Nonlinear Analysis: Modelling and Control, Vol. 29, No. 2, 244–264
https://doi.org/10.15388/namc.2024.29.34196

Vilnius University Press

On the unique weak solvability of second-order unconditionally stable difference scheme for the system of sine-Gordon equations

Ozgur Yildirim

Department of Mathematics, Yildiz Technical University,
34220, Istanbul, Turkey
ozgury@yildiz.edu.tr

Received: January 15, 2023 / **Revised:** November 16, 2023 / **Published online:** January 15, 2024

Abstract. In the present paper, a nonlinear system of sine-Gordon equations that describes the DNA dynamics is considered. A novel unconditionally stable second-order accuracy difference scheme corresponding to the system of sine-Gordon equations is presented. In this work, for the first time in the literature, weak solution of this difference scheme is studied. The existence and uniqueness of the weak solution for the difference scheme are proved in the space of distributions, and the methods of variational calculus are applied. The finite-difference method and the fixed point theory are used in combination to perform numerical experiments that verify the theoretical statements.

Keywords: weak solvability, stability, difference schemes, fixed point theory.

1 Introduction

In many branches of science such as mathematics, physics, biology, engineering, in particular, relativistic quantum mechanics, acoustics, biomedical engineering, and field theory problems, the wave equations are of great interest (see [2, 11, 17, 20, 28]). In the literature, many scientists have studied the theoretical and numerical aspects of the system of nonlinear wave equations such as sine-Gordon, Klein–Gordon, and coupled sine-Gordon equations (see [10, 17, 20]). In recent decades, these types of problems have had more attention due to the presence of soliton solutions. Soliton-type nonlinear equations describe waves that occur in proteins, signal conduction between neurons and deoxyribonucleic acid (DNA) (see [28] and the references given therein).

In the study of partial differential equations (PDEs), the weak solutions play an important role since many real-world biological models yield problems with nonsmooth solutions (see [9, 20–27]). In the case of low regularity of coefficients and source functions, weak solutions are of great interest in many problems, including coupled sine-Gordon equations, and are often easier than finding smooth solutions. They allow us to study systems that would be difficult or impossible to study using smooth solutions. This is because the constraints on weak solutions are less stringent. The weak solution may have discontinuities or other irregularities, but it still satisfies the problem in a small region. Weak solutions can model the spread of cancer in the body, the dynamics of populations of competing species, and the spread of viruses and other diseases in populations (see [12,18, 30]). In mathematical biology, the existence of traveling wave front solution and analysis for Nicholson's blowflies equation are studied in [14] and [15]. The finite-time stability for cellular neural networks with neutral proportional delays and time-varying leakage delays are studied by using the differential inequality technique in [19]. In the theory of PDEs, these types of solutions are generally obtained in the space of distributions by the energy method also known as the variational method (see [9, 13, 20–26]).

In the present work, the methods of Roger Temam et al. (see [7, 21, 24–27]) are used. The study is based on the following second-order hyperbolic evolution equations. Let $\Omega_T = \Omega \times (0,T]$ with $T > 0$ and $S = [0,T] \times \Gamma$ for $\Gamma = \partial\Omega$, $\overline{\Omega} = \Omega \times \Gamma$, and $\Omega \subset \mathbb{R}^n$ be an open and bounded set. A widely-known (see [10]) initial/boundary-value problem is

$$\begin{aligned} &w_{tt} + Lw = f, \quad \text{in } \Omega_T, \\ &w = 0 \quad \text{on } S, \\ &w = g, w_t = h \quad \text{on } \overline{\Omega} \end{aligned} \tag{1}$$

with given functions $f : \Omega_T \to \mathbb{R}$, $g, h : \overline{\Omega} \to \mathbb{R}$, and $w : \overline{\Omega}_T \to \mathbb{R}$, the unknown $w(x,t)$. Here L denotes a differential operator for each time t in the form

$$Lw = -\sum_{i,j=1}^{n} a^{ij}(x,t) w_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t) w_{x_i} + c(x,t) w$$

with coefficients a^{ij}, b^i, c $(i,j = 1,\dots,n)$. In the present study, the second-order differential operator

$$Lw = -\sum_{i,j=1}^{2} a^{ij} w_{xx} + \sum_{i=1}^{2} b^i w_x + cw$$

to a coupled system (1) for Ω_T and $\Omega \subset \mathbb{R}^2$, $f\colon \Omega_T \to \mathbb{R}$, $g,h\colon \overline{\Omega} \to \mathbb{R}$, and $w : \overline{\Omega}_T \to \mathbb{R}$ is considered. Here a^{ij}, b^i, c $(i,j = 1,2)$ are given constant coefficients.

In this paper, the weak solution of second-order unconditionally stable difference scheme corresponding to the nonlinear system of coupled sine-Gordon equations

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \alpha_{11}\frac{\partial u}{\partial t} + \alpha_{12}\frac{\partial v}{\partial t} - \beta_1 \Delta u + \gamma_1 \sin(\delta_{11} u + \delta_{12} v) \\ &\quad + \rho_{11} u + \rho_{12} v = f, \quad f \in \Omega_T, \\ &\frac{\partial^2 v}{\partial t^2} + \alpha_{21}\frac{\partial u}{\partial t} + \alpha_{22}\frac{\partial v}{\partial t} - \beta_2 \Delta v + \gamma_2 \sin(\delta_{21} u + \delta_{22} v) \\ &\quad + \rho_{21} u + \rho_{22} v = g, \quad g \in \Omega_T, \end{aligned} \tag{2}$$

with boundary conditions

$$u = 0 \quad \text{and} \quad v = 0 \quad \text{on } S \tag{3}$$

and initial conditions

$$u(0,x) = \varphi_1(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(0,x) = \psi_1(x) \quad \text{in } \overline{\Omega}, \tag{4}$$

$$v(0,x) = \varphi_2(x) \quad \text{and} \quad \frac{\partial v}{\partial t}(0,x) = \psi_2(x) \quad \text{in } \overline{\Omega} \tag{5}$$

is studied. Here $\Omega \subset \mathbb{R}^2$ is a bounded open set, and Δ is the Laplacian. The coefficients α_{ij}, β_i, γ_i, δ_{ij}, ρ_{ij} are bounded nonzero real numbers for $i, j = 1, \dots$. Let us denote the source functions as

$$\begin{aligned}
\tilde{f}(t,x,u,v,u_t,v_t) &= f(t,x) - \gamma_1 \sin(\delta_{11}u + \delta_{12}v) \\
&\quad - \rho_{11}u - \rho_{12}v - \alpha_{11}u_t - \alpha_{12}v_t, \\
\tilde{g}(t,x,u,v,u_t,v_t) &= g(t,x) - \gamma_2 \sin(\delta_{21}u + \delta_{22}v) \\
&\quad - \rho_{21}u - \rho_{22}v - \alpha_{21}u_t - \alpha_{22}v_t.
\end{aligned}$$

