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Controllability of ψ-Hilfer fractional differential equations with infinite delay via measure of noncompactness

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Abstract. In this article, we study the controllability of ψ-Hilfer fractional differential equations with infinite delay. Sufficient conditions for controllability results are obtained by using the notion of the measure of noncompactness and the Mönch fixed point theorem. The novel feature of this study is to inquire into the controllability notion by using ψ-Hilfer fractional derivative, the generalized variant of the Hilfer derivative. Finally, we provide a numerical example to illustrate our main result.

Keywords: controllability, ψ-Hilfer fractional differential equations, measure of noncompactness, fixed point theorem.

1 Introduction

The techniques of fractional calculus help in the development of mathematical models that frequently more accurately describe the dynamic response of live systems to mechanical, chemical, and electrical stimuli [25]. By using these techniques more frequently, bioengineers may be better able to create, describe, and control biomedical devices. The research publications [1, 12, 23] can be study by the readers on the theory of fractional differential systems. The Hilfer fractional derivative [14] has technical property that makes it significantly more relevant than other fractional derivatives since it unifies the

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Riemann–Liouville and Caputo fractional derivatives. Due to this reason, Hilfer fractional derivatives are a stronger mathematical tool for studying real-world occurrences and the resulting technical advancements [13, 27]. A new fractional derivative was introduced by Sousa et al. [4] called "ψ-Hilfer fractional derivative", which generalized a number of earlier fractional derivatives. The advantage of this type of fractional derivative is the flexibility to choosing the kernel ψ, which enables unification and recovery of the majority of earlier studied fractional differential equations [6]. The importance of ψ-Hilfer fractional differential equations has made it essential to study these kinds of equations [26].

One of the fundamental concepts in mathematical control theory is controllability, and this plays an important role in solving many control problems such as stabilising unstable systems through feedback or optimal control [8–10]. Controllability of nonlinear systems in finite-dimensional spaces has been studied extensively by using fixed point theorems [2, 19, 20]. Compactness and boundedness of the corresponding operator are required for controllability results for fractional differential equations obtained by the Schauder fixed point theorem. Therefore, many researchers have worked to find sufficient conditions to ensure the controllability results of various systems without involving the compactness of the operator [21, 24]. Very recently, the authors Wang and Zhou [29] found some conditions guaranteeing the complete controllability of fractional evolution systems without assuming the compactness of characteristic solution operators by means of the Mönch fixed point technique and the measures of noncompactness. Wang et al. [28] established two sufficient conditions for nonlocal controllability for fractional evolution systems. These theorems guarantee the effectiveness of controllability results under some weakly noncompactness conditions.

Kavitha et al. [17] discussed the approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay, and they [16] studied the results on controllability of Hilfer fractional differential equations with infinite delay via measure of noncompactness by means of the Mönch fixed point theorem. Also, Kavitha et al. [15] discussed the controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness. Yet, to our knowledge, no research on the controllability of ψ-Hilfer fractional derivative has been published. Therefore, in this paper, we study the controllability of ψ-Hilfer fractional differential equations with infinite delay via measure of noncompactness. This result can cover the results of controllability involving Hilfer fractional differential equations by the appropriate choice of ψ.

Let us take the ψ-Hilfer fractional differential equations with control having the form

$$\mathrm{D}_{0+}^{\delta,\gamma;\psi} x(s) = Ax(s) + g(s, x_s) + Bu(s), \quad s \in \mathrm{I} = (0, d], \tag{1}$$

$$\mathrm{I}_{0+}^{(1-\delta)(1-\gamma);\psi} x(s) = \hbar(s) \in \mathfrak{B}_h, \tag{2}$$

where $\delta \in (1/2, 1)$, $\gamma \in [0, 1]$, $\mathrm{D}_{0+}^{\delta,\gamma;\psi}$ is ψ-Hilfer fractional derivative operator of order δ and type γ, $\mathrm{I}_{0+}^{(1-\delta)(1-\gamma);\psi}(\cdot)$ is ψ-Riemann–Liouville integral of order $(1-\delta)(1-\gamma)$, and $x(\cdot)$ takes the values in Banach space Z with $\|\cdot\|$, the control function $u(\cdot) \in L^2(\mathrm{I}, \mathrm{U})$, Banach space of admissible control functions, with U as a Banach space, and $B: L^2(\mathrm{I}, \mathrm{U}) \to L^2(\mathrm{I}, \mathrm{Z})$ is a bounded linear operator. The operator $A : D(A) \subset \mathrm{Z} \to \mathrm{Z}$ is the infinitesimal generator of analytic semigroup $\{T(s)\}_{s\geqslant 0}$ on Z. For analytic semigroup $\{T(s)\}_{s\geqslant 0}$, there is a constant $M \geqslant 1$ such that $M := \sup_{s\in[0,\infty)} |T(s)| < \infty$, and $g : \mathrm{I} \times \mathfrak{B}_h \to \mathrm{Z}$ is a given function, where $\mathfrak{B}_h$ is an abstract phase space to be defined later. The histories $x_s : (-\infty, 0] \to X$ defined by $x_s(r) = x(s+r)$, $r \leqslant 0$, belong to $\mathfrak{B}_h$. Our main contribution is the controllability of the ψ-Hilfer fractional differential equation.

The paper is organized as follows. In Section 2, we briefly present some basic notations and preliminaries. In Section 3, we establish some sufficient conditions for controllability of ψ-Hilfer fractional differential equations with infinite delay via measure of noncompactness. Finally, an example is given to illustrate the results in Section 4.

2 Preliminaries

In this section, we discuss the notations, definitions, lemmas, and introductory information that are necessary to establish our main results.

Let $C(\mathrm{I},\mathrm{Z})$, $C^1(\mathrm{I},\mathrm{Z})$, $AC^k(\mathrm{I},\mathrm{Z})$, $C^k(\mathrm{I},\mathrm{Z})$ be the spaces of continuous functions, continuously differentiable functions, k-times absolutely continuous, and k-times continuously differentiable functions from $\mathrm{I} \to \mathrm{Z}$, respectively. Suppose that $\sigma = \delta + \gamma - \gamma\delta$, we define $C_{1-\sigma,\psi}(\mathrm{I},\mathrm{Z}) = \{x\colon (\psi(s) - \psi(0))^{1-\delta} x(s) \in C(\mathrm{I},\mathrm{Z})\}$ with norm $\|x\|_{\sigma,\psi} = \sup\{(\psi(s) - \psi(0))^{1-\sigma}\|x\|\}$. Clearly, $C_{1-\sigma,\psi}(\mathrm{I},\mathrm{Z})$ is a Banach space. Also, let $L^p(\mathrm{I},\mathrm{Z})$ be the Banach space of functions $g : \mathrm{I} \times \mathfrak{B}_h \times \mathrm{Z} \to \mathrm{Z}$, which are Bochner integrable, normed with $\|g\|_{L^p(\mathrm{I},\mathrm{Z})}$, and we use g with norm $\|g\|_{L^p(\mathrm{I},\mathbb{R}^+)}$ whenever $g \in L^p(\mathrm{I},\mathbb{R}^+)$ for some p with $1 \leqslant p \leqslant \infty$.

Now, we discuss the abstract phase $\mathfrak{B}_h$ [7]. Let $h : (-\infty, 0] \to (0,\infty)$ be continuous function with $\mathrm{I}_h = \int_{-\infty}^0 h(z)\,\mathrm{d}z < +\infty$. For each $a > 0$,

$$\mathfrak{B} = \bigl\{\xi : [-a, 0] \to \mathrm{Z} \bigm| \xi(z) \text{ is bounded and measurable}\bigr\},$$

with

$$\|\xi\|_{[-a,0]} = \sup_{z\in[-a,0]} \bigl\|\xi(z)\bigr\| \quad \forall \xi \in \mathfrak{B}.$$

Now, we define

$$\mathfrak{B}_h = \left\{\xi : (-\infty, 0] \to \mathrm{Z} \;\middle|\; \text{for any } a > 0,\ \xi|_{[-a,0]} \in \mathfrak{B} \text{ and } \int_{-\infty}^{0} h(z)\|\xi\|_{[z,0]}\,\mathrm{d}z < \infty\right\}$$

with

$$\|\xi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(z)\|\xi\|_{[z,0]}\,\mathrm{d}z \quad \forall z \in \mathfrak{B}_h.$$

Hence, $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a Banach space.

