G(1)} \int_0^1 \vartheta(t)\,\mathrm{d}\gamma(t), \quad \vartheta \in E,\ t \in [0,1].$$

It is easy to find that if Ψ has a fixed point ϑ^* in E, i.e., $\Psi\vartheta^* = \vartheta^*$, then ϑ^* is a solution for (1). Note that

$$\vartheta^*(t) = \int_0^1 K(t,s) f\big(s, \vartheta^*(s)\big)\,\mathrm{d}s + \frac{G(t)}{G(1)} \int_0^1 \vartheta^*(t)\,\mathrm{d}\gamma(t).$$

From Lemma 1 we have

$$\vartheta^*(t) = \int_0^1 \Theta(t,s) f\big(s, \vartheta^*(s)\big)\,\mathrm{d}s.$$

Hence, ϑ^* is also a fixed point of operator $L \circ F$, which can be denoted by Ψ, where $F: E \to E$ is the Nemytskii operator defined by $(F\vartheta)(t) := f(t, \vartheta(t))$, and

$$(L\vartheta)(t) = \int_0^1 \Theta(t,s)\vartheta(s)\,\mathrm{d}s.$$

Lemma 3. *Let* $P_{01} = \{\vartheta \in P\colon \vartheta(t) \geqslant (\kappa_1/\kappa_2)t^{\alpha-1}\|\vartheta\|,\ t \in [0,1]\}$. *Then* $L(P) \subset P_{01}$.

Proof. If $\vartheta \in P$, from Lemma 2(i), (ii) we have

$$(L\vartheta)(t) \leqslant \int_0^1 \kappa_2 s(1-s)^{\alpha-1}\vartheta(s)\,\mathrm{d}s$$

and

$$\begin{aligned}(L\vartheta)(t) &\geqslant \int_0^1 \kappa_1 s(1-s)^{\alpha-1} t^{\alpha-1}\vartheta(s)\,\mathrm{d}s = \frac{\kappa_1}{\kappa_2} t^{\alpha-1} \int_0^1 \kappa_2 s(1-s)^{\alpha-1}\vartheta(s)\,\mathrm{d}s \\ &\geqslant \frac{\kappa_1}{\kappa_2} t^{\alpha-1}\|L\vartheta\|.\end{aligned}$$

This completes the proof. □

Lemma 4. *Let*

$$(L_\xi\vartheta)(t) = \xi \int_0^1 K(t,s)\vartheta(s)\,\mathrm{d}s + \frac{G(t)}{G(1)} \int_0^1 \vartheta(t)\,\mathrm{d}\gamma(t), \quad \xi > 0.$$

Then the spectral radius of L_ξ*, denoted by* $r(L_\xi)$*, is positive.*

Note that $r(L_\xi) \geqslant \xi r(L_K)$. So, we only need to prove $r(L_K) > 0$, where $r(L_K)$ is the spectral radius of the operator L_K defined as follows:

$$(L_K\vartheta)(t) = \int_0^1 K(t,s)\vartheta(s)\,\mathrm{d}s.$$

Moreover, we can use the similar method in [21, Lemma 5] to obtain $r(L_K) > 0$, so we omit its proof here. Therefore, the Krein–Rutman theorem [11] implies that there exists $\zeta_\xi \in P \setminus \{0\}$ such that

$$L_\xi\zeta_\xi = r(L_\xi)\zeta_\xi. \tag{8}$$

Lemma 5. (*See* [7].) *Let E be a Banach space, $\Omega \subset E$ a bounded open set, and $T: \Omega \to E$ a continuous compact operator. If there exists $x_0 \in E \setminus \{0\}$ such that*

$$x - Tx \neq \mu x_0 \quad \forall x \in \partial\Omega,\ \mu \geqslant 0,$$

then the topological degree $\deg(I - T,\ \Omega,\ 0) = 0$.

Lemma 6. (*See* [7].) *Let E be a Banach space, $\Omega \subset E$ a bounded open set with $0 \in \Omega$, and $T: \Omega \to E$ a continuous compact operator. If*

$$Tx \neq \mu x \quad \forall x \in \partial\Omega,\ \mu \geqslant 1,$$

then the topological degree $\deg(I - T,\ \Omega,\ 0) = 1$.

3 Main results

Now, we list some assumptions on f, which we need in this section.

- (H6) There exists $\xi_1 > 0$ with $r(L_{\xi_1}) \geqslant 1$ such that $\liminf_{|x|\to+\infty} f(t,x)/|x| > \xi_1$ uniformly for $t \in [0,1]$,
- (H7) There exists $\xi_2 > 0$ with $r(L_{\xi_2}) < 1$ such that $\limsup_{|x|\to 0} |f(t,x)|/|x| < \xi_2$ uniformly for $t \in [0,1]$.

Theorem 1. *Suppose that* (H1)–(H7) *hold. Then* (1) *has at least one nontrivial solution.*

Before the proof of Theorem 1, we present a lemma.