The functions $\tilde{f}$ and $\tilde{g}$ satisfy the Lipschitz conditions

$$\begin{aligned}
&\bigl|\tilde{f}(t,x,u_1,u_2,u_t,v_t) - \tilde{f}(t,x,v_1,v_2,u_t,v_t)\bigr| \\
&\quad \leqslant l\bigl[|u_1 - v_1| + |u_2 - v_2| + \bigl|(u_1)_t - (v_1)_t\bigr| + \bigl|(u_2)_t - (v_2)_t\bigr|\bigr]
\end{aligned}$$

on Ω_T, where l is a positive constant.

Let $A = -\Delta$ be a self-adjoint, positive-definite, unbounded operator in a Hilbert space H. One can write problem (2)–(5) in the following form:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + \alpha_{11}\frac{\partial u}{\partial t} + \alpha_{12}\frac{\partial v}{\partial t} + \beta_1 Au + \gamma_1 \sin(\delta_{11}u + \delta_{12}v) \\
&\quad + \rho_{11}u + \rho_{12}v = f, \quad 0 < t < T, \\
&\frac{\partial^2 v}{\partial t^2} + \alpha_{21}\frac{\partial u}{\partial t} + \alpha_{22}\frac{\partial v}{\partial t} + \beta_2 Av + \gamma_2 \sin(\delta_{21}u + \delta_{22}v) \\
&\quad + \rho_{21}u + \rho_{22}v = g, \quad 0 < t < T, \\
&u(0) = \varphi_1 \in V, \qquad u'(0) = \psi_1 \in H, \\
&v(0) = \varphi_2 \in V, \qquad v'(0) = \psi_2 \in H.
\end{aligned} \tag{6}$$

Here V is the Hilbert space satisfying the relation $V \subset H$. A simpler form of system (6) is

$$\begin{aligned}
u_{tt} - u_{xx} &= -\delta^2 \sin(u - v), \\
v_{tt} - v_{xx} &= \sin(u - v).
\end{aligned}$$

This type of system models the open states in DNA double helices and is studied by many scientists (see [17, 29]).

Existence and uniqueness of problem (6) is presented as the limit of second-order accuracy unconditionally stable difference scheme

$$\begin{aligned}
&\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{\beta_1}{4}A(u_{k+1}+2u_k+u_{k-1})\\
&\quad+\frac{\alpha_{11}}{2\tau}(u_{k+1}-u_{k-1})+\frac{\alpha_{12}}{2\tau}(v_{k+1}-v_{k-1})+\gamma_1\sin(\delta_{11}u_k+\delta_{12}v_k)\\
&\quad+\frac{\rho_{11}}{4}(u_{k+1}+2u_k+u_{k-1})+\frac{\rho_{12}}{4}(v_{k+1}+2v_k+v_{k-1})=f_k,\\
&f_k=f(t_k,u_k,v_k),\quad t_k=k\tau,\ 1\leqslant k\leqslant N-1,\ N\tau=1,\\
&\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+\frac{\beta_2}{4}A(v_{k+1}+2v_k+v_{k-1})\\
&\quad+\frac{\alpha_{21}}{2\tau}(u_{k+1}-u_{k-1})+\frac{\alpha_{22}}{2\tau}(v_{k+1}-v_{k-1})+\gamma_2\sin(\delta_{21}u_k+\delta_{22}v_k)\\
&\quad+\frac{\rho_{21}}{4}(u_{k+1}+2u_k+u_{k-1})+\frac{\rho_{22}}{4}(v_{k+1}+2v_k+v_{k-1})=g_k,\\
&g_k=g(t_k,u_k,v_k),\quad t_k=k\tau,\ 1\leqslant k\leqslant N-1,\ N\tau=1,\\
&u(0)=u_0=\varphi_1,\qquad v(0)=v_0=\varphi_2,\\
&u'(0)=\left(I+\frac{\tau^2A}{4}\right)\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(\widetilde{f}_0-Au_0)=\psi_1,\\
&v'(0)=\left(I+\frac{\tau^2A}{4}\right)\tau^{-1}(v_1-v_0)-\frac{\tau}{2}(\widetilde{g}_0-Av_0)=\psi_2,\\
&\widetilde{f}(0)=\widetilde{f}_0,\widetilde{g}_0(0)=\widetilde{g}_0
\end{aligned}\tag{7}$$

with a modification to a damped nonlinear system. For the solution of problem (7), we consider the set of a family of grid points

$$\begin{aligned}
\Omega_h&=[0,T]_\tau\times[0,L]_h\\
&=\bigl\{(t_k,x_n)\colon t_k=k\tau,\ 0\leqslant k\leqslant N,\ N\tau=T,\\
&\qquad x_n=nh,\ 0\leqslant n\leqslant M,\ Mh=L\bigr\}
\end{aligned}\tag{8}$$

with step sizes τ, h and constants T, L. Here f_k, g_k, φ_1, φ_2, ψ_1, and ψ_2 are given nonzero functions. Unconditional stability and the convergence of linear undamped form of difference scheme (7) are presented in [2–4].

The weak and global solutions, nonlinear dynamics of PDEs, finite-difference and finite-element methods are extensively studied by many scientists (see [1,6,13–23,28,32] and the references given therein).