Now, we discuss

$$\mathfrak{B}'_h = \bigl\{x : (-\infty, d] \to \mathrm{Z} \bigm| x|_{\mathrm{I}} \in C(\mathrm{I},\mathrm{Z}),\ x_0 = \hbar \in \mathfrak{B}_h\bigr\}.$$

Set $\|\cdot\|_d$ be seminorm in $\mathfrak{B}'_h$ defined by

$$\|x\|_d = \|\hbar\|_{\mathfrak{B}_h} + \sup\{\|x(s)\|:\ s \in [0,d]\},\ x \in \mathfrak{B}'_h.$$

Lemma 1. (*See* [19].) *Suppose $x \in \mathfrak{B}'_h$, then for $\beta \in \mathrm{I}$, $x_\beta \in \mathfrak{B}_h$ and*

$$k|x(\beta)| \leqslant \|x_\beta\|_{\mathfrak{B}_h} \leqslant \|\hbar\|_{\mathfrak{B}_h} + k \sup_{\zeta \in [0,\beta]} |x(\zeta)|,$$

where $k = \int_{-\infty}^{0} h(\beta) d\beta < \infty$.

Definition 1. (See [18].) Left-sided Riemann–Liouville fractional integral of order $\delta > 0$ of integrable function w defined on $[a,b]$ is defined by

$$I_{a^+}^{\delta} w(s) = \frac{1}{\Gamma(\delta)} \int_a^s (s-r)^{\delta-1} w(r)\,\mathrm{d}r, \quad s > a.$$

Definition 2. (See [18].) Left-sided Riemann–Liouville fractional derivative of order $\delta > 0$ is defined by

$$\mathrm{D}_{a^+}^{\delta} w(s) = \frac{\mathrm{d}^k}{\mathrm{d}s^k} I_{a^+}^{k-\delta} w(s),$$

where $k = [\delta] + 1$, $w \in AC^k[a,b]$.

Definition 3. (See [18].) The left-sided Caputo fractional derivatives of order δ defined by

$${}^c\mathrm{D}_{a^+}^{\delta} w(s) = \left(I_{a^+}^{k-\delta} \frac{\mathrm{d}^k}{\mathrm{d}s^k} w\right)(s), \quad s > a,$$

where $k = [\delta] + 1$ and $w \in AC^k[a,b]$.

Definition 4. (See [14].) Left-sided Hilfer fractional derivatives of order δ and type γ $(0 \leqslant \gamma \leqslant 1)$ is defined by

$$\mathrm{D}_{a^+}^{\delta,\gamma} w(s) = \left(I_{a^+}^{\gamma(k-\delta)} D_{a^+}^{\delta+\gamma(k-\delta)} w\right)(s), \quad s > a,$$

where $k = [\delta] + 1$, $w \in AC^k[a,b]$.

Definition 5. (See [18].) Let $\psi'(x) \in C^1([a,b])$ with $\psi'(x) > 0$ for all $x \in (a,b)$. For $\delta > 0$, ψ-Riemann–Liouville fractional integral of a function w of order δ is defined by

$$I_{a^+}^{\delta;\psi} w(s) = \frac{1}{\Gamma(\delta)} \int_a^s \psi'(r)\big(\psi(s) - \psi(r)\big)^{\delta-1} w(r)\,\mathrm{d}r, \quad s > a,\ \delta > 0.$$

Table 1. Particular cases of ψ-Hilfer fractional derivatives.

$\mathrm{D}_{a+}^{\delta,\gamma;\psi}$		Particular cases
$\psi(s)$	γ	
$\psi(s)$	0	ψ-Riemann–Liouville derivative
$\psi(s)$	1	ψ-Caputo derivative
s	0	Riemann–Liouville derivative
s	1	Caputo derivative
s	γ	Hilfer derivative
$\log s$	0	Hadamard derivative
$\log s$	1	Caputo–Hadamard derivative
$\log s$	γ	Hilfer–Hadamard derivative

Definition 6. (See [18].) Let $\psi'(x) \in C^1([a,b])$ with $\psi'(x) > 0$ for all $x \in (a,b)$. For $\delta > 0$, ψ-Riemann–Liouville fractional derivative of a function w of order δ is defined by

$$\begin{aligned}\mathrm{D}_{a^+}^{\delta;\psi} w(s) &= \left(\frac{1}{\psi'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\right)^k I_{a^+}^{k-\delta;\psi} w(s) \\ &= \frac{1}{\Gamma(k-\delta)}\left(\frac{1}{\psi'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\right)^k \int_a^s \psi'(r)\big(\psi(s)-\psi(r)\big)^{k-\delta-1} w(r)\,\mathrm{d}r, \quad s>a,\ \delta>0,\end{aligned}$$

where $k-1=[\delta]$.

Definition 7. (See [4].) Let $\psi \in C^k([a,b])$ be positive function on $[a,b]$ such that $\psi'(x)$ is continuous and $\psi'(x) > 0$ for all $x \in (a,b)$, and let $w \in C^k([a,b])$. Then the left ψ-Hilfer fractional derivative of w of order δ and type γ is defined by

$$\mathrm{D}_{a^+}^{\delta,\gamma;\psi} w(s) = I_{a^+}^{\gamma(k-\delta);\psi}\left(\frac{1}{\psi'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\right)^k I_{a^+}^{(1-\gamma)(k-\delta);\psi} w(s),$$

where $k-1=[\delta]$.

Remark 1. We provide Table 1 showing the particular cases of Definition 7.

Lemma 2. *A function $\mathcal{L}:(-0,d]\to \mathrm{Z}$ is an integral solution of system* (1)–(2) *if $\mathcal{L}$ satisfies the following*:

(i) *$\mathcal{L}:[0,d]\to \mathrm{Z}$ is continuous,*
(ii) *$I_{0+}^{\delta;\psi} x(s) \in D(A)$ for $s \in [0,d]$, and*
(iii) *system* (1)–(2) *is equivalent to* [5]

$$\begin{aligned} x(s) = {}& \frac{\hbar(0)}{\Gamma(\delta(1-\gamma)+\gamma)}\big(\psi(s)-\psi(0)\big)^{(\delta-1)(1-\gamma)} \\ &+ \frac{1}{\Gamma(\delta)}\int_a^s \psi'(r)\big(\psi(s)-\psi(r)\big)^{\delta-1} g(r,x_r)\,\mathrm{d}r \\ &+ \frac{1}{\Gamma(\delta)}\int_a^s \psi'(r)\big(\psi(s)-\psi(r)\big)^{\delta-1} Bu(r)\,\mathrm{d}r, \quad s\in \mathrm{I}. \end{aligned} \tag{3}$$

We introduce the mild solution of fractional differential system (1)–(2) by using the Wright function $M_\delta(z)$ defined by

$$M_\delta(z) = \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!\Gamma(1-\delta n)}, \quad 0 < \delta < 1, \ z \in \mathbb{C},$$

which satisfies the following equality:

$$\int_0^\infty \theta^\tau M_\delta(\theta)\,\mathrm{d}\theta = \frac{\Gamma(1+\tau)}{\Gamma(1+\delta\tau)} \quad \text{for } \theta \geqslant 0.$$

Lemma 3. *If* (3) *holds, then we have*

$$x(s) = \mathcal{S}_{\delta,\gamma;\psi}(s,0)\hbar(0) + \int_0^s \mathcal{P}_{\delta,\gamma;\psi}(s,r)g(r,x_r)\psi'(r)\,\mathrm{d}r + \int_0^s \mathcal{P}_{\delta,\gamma;\psi}(s,r)Bu(r)\psi'(r)\,\mathrm{d}r, \quad s \in \mathrm{I},$$

where the operators $\mathcal{S}_{\delta,\gamma;\psi}(s,r)$ *and* $\mathcal{P}_{\delta,\gamma;\psi}(s,r)$ *defined by*

$$\mathcal{P}_{\delta,\gamma;\psi}(s,r) = \big(\psi(s) - \psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r),$$

$$\mathcal{Q}_{\delta,\gamma;\psi}(s,r) = \delta\int_0^\infty \theta M_\delta(\theta)T\big(\big(\psi(s)-\psi(r)\big)^\delta\theta\big)\,\mathrm{d}\theta$$

and

$$\mathcal{S}_{\delta,\gamma;\psi}(s,r) = \mathrm{I}_{0+}^{(1-\delta)\gamma;\psi}\mathcal{K}_{\delta,\gamma;\psi}(s,r).$$

Proof. The proof is similar to [11]. □

Due to Lemma 3, we give the following definition of the mild solution of (1)–(2).