Lemma 7. *Suppose that all assumptions in Theorem* 1 *hold. Let $P_{02} = \{\vartheta \in P\colon \vartheta(t) \geqslant \kappa_3 t^{\alpha-1}\|\vartheta\|,\ t \in [0,1]\}$. Then $\zeta_{\xi_1} \in P_{02}$, where κ_3 and ζ_{ξ_1} are given in the proof.*

Proof. By Lemma 4 and (8) we have

$$(L_{\xi_1}\vartheta)(t) = \xi_1\int_0^1 K(t,s)\vartheta(s)\,\mathrm{d}s + \frac{G(t)}{G(1)}\int_0^1 \vartheta(t)\,\mathrm{d}\gamma(t),$$

and there exists $\zeta_{\xi_1} \in P \setminus \{0\}$ such that

$$L_{\xi_1}\zeta_{\xi_1} = r(L_{\xi_1})\zeta_{\xi_1}. \tag{9}$$

Hence, we obtain

$$\xi_1 \int_0^1 K(t,s)\zeta_{\xi_1}(s)\,\mathrm{d}s + \frac{G(t)}{G(1)} \int_0^1 \zeta_{\xi_1}(t)\,\mathrm{d}\gamma(t) = r(L_{\xi_1})\zeta_{\xi_1}(t).$$

Note that $r(L_{\xi_1}) \geqslant 1$, and (H5) enables us to obtain

$$\int_0^1 \zeta_{\xi_1}(t)\,\mathrm{d}\gamma(t) = \frac{\xi_1 G(1)\int_0^1\int_0^1 K(t,s)\zeta_{\xi_1}(s)\,\mathrm{d}s\,\mathrm{d}\gamma(t)}{r(L_{\xi_1})G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)},$$

and then

$$r(L_{\xi_1})\zeta_{\xi_1}(t) = \xi_1 \int_0^1 K(t,s)\zeta_{\xi_1}(s)\,\mathrm{d}s + \frac{\xi_1 G(t)\int_0^1\int_0^1 K(t,s)\zeta_{\xi_1}(s)\,\mathrm{d}s\,\mathrm{d}\gamma(t)}{r(L_{\xi_1})G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}.$$

Consequently, we have

$$\zeta_{\xi_1}(t) = \int_0^1 \varLambda(t,s)\zeta_{\xi_1}(s)\,\mathrm{d}s,$$

where

$$\varLambda(t,s) = \frac{\xi_1}{r(L_{\xi_1})}\left[K(t,s) + \frac{G(t)\int_0^1 K(t,s)\,\mathrm{d}\gamma(t)}{r(L_{\xi_1})G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}\right].$$

Note that (6) and (7) imply that

$$\zeta_{\xi_1}(t) \leqslant \int_0^1 \frac{\xi_1[G'(1)]^2}{r(L_{\xi_1})G(1)s^*}\left[1 + \frac{G(1)\gamma(1)}{r(L_{\xi_1})G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}\right] s(1-s)^{\alpha-1}\zeta_{\xi_1}(s)\,\mathrm{d}s$$

and

$$\begin{aligned}
\zeta_{\xi_1}(t) &\geqslant \frac{\xi_1}{r(L_{\xi_1})} \int_0^1 \frac{G(t)\int_0^1 K(t,s)\,\mathrm{d}\gamma(t)}{r(L_{\xi_1})G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)}\zeta_{\xi_1}(s)\,\mathrm{d}s \\
&\geqslant t^{\alpha-1} \frac{\int_0^1 (1-t)t^{\alpha-1}\,\mathrm{d}\gamma(t)}{G(1)[\Gamma(\alpha)]^3[r(L_{\xi_1})G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)]} \\
&\quad \times \frac{\xi_1}{r(L_{\xi_1})} \int_0^1 s(1-s)^{\alpha-1}\zeta_{\xi_1}(s)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= t^{\alpha-1}\frac{s^*\int_0^1(1-t)t^{\alpha-1}\,\mathrm{d}\gamma(t)}{[G'(1)]^2[\Gamma(\alpha)]^3[r(L_{\xi_1})G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)]}\\
&\quad\times\left[1+\frac{G(1)\gamma(1)}{r(L_{\xi_1})G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)}\right]^{-1}\\
&\quad\times\int\limits_0^1\frac{\xi_1[G'(1)]^2}{r(L_{\xi_1})G(1)s^*}\left[1+\frac{G(1)\gamma(1)}{r(L_{\xi_1})G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)}\right]\\
&\qquad\times s(1-s)^{\alpha-1}\zeta_{\xi_1}(s)\,\mathrm{d}s\\
&\geqslant \kappa_3 t^{\alpha-1}\|\zeta_{\xi_1}\|,
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_3 = {}&\frac{s^*\int_0^1(1-t)t^{\alpha-1}\,\mathrm{d}\gamma(t)}{[G'(1)]^2[\Gamma(\alpha)]^3[r(L_{\xi_1})G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)]}\\
&\times\left[1+\frac{G(1)\gamma(1)}{r(L_{\xi_1})G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)}\right]^{-1}.
\end{aligned}
$$