2 Theoretical background and problem settings

In the present section, some theoretical preliminaries that are necessary in the sequel will be presented. For the overall literature on the elementary spectral theory and bilinear forms, we refer to [5, 7, 16, 22, 25, 31]. Let us denote Hilbert spaces H, V and the dual space V' as $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and $V' = H_0^{-1}(\Omega)$, respectively. These spaces are equipped with the following inner products and norms:

$$(\psi, \phi) = \int_\Omega \psi(x)\phi(x)\,\mathrm{d}x, \quad |\psi| = (\psi, \psi)^{1/2} \quad \forall \phi, \psi \in L^2(\Omega),$$

$$(\!(\psi, \phi)\!) = \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\psi(x)\frac{\partial}{\partial x_i}\phi(x)\,\mathrm{d}x, \quad \|\psi\| = (\!(\psi, \psi)\!)^{1/2} \quad \forall \phi, \psi \in H_0^1(\Omega).$$

The pair (V, H) is a Gelfand triple space with $V \hookrightarrow H \equiv H' \hookrightarrow V'$ for $V' = H^{-1}(\Omega)$. The embeddings $V \subset H$ and $H \subset V'$ are continuous, dense, and compact. The unique solvability results are presented in the setting of the triple space. The bilinear form

$$a(\phi, \varphi) = \int_\Omega \nabla\phi \cdot \nabla\varphi\,\mathrm{d}x = (\!(\phi, \varphi)\!) \quad \forall \phi, \varphi \in V = H_0^1(\Omega)$$

will be used in variational formulation. This form is bounded, symmetric on $V \times V = H_0^1(\Omega)^2$, and coercive, that is,

$$a(\phi, \phi) \geqslant \mu_i \|\phi\|^2, \quad i = 1, \dots, 6,\ \phi \in V. \tag{9}$$

Letting $A = -\Delta$, we have

$$A(\phi, \varphi) = a(\!(\phi, \varphi)\!) \tag{10}$$

for the operator A that is an isomorphism from V onto V'. Operator A is an unbounded self-adjoint operator in H with dense domain $D(A) = \{\phi \in V\colon A\phi \in H\}$ in V and in H. If the bilinear form a is symmetric, then the operator A is self-adjoint:

$$\langle Au, v\rangle = \langle Av, u\rangle = a(\!(u, v)\!) \quad \forall u, v \in V$$

(from V into V' and as an unbounded operator in H), and moreover, inverse A^{-1} is also self-adjoint (in H).

The complete theory of the function spaces $D(A)$, V, and H, as well as the operators A and a, are given in the references (see, e.g., [7, 24, 25]).

The solution space, which is the space of distributions, can be expressed in the following form:

$$W(0,T) = \bigl\{g\colon g \in L^2\bigl(0,T; H_0^1\bigr),\ g' \in L^2(0,T;H),\ g'' \in L^2\bigl(0,T; H_0^{-1}\bigr)\bigr\}.$$

In this article, we assume that $f, g \in L^\infty(\mathbb{R}_+; L^2(\Omega)^2)$, and we set

$$|f|_\infty := |f|_{L^\infty(\mathbb{R}_+; L^2(\Omega)^2)}, \qquad |g|_\infty := |g|_{L^\infty(\mathbb{R}_+; L^2(\Omega)^2)}.$$

The definition of the weak solution for (7) can be stated in the following lemma.

Lemma 1. (*See* [24].) *Let X be a given Banach space, X' be the dual, and let v and g be functions in $L^1(a, b; X)$. The following conditions are equivalent*:

(i) *v is a.e. equal to the primitive function of g, i.e., there exists $w \in X$ with $v(t) = w + \int_0^t g(s)\,\mathrm{d}s$ for a.e. $t \in [a, b]$.*

(ii) *For each test function $\varphi \in B(]a, b[)$, $\int_a^b v(t)\varphi'(t)\,\mathrm{d}t = -\int_a^b g(t)\varphi(t)\,\mathrm{d}t$, where $\varphi'(t) = \mathrm{d}\varphi/\mathrm{d}t$.*

(iii) *For every $\eta \in X'$, $\mathrm{d}/\mathrm{d}t\langle v, \eta\rangle = \langle g, \eta\rangle$ on $]a, b[$ in the scalar distribution sense.*

When conditions (i)–(iii) *are satisfied, then g is said to be the* (X*-valued*) *distribution derivative of v, and v is a.e. equal to a function from $[a, b]$ into X, which is continuous.*

Next, the weak solutions that are established via variational formulation for a nonlinear coupled system of difference equations (7) will be studied. Some results on the strong convergence of the sequences will be derived using the theorems on the compactness.

3 Weak solution of the second-order accuracy difference scheme generated by A

In the present section, some theoretical statements on the weak approximate solution of (6) is established for the unconditionally stable difference scheme (7). Applying variational formulation, it will be shown that the difference problem (7) converges to a unique weak solution.

Note that, throughout this paper, K_i, $\widetilde{K}_i$, c_i, d_i, μ_i represent generic constants, which may have different values at different places.

Definition 1. The set of mesh functions $\{u_k^h\}$ and $\{v_k^h\}$ are said to be the approximate weak solutions of (7) if $u_k^h, v_k^h \in V^h$ satisfy the weak formulation of (7). The family of grid points (8) are used to present Hilbert space

$$L_{2h}(\Omega) = L_{2h}(\Omega_h).$$

The space is equipped with the following norm:

$$\|u_k\|_{L_{2h}(\Omega)} = \left(\sum_{j=1}^{N} |u_k^j|^2 h\right)^{1/2}.$$

Now, we will give our main theorem on the weak solvability of (7). By the theorem the solutions u_k^h and v_k^h of (7) will be proved to be bounded.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with piecewise smooth boundary, $u_0 \in L^2(\Omega)$, $u_0' \in H_0^1(\Omega)$, and $f, g \in L^\infty(\mathbb{R}_+; L^2(\Omega)^2)$. Then the solutions u_k^h and v_k^h of* (7) *are bounded in the following sense*:

$$\left|u_k^h\right|^2 \leqslant C, \quad \left|v_k^h\right|^2 \leqslant C, \quad k = 0, \dots, N, \tag{11}$$

$$\sum_{k=1}^{N-1} \left|u_{k+1}^h - u_k^h\right|_h^2 \leqslant C, \qquad \sum_{k=1}^{N-1} \left|v_{k+1}^h - v_k^h\right|_h^2 \leqslant C, \tag{12}$$

$$\begin{aligned}
&\tau^2\sum_{k=1}^{N-1}\left\|u_{k+1}^h\right\|_h^2 \leqslant \frac{1}{\beta_1\mu_1}C, \qquad \tau^2\sum_{k=1}^{N-1}\left\|u_k^h\right\|_h^2 \leqslant \frac{1}{\beta_1\mu_2}C,\\
&\tau^2\sum_{k=1}^{N-1}\left\|u_{k-1}^h\right\|_h^2 \leqslant \frac{2}{\beta_1\mu_3}C,
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\tau^2\sum_{k=1}^{N-1}\left\|v_{k+1}^h\right\|_h^2 \leqslant \frac{1}{\beta_1\mu_1}C, \qquad \tau^2\sum_{k=1}^{N-1}\left\|v_k^h\right\|_h^2 \leqslant \frac{1}{\beta_1\mu_2}C,\\
&\tau^2\sum_{k=1}^{N-1}\left\|v_{k-1}^h\right\|_h^2 \leqslant \frac{2}{\beta_1\mu_3}C,
\end{aligned} \tag{14}$$

where $C = |u_0^h|_h^2 + |v_0^h|_h^2 + d_7\int_0^{\mathrm{T}} |f(s)|^2\,\mathrm{d}s + d_8\int_0^{\mathrm{T}} |g(s)|^2\,\mathrm{d}s$.