Definition 8. A function $x : (-\infty, d] \to \mathrm{Z}$ is called mild solution of fractional differential system (1)–(2) if x is continuous with $x_0 = \hbar(0) \in \mathfrak{B}_h$ on $(0, -\infty]$ and satisfies

$$x(s) = \mathcal{S}_{\delta,\gamma;\psi}(s,0)\hbar(0) + \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)g(r,x_r)\psi'(r)\,\mathrm{d}r + \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)Bu(r)\psi'(r)\,\mathrm{d}r, \quad s \in \mathrm{I},$$

where the characteristic solution operators are given by

$$\mathcal{S}_{\delta,\gamma;\psi}(s,r) = \int_0^\infty \phi_\delta(\theta) T\big(\big(\psi(s)-\psi(r)\big)^\delta \theta\big)\,\mathrm{d}\theta$$

and

$$\mathcal{Q}_{\delta,\gamma;\psi}(s,r) = \delta \int_0^\infty \theta\phi_\delta(\theta) T\big(\big(\psi(s)-\psi(r)\big)^\delta \theta\big)\,\mathrm{d}\theta,$$

where $\phi_\delta(\theta) = (1/\delta)\theta^{-1-1/\delta}\rho_\delta(\theta^{-1/\delta})$ is the probability density function defined on $\theta \in (0,\infty)$, that is,

$$\phi_\delta(\theta) \geqslant 0, \qquad \int_0^\infty \phi_\delta(\theta)\,\mathrm{d}\theta = 1$$

and

$$\rho_\delta(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-k\delta-1}\frac{\Gamma(k\delta+1)}{k!}\sin(k\pi\delta).$$

Remark 2. (See [30].) For $\mu \in [0,1]$,

$$\int_0^\infty \theta^\mu \phi_\delta(\theta)\,\mathrm{d}\theta = \int_0^\infty \theta^{-\delta\mu}\rho_\delta(\theta) = 1.$$

Lemma 4. *The operators $\mathcal{S}_{\delta,\gamma;\psi}(s,r)$ and $\mathcal{Q}_{\delta,\gamma;\psi}(s,r)$ have the following properties*:

(i) *For any fixed $s \geqslant r \geqslant 0$, $\mathcal{S}_{\delta,\gamma;\psi}(s,r)$ and $\mathcal{Q}_{\delta,\gamma;\psi}(s,r)$ are bounded linear operators with*

$$\big\|\mathcal{S}_{\delta,\gamma;\psi}(s,r)x\big\| \leqslant \frac{M(\psi(s)-\psi(0))^{(\delta-1)ko(\gamma-1)}}{\Gamma(\delta(1-\gamma)+\gamma)}\|x\|$$

and

$$\big\|\mathcal{Q}_{\delta,\gamma;\psi}(s,r)x\big\| \leqslant \frac{M}{\Gamma(\delta)}\|x\|$$

for all $x \in \mathrm{Z}$.

(ii) *The operators $\mathcal{S}_{\delta,\gamma;\psi}(s,r)$ and $\mathcal{Q}_{\delta,\gamma;\psi}(s,r)$ are strongly continuous for all $s \geqslant r \geqslant 0$, that is, for every $x \in \mathrm{Z}$ and $0 \leqslant r \leqslant s_1 < s_2$, $s_1, s_2 \in \mathrm{I}$, we have*

$$\big\|\mathcal{S}_{\delta,\gamma;\psi}(s_2,r)x - \mathcal{S}_{\delta,\gamma;\psi}(s_1,r)x\big\| \to 0$$

and

$$\big\|\mathcal{Q}_{\delta,\gamma;\psi}(s_2,r)x - \mathcal{Q}_{\delta,\gamma;\psi}(s_1,r)x\big\| \to 0$$

as $s_1 \to s_2$.

Proof. The proof is similar to [30]. □

Definition 9. The fractional differential system (1)–(2) is said to be controllable on I if for each continuous initial functions $\hbar \in \mathfrak{B}_h$, $x^1 \in \mathrm{Z}$, there exists $u \in L^2(\mathrm{I}, \mathrm{U})$ such that the mild solution $x(s)$ of (1) with (2) satisfies $x(d) = x^1$.

Now, let us discuss some of the definitions and properties of the measure of noncompactness.

Let us denote by $\mathfrak{M}_{\mathrm{Z}}$ the family of nonempty bounded subsets of Z such that for $Q \in \mathfrak{M}_{\mathrm{Z}}$, $\overline{\mathrm{co}}\, Q$ is the closed convex hull of Q.

Definition 10. (See [24].) Let Y^+ be a positive cone of an ordered Banach space $(\mathrm{Y}, \leqslant)$. A mapping ν defined on $\mathfrak{M}_{\mathrm{Z}}$ with values in Y^+ is called a measure of noncompactness on Z iff $\nu(\overline{\mathrm{co}}\, Q) = \nu(Q)$ for all $Q \in \mathfrak{M}_{\mathrm{Z}}$.

The measure of noncompactness ν is said to be:

(i) regular iff $\nu(Q) = 0 \Leftrightarrow Q$ is relatively compact in Z for all $Q \in \mathfrak{M}_{\mathrm{Z}}$;
(ii) monotone iff $(Q_1 \subseteq Q_2) \Rightarrow \nu(Q_1) \leqslant \nu(Q_2)$ for all $Q_1, Q_2 \in \mathfrak{M}_{\mathrm{Z}}$;
(iii) nonsingular if $\nu(\{a\} \cup Q) = \nu(Q)$ for each $a \in \mathrm{Z}$, $Q \in \mathfrak{M}_{\mathrm{Z}}$.

Another important measure of noncompactness is the Hausdorff (or ball) measure of noncompactness for Q defined as follows:

$$\chi(Q) = \inf\bigl\{\epsilon > 0\colon\ Q \text{ has a finite } \epsilon\text{-net in Z}\bigr\}.$$

Banaś and Goebel [3] have presented some basic properties of the measure of noncompactness χ. For $Q, Q_1, Q_2 \in \mathfrak{M}_{\mathrm{Z}}$, we have:

(A1) $\chi(Q_1 + Q_2) \leqslant \chi(Q_1) + \chi(Q_2)$;
(A2) $\chi(Q_1 \cup Q_2) \leqslant \max\{\chi(Q_1), \chi(Q_2)\}$;
(A3) $\chi(\lambda Q) \leqslant |\lambda|\chi(Q)$ for any $\lambda \in \mathbb{R}$;
(A4) If the function $\mathcal{L} : D(\mathcal{L}) \subseteq \mathrm{Z} \to \mathrm{W}$ is Lipschitz continuous with constant k, then $\chi_{\mathrm{W}}(\mathcal{L}Q) \leqslant k\chi(Q)$ for any bounded subset $Q \subseteq D(\mathcal{L})$, where W is a Banach space.

Lemma 5. (*See* [3].) *If $\mathcal{S} \subset C(\mathrm{I}, \mathrm{Z})$ is bounded and equicontinuous, then $\chi(\mathcal{S})$ is continuous for $s \in \mathrm{I}$, and*

$$\chi(\mathcal{S}) = \sup\bigl\{\chi(\mathcal{S}(s)),\ s \in \mathrm{I}\bigr\}, \quad \text{where } \mathcal{S}(s) = \bigl\{z(s),\ z \in \mathcal{S}\bigr\} \subseteq \mathrm{Z}.$$

Theorem 1. (*See* [3, 28].) *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of Bochner integrable functions from I into Z with $\|a_k\| \leqslant \mu(s)$ for all $s \in \mathrm{I}$ and for every $k \geqslant 1$, where $\mu \in L^1(\mathrm{I}, \mathbb{R})$. Then the function $\eta(r) = \nu(\{a_k(r),\ k \geqslant 1\}) \in L^1(\mathrm{I}, \mathbb{R})$ and satisfies $\nu(\{\int_0^s \eta(r)\,\mathrm{d}r,\ k \geqslant 1\}) \leqslant 2\int_0^s \eta(r)\,\mathrm{d}r$.*

Lemma 6. (*See* [22].) *Let $\mathcal{C}$ be convex and closed subset of a Banach space Z and $0 \in \mathcal{C}$. Suppose that $\mathcal{T} : \mathcal{C} \to \mathrm{Z}$ is continuous function and satisfies Mönch's condition ($\mathcal{D} \subseteq \mathcal{C}$ is countable, $\mathcal{D} \subseteq \overline{\mathrm{co}}(\{0\} \cup \mathcal{T}(\mathcal{D})) \Leftrightarrow \overline{\mathcal{D}}$ is compact). Then $\mathcal{T}$ has a fixed point in $\mathcal{C}$.*

3 Controllability

We assume the following hypotheses:

(A1) For all bounded subsets $\mathcal{C} \subseteq \mathrm{Z}$ and $x \in \mathcal{C}$,

$$\left\| T\big(\psi(s_2) - \psi(0)\big)^{\delta} w\big)x - T\big(\psi(s_1) - \psi(0)\big)^{\delta} w\big)x \right\| \to 0 \quad \text{as } s_2 \to s_1$$

for each fixed $w \in (0, \infty)$.