This completes the proof. □

Note that

$$
\frac{\kappa_1}{\kappa_2}=\frac{s^*\int_0^1(1-t)t^{\alpha-1}\,\mathrm{d}\gamma(t)}{[G'(1)]^2[\Gamma(\alpha)]^3[G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)]}\left[1+\frac{G(1)\gamma(1)}{G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)}\right]^{-1}\geqslant\kappa_3,
$$

and let

$$
P_0=\bigl\{\vartheta\in P\colon\ \vartheta(t)\geqslant\kappa_4 t^{\alpha-1}\|\vartheta\|,\ t\in[0,1]\bigr\},
$$

where $\kappa_4=\min\{\kappa_3, 1/(G(1)\Gamma(\alpha))\}$. Then from Lemmas 3 and 7 we obtain

$$
L(P)\subset P_0 \tag{10}
$$

and

$$
\zeta_{\xi_1}\in P_0,\quad \text{i.e.,}\quad \zeta_{\xi_1}(t)\geqslant\kappa_4 t^{\alpha-1}\|\zeta_{\xi_1}\|,\quad t\in[0,1]. \tag{11}
$$

Proof of Theorem 1. As discussed in Section 2, we only need to prove that Ψ has a fixed point. By (H6) there exist $\varepsilon_0>0$ and $X_0>0$ such that

$$
f(t,x)\geqslant(\xi_1+\varepsilon_0)|x|\quad\text{for } |x|>X_0,\ t\in[0,1].
$$

For any fixed ε with $\varepsilon_0-\|\delta\|\varepsilon>0$, from (H3) there exists $X_1>X_0$ such that

$$
\mathcal{K}(x)\leqslant\varepsilon|x|\quad\text{for } |x|>X_1.
$$

Note (H2), and we obtain

$$\begin{aligned} f(t,x) &\geqslant (\xi_1+\varepsilon_0)|x| - \sigma(t) - \delta(t)\mathcal{K}(x) \\ &\geqslant \bigl(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\bigr)|x| - \sigma(t), \quad t\in[0,1],\ |x|>X_1. \end{aligned}$$

Let

$$C_{X_1} = \bigl(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\bigr)X_1 + \max_{t\in[0,1],\,|x|\leqslant X_1}\bigl|f(t,x)\bigr|, \qquad \mathcal{K}^* = \max_{|x|\leqslant X_1}\mathcal{K}(x),$$

and we have

$$\begin{aligned} f(t,x) &\geqslant \bigl(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\bigr)|x| - \sigma(t) - C_{X_1}, \\ \mathcal{K}(x) &\leqslant \varepsilon|x| + \mathcal{K}^*, \quad t\in[0,1],\ x\in\mathbb{R}. \end{aligned} \tag{12}$$

Note that ε can be chosen arbitrarily small, and we let

$$\begin{aligned} \mathcal{R} > \max\biggl\{ &\frac{\kappa_2[\|\sigma\|+\|\delta\|\mathcal{K}^*+C_{X_1}]\Gamma(\alpha)}{\Gamma(\alpha+2)-\varepsilon\kappa_2\|\delta\|\Gamma(\alpha)}, \\ &\frac{[\|\sigma\|+\|\delta\|\mathcal{K}^*+C_{X_1}][\frac{\kappa_2\kappa_4(\varepsilon_0-\varepsilon\|\delta\|)\Gamma(\alpha)}{\Gamma(\alpha+2)}+\frac{(\xi_1+\varepsilon_0-\varepsilon\|\delta\|)G^2(1)}{\alpha[G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)]}]}{\kappa_4(\varepsilon_0-\varepsilon\|\delta\|)[1-\varepsilon\frac{\|\delta\|\kappa_2\Gamma(\alpha)}{\Gamma(\alpha+2)}]-\varepsilon\frac{\|\delta\|(\xi_1+\varepsilon_0-\varepsilon\|\delta\|)G^2(1)}{\alpha[G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)]}}\biggr\}. \end{aligned} \tag{13}$$

In what follows, we prove that

$$\vartheta - \Psi\vartheta \neq \mu\zeta_{\xi_1}, \quad \vartheta\in\partial B_{\mathcal{R}},\ \mu\geqslant 0, \tag{14}$$

where ζ_{ξ_1} is defined in (9), and $B_{\mathcal{R}} = \{\vartheta\in E\colon \|\vartheta\|<\mathcal{R}\}$. Suppose the contrary. Then there exist $\vartheta_1\in\partial B_{\mathcal{R}}$, $\mu_1\geqslant 0$ such that

$$\vartheta_1 - \Psi\vartheta_1 = \mu_1\zeta_{\xi_1}. \tag{15}$$

Note that $\mu_1\neq 0$, otherwise, ϑ_1 is a solution for (1), and the theorem is proved. Let

$$\widetilde{\vartheta}_1(t) = \int_0^1 \Theta(t,s)\bigl[\sigma(s)+\delta(s)\mathcal{K}\bigl(\vartheta_1(s)\bigr)+C_{X_1}\bigr]\,\mathrm{d}s, \quad t\in[0,1],$$

where $\Theta(t,s)$ is defined by Lemma 1. From (2), (3), $\widetilde{\vartheta}_1$ can also be expressed by

$$\begin{aligned} \widetilde{\vartheta}_1(t) = &\int_0^1 K(t,s)\bigl[\sigma(s)+\delta(s)\mathcal{K}\bigl(\vartheta_1(s)\bigr)+C_{X_1}\bigr]\,\mathrm{d}s \\ &+ \frac{G(t)}{G(1)}\int_0^1 \widetilde{\vartheta}_1(t)\,\mathrm{d}\gamma(t), \quad t\in[0,1]. \end{aligned} \tag{16}$$