Proof. In the proof, it will be shown that $|u_k^h|_h$, $\|u_k^h\|_h$ are uniformly bounded. For that purpose, we take the inner product of the equations in (7) with $2u_{k+1}$ and $2v_{k+1}$, respectively, and we obtain

$$\begin{aligned}
&\left(u_{k+1}^h - u_k^h, 2u_{k+1}^h\right) + \left(u_{k-1}^h - u_k^h, 2u_{k+1}^h\right)\\
&\quad - \frac{\beta_1}{4}\tau^2\left[\left(\nabla^2 u_{k+1}^h, 2u_{k+1}^h\right) + 2\left(\nabla^2 u_k^h, 2u_{k+1}^h\right) + \left(\nabla^2 u_{k-1}^h, 2u_{k+1}^h\right)\right]\\
&\quad + \tau^2\gamma_1\left(\sin\left(\delta_{11}u_k^h + \delta_{12}v_k^h\right),\, 2u_{k+1}^h\right) + \frac{\alpha_{11}\tau}{2}\left[\left(u_{k+1}^h, 2u_{k+1}^h\right) - \left(u_{k-1}^h, 2u_{k+1}^h\right)\right]\\
&\quad + \frac{\alpha_{12}\tau}{2}\left[\left(v_{k+1}^h, 2u_{k+1}^h\right) - \left(v_{k-1}^h, 2u_{k+1}^h\right)\right]\\
&\quad + \frac{\tau^2\rho_{11}}{4}\left[\left(u_{k+1}^h, 2u_{k+1}^h\right) + 2\left(u_k^h, 2u_{k+1}^h\right) + \left(u_{k-1}^h, 2u_{k+1}^h\right)\right]\\
&\quad + \frac{\tau^2\rho_{12}}{4}\left[\left(v_{k+1}^h, 2u_{k+1}^h\right) + 2\left(v_k^h, 2u_{k+1}^h\right) + \left(v_{k-1}^h, 2u_{k+1}^h\right)\right]\\
&= \left(\tau^2 f_k, 2u_{k+1}^h\right),
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\left(v_{k+1}^h - v_k^h, 2v_{k+1}^h\right) + \left(v_{k-1}^h - v_k^h, 2v_{k+1}^h\right)\\
&\quad - \frac{\beta_2}{4}\tau^2\left[\left(\nabla^2 v_{k+1}^h, 2v_{k+1}^h\right) + 2\left(\nabla^2 v_k^h, 2v_{k+1}^h\right) + \left(A v_{k-1}^h, 2v_{k+1}^h\right)\right]\\
&\quad + \tau^2\gamma_2\left(\sin\left(\delta_{21}u_k^h + \delta_{22}v_k^h\right),\, 2v_{k+1}^h\right) + \frac{\alpha_{21}\tau}{2}\left[\left(u_{k+1}^h, 2v_{k+1}^h\right) - \left(u_{k-1}^h, 2v_{k+1}^h\right)\right]\\
&\quad + \frac{\alpha_{22}\tau}{2}\left[\left(v_{k+1}^h, 2v_{k+1}^h\right) - \left(v_{k-1}^h, 2v_{k+1}^h\right)\right]\\
&\quad + \frac{\tau^2\rho_{21}}{4}\left[\left(u_{k+1}^h, 2v_{k+1}^h\right) + 2\left(u_k^h, 2v_{k+1}^h\right) + \left(u_{k-1}^h, 2v_{k+1}^h\right)\right]\\
&\quad + \frac{\tau^2\rho_{22}}{4}\left[\left(v_{k+1}^h, 2v_{k+1}^h\right) + 2\left(v_k^h, 2v_{k+1}^h\right) + \left(v_{k-1}^h, 2v_{k+1}^h\right)\right]\\
&= \left(\tau^2 g_k, 2v_{k+1}^h\right).
\end{aligned} \tag{16}$$

Using relations

$$2(\varphi-\psi,\varphi)_h=|\varphi|_h^2-|\psi|_h^2+|\varphi-\psi|_h^2\quad\forall\varphi,\psi\in v_h^h,$$
$$2(\varphi-\psi,\psi)_h=|\varphi|_h^2-|\psi|_h^2-|\varphi-\psi|_h^2\quad\forall\varphi,\psi\in v_h^h$$