(A2) The mapping $g : \mathrm{I} \times \mathfrak{B}_h \to \mathrm{Z}$ satisfies:

(i) $g(\cdot, \hbar)$ is measurable for all $\hbar \in \mathfrak{B}_h$, $g(s, \cdot)$ is continuous for a.e. $s \in \mathrm{I}$, and $\mathfrak{B}_h, g(s, \cdot) : [0, d] \to \mathrm{Z}$ is strongly measurable;

(ii) there exists a constant $\delta_1 \in (0, \delta)$, $m_1 \in L^{1/\delta_1}(\mathrm{I}, \mathbb{R}^+)$, and a nondecreasing continuous function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $(s, \hbar) \in \mathrm{I} \times \mathfrak{B}_h$, $\|g(s, \hbar)\| \leqslant m_1(s)\Theta(\|\hbar\|)$, where Θ satisfies $\liminf_{k\to\infty} \Theta(k)/k = 0$.

(A3) There exist $\delta_2 \in (0, \delta)$ and $m_2 \in L^{1/\delta_2}(\mathrm{I}, \mathbb{R}^+)$ such that, for any bounded subset $\mathfrak{B}_1 \subset \mathfrak{B}_h$,

$$\chi(g(s, \mathfrak{B}_1)) \leqslant m_2(s)\Big[\sup_{-\infty<\omega\leqslant 0} \chi\big(\mathfrak{B}_1(\omega)\big)\Big]$$

for a.e. $s \in \mathrm{I}$.

(A4) Let the bounded operator $\mathcal{K} : L^2(\mathrm{I}, \mathrm{U}) \to \mathrm{Z}$ defined by

$$\mathcal{K}u = \int_0^d \big(\psi(d) - \psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s, r)Bu(r)\psi'(r)\,\mathrm{d}r,$$

and the following hold:

(i) $\mathcal{K}$ having an inverse $\mathcal{K}^{-1}$ takes values in $L^2(\mathrm{I}, \mathrm{U})/\ker \mathcal{K}$, and there exist the constants $\theta_1, \theta_2 > 0$ such that $\|B\| \leqslant \theta_1$ and $\|\mathcal{K}^{-1}\| \leqslant \theta_2$;

(ii) for $\delta_3 \in (0, \delta)$ and for any bounded subset $\mathrm{F} \subset \mathrm{Z}$, there exists $m_3 \in L^{1/\delta_3}(\mathrm{I}, \mathbb{R}^+)$ such that $\chi((\mathcal{K}^{-1}\mathrm{F})(s)) \leqslant m_3(s)\chi(\mathrm{F})$.

For our convenience, let us take

$$\mathcal{R}_1 = \tau_1\|m_1\|_{L^{1/\delta_1}(\mathrm{I},\mathbb{R}^+)}, \qquad \mathcal{R}_2 = \tau_2\|m_2\|_{L^{1/\delta_2}(\mathrm{I},\mathbb{R}^+)}, \qquad \mathcal{R}_3 = \tau_3\|m_3\|_{L^{1/\delta_3}(\mathrm{I},\mathbb{R}^+)},$$

$$\tau_i = \left[\left(\frac{1-\delta_i}{\delta-\delta_i}\right)\big(\psi(d) - \psi(0)\big)^{(\delta-\delta_i)/(1-\delta_i)}\right]^{1-\delta_i}, \quad i = 1, 2, 3,\ \kappa = \frac{\delta-1}{1-\delta'},$$

$$\mathcal{R}^* = \frac{(\psi(d) - \psi(0))^{(1+\kappa)(1-\delta')}}{(1+\kappa)^{1-\delta'}}, \quad \delta' \in (0, \delta).$$

Theorem 2. *Under hypotheses* (A1)–(A3)*, system* (1)–(2) *is controllable on* $[0, d]$ *if*

$$\mathcal{S}^* = \frac{2M\mathcal{R}_2(\psi(d) - \psi(0))^{1-\sigma}}{\Gamma(\delta)}\left[1 + \frac{2\mathrm{M}\theta_1\mathcal{R}_3}{\Gamma(\delta)}\right] < 1 \quad \textit{for some } \frac{1}{2} < \delta < 1. \tag{4}$$

Proof. Using (A4), we define control function $u_x(s)$ by

$$u_x(s) = \mathcal{K}^{-1}\Bigg[x^1 - \mathcal{S}_{\delta,\gamma;\psi}(s,r)\hbar(0) - \int_0^d \big(\psi(s)-\psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)g(r,x_r)\psi'(r)\,\mathrm{d}r\Bigg](s).$$

By using this control function, we show that the operator $\mathcal{H}:\mathfrak{B}'_h \to \mathfrak{B}'_h$ defined by

$$\mathcal{H}x(s) = \begin{cases} \hbar(s), & s\in(-\infty,0],\\ \mathcal{S}_{\delta,\gamma;\psi}(s,0)\hbar(0) + \int_0^s(\psi(s)-\psi(r))^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)g(r,x_r)\psi'(r)\,\mathrm{d}r \\ \quad + \int_0^s(\psi(s)-\psi(r))^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)Bu_x(r)\psi'(r)\,\mathrm{d}r, & s\in \mathrm{I},\end{cases}$$

possesses a fixed point, which is a mild solution of system (1)–(2). It is easy to see that $\mathcal{H}x(d) = x^1$. So, system (1)–(2) is controllable on $[0,d]$.

For $\hbar\in\mathfrak{B}_h$, we define $\hat{h}$ by

$$\hat{h}(s) = \begin{cases} \hbar(s), & s\in(-\infty,0],\\ \mathcal{S}_{\delta,\gamma;\psi}(s,r)\hbar(0), & s\in \mathrm{I},\end{cases}$$

then $\hat{h}\in\mathfrak{B}'_h$. Let $x(s) = [z(s)+\hat{h}(s)]$, $-\infty < s \leqslant d$. It is easy to see that x satisfies Eq. (3) if and only if z satisfies $z_0 = 0$ and

$$z(s) = \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)g\big(r,[z_r+\hat{h}_r]\big)\psi'(r)\,\mathrm{d}r + \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)Bu_z(r)\psi'(r)\,\mathrm{d}r,$$

where

$$u_z(s) = \mathcal{K}^{-1}\Bigg[x^1 - \mathcal{S}_{\delta,\gamma;\psi}(s,r)\hbar(0) - \int_0^d \big(\psi(d)-\psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)g\big(r,[z_r+\hat{h}_r]\big)\psi'(r)\,\mathrm{d}r\Bigg](s).$$

Let $\mathfrak{B}''_h = \{z\in\mathfrak{B}'_h\colon z_0 = 0\in\mathfrak{B}_h\}$. For any $z\in\mathfrak{B}''_h$,

$$\begin{aligned}\|z\|_d &= \|z_0\|_{\mathfrak{B}_h} + \sup\big\{\big\|z(r)\big\|,\ 0\leqslant r\leqslant d\big\}\\ &= \sup\big\{\big\|z(r)\big\|,\ 0\leqslant r\leqslant d\big\}.\end{aligned}$$

Hence, $(\mathfrak{B}_h'', \|\cdot\|_d)$ is a Banach space. Now, $r_0 > 0$, choose $\mathbb{B}_{r_0} = \{z \in \mathfrak{T}_h'' \colon \|z\|_d \leqslant r_0\}$, then $\mathbb{B}_{r_0} \subseteq \mathfrak{B}_h''$ is uniformly bounded, and for $z \in \mathbb{B}_{r_0}$, from Lemma 1

$$\begin{aligned}\|z_s + \hat{h}_s\| &\leqslant \|z_s\|_{\mathfrak{B}_h} + \|\hat{h}_s\|_{\mathfrak{B}_h} \\ &\leqslant k\left(r_0 + \frac{M|h|}{\Gamma(\delta(1-\gamma)+\gamma)}\right) + \|h\|_{\mathfrak{B}_h} = r_0'.\end{aligned}$$

Consider an operator $\tilde{\mathcal{H}} : \mathfrak{B}_h'' \to \mathfrak{B}_h''$ defined by

$$\tilde{\mathcal{H}}z(s) = \begin{cases} 0, & s \in (-\infty, 0], \\ \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s,r) g(r, [z_r + \hat{h}_r]) \psi'(r)\,\mathrm{d}r \\ \quad + \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s,r) B u_z(r) \psi'(r)\,\mathrm{d}r, & s \in \mathrm{I}. \end{cases}$$

Clearly, the existence of a fixed point of the operator $\mathcal{H}$ is equivalent to $\tilde{\mathcal{H}}$ has one. Now, we show that $\tilde{\mathcal{H}}$ has a fixed point. The following steps are used to obtain the proof.

Step 1. There exists $r_0 > 0$ such that $\tilde{\mathcal{H}}(\mathbb{B}_{r_0}) \subseteq \mathbb{B}_{r_0}$.

If this is not true, then for each $r_0 > 0$, there exists $z^{r_0} \in \mathbb{B}_{r_0}$. But $\tilde{\mathcal{H}}(z^{r_0}) \notin \mathbb{B}_{r_0}$, i.e., $\|\tilde{\mathcal{H}}(z^{r_0})(s)\| > r_0$ for all $s \in \mathrm{I}$.