Recall that $\sigma(s)+\delta(s)\mathcal{K}(\vartheta_1(s))+C_{X_1}\in P$ for $s\in[0,1]$, then $\widetilde{\vartheta}_1\in P$. Furthermore, (15) enables us to obtain

$$\begin{aligned}\vartheta_1(t)+\widetilde{\vartheta}_1(t)&=\mu_1\zeta_{\xi_1}(t)+(\Psi\vartheta_1)(t)+\widetilde{\vartheta}_1(t)\\&=\int_0^1 K(t,s)\bigl[f\bigl(s,\vartheta_1(s)\bigr)+\sigma(s)+\delta(s)\mathcal{K}\bigl(\vartheta_1(s)\bigr)+C_{X_1}\bigr]\,\mathrm{d}s\\&\quad+\frac{G(t)}{G(1)}\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)+\mu_1\zeta_{\xi_1}(t).\end{aligned}$$

From (H5) we have

$$\begin{aligned}&\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)\\&\quad=\int_0^1\int_0^1 K(t,s)\bigl[f\bigl(s,\vartheta_1(s)\bigr)+\sigma(s)+\delta(s)\mathcal{K}\bigl(\vartheta_1(s)\bigr)+C_{X_1}\bigr]\,\mathrm{d}s\,\mathrm{d}\gamma(t)\\&\qquad+\int_0^1\frac{G(t)}{G(1)}\,\mathrm{d}\gamma(t)\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)+\mu_1\int_0^1\zeta_{\xi_1}(t)\,\mathrm{d}\gamma(t)\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)\\&\quad=\biggl\{G(1)\int_0^1\int_0^1 K(t,s)\bigl[f\bigl(s,\vartheta_1(s)\bigr)+\sigma(s)+\delta(s)\mathcal{K}\bigl(\vartheta_1(s)\bigr)+C_{X_1}\bigr]\,\mathrm{d}s\,\mathrm{d}\gamma(t)\\&\qquad+\mu_1G(1)\int_0^1\zeta_{\xi_1}(t)\,\mathrm{d}\gamma(t)\biggr\}\frac{1}{G(1)-\int_0^1G(t)\,\mathrm{d}\gamma(t)}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\vartheta_1(t)+\widetilde{\vartheta}_1(t)\\&\quad=\int_0^1 K(t,s)\bigl[f\bigl(s,\vartheta_1(s)\bigr)+\sigma(s)+\delta(s)\mathcal{K}\bigl(\vartheta_1(s)\bigr)+C_{X_1}\bigr]\,\mathrm{d}s+\mu_1\zeta_{\xi_1}(t)\\&\qquad+\frac{\mu_1G(t)}{G(1)-\int_0^1G(t)\,\mathrm{d}\gamma(t)}\int_0^1\zeta_{\xi_1}(t)\,\mathrm{d}\gamma(t)\end{aligned}$$

$$+ \frac{G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \times \int_0^1 \int_0^1 K(t,s)\big[f\big(s, \vartheta_1(s)\big) + \sigma(s) + \delta(s)\mathcal{K}\big(\vartheta_1(s)\big) + C_{X_1}\big]\,\mathrm{d}s\,\mathrm{d}\gamma(t)$$

$$= \int_0^1 \Theta(t,s)\big[f\big(s, \vartheta_1(s)\big) + \sigma(s) + \delta(s)\mathcal{K}\big(\vartheta_1(s)\big) + C_{X_1}\big]\,\mathrm{d}s + \mu_1\zeta_{\xi_1}(t) + \frac{\mu_1 G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 \zeta_{\xi_1}(t)\,\mathrm{d}\gamma(t).$$

Note that

$$f\big(s, \vartheta_1(s)\big) + \sigma(s) + \delta(s)\mathcal{K}\big(\vartheta_1(s)\big) + C_{X_1} \geqslant 0, \quad s \in [0,1],$$

and

$$\frac{\mu_1 G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 \zeta_{\xi_1}(t)\,\mathrm{d}\gamma(t) \geqslant \frac{t^{\alpha-1}}{G(1)\Gamma(\alpha)} \frac{\mu_1 G(1)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 \zeta_{\xi_1}(t)\,\mathrm{d}\gamma(t), \quad t \in [0,1].$$

Then (10) and (11) imply that

$$\vartheta_1 + \widetilde{\vartheta}_1 \in P_0. \tag{17}$$

From (16) we obtain

$$\widetilde{\vartheta}_1(t) = \int_0^1 \Theta(t,s)\big[\sigma(s) + \delta(s)\mathcal{K}\big(\vartheta_1(s)\big) + C_{X_1}\big]\,\mathrm{d}s.$$

Note that

$$\sigma(s) + \delta(s)\mathcal{K}\big(\vartheta_1(s)\big) + C_{X_1} \geqslant 0, \quad s \in [0,1],$$

and we have $\widetilde{\vartheta}_1 \in P_0$. Moreover, note that the choice of $\mathcal{R}$ in (13), Lemma 2(ii), and (12) imply that

$$\|\widetilde{\vartheta}_1\| \leqslant \int_0^1 \kappa_2 s(1-s)^{\alpha-1}\,\mathrm{d}s\big[\|\sigma\| + \|\delta\|(\varepsilon\mathcal{R} + \mathcal{K}^*) + C_{X_1}\big] < \mathcal{R}. \tag{18}$$