and denoting $\Delta=-A$, we obtain

$$\begin{aligned}
&\left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2+\left(u_{k-1}^h,2u_{k+1}^h\right)-\left(u_k^h,2u_{k+1}^h\right)\\
&\quad+\frac{\beta_1}{4}\tau^2\left[\left(Au_{k+1}^h,2u_{k+1}^h\right)+2\left(Au_k^h,2u_{k+1}^h\right)+\left(Au_{k-1}^h,2u_{k+1}^h\right)\right]\\
&\quad+\tau^2\gamma_1\left(\sin\left(\delta_{11}u_k^h+\delta_{12}v_k^h\right),\,2u_{k+1}^h\right)+\frac{\alpha_{11}\tau}{2}\left[\left(u_{k+1}^h,2u_{k+1}^h\right)-\left(u_{k-1}^h,2u_{k+1}^h\right)\right]\\
&\quad+\frac{\alpha_{12}\tau}{2}\left[\left(v_{k+1}^h,2u_{k+1}^h\right)-\left(v_{k-1}^h,2u_{k+1}^h\right)\right]\\
&\quad+\frac{\tau^2\rho_{11}}{4}\left[\left(u_{k+1}^h,2u_{k+1}^h\right)+2\left(u_k^h,2u_{k+1}^h\right)+\left(u_{k-1}^h,2u_{k+1}^h\right)\right]\\
&\quad+\frac{\tau^2\rho_{12}}{4}\left[\left(v_{k+1}^h,2u_{k+1}^h\right)+2\left(v_k^h,2u_{k+1}^h\right)+\left(v_{k-1}^h,2u_{k+1}^h\right)\right]\\
&=\left(\tau^2f_k,2u_{k+1}^h\right),\\
&\left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2+\left(v_{k-1}^h,2v_{k+1}^h\right)-\left(v_k^h,2v_{k+1}^h\right)\\
&\quad+\frac{\beta_2}{4}\tau^2\left[\left(Av_{k+1}^h,2v_{k+1}^h\right)+2\left(Av_k^h,2v_{k+1}^h\right)+\left(Av_{k-1}^h,2v_{k+1}^h\right)\right]\\
&\quad+\tau^2\gamma_2\left(\sin\left(\delta_{21}u_k^h+\delta_{22}v_k^h\right),\,2v_{k+1}^h\right)+\frac{\alpha_{21}\tau}{2}\left[\left(u_{k+1}^h,2v_{k+1}^h\right)-\left(u_{k-1}^h,2v_{k+1}^h\right)\right]\\
&\quad+\frac{\alpha_{22}\tau}{2}\left[\left(v_{k+1}^h,2v_{k+1}^h\right)-\left(v_{k-1}^h,2v_{k+1}^h\right)\right]\\
&\quad+\frac{\tau^2\rho_{21}}{4}\left[\left(u_{k+1}^h,2v_{k+1}^h\right)+2\left(u_k^h,2v_{k+1}^h\right)+\left(u_{k-1}^h,2v_{k+1}^h\right)\right]\\
&\quad+\frac{\tau^2\rho_{22}}{4}\left[\left(v_{k+1}^h,2v_{k+1}^h\right)+2\left(v_k^h,2v_{k+1}^h\right)+\left(v_{k-1}^h,2v_{k+1}^h\right)\right]\\
&=\left(\tau^2g_k,2v_{k+1}^h\right).
\end{aligned}\tag{17}$$

Replacing the operator A with bilinear form (10) and rewriting the equations of system (17) separately, we get

$$\begin{aligned}
&\left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2\\
&\quad+\frac{\beta_1}{2}\tau^2\left[a\left(\left(u_{k+1}^h,u_{k+1}^h\right)\right)+2a\left(\left(u_k^h,u_{k+1}^h\right)\right)+a\left(\left(u_{k-1}^h,u_{k+1}^h\right)\right)\right]\\
&=\left(\tau^2f_k,2u_{k+1}^h\right)-\left(\alpha_{12}\tau+\frac{1}{2}\tau^2\rho_{12}\right)\left(v_{k+1}^h,u_{k+1}^h\right)\\
&\quad+\left(\alpha_{12}\tau-\frac{1}{2}\tau^2\rho_{12}\right)\left(v_{k-1}^h,u_{k+1}^h\right)-\left(\alpha_{11}\tau+\frac{1}{2}\tau^2\rho_{11}\right)\left(u_{k+1}^h,u_{k+1}^h\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\alpha_{11}\tau - \frac{1}{2}\tau^2\rho_{11} - 2\right)\left(u_{k-1}^h, u_{k+1}^h\right) - \tau^2\rho_{12}\left(v_k^h, u_{k+1}^h\right) \\
&+ \left(2 - \tau^2\rho_{11}\right)\left(u_k^h, u_{k+1}^h\right) - \tau^2\gamma_1\left(\sin\left(\delta_{11}u_k^h + \delta_{12}v_k^h\right),\, 2u_{k+1}^h\right), \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
&\left|v_{k+1}^h\right|_h^2 - \left|v_k^h\right|_h^2 + \left|v_{k+1}^h - v_k^h\right|_h^2 \\
&\quad + \frac{\beta_2}{2}\tau^2\left[a\left(\left(v_{k+1}^h, v_{k+1}^h\right)\right) + 2a\left(\left(v_k^h, v_{k+1}^h\right)\right) + a\left(\left(v_{k-1}^h, v_{k+1}^h\right)\right)\right] \\
&= \left(\tau^2 g_k, 2v_{k+1}^h\right) - \left(\alpha_{21}\tau + \frac{1}{2}\tau^2\rho_{21}\right)\left(u_{k+1}^h, v_{k+1}^h\right) \\
&\quad + \left(\alpha_{21}\tau - \frac{1}{2}\tau^2\rho_{21}\right)\left(u_{k-1}^h, v_{k+1}^h\right) - \left(\alpha_{22}\tau + \frac{1}{2}\tau^2\rho_{22}\right)\left(v_{k+1}^h, v_{k+1}^h\right) \\
&\quad + \left(\alpha_{22}\tau - \frac{1}{2}\tau^2\rho_{22} - 2\right)\left(v_{k-1}^h, v_{k+1}^h\right) - \tau^2\rho_{21}\left(u_k^h, v_{k+1}^h\right) \\
&\quad + \left(2 - \tau^2\rho_{22}\right)\left(v_k^h, v_{k+1}^h\right) - \tau^2\gamma_2\left(\sin\left(\delta_{21}u_k^h + \delta_{22}v_k^h\right),\, 2v_{k+1}^h\right). \qquad (19)
\end{aligned}
$$