Choose $r_0 > 0$, and let $\{\mathbb{B}_{r_0} = x \in \mathrm{Z}\colon \|x\|_{\sigma,\psi} \leqslant r_0\}$. It is easy to see that $\mathbb{B}_{r_0}$ is a closed, bounded, and convex set of Z. Therefore,

$$\big\|\tilde{\mathcal{H}}\big(x^{r_0}\big)\big\|_{\sigma,\psi} = \sup\big\{\big(\psi(s) - \psi(0)\big)^{1-\sigma} \big\|\tilde{\mathcal{H}}\big(x^{r_0}\big)(s)\big\|,\ s \in \mathrm{I}\colon \big\|\tilde{\mathcal{H}}\big(x^{r_0}\big)(s)\big\| \geqslant r_0\big\}.$$

Then by hypotheses (A2)(ii), (A3), (A4), Lemma 4, and Hölder's inequality,

$$\begin{aligned} r_0 &< \big(\psi(d) - \psi(0)\big)^{1-\sigma} \big\|(\tilde{\mathcal{H}} z^{r_0})(s)\big\|, \\ &\leqslant \big(\psi(d) - \psi(0)\big)^{1-\sigma} \left\| \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s,r) g\big(r, [z_r + \hat{h}_r]\big) \psi'(r)\,\mathrm{d}r \right\| \\ &\quad + \big(\psi(d) - \psi(0)\big)^{1-\sigma} \left\| \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s,r) B u_z(r) \psi'(r)\,\mathrm{d}r \right\| \\ &\leqslant \frac{M(\psi(d) - \psi(0))^{1-\sigma}}{\Gamma(\delta)} \left\| \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} g\big(r, [z_r + \hat{h}_r]\big) \psi'(r)\,\mathrm{d}r \right\| \\ &\quad + \frac{M(\psi(d) - \psi(0))^{1-\sigma}}{\Gamma(\delta)} \left\| \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} B u_z(r) \psi'(r)\,\mathrm{d}r \right\| \\ &\leqslant \frac{M(\psi(d) - \psi(0))^{1-\sigma}}{\Gamma(\delta)} \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} m_1(r) \varTheta(r_0') \psi'(r)\,\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&+ \frac{M(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} \\
&\times \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1} \bigg\| B\mathcal{K}^{-1}\bigg[x(d) - \mathcal{S}_{\delta,\gamma;\psi}(s,r)h(0) \\
&\quad - \int_0^d \big(\psi(s)-\psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s,r) g\big(r,[z_r+\hat{h}_r]\big)\psi'(r)\,\mathrm{d}r\bigg](r)\bigg\| \psi'(r)\,\mathrm{d}r \\
&\leqslant \frac{M(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1} m_1(r)\Theta(r_0')\psi'(r)\,\mathrm{d}r \\
&+ \frac{M\theta_1\theta_2(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} \\
&\times \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1} \bigg[\big\|x^1\big\| + \frac{M(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\gamma(1-\delta)+\delta)}\big\|h(0)\big\| \\
&\quad + \frac{M}{\Gamma(\delta)} \int_0^d \big(\psi(d)-\psi(r)\big)^{\delta-1} m_1(r)\Theta(r_0')\psi'(r)\,\mathrm{d}r\bigg]\psi'(r)\,\mathrm{d}r \\
&\leqslant \frac{M\mathcal{R}_1(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)}\Theta(r_0') + \frac{M\theta_1\theta_2(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} \\
&\times \mathcal{R}^*\bigg[\big\|x^1\big\| + \frac{M(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\gamma(1-\delta)+\delta)}\big\|h(0)\big\| + \frac{M\mathcal{R}_1}{\Gamma(\delta)}\Theta(r_0')\bigg]. \qquad (5)
\end{aligned}
$$

Now, dividing both sides of Eq. (5) by r_0 and taking the limit $r_0 \to \infty$, we have

$$
1 \leqslant \frac{M\mathcal{R}_1(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)}\Theta(q')\bigg(1 + \frac{M\theta_1\theta_2}{\Gamma(\delta)}\mathcal{R}^*\bigg).
$$

Then by (A2), we get $1 \leqslant 0$, which is a contradiction. Hence, for $r_0 > 0$, $\tilde{\mathcal{H}}(\mathbb{B}_{r_0}) \subseteq \mathbb{B}_{r_0}$.

Step 2. $\tilde{\mathcal{H}}$ is continuous on $\mathbb{B}_{r_0}$.

For any z^k, $z \in \mathbb{B}_{r_0}$, $k = 0, 1, 2, \ldots$, with $\lim_{k\to\infty} z^k = z$, then $\lim_{k\to\infty} z^k(s) = z(s)$ and $\lim_{k\to\infty}(\psi(s)-\psi(0))^{1-\sigma} z^k(s) = (\psi(s)-\psi(0))^{1-\sigma} z(s)$.

Let $x(s) = (\psi(s)-\psi(0))^{1-\sigma}[z(s)+\hat{h}(s)]$. Then $\{z^k+\hat{h}\} \subset \mathbb{B}_{r_0}$ with $z^k+\hat{h} \to z+\hat{h}$ in $\mathbb{B}_{r_0}$ as $k \to \infty$, and we have

$$
\begin{aligned}
g\big(s, x^k(s)\big) &= g\big(s, \big(\psi(s)-\psi(0)\big)^{1-\sigma}\big[z^k(s)+\hat{h}(s)\big]\big) \\
&\to g\big(s, \big(\psi(s)-\psi(0)\big)^{1-\sigma}\big[z(s)+\hat{h}(s)\big]\big) \\
&= g\big(s, x(s)\big) \quad \text{as } k \to \infty,
\end{aligned}
$$

where, $g(r, (\psi(r)-\psi(0))^{1-\sigma}[z_r^{(k)}+\hat{h}_r]) = \mathcal{G}_k(r)$ and $g(r, (\psi(r)-\psi(0))^{1-\sigma}[z_r+\hat{h}_r]) = \mathcal{G}(r)$.

Then, using the (A2)(i), (ii) and Lebesgue's dominated convergence theorem, we have

$$\int_0^s \big(\psi(s)-\psi(r)\big)^{1-\delta} \big\|\mathcal{G}_k(r)-\mathcal{G}(r)\big\| \psi'(r)\,\mathrm{d}r \to 0 \quad \text{as } k\to\infty,\ s\in \mathrm{I}. \tag{6}$$

Now by (A2),

$$\begin{aligned}
&\big\|\tilde{\mathcal{H}}z^k-\tilde{\mathcal{H}}z\big\|_{\sigma,\psi}\\
&\quad\leqslant \big(\psi(d)-\psi(0)\big)^{1-\sigma}\\
&\qquad\times\left\|\int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)\big[g\big(r,\big[z_r^k+\hat{h}_r\big]\big)-g\big(r,[z_r+\hat{h}_r]\big)\big]\psi'(r)\,\mathrm{d}r\right\|\\
&\qquad+\big(\psi(s)-\psi(0)\big)^{1-\sigma}\|B\|\\
&\qquad\times\left\|\int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)\big[u_{z^k}(r)-u_z(r)\big]\psi'(r)\,\mathrm{d}r\right\|\\
&\quad\leqslant \frac{M(\psi(s)-\psi(0))^{1-\sigma}}{\Gamma(\delta)}\\
&\qquad\times\left\|\int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\big[g\big(r,\big[z_r^k+\hat{h}_r\big]\big)-g\big(r,[z_r+\hat{h}_r]\big)\big]\psi'(r)\,\mathrm{d}r\right\|\\
&\qquad+\frac{M(\psi(s)-\psi(0))^{1-\sigma}\theta_1}{\Gamma(\delta)}\left\|\int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\big[u_{z^k}(r)-u_z(r)\big]\psi'(r)\,\mathrm{d}r\right\|\\
&\quad\leqslant \frac{M(\psi(s)-\psi(0))^{1-\sigma}}{\Gamma(\delta)}\int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\big[\mathcal{G}_k(r)-\mathcal{G}(r)\big]\psi'(r)\,\mathrm{d}r\\
&\qquad+\frac{M^2(\psi(s)-\psi(0))^{1-\sigma}\theta_1\theta_2}{\Gamma(\delta)^2}\\
&\qquad\times\int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1}\left(\int_0^d (\psi(d)-\psi(r))^{\delta-1}\big[\mathcal{G}_k(r)-\mathcal{G}(r)\big]\psi'(r)\,\mathrm{d}r\right)\psi'(r)\,\mathrm{d}r.
\end{aligned} \tag{7}$$

Observing (6)–(7), we have

$$\big\|\tilde{\mathcal{H}}z^k-\tilde{\mathcal{H}}z\big\|\to 0 \quad \text{as } k\to\infty.$$

Hence, $\tilde{\mathcal{H}}$ is continuous on $\mathbb{B}_{r_0}$.