From (12) and (H2) we have

$$\begin{aligned}
&(\Psi\vartheta_1)(t)+\widetilde{\vartheta}_1(t)\\
&\quad=\int_0^1 K(t,s)\bigl[f\bigl(s,\vartheta_1(s)\bigr)+\sigma(s)+\delta(s)\mathcal{K}\bigl(\vartheta_1(s)\bigr)+C_{X_1}\bigr]\,\mathrm{d}s\\
&\qquad+\frac{G(t)}{G(1)}\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)\\
&\quad\geqslant\int_0^1 K(t,s)\bigl[\bigl(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\bigr)\bigl|\vartheta_1(s)\bigr|-\sigma(s)-C_{X_1}+\sigma(s)+C_{X_1}\bigr]\,\mathrm{d}s\\
&\qquad+\frac{G(t)}{G(1)}\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)\\
&\quad\geqslant\bigl(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\bigr)\int_0^1 K(t,s)\vartheta_1(s)\,\mathrm{d}s+\frac{G(t)}{G(1)}\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)\\
&\quad=\bigl(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\bigr)\int_0^1 K(t,s)\bigl[\vartheta_1(s)+\widetilde{\vartheta}_1(s)\bigr]\,\mathrm{d}s+\frac{G(t)}{G(1)}\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)\\
&\qquad-\bigl(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\bigr)\int_0^1 K(t,s)\widetilde{\vartheta}_1(s)\,\mathrm{d}s\\
&\quad\geqslant\xi_1\int_0^1 K(t,s)\bigl[\vartheta_1(s)+\widetilde{\vartheta}_1(s)\bigr]\,\mathrm{d}s+\frac{G(t)}{G(1)}\int_0^1\bigl[\vartheta_1(t)+\widetilde{\vartheta}_1(t)\bigr]\,\mathrm{d}\gamma(t)
\end{aligned}$$

using the fact that

$$\begin{aligned}
&\bigl(\varepsilon_0-\varepsilon\|\delta\|\bigr)\int_0^1 K(t,s)\bigl[\vartheta_1(s)+\widetilde{\vartheta}_1(s)\bigr]\,\mathrm{d}s\\
&\qquad-\bigl(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\bigr)\int_0^1 K(t,s)\widetilde{\vartheta}_1(s)\,\mathrm{d}s\geqslant 0,\quad t\in[0,1].
\end{aligned}$$

Indeed, from (17) and (18) we get

$$\vartheta_1(t)+\widetilde{\vartheta}_1(t)\geqslant\kappa_4 t^{\alpha-1}\|\vartheta_1+\widetilde{\vartheta}_1\|\geqslant\kappa_4 t^{\alpha-1}\bigl(\|\vartheta_1\|-\|\widetilde{\vartheta}_1\|\bigr),\quad t\in[0,1].$$

Then note that the choice of $\mathcal{R}$ in (13), Lemma 2(iii), and (18) imply that

$$\begin{aligned}
&\big(\varepsilon_0-\varepsilon\|\delta\|\big)\int_0^1 K(t,s)\big[\vartheta_1(s)+\widetilde{\vartheta}_1(s)\big]\,\mathrm{d}s-\big(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\big)\int_0^1 K(t,s)\widetilde{\vartheta}_1(s)\,\mathrm{d}s\\
&\geqslant\big(\varepsilon_0-\varepsilon\|\delta\|\big)\int_0^1 K(t,s)\kappa_4 s^{\alpha-1}\big(\|\vartheta_1\|-\|\widetilde{\vartheta}_1\|\big)\,\mathrm{d}s-\big(\xi_1+\varepsilon_0-\varepsilon\|\delta\|\big)\\
&\quad\times\int_0^1 K(t,s)\int_0^1\frac{G^2(1)}{G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)}s^{\alpha-1}(1-\tau)^{\alpha-1}\\
&\qquad\times\big[\sigma(\tau)+\delta(\tau)\mathcal{K}\big(\vartheta_1(\tau)\big)+C_{X_1}\big]\,\mathrm{d}\tau\,\mathrm{d}s\\
&\geqslant\int_0^1 K(t,s)s^{\alpha-1}\,\mathrm{d}s\\
&\quad\times\Bigg[\kappa_4\big(\varepsilon_0-\varepsilon\|\delta\|\big)\bigg(\mathcal{R}-\frac{\kappa_2\Gamma(\alpha)}{\Gamma(\alpha+2)}\big[\|\sigma\|+\|\delta\|(\varepsilon\mathcal{R}+\mathcal{K}^*)+C_{X_1}\big]\bigg)\\
&\quad-\frac{(\xi_1+\varepsilon_0-\varepsilon\|\delta\|)G^2(1)}{\alpha[G(1)-\int_0^1 G(t)\,\mathrm{d}\gamma(t)]}\big[\|\sigma\|+\|\delta\|(\varepsilon\mathcal{R}+\mathcal{K}^*)+C_{X_1}\big]\Bigg]\\
&\geqslant 0.
\end{aligned}$$