The estimates

$$
c_i\left|u_k^h\right|_h \leqslant \left\|u_k^h\right\|_h, \quad i = 1, \dots, 9,\ u_k^h \in V^h, \qquad (20)
$$

and

$$
\begin{aligned}
&\left|\left(\sin\left(\delta_{11}u_k^h + \delta_{12}v_k^h\right),\, u_{k+1}^h\right)\right| \\
&\quad \leqslant \left|\sin\left(\delta_{11}u_k^h + \delta_{12}v_k^h\right)\right|\left|u_{k+1}^h\right| \\
&\quad \leqslant \frac{c_1}{2}\left(|\delta_{11}|M_1\left(\left|u_k^h\right|^2 + \left|u_{k+1}^h\right|^2\right) + |\delta_{12}|\left(\left|v_k^h\right|^2 + \left|u_{k+1}^h\right|^2\right)\right) \qquad (21)
\end{aligned}
$$

are used to obtain a priori estimates for the nonnegativity and boundedness of the terms in Eqs. (18), (19) of system (16) (see, e.g., [24]). Next, using the coercivity estimate (9), spectral properties of the operator A, estimates (20), (21), Cauchy–Schwarz inequality, Young's inequality, triangle inequality, and some simple identities together, the following inequalities are obtained:

$$
\begin{aligned}
&\left|u_{k+1}^h\right|_h^2 - \left|u_k^h\right|_h^2 + \left|u_{k+1}^h - u_k^h\right|_h^2 \\
&\quad + \frac{\beta_1}{2}\tau^2\mu_1\left\|u_{k+1}^h\right\|^2 + \beta_1\tau^2\mu_2\left\|u_k^h\right\|^2 + \frac{\beta_1}{2}\tau^2\mu_3\left\|u_{k-1}^h\right\|^2 \\
&\leqslant 2\tau^2|f_k|_\infty\left|u_{k+1}^h\right|_h + \frac{1}{2}\left|\alpha_{12}\tau + \frac{1}{2}\tau^2\rho_{12}\right||M_2|\left(\left|v_{k+1}^h\right|_h^2 + \left|u_{k+1}^h\right|_h^2\right) \\
&\quad + \frac{1}{2}\left|\alpha_{12}\tau - \frac{1}{2}\tau^2\rho_{12}\right||M_3|\left(\left|v_{k-1}^h\right|_h^2 + \left|u_{k+1}^h\right|_h^2\right) + \left|\alpha_{11}\tau + \frac{1}{2}\tau^2\rho_{11}\right|\left|u_{k+1}^h\right|_h^2 \\
&\quad + \frac{1}{2}\left|\alpha_{11}\tau - \frac{1}{2}\tau^2\rho_{11} - 2\right||M_4|\left(\left|u_{k-1}^h\right|_h^2 + \left|u_{k+1}^h\right|_h^2\right) \\
&\quad + \frac{1}{2}\tau^2|\rho_{12}||M_5|\left(\left|v_k^h\right|_h^2 + \left|u_{k+1}^h\right|_h^2\right) + \frac{1}{2}\left|2 - \tau^2\rho_{11}\right||M_6|\left(\left|u_k^h\right|_h^2 + \left|u_{k+1}^h\right|_h^2\right) \\
&\quad + \tau^2|\gamma_1||c_2||\delta_{11}||M_6|\left(\left|u_k^h\right|_h^2 + \left|u_{k+1}^h\right|_h^2\right) + \tau^2|\gamma_1||c_2||\delta_{12}||M_7|\left(\left|v_k^h\right|_h^2 + \left|u_{k+1}^h\right|_h^2\right)
\end{aligned}
$$

and

$$\begin{aligned}
&\left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2 \\
&\quad+\frac{\beta_2}{2}\tau^2\mu_1\left\|v_{k+1}^h\right\|^2+\beta_2\tau^2\mu_2\left\|v_k^h\right\|^2+\frac{\beta_2}{2}\tau^2\mu_3\left\|v_{k-1}^h\right\|^2 \\
&\leqslant 2\tau^2|g_k|_\infty\left|v_{k+1}^h\right|_h+\frac{1}{2}\left|\alpha_{21}\tau+\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_2|\left(\left|u_{k+1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\left|\alpha_{21}\tau-\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_3|\left(\left|u_{k-1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\left|\alpha_{22}\tau+\frac{1}{2}\tau^2\rho_{22}\right|\left|v_{k+1}^h\right|_h^2 \\
&\quad+\frac{1}{2}\left|\alpha_{22}\tau-\frac{1}{2}\tau^2\rho_{22}-2\right||\widetilde{M}_4|\left(\left|v_{k-1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\tau^2|\rho_{21}||\widetilde{M}_5|\left(\left|u_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\frac{1}{2}\left|2-\tau^2\rho_{22}\right||\widetilde{M}_6|\left(\left|v_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
&\quad+\tau^2|\gamma_2||c_4||\delta_{21}||\widetilde{M}_7|\left(\left|u_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\tau^2|\gamma_2||c_4||\delta_{22}||\widetilde{M}_7|\left(\left|v_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right).
\end{aligned}$$

Using estimate (20), we have

$$\begin{aligned}
&\left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2 \\
&\quad+\frac{\beta_1}{2}\tau^2\mu_1\left\|u_{k+1}^h\right\|_h^2+\beta_1\tau^2\mu_2\left\|u_k^h\right\|_h^2+\frac{\beta_1}{2}\tau^2\mu_3\left\|u_{k-1}^h\right\|_h^2 \\
&\leqslant 2\tau^2c_1|f_k|_\infty\left\|u_{k+1}^h\right\|_h+\frac{1}{2}\left|\alpha_{12}\tau+\frac{1}{2}\tau^2\rho_{12}\right||M_2|\left(\left|v_{k+1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\left|\alpha_{12}\tau-\frac{1}{2}\tau^2\rho_{12}\right||M_3|\left(\left|v_{k-1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right)+\left|\alpha_{11}\tau+\frac{1}{2}\tau^2\rho_{11}\right||u_{k+1}^h|_h^2 \\
&\quad+\frac{1}{2}\left|\alpha_{11}\tau-\frac{1}{2}\tau^2\rho_{11}-2\right||M_4|\left(\left|u_{k-1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\tau^2|\rho_{12}||M_5|\left(\left|v_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right)+\frac{1}{2}\left|2-\tau^2\rho_{11}\right||M_6|\left(\left|u_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
&\quad+\tau^2|\gamma_1||c_2||\delta_{11}||M_6|\left(\left|u_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right)+\tau^2|\gamma_1||c_2||\delta_{12}||M_7|\left(\left|v_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2 \\
&\quad+\frac{\beta_2}{2}\tau^2\mu_1\left\|v_{k+1}^h\right\|^2+\beta_2\tau^2\mu_2\left\|v_k^h\right\|^2+\frac{\beta_2}{2}\tau^2\mu_3\left\|v_{k-1}^h\right\|^2 \\
&\leqslant 2\tau^2c_3|g_k|_\infty\left\|v_{k+1}^h\right\|_h^2+\frac{1}{2}\left|\alpha_{21}\tau+\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_2|\left(\left|u_{k+1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\left|\alpha_{21}\tau-\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_3|\left(\left|u_{k-1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\left|\alpha_{22}\tau+\frac{1}{2}\tau^2\rho_{22}\right|\left|v_{k+1}^h\right|_h^2 \\
&\quad+\frac{1}{2}\left|\alpha_{22}\tau-\frac{1}{2}\tau^2\rho_{22}-2\right||\widetilde{M}_4|\left(\left|v_{k-1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)
\end{aligned}$$

$$+\frac{1}{2}\tau^2|\rho_{21}||\widetilde{M}_5|\left(\left|u_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\frac{1}{2}\left|2-\tau^2\rho_{22}\right||\widetilde{M}_6|\left(\left|v_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)$$
$$+\tau^2|\gamma_2||c_4||\delta_{21}||\widetilde{M}_7|\left(\left|u_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\tau^2|\gamma_2||c_4||\delta_{22}||\widetilde{M}_7|\left(\left|v_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right).$$