Step 3. $\tilde{\mathcal{H}}(\mathbb{B}_{r_0})$ is equicontinuous on I.
Let $w \in \tilde{\mathcal{H}}(\mathbb{B}_{r_0})$. Then there is $z \in \mathbb{B}_{r_0}$ such that

$$w(s) = \mathcal{S}_{\delta,\gamma;\psi}(s,0)\hbar(0) + \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s,r)\mathcal{G}(r)\psi'(r)\,\mathrm{d}r + \int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s,r)Bu_z(r)\psi'(r)\,\mathrm{d}r.$$

Let $0 < \epsilon < s$ and $0 \leqslant s_1 < s_2 \leqslant d$. Now,

$$\begin{aligned}
&\big\|w(s_2)-w(s_1)\big\|_{\sigma,\psi} \\
&= \Bigg\|\big(\psi(s_2) - \psi(0)\big)^{1-\sigma} \int_0^{s_2} \big(\psi(s_2)-\psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s_2,r)\big[\mathcal{G}(r)+Bu_z(r)\big]\psi'(r)\,\mathrm{d}r \\
&\quad - \big(\psi(s_1)-\psi(0)\big)^{1-\sigma} \int_0^{s_1} \big(\psi(s_1)-\psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s_1,r)\big[\mathcal{G}(r)+Bu_z(r)\big]\psi'(r)\,\mathrm{d}r\Bigg\| \\
&\leqslant \Bigg\|\big(\psi(s_2)-\psi(0)\big)^{1-\sigma} \int_{s_1}^{s_2} \big(\psi(s_2)-\psi(r)\big)^{\delta-1} \mathcal{Q}_{\delta,\gamma;\psi}(s_2,r)\big[\mathcal{G}(r)+Bu_z(r)\big]\psi'(r)\,\mathrm{d}r\Bigg\| \\
&\quad + \Bigg\|\big(\psi(s_2)-\psi(0)\big)^{1-\sigma} \int_{s_1-\epsilon}^{s_1} \big(\psi(s_2)-\psi(r)\big)^{\delta-1} \\
&\qquad \times \big[\mathcal{Q}_{\delta,\gamma;\psi}(s_2,r)-\mathcal{Q}_{\delta,\gamma;\psi}(s_1,r)\big]\big[\mathcal{G}(r)+Bu_z(r)\big]\psi'(r)\,\mathrm{d}r\Bigg\| \\
&\quad + \Bigg\|\big(\psi(s_2)-\psi(0)\big)^{1-\sigma} \int_{s_1-\epsilon}^{s_1} \big[\big(\psi(s_2)-\psi(r)\big)^{\delta-1} - \big(\psi(s_1)-\psi(r)\big)^{\delta-1}\big] \\
&\qquad \times \mathcal{Q}_{\delta,\gamma;\psi}(s_1,r)\big[\mathcal{G}(r)+Bu_z(r)\big]\psi'(r)\,\mathrm{d}r\Bigg\| \\
&\quad + \Bigg\|\big(\psi(s_2)-\psi(0)\big)^{1-\sigma} \int_0^{s_1-\epsilon} \big(\psi(s_2)-\psi(r)\big)^{\delta-1} \big[\mathcal{Q}_{\delta,\gamma;\psi}(s_2,r)-\mathcal{Q}_{\delta,\gamma;\psi}(s_1,r)\big] \\
&\qquad \times \big[\mathcal{G}(r)+Bu_z(r)\big]\psi'(r)\,\mathrm{d}r\Bigg\| \\
&\quad + \Bigg\|\big(\psi(s_2)-\psi(0)\big)^{1-\sigma} \int_0^{s_1-\epsilon} \big[\big(\psi(s_2)-\psi(r)\big)^{\delta-1} - \big(\psi(s_1)-\psi(r)\big)^{\delta-1}\big] \\
&\qquad \times \mathcal{Q}_{\delta,\gamma;\psi}(s_1,r)\big[\mathcal{G}(r)+Bu_z(r)\big]\psi'(r)\,\mathrm{d}r\Bigg\|.
\end{aligned}$$

Using the Lebesgue integral dominance convergence theorem, for ϵ sufficiently small, $\|w(s_2) - w(s_1)\|_{\sigma,\psi} \to 0$ as $s_2 \to s_1$.

Hence, $\tilde{\mathcal{H}}(\mathbb{B}_{r_0})$ is equicontinuous on I.

Step 4. The Mönch's condition holds.

Let $z^0(s) + \hat{h}(s) = (\psi(s) - \psi(0))^{1-\sigma}\mathcal{S}_{\delta,\gamma;\psi}(s,0)\hat{h}_0$ for all $s \in$ I and $z^{n+1} + \hat{h}(s) = \tilde{\mathcal{H}}[w^n + \hat{h}(s)]$, $n = 0, 1, 2, \dots$.

Let $\mathcal{P} \subseteq \mathbb{B}_{r_0}$ be countable and $\mathcal{P} \subseteq \operatorname{conv}(\{0\} \cup \tilde{\mathcal{H}}(\mathcal{P}))$. We show that $\chi(\mathcal{P}) = 0$. Suppose $\mathcal{P} = \{z^n + \hat{h}\}_{n=1}^{\infty}$. Now, we have to show that $\tilde{\mathcal{H}}(\mathcal{P})(s)$ is relatively compact in Z for all $s \in$ I.

By using Theorem 1, we have

$$\begin{aligned}\chi(\mathcal{P}(s)) &= \chi\big(\big\{z^n + \hat{h}\big\}_{n=0}^{\infty}\big) = \chi\big(\big\{z^0 + \hat{h}\big\} \cup \big\{z^n + \hat{h}\big\}_{n=1}^{\infty}\big)\\ &= \chi\big(\big\{z^n(s) + \hat{h}(s)\big\}_{n=1}^{\infty}\big)\end{aligned}$$

and

$$\begin{aligned}&\chi\big(\big\{\tilde{\mathcal{H}}z^n(s)\big\}_{n=1}^{\infty}\big)\\ &\quad= \chi\Bigg(\Bigg\{\big(\psi(s) - \psi(0)\big)^{1-\sigma}\\ &\qquad\times\int_0^s \big(\psi(s) - \psi(r)\big)^{1-\delta}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)\big[\mathcal{G}_n(r) + Bu_{z^n}(r)\big]\psi'(r)\,\mathrm{d}r\Bigg\}_{n=1}^{\infty}\Bigg)\\ &\quad\leqslant J_1 + J_2,\end{aligned}$$

where

$$\begin{aligned}J_1 &= \chi\Bigg(\Bigg\{\big(\psi(s) - \psi(0)\big)^{1-\sigma}\int_0^s \big(\psi(s) - \psi(r)\big)^{1-\delta}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)\mathcal{G}_n(r)\Bigg\}\Bigg),\\ J_2 &= \chi\Bigg(\Bigg\{\big(\psi(s) - \psi(0)\big)^{1-\sigma}\int_0^s \big(\psi(s) - \psi(r)\big)^{1-\delta}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)Bu_{z^n}(r)\Bigg\}\Bigg).\end{aligned}$$

Now, again by using by Theorem 1, (A3), and (A4),

$$\begin{aligned}J_1 &= \chi\Bigg(\Bigg\{\big(\psi(s) - \psi(0)\big)^{1-\sigma}\int_0^s \big(\psi(s) - \psi(r)\big)^{1-\delta}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)\mathcal{G}_n(r)\Bigg\}\Bigg)\\ &\leqslant \frac{2M(\psi(d) - \psi(0))^{1-\sigma}}{\Gamma(\delta)}\int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1}\chi\big(\big\{\mathcal{G}_n(r)\big\}_{n=1}^{\infty}\big)\psi'(r)\,\mathrm{d}r\\ &\leqslant \frac{2M(\psi(d) - \psi(0))^{1-\sigma}}{\Gamma(\delta)}\int_0^s \big(\psi(s) - \psi(r)\big)^{\delta-1}\chi\big(\big\{g\big(r, [z_r^n + \hat{h}_r]\big)\big\}_{n=1}^{\infty}\big)\psi'(r)\,\mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{2M(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} \\
&\quad\times \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1} m_2(r) \sup_{-\infty<\theta\leqslant 0} \chi\big(\big\{\big[z^n(r+\theta)+\hat{h}(r+\theta)\big]\big\}_{n=1}^{\infty}\big)\psi'(r)\,\mathrm{d}r \\
&\leqslant \frac{2M(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1} m_2(r) \sup_{0<\omega\leqslant r} \chi\big(\mathcal{P}(\omega)\big)\psi'(r)\,\mathrm{d}r, \\
J_2 &= \chi\Bigg(\Bigg\{(\psi(s)-\psi(0))^{1-\sigma}\int_0^s \big(\psi(s)-\psi(r)\big)^{1-\delta}\mathcal{Q}_{\delta,\gamma;\psi}(s,r)Bu_{z^n}(r)\Bigg\}\Bigg) \\
&\leqslant \frac{2M\theta_1(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1} \chi\big(\{u_{z^n}(r)\}_{n=1}^{\infty}\big)\psi'(r)\,\mathrm{d}r \\
&\leqslant \frac{2M\theta_1(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} \\
&\quad\times \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1} \\
&\qquad\times \Bigg[\frac{2M}{\Gamma(\delta)}\int_0^d \big(\psi(d)-\psi(r)\big)^{\delta-1}\chi\big(\big\{g\big(r,[z_r^n+\hat{h}_r]\big)\big\}_{n=1}^{\infty}\big)\psi'(r)\,\mathrm{d}r\Bigg]\psi'(r)\,\mathrm{d}r \\
&\leqslant \frac{4M^2\theta_1(\psi(d)-\psi(0))^{1-\sigma}}{(\Gamma(\delta))^2} \\
&\quad\times \int_0^s \big(\psi(s)-\psi(r)\big)^{\delta-1} m_3(r) \\
&\qquad\times \Bigg[\Bigg(\int_0^d \big(\psi(d)-\psi(r)\big)^{\delta-1} m_2(r)\Bigg) \sup_{0<\omega\leqslant r}\chi\big(\mathcal{P}(\omega)\big)\psi'(r)\,\mathrm{d}r\Bigg]\psi'(r)\,\mathrm{d}r,
\end{aligned}
$$