Therefore, we have

$$(\Psi\vartheta_1)(t)+\widetilde{\vartheta}_1(t)\geqslant L_{\xi_1}(\vartheta_1+\widetilde{\vartheta}_1)(t),\quad t\in[0,1].$$

Together with (15), we have

$$\begin{aligned}
\vartheta_1(t)+\widetilde{\vartheta}_1(t)&=\mu_1\zeta_{\xi_1}(t)+(\Psi\vartheta_1)(t)+\widetilde{\vartheta}_1(t)\geqslant\mu_1\zeta_{\xi_1}(t)+L_{\xi_1}(\vartheta_1+\widetilde{\vartheta}_1)(t)\\
&\geqslant\mu_1\zeta_{\xi_1}(t),\quad t\in[0,1].
\end{aligned}$$

Define a set $W=\{\mu\colon\ \vartheta_1+\widetilde{\vartheta}_1\geqslant\mu\zeta_{\xi_1}\}$ and $\mu^*=\sup W$. Then $\mu_1\in W$ and $\mu^*\geqslant\mu_1$. Hence, note that $L_{\xi_1}:P\to P$, by (9) we have

$$\begin{aligned}
\vartheta_1(t)+\widetilde{\vartheta}_1(t)&\geqslant\mu_1\zeta_{\xi_1}(t)+L_{\xi_1}(\vartheta_1+\widetilde{\vartheta}_1)(t)\geqslant\mu_1\zeta_{\xi_1}(t)+(L_{\xi_1}\mu^*\zeta_{\xi_1})(t)\\
&=\mu_1\zeta_{\xi_1}(t)+\mu^*r(L_{\xi_1})\zeta_{\xi_1}(t)\geqslant(\mu^*+\mu_1)\zeta_{\xi_1}(t),
\end{aligned}$$

which contradicts the definition of μ^*. As a result, (14) holds, and Lemma 5 implies that

$$\deg(I-\Psi,\,B_{\mathcal{R}},\,0)=0. \tag{19}$$

From (H7) there exists a sufficient small $\varrho\in(0,\mathcal{R})$ such that

$$\big|f(t,x)\big|\leqslant\xi_2|x|\quad\text{for } x\in[0,\varrho],\ t\in[0,1]. \tag{20}$$

For this ϱ we claim that

$$\Psi\vartheta \neq \mu\vartheta, \quad \vartheta \in \partial B_\varrho,\ \mu \geqslant 1,$$

where $B_\varrho = \{\vartheta \in E\colon \|\vartheta\| < \varrho\}$ for $\varrho > 0$. If the claim is false and there exist $\vartheta_2 \in \partial B_\varrho$, $\mu_2 \geqslant 1$ such that

$$\Psi\vartheta_2 = \mu_2\vartheta_2,$$

then (20) implies

$$\begin{aligned}
|\vartheta_2| &= \frac{1}{\mu_2}|\Psi\vartheta_2| \leqslant |\Psi\vartheta_2| \to \bigl|\vartheta_2(t)\bigr| \\
&\leqslant \int_0^1 K(t,s)\bigl|f\bigl(s,\vartheta_2(s)\bigr)\bigr|\,\mathrm{d}s + \frac{G(t)}{G(1)}\int_0^1 \bigl|\vartheta_2(t)\bigr|\,\mathrm{d}\gamma(t) \\
&\leqslant \xi_2\int_0^1 K(t,s)\bigl|\vartheta_2(s)\bigr|\,\mathrm{d}s + \frac{G(t)}{G(1)}\int_0^1 \bigl|\vartheta_2(t)\bigr|\,\mathrm{d}\gamma(t) = \bigl(L_{\xi_2}|\vartheta_2|\bigr)(t).
\end{aligned}$$

Note that $r(L_{\xi_2}) < 1$, which implies that $(I - L_{\xi_2})^{-1}$ exists, and

$$(I - L_{\xi_2})^{-1} = I + L_{\xi_2} + L_{\xi_2}^2 + \cdots + L_{\xi_2}^n + \cdots .$$

Consequently, note that $(I - L_{\xi_2})^{-1} : P \to P$, and we have

$$\bigl((I - L_{\xi_2})|\vartheta_2|\bigr)(t) \leqslant 0 \to \bigl|\vartheta_2(t)\bigr| \leqslant (I - L_{\xi_2})^{-1}0 = 0, \tag{21}$$

which implies that $\vartheta_2(t) \equiv 0$, $t \in [0,1]$, and contradicts $\vartheta_2 \in \partial B_\varrho$. Consequently, Lemma 6 implies that

$$\deg(I - \Psi,\ B_\varrho,\ 0) = 1.$$

Combining this with (19), we have

$$\deg(I - \Psi,\ B_{\mathcal{R}} \setminus \overline{B}_\varrho,\ 0) = \deg(I - \Psi,\ B_{\mathcal{R}},\ 0) - \deg(I - \Psi,\ B_\varrho,\ 0) = -1.$$

Therefore, the operator Ψ has at least one fixed point in $B_{\mathcal{R}} \setminus \overline{B}_\varrho$. Equivalently, (1) has at least one nontrivial solution. This completes the proof. □