By the Hölder inequality we get

$$\begin{aligned}
&\left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2 \\
&\quad+\frac{\beta_1}{4}\tau^2\mu_1 2\left\|u_{k+1}^h\right\|_h^2+\frac{\beta_1}{2}\tau^2\mu_2 2\left\|u_k^h\right\|_h^2+\frac{\beta_1}{4}\tau^2\mu_3 2\left\|u_{k-1}^h\right\|_h^2 \\
&\leqslant \frac{\beta_1}{4}\mu_1\left\|u_{k+1}^h\right\|_h^2+\frac{4}{\beta_1\mu_1}c_1^2|f_k|_\infty^2 \\
&\quad+\frac{1}{2}\left|\alpha_{12}\tau+\frac{1}{2}\tau^2\rho_{12}\right||M_2|\left(\left|v_{k+1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\left|\alpha_{12}\tau-\frac{1}{2}\tau^2\rho_{12}\right||M_3|\left(\left|v_{k-1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
&\quad+\left|\alpha_{11}\tau+\frac{1}{2}\tau^2\rho_{11}\right|\left|u_{k+1}^h\right|_h^2+\frac{1}{2}\left|\alpha_{11}\tau-\frac{1}{2}\tau^2\rho_{11}-2\right||M_4|\left(\left|u_{k-1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\tau^2|\rho_{12}||M_5|\left(\left|v_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right)+\frac{1}{2}\left|2-\tau^2\rho_{11}\right||M_6|\left(\left|u_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
&\quad+\tau^2|\gamma_1||c_2||\delta_{11}||M_6|\left(\left|u_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
&\quad+\tau^2|\gamma_1||c_2||\delta_{12}||M_7|\left(\left|v_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right)
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
&\left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2 \\
&\quad+\frac{\beta_2}{4}\tau^2\mu_1 2\left\|v_{k+1}^h\right\|^2+\frac{\beta_2}{2}\tau^2\mu_2 2\left\|v_k^h\right\|^2+\frac{\beta_2}{4}\tau^2\mu_3 2\left\|v_{k-1}^h\right\|^2 \\
&\leqslant \frac{\beta_2}{4}\mu_1\left\|v_{k+1}^h\right\|_h^2+\frac{4}{\beta_2\mu_1}c_1^2|g_k|_\infty^2 \\
&\quad+\frac{1}{2}\left|\alpha_{21}\tau+\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_2|\left(\left|u_{k+1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\left|\alpha_{21}\tau-\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_3|\left(\left|u_{k-1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\left|\alpha_{22}\tau+\frac{1}{2}\tau^2\rho_{22}\right|\left|v_{k+1}^h\right|_h^2 \\
&\quad+\frac{1}{2}\left|\alpha_{22}\tau-\frac{1}{2}\tau^2\rho_{22}-2\right||\widetilde{M}_4|\left(\left|v_{k-1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
&\quad+\frac{1}{2}\tau^2|\rho_{21}||\widetilde{M}_5|\left(\left|u_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\frac{1}{2}\left|2-\tau^2\rho_{22}\right||\widetilde{M}_6|\left(\left|v_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
&\quad+\tau^2|\gamma_2||c_4||\delta_{21}||\widetilde{M}_7|\left(\left|u_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
&\quad+\tau^2|\gamma_2||c4||\delta_{22}||\widetilde{M}_7|\left(\left|v_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right).
\end{aligned}\tag{23}$$