$$
\begin{aligned}
J_1+J_2 &= \Bigg[\frac{2M\mathcal{R}_2(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)} + \frac{4M^2\mathcal{R}_2\mathcal{R}_3\theta_1(\psi(d)-\psi(0))^{1-\sigma}}{(\Gamma(\delta))^2}\Bigg] \\
&\quad\times \sup_{0<\omega\leqslant r}\chi\big(\mathcal{P}(\omega)\big) \\
&\leqslant \frac{2M\mathcal{R}_2(\psi(d)-\psi(0))^{1-\sigma}}{\Gamma(\delta)}\Bigg[1+\frac{2M\mathcal{R}_3\theta_1}{\Gamma(\delta)}\Bigg] \sup_{0<\omega\leqslant r}\chi\big(\mathcal{P}(\omega)\big).
\end{aligned}
$$

Hence, by Lemma 5,

$$
\chi\big(\tilde{\mathcal{H}}(\mathcal{P})\big) \leqslant \mathcal{S}^*\chi(\mathcal{P}),
$$

where $\mathcal{S}^*$ is defined in (4). Thus, from Mönch's condition we have

$$\chi(\mathcal{P}) \leqslant \chi\big(\operatorname{conv}\big(\{0\} \cup \tilde{\mathcal{H}}(\mathcal{P})\big)\big) = \chi\big(\tilde{\mathcal{H}}(\mathcal{P})\big) \leqslant \mathcal{S}^* \chi(\mathcal{P}),$$

which implies that $\chi(\mathcal{P}) = 0$. Hence, $\mathcal{P}$ is relatively compact.

Hence, using Lemma 6, $\tilde{\mathcal{H}}$ has a fixed point z in $\mathbb{B}_{r_0}$. Therefore, $x = z + \hat{h}$ is a fixed point of operator $\mathcal{H}$, which is the mild solution of system (1)–(2) satisfying $x(d) = x^1$. Thus, system (1)–(2) is controllable on I. Hence, the proof is complete. □

4 An example

Let us take the following ψ-Hilfer fractional differential control system:

$$\mathrm{D}_{0+}^{3/4,\gamma;s} x(s,z) = \frac{\partial^2}{\partial z^2} x(s,z) + \mathcal{K}\varphi(s,z) + \frac{\mathrm{e}^{-s}}{4} \mathrm{e}^{-x(s,z)}, \tag{8}$$

$$\mathrm{I}_{0+}^{(1-\gamma)/4;s} x(s,z)\big|_{s=0} = x_0(z), \quad z \in [0,\pi], \tag{9}$$

$$x(s,0) = x(s,\pi) = 0, \quad s \in (0,1], \tag{10}$$

$$x(0,z) = \hbar(s,z), \quad z \in [0,\pi], \tag{11}$$

where $\mathrm{D}_{0+}^{3/4,\gamma;s}$ is ψ-Hilfer fractional derivative operator, $\mathrm{I}_{0+}^{(1-\gamma)/4;s}$ is left-sided ψ-Riemann–Liouville fractional integration operator, $\hbar \in \mathfrak{B}_h$, $\mathcal{K}_\varphi(s,z) : \mathrm{I} \times [0,\pi] \to \mathbb{R}$ is continuous, $\delta = 3/4$, and $\psi(s) = s$.

Let $\mathrm{Z} = \mathrm{U} = L^2[0,\pi]$ with the norm $\|\cdot\|_{L^2}$, and the operator $A : D(A) \subset \mathrm{Z} \to \mathrm{Z}$ is defined by

$$Ax = \frac{\partial^2}{\partial z^2} x(s,z), \quad x \in D(A),$$

and

$$D(A) = \big\{x \in \mathrm{Z}\colon\ x'' \in \mathrm{Z},\ x(s,0) = x(s,\pi) = 0\big\}.$$

Here A is the infinitesimal generator of analytic semigroup $\{T(s)\}_{s\geqslant 0}$ on Z and is given by $T(s)w(r) = w(s+r)$ for $w \in \mathrm{Z}$, $T(s)$ is not a compact semigroup on Z with $\chi(T(s)D) \leqslant \chi(D)$, and there exists $M \geqslant 1$ such that $\sup_{s\in(0,1]} \|T(s)\| \leqslant M$. Also, $s \to w(s^{3/4}\theta + r)x$ is equicontinuous, $s \geqslant 0$, and $\theta \in (0,\infty)$. Hence, it is easy to see that T satisfies asssumption (A1).

Define

$$\mathrm{D}_{0+}^{3/4,\gamma;s} x(s)(z) = \frac{\partial^{3/4}}{\partial r^{3/4}} x(s,z), \qquad x(s)(z) = x(s,z).$$

Now, we define the control operator $\mathrm{B} : \mathrm{Z} \to \mathrm{Z}$ by

$$(\mathrm{B}u)(s)(z) = \mathcal{K}\varphi(s,z), \quad z \in [0,\pi].$$

For $z \in (0, \pi)$, $\mathcal{K}$ is given by

$$\mathcal{K}u(z) = \int_0^1 (1-s)^{-1/4} \mathcal{Q}_{\delta,\gamma;\psi}(s,1) w\varphi(s,z)\,\mathrm{d}s,$$

where

$$\mathcal{Q}_{\frac{3}{4},\gamma;s}(s)w(r) = \frac{3}{4}\int_0^\infty \theta\phi_{3/4}(\theta) w\big(s^{3/4}\theta + r\big)\,\mathrm{d}\theta$$

and

$$\phi_{3/4}(\theta) = \frac{4}{3}\theta^{-1-4/3}\rho_{3/4}\big(\theta^{-3/4}\big),$$

$$\rho_{3/4}(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-3k/4-1}\frac{\Gamma(\frac{3}{4}k+1)}{k!}\sin\left(\frac{3}{4}k\pi\right), \quad \theta \in (0,\infty).$$

We assume that $\mathcal{K}$ has an inverse and satisfies (A4)(i), (ii) with $\theta_1 = 1$, $m_3(s) = \mathrm{e}^{-s}/4$, and

$$g\big(s, x(s,z)\big) = \frac{\mathrm{e}^{-s}}{4}\mathrm{e}^{-x(s,z)}.$$

By this choice of g, A, and B, system (8) can be written as

$$\mathrm{D}_{0^+}^{3/4,\gamma;s} x(s) = Ax(s) + g(s, x_s) + Bu(s), \quad s \in \mathrm{I} = (0,1],$$
$$\mathrm{I}_{0^+}^{(1-\gamma)/4;s} x(s,z)\big|_{s=0} = \hbar(s), \quad s \in (-\infty, 0].$$

Let us define phase space $\mathfrak{B}_h$ with the norm

$$\|x\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s)\|x\|_{[s,0]}\,\mathrm{d}s \quad \forall x \in \mathfrak{B}_h,$$

where $h(s) = \mathrm{e}^{2s}$ for $s < 0$, $\int_{-\infty}^{0} h(s)\,\mathrm{d}s = 1/2$. Hence, $\|x\|_{\mathfrak{B}_h} = \|x\|/2$.