From [16] we obtain that the conjugate space of E, denoted by E^*, is

$$E^* = \{\gamma\colon \gamma \text{ has bounded variation on } [0,1]\}.$$

Moreover, the dual cone of P and the bounded linear functional on E can be expressed by

$$P^* := \bigl\{\gamma \in E^*\colon \gamma \text{ is nondecreasing on } [0,1]\bigr\}$$

and

$$\langle\gamma,\vartheta\rangle = \int_0^1 \vartheta(t)\,\mathrm{d}\gamma(t), \quad \vartheta \in E,\ \gamma \in E^*.$$

Note that $r(L_\xi) > 0$ in Lemma 4, and there exists $\psi_\xi \in P^* \setminus \{0\}$ such that

$$L_\xi^* \psi_\xi = r(L_\xi)\psi_\xi,$$

where $L_\xi^* : E^* \to E^*$ is the conjugate operator of L_ξ denoted by

$$(L_\xi^* \theta)(t) := \xi \int_0^t \mathrm{d}s \int_0^1 K(\tau, s)\,\mathrm{d}\theta(\tau) + \gamma(t) \int_0^1 \frac{G(\tau)}{G(1)}\,\mathrm{d}\theta(\tau), \quad \theta \in E^*.$$

Now, we list some assumptions when the nonlinearity f grows sublinearly:

(H8) There exists $\xi_3 > 0$ with $r(L_{\xi_3}) > 1$ such that $\liminf_{|x|\to 0} f(t,x)/|x| \geqslant \xi_3$ uniformly for $t \in [0,1]$.

(H9) There exists $\xi_4 > 0$ with $r(L_{\xi_4}) < 1$ such that $\limsup_{|x|\to +\infty} |f(t,x)|/|x| \leqslant \xi_4$ uniformly for $t \in [0,1]$.

Theorem 2. *Suppose that* (H1)*,* (H4)*,* (H5)*,* (H8)*, and* (H9) *hold. Then* (1) *has at least one nontrivial solution.*

Proof. From (H8) there exists a sufficient small $\varrho_1 > 0$ such that

$$f(t,x) \geqslant \xi_3|x|, \quad |x| \leqslant \varrho_1,\ t \in [0,1]. \tag{22}$$

In what follows, we prove

$$\vartheta - \Psi\vartheta \neq \mu\varpi_1, \quad \mu \geqslant 0, \tag{23}$$

where ϖ_1 is a fixed element in P with $\varpi_1(t) \not\equiv 0, t \in [0,1]$. Suppose the contrary. Then there exist $\vartheta_3 \in \partial B_{\varrho_1}$, $\mu_3 \geqslant 0$ such that

$$\vartheta_3 - \Psi\vartheta_3 = \mu_3\varpi_1,$$

where $B_{\varrho_1} = \{\vartheta \in E\colon \|\vartheta\| < \varrho_1\}$. This equation can also be expressed by

$$\vartheta_3(t) = \int_0^1 K(t,s) f\bigl(s, \vartheta_3(s)\bigr)\,\mathrm{d}s + \frac{G(t)}{G(1)} \int_0^1 \vartheta_3(t)\,\mathrm{d}\gamma(t) + \mu_3\varpi_1(t), \quad t \in [0,1].$$

Now (H5) enables us to obtain

$$\begin{aligned}
\vartheta_3(t) &= \int_0^1 \Theta(t,s) f\bigl(s, \vartheta_3(s)\bigr)\,\mathrm{d}s + \mu_3\varpi_1(t) + \frac{\mu_3 G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 \varpi_1(t)\,\mathrm{d}\gamma(t) \\
&= \bigl((L \circ F)\vartheta_3\bigr)(t) + \mu_3\varpi_1(t) + \frac{\mu_3 G(t)}{G(1) - \int_0^1 G(t)\,\mathrm{d}\gamma(t)} \int_0^1 \varpi_1(t)\,\mathrm{d}\gamma(t).
\end{aligned}$$

This, combining with (22), implies that

$$\vartheta_3 \in P.$$

Consequently, we have

$$\begin{aligned}\vartheta_3(t) &\geqslant \xi_3 \int_0^1 K(t,s)\vartheta_3(s)\,\mathrm{d}s + \frac{G(t)}{G(1)}\int_0^1 \vartheta_3(t)\,\mathrm{d}\gamma(t)\\ &= (L_{\xi_3}\vartheta_3)(t), \quad t\in[0,1]. \end{aligned}\tag{24}$$

Note that $r(L_{\xi_3}) > 1$, and thus there exists $\psi_{\xi_3} \in P^*\backslash\{0\}$ such that $L^*_{\xi_3}\psi_{\xi_3} = r(L_{\xi_3})\psi_{\xi_3}$. Therefore, multiplying both sides of (24) by $\mathrm{d}\psi_{\xi_3}(t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 \vartheta_3(t)\,\mathrm{d}\psi_{\xi_3}(t)\\ &\quad\geqslant \int_0^1 (L_{\xi_3}\vartheta_3)(t)\,\mathrm{d}\psi_{\xi_3}(t)\\ &\quad= \int_0^1 \mathrm{d}\psi_{\xi_3}(t)\left(\xi_3\int_0^1 K(t,s)\vartheta_3(s)\,\mathrm{d}s + \frac{G(t)}{G(1)}\int_0^1 \vartheta_3(t)\,\mathrm{d}\gamma(t)\right)\\ &\quad= \int_0^1 \vartheta_3(s)\,\mathrm{d}\left(\xi_3\int_0^s \mathrm{d}\tau\int_0^1 K(t,\tau)\,\mathrm{d}\psi_{\xi_3}(t) + \gamma(s)\int_0^1 \frac{G(t)}{G(1)}\,\mathrm{d}\psi_{\xi_3}(t)\right)\\ &\quad= \langle L^*_{\xi_3}\psi_{\xi_3},\vartheta_3\rangle = r(L_{\xi_3})\int_0^1 \vartheta_3(t)\,\mathrm{d}\psi_{\xi_3}(t). \end{aligned}\tag{25}$$

Note that $\vartheta_3 \in P$ and $\psi_{\xi_3} \in P^*\setminus\{0\}$, and from the definition of the Riemann–Stieltjes integral

$$\int_0^1 \vartheta_3(t)\,\mathrm{d}\psi_{\xi_3}(t) = \lim_{\lambda\to 0}\sum_{i=1}^n \vartheta_3(\xi_i)\big[\psi_{\xi_3}(t_i) - \psi_{\xi_3}(t_{i-1})\big] \geqslant 0, \tag{26}$$

where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, $\lambda = \max_{1\leqslant i\leqslant n}(t_i - t_{i-1})$ for all $\xi_i \in [t_{i-1}, t_i]$, $i = 1,2,\dots,n$. Note that $r(L_{\xi_3}) > 1$, and by (25), (26) we have

$$\int_0^1 \vartheta_3(t)\,\mathrm{d}\psi_{\xi_3}(t) = 0. \tag{27}$$

Note that for all divisions t_i, (27) holds, we only obtain $\vartheta_3(t) \equiv 0$, $t \in [0,1]$. Therefore, this contradicts $\vartheta_3 \in \partial B_{\varrho_1}$, and thus (23) holds. Lemma 5 implies that

$$\deg(I-\Psi,\, B_{\varrho_1},\, 0) = 0. \tag{28}$$

From (H9) there exists $C_2 > 0$ such that

$$\left|f(t,x)\right| \leqslant \xi_4|x| + C_2, \quad x \in \mathbb{R},\ t \in [0,1].$$

Define a set $S = \{\vartheta \in E\colon \varPsi\vartheta = \mu\vartheta, \mu \geqslant 1\}$. Now we prove that S is a bounded set in E. Indeed, if $\vartheta_4 \in S$, then there exists $\mu_4 \geqslant 1$ such that

$$\begin{aligned}
\left|\vartheta_4(t)\right| &= \frac{1}{\mu_4}\left|(\varPsi\vartheta_4)(t)\right| \\
&\leqslant \int_0^1 K(t,s)\left|f\left(s,\vartheta_4(s)\right)\right| \mathrm{d}s + \frac{G(t)}{G(1)} \int_0^1 \left|\vartheta_4(t)\right| \mathrm{d}\gamma(t) \\
&\leqslant \xi_4 \int_0^1 K(t,s)\left|\vartheta_4(s)\right| \mathrm{d}s + \frac{G(t)}{G(1)} \int_0^1 \left|\vartheta_4(t)\right| \mathrm{d}\gamma(t) \\
&\quad + C_2 \int_0^1 \frac{[G'(1)]^2}{G(1)s^*} s(1-s)^{\alpha-1}\, \mathrm{d}s \\
&= \left(L_{\xi_4}|\vartheta_4|\right)(t) + \frac{[G'(1)]^2 C_2 \Gamma(\alpha)}{G(1)s^*\Gamma(\alpha+2)}.
\end{aligned}$$

Note that $r(L_{\xi_4}) < 1$, and we use the similar method in (21) to obtain

$$\left|\vartheta_4(t)\right| \leqslant (I - L_{\xi_4})^{-1} \frac{[G'(1)]^2 C_2 \Gamma(\alpha)}{G(1)s^*\Gamma(\alpha+2)}, \quad t \in [0,1].$$

From the definition of our norm we obtain that there exists a $\mathcal{M} > 0$ such that

$$\|\vartheta_4\| \leqslant \mathcal{M}.$$

This implies that S is a bounded set in E as required. If we take $\mathcal{R}_1 > \sup S$ and $\mathcal{R}_1 > \varrho_1$, we have

$$\varPsi\vartheta \neq \mu\vartheta, \quad \vartheta \in \partial B_{\mathcal{R}_1},\ \mu \geqslant 1,$$

where $B_{\mathcal{R}_1} = \{\vartheta \in E\colon \|\vartheta\| < \mathcal{R}_1\}$. Consequently, Lemma 6 implies that

$$\deg(I - \varPsi,\ B_{\mathcal{R}_1},\ 0) = 1.$$

Combining this with (28), we have

$$\deg(I - \varPsi,\ B_{\mathcal{R}_1} \setminus \overline{B}_{\varrho_1},\ 0) = \deg(I - \varPsi,\ B_{\mathcal{R}_1},\ 0) - \deg(I - \varPsi,\ B_{\varrho_1},\ 0) = 1.$$

Therefore the operator $\varPsi$ has at least one fixed point in $B_{\mathcal{R}_1} \setminus \overline{B}_{\varrho_1}$. Equivalently, (1) has at least one nontrivial solution. This completes the proof. □

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