Now, $g(s, x(s)) = (\mathrm{e}^{-s}/4)\mathrm{e}^{-x(s)}$, and we have

$$\big\|g\big(s, x(s,z)\big)\big\| \leqslant \frac{\mathrm{e}^{-s}}{4}\left\|\frac{1}{x(s)}\right\| = \frac{\mathrm{e}^{-s}}{4}\frac{1}{2\|x(s)\|_{\mathfrak{B}_h}}, \quad x(s) \neq 0,$$

with $m_1 = \mathrm{e}^{-s}/4$ and $\Theta(\|x(s)\|_{\mathfrak{B}_h}) = 1/(2\|x(s)\|_{\mathfrak{B}_h})$. We get $\liminf_{k\to\infty}\Theta(k)/k = 0$. Hence, conditions (A2)(i), (ii) are satisfied. Further, for any bounded subset $\mathfrak{B}_1 \subset \mathfrak{B}_h$,

$$\chi\big(g(s, \mathfrak{B}_1)\big) \leqslant \frac{\mathrm{e}^{-s}}{4}s\Big[\sup_{-\infty<\omega\leqslant 0}\chi\big(\mathfrak{B}_1(\omega)\big)\Big].$$

So, $m_2(s) = \mathrm{e}^{-s}/4$. Hence, assumption (A3) is satisfied. Let $M = 1$ and $\delta_1 = \delta_2 = \delta_3 = 1/2$, then $m_1(s) = m_2(s) = \mathrm{e}^{-s}/4 \in L^2([0,1],\mathbb{R}^+)$, $\delta_i \in (0,3/4)$, $i = 1,2,3$, and $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = \sqrt{2}/8$. Moreover, substituting the values in (4), we have $\mathcal{S} \approx 0.373 < 1$.

Hence, all the conditions of Theorem 2 are satisfied. Therefore, system (8)–(11) is controllable on I.

5 Conclusion

In this article, we studied the controllability results of a fractional differential system involving ψ-Hilfer fractional derivative with infinite delay using a noncompactness measure. Sufficient conditions for controllability results are obtained by using some weakly compactness criteria, appropriate assumptions, and techniques of semigroup theory, fractional calculus, and Mönch's fixed point theorem via a measure of noncompactness. This study of controllability of ψ-Hilfer fractional derivative gives the controllability results for many other distinct fractional derivatives stated in Table 1. The study on approximation theorem for controllability problem governed by ψ-Hilfer fractional differential equation can be useful to future work. Also, our result can be extended to the controllability of (k,ψ)-Hilfer fractional differential equations with infinite delay via a measure of noncompactness, which is our future research plan. An example is presented to illustrate the main result.

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References

1. S. Abbas, M. Benchohra, J.E. Lazreg, J.J. Nieto, Y. Zhou, *Fractional Differential Equations and Inclusions: Classical and Advanced Topics*, Ser. Anal. Appl. Comput., Vol. 10, World Scientific, Singapore, 2023, https://doi.org/10.1142/12993.
2. K. Balachandran, J.P. Dauer, Controllability of nonlinear systems via fixed-point theorems, *J. Optim. Theory Appl.*, **53**:345–352, 1987, https://doi.org/10.1007/BF00938943.
3. J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lect Notes Pure Appl. Math., Vol. 60, Marcel Dekker, New York, 1980.
4. J.V. da C. Sousa, E.C. de Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.*, **60**:72–91, 2018, https://doi.org/10.1016/j.cnsns.2018.01.005.

5. J.V. da C. Sousa, E.C. de Oliveira, A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator, *Diff. Equ. Appl.*, **11**:87–106, 2019, https://doi.org/10.7153/dea-2019-11-02.
6. J.V. da C. Sousa, E.C. Oliveira, Leibniz type rule: ψ-Hilfer fractional operator, *Commun. Nonlinear Sci. Numer. Simul.*, **77**:305–311, 2019, https://doi.org/10.1016/j.cnsns.2019.05.003.
7. Y.K. Chang, Controllability of impulsive functional differential systems with infinite delay in Banach spaces, *Chaos Solitons Fractals*, **33**(5):1601–1609, 2007, https://doi.org/10.1016/j.chaos.2006.03.006.
8. R. Dhayal, M. Malik, S. Abbas, A. Kumar, R. Sakthivel, Approximation theorems for controllability problem governed by fractional differential equation, *Evol. Equ. Control Theory*, **10**(2): 411, 2021, https://doi.org/10.3934/eect.2020073.
9. X.-L. Ding, I. Area, J.J. Nieto, Controlled singular evolution equations and Pontryagin type maximum principle with applications, *Evol. Equ. Control Theory*, **11**:1655–1679, 2022, https://doi.org/10.3934/eect.2021059.
10. A. Diop, G.S.F. Frederico, J.V. da C. Sousa, On controllability for a class of multi-term time-fractional random differential equations with state-dependent delay, *Ann. Funct. Anal.*, **13**(2): 20, 2022, https://doi.org/10.1007/s43034-022-00165-w.
11. H. Gu, J.J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Appl. Math. Comput.*, **257**:344–354, 2015, https://doi.org/10.1016/j.amc.2014.10.083.
12. I. Haque, J. Ali, M. Mursaleen, Solvability of implicit fractional order integral equation in ℓ_p $(1 \leqslant p < \infty)$ space via generalized Darbo's fixed point theorem, *J. Funct. Spaces*, **2022**: 1674243, 2022, https://doi.org/10.1155/2022/1674243.
13. I. Haque, J. Ali, M. Mursaleen, Existence of solutions for an infinite system of Hilfer fractional boundary value problems in tempered sequence spaces, *Alexandria Eng. J.*, **65**:575–583, 2023, https://doi.org/10.1016/j.aej.2022.09.032.
14. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, River Edge, NJ, 2000, https://doi.org/10.1142/3779.
15. K. Kavitha, V. Vijayakumar, R. Udhayakumar, Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness, *Chaos Solitons Fractals*, **139**:110035, 2020, https://doi.org/10.1016/j.chaos.2020.110035.
16. K. Kavitha, V. Vijayakumar, R. Udhayakumar, C. Ravichandran, Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness, *Asian J. Control*, **24**(3):1406–1415, 2022, https://doi.org/10.1002/asjc.2549.
17. K. Kavitha, V. Vijayakumar, R. Udhayakumar, N. Sakthivel, K.S. Nisar, A note on approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay, *Math. Methods Appl. Sci.*, **44**(6):4428–4447, 2021, https://doi.org/10.1002/mma.7040.
18. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Math. Stud., Vol. 204, Elsevier, Amsterdam, 2006.
19. J. Klamka, Schauder's fixed-point theorem in nonlinear controllability problems, *Control Cybern.*, **29**:153–165, 2000, https://api.semanticscholar.org/CorpusID:119040653.

20. S. Kumar, R. Sakthivel, Constrained controllability of second order retarded nonlinear systems with nonlocal condition, *IMA J. Math. Control Inform.*, **37**(2):441–454, 2020, https://doi.org/10.1093/imamci/dnz007.

21. J.A. Machado, C. Ravichandran, M. Rivero, J.J. Trujillo, Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions, *Fixed Point Theory Appl.*, **2013**(1):1–16, 2013, https://doi.org/10.1186/1687-1812-2013-66,.

22. H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal., Theory Methods Appl.*, **4**(5):985–999, 1980, https://doi.org/10.1016/0362-546X(80)90010-3.

23. J.J Nieto, Fractional euler numbers and generalized proportional fractional logistic differential equation, *Fract. Calc. Appl. Anal.*, **25**(3):876–886, 2022, https://doi.org/10.1007/s13540-022-00044-0.

24. J. Shaochun, L. Gang, W. Min, Controllability of impulsive differential systems with nonlocal conditions, *Appl. Math. Comput.*, **217**(16):6981–6989, 2011, https://doi.org/10.1016/j.amc.2011.01.107.

25. H. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Chen, A new collection of real world applications of fractional calculus in science and engineering, *Commun. Nonlinear Sci. Numer. Simul.*, **64**:213–231, 2018, https://doi.org/10.1016/j.cnsns.2018.04.019.

26. M. Vellappandi, V. Govindaraj, J.V. da C. Sousa, Fractional optimal reachability problems with ψ-Hilfer fractional derivative, *Math. Methods Appl. Sci.*, **45**(10):6255–6267, 2022, https://doi.org/10.1002/mma.8168.

27. V. Vijayakumar, R. Udhayakumar, A new exploration on existence of Sobolev-type Hilfer fractional neutral integro-differential equations with infinite delay, *Numer. Methods Partial Diff. Equations*, **37**(1):750–766, 2021, https://doi.org/10.1002/num.22550.

28. J.R. Wang, Z. Fan, Y. Zhou, Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces, *J. Optim. Theory Appl.*, **154**(1):292–302, 2012, https://doi.org/10.1007/s10957-012-9999-3.

29. J.R. Wang, Y. Zhou, Complete controllability of fractional evolution systems, *Commun. Nonlinear Sci. Numer. Simul.*, **17**(11):4346–4355, 2012, https://doi.org/10.1016/j.cnsns.2012.02.029.

30. Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.*, **59**:1063–1077, 2010, https://doi.org/10.1016/j.camwa.2009.06.026.