Collecting the like terms of Eqs. (22), (23), respectively, we obtain

$$
\begin{aligned}
&\left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2+\frac{\beta_1}{4}\tau^2\mu_1\left\|u_{k+1}^h\right\|_h^2+\beta_1\tau^2\mu_2\left\|u_k^h\right\|_h^2+\frac{\beta_1}{2}\tau^2\mu_3\left\|u_{k-1}^h\right\|_h^2\\
&\quad\leqslant \frac{4}{\beta_1\mu_1}c_1^2\tau^2|f_k|_\infty^2+\biggl(\tau^2|M_1|+\frac{1}{2}\left|\alpha_{12}\tau+\frac{1}{2}\tau^2\rho_{12}\right||M_2|\\
&\qquad+\frac{1}{2}\left|\alpha_{11}\tau-\frac{1}{2}\tau^2\rho_{11}-2\right||M_4|+\frac{1}{2}\left|\alpha_{12}\tau-\frac{1}{2}\tau^2\rho_{12}\right||M_3|\\
&\qquad+\left|\alpha_{11}\tau+\frac{1}{2}\tau^2\rho_{11}\right|+\frac{1}{2}\tau^2|\rho_{12}||M_5|+\frac{1}{2}\left|2-\tau^2\rho_{11}\right||M_6|\\
&\qquad+\tau^2|\gamma_1||c_2||\delta_{11}||M_7|+\tau^2|\gamma_1||c_2||\delta_{12}||M_7|\biggr)\left|u_{k+1}^h\right|_h^2\\
&\qquad+\left(\frac{1}{2}\left|\alpha_{11}\tau-\frac{1}{2}\tau^2\rho_{11}-2\right||M_4|\right)\left|u_{k-1}^h\right|_h^2\\
&\qquad+\left(\frac{1}{2}\left|2-\tau^2\rho_{11}\right||M_6|+\tau^2|\gamma_1||c_1||\delta_{11}||M_7|\right)\left|u_k^h\right|_h^2\\
&\qquad+\frac{1}{2}\left|\alpha_{12}\tau+\frac{1}{2}\tau^2\rho_{12}\right||M_2|\left|v_{k+1}^h\right|_h^2+\frac{1}{2}\left|\alpha_{12}\tau-\frac{1}{2}\tau^2\rho_{12}\right||M_3|\left|v_{k-1}^h\right|_h^2\\
&\qquad+\left(\frac{1}{2}\tau^2|\rho_{12}||M_5|+\tau^2|\gamma_1||c_2||\delta_{12}||M_7|\right)\left|v_k^h\right|_h^2,
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
&\left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2+\frac{\beta_2}{4}\tau^2\mu_1\left\|v_{k+1}^h\right\|^2+\beta_2\tau^2\mu_2\left\|v_k^h\right\|^2+\frac{\beta_2}{2}\tau^2\mu_3\left\|v_{k-1}^h\right\|^2\\
&\quad\leqslant \tau^2\frac{4}{\beta_2\mu_1}c_3^2|g_k|_\infty^2+\biggl(\tau^2|\widetilde{M}_1|+\frac{1}{2}\left|\alpha_{21}\tau+\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_2|\\
&\qquad+\frac{1}{2}\left|\alpha_{22}\tau-\frac{1}{2}\tau^2\rho_{22}-2\right||\widetilde{M}_4|+\frac{1}{2}\left|\alpha_{21}\tau-\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_3|\\
&\qquad+\left|\alpha_{22}\tau+\frac{1}{2}\tau^2\rho_{22}\right|+\frac{1}{2}\tau^2|\rho_{21}||\widetilde{M}_5|+\frac{1}{2}\left|2-\tau^2\rho_{22}\right||\widetilde{M}_6|\\
&\qquad+\tau^2|\gamma_2||c_4||\delta_{21}||\widetilde{M}_7|+\tau^2|\gamma_2||c_4||\delta_{22}||\widetilde{M}_7|\biggr)\left|v_{k+1}^h\right|_h^2\\
&\qquad+\left(\frac{1}{2}\left|\alpha_{22}\tau-\frac{1}{2}\tau^2\rho_{22}-2\right||\widetilde{M}_4|\right)\left|v_{k-1}^h\right|_h^2\\
&\qquad+\left(\frac{1}{2}\left|2-\tau^2\rho_{22}\right||\widetilde{M}_6|+\tau^2|\gamma_2||c_4||\delta_{22}||\widetilde{M}_7|\right)\left|v_k^h\right|_h^2\\
&\qquad+\frac{1}{2}\left|\alpha_{21}\tau+\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_2|\left|u_{k+1}^h\right|_h^2+\frac{1}{2}\left|\alpha_{21}\tau-\frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_3||u_{k-1}^h|_h^2\\
&\qquad+\left(\frac{1}{2}\tau^2|\rho_{21}||\widetilde{M}_5|+\tau^2|\gamma_2||c_4||\delta_{21}||\widetilde{M}_7|\right)\left|u_k^h\right|_h^2.
\end{aligned}
\tag{25}
$$

Denoting

$$\begin{aligned}
K_1 &= \tau^2|M_1| + \frac{1}{2}\left|\alpha_{12}\tau + \frac{1}{2}\tau^2\rho_{12}\right||M_2| + \frac{1}{2}\left|\alpha_{11}\tau - \frac{1}{2}\tau^2\rho_{11} - 2\right||M_4| \\
&\quad + \frac{1}{2}\left|\alpha_{12}\tau - \frac{1}{2}\tau^2\rho_{12}\right||M_3| + \left|\alpha_{11}\tau + \frac{1}{2}\tau^2\rho_{11}\right| + \frac{1}{2}\tau^2|\rho_{12}||M_5| \\
&\quad + \frac{1}{2}|2 - \tau^2\rho_{11}||M_6| + \tau^2|\gamma_1||c_2||\delta_{11}||M_7| + \tau^2|\gamma_1||c_2||\delta_{12}||M_7|, \\
K_2 &= \frac{1}{2}\left|\alpha_{11}\tau - \frac{1}{2}\tau^2\rho_{11} - 2\right||M_4|, \\
K_3 &= \frac{1}{2}\left|2 - \tau^2\rho_{11}\right||M_6| + \tau^2|\gamma_1||c_1||\delta_{11}||M_7|, \\
K_4 &= \frac{1}{2}\left|\alpha_{12}\tau + \frac{1}{2}\tau^2\rho_{12}\right||M_2|, K_5 = \frac{1}{2}\left|\alpha_{12}\tau - \frac{1}{2}\tau^2\rho_{12}\right||M_3| \\
K_6 &= \frac{1}{2}\tau^2|\rho_{12}||M_5| + \tau^2|\gamma_1||c_1||\delta_{12}||M_7|
\end{aligned}$$

at Eq. (24) and

$$\begin{aligned}
\widetilde{K}_1 &= \tau^2|\widetilde{M}_1| + \frac{1}{2}\left|\alpha_{21}\tau + \frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_2| + \frac{1}{2}\left|\alpha_{22}\tau - \frac{1}{2}\tau^2\rho_{22} - 2\right||\widetilde{M}_4| \\
&\quad + \frac{1}{2}\left|\alpha_{21}\tau - \frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_3| + \left|\alpha_{22}\tau + \frac{1}{2}\tau^2\rho_{22}\right| + \frac{1}{2}\tau^2|\rho_{21}||\widetilde{M}_5|, \\
&\quad + \frac{1}{2}|2 - \tau^2\rho_{22}||\widetilde{M}_6| + \tau^2|\gamma_2||c_4||\delta_{21}||\widetilde{M}_7| + \tau^2|\gamma_2||c_4||\delta_{22}||\widetilde{M}_7|, \\
\widetilde{K}_2 &= \frac{1}{2}\left|\alpha_{22}\tau - \frac{1}{2}\tau^2\rho_{22} - 2\right||\widetilde{M}_4|, \\
\widetilde{K}_3 &= \frac{1}{2}|2 - \tau^2\rho_{22}||\widetilde{M}_6| + \tau^2|\gamma_2||c_1||\delta_{22}||\widetilde{M}_7|, \\
\widetilde{K}_4 &= \frac{1}{2}|\alpha_{21}\tau + \frac{1}{2}\tau^2\rho_{21}||\widetilde{M}_2|, \qquad \widetilde{K}_5 = \frac{1}{2}\left|\alpha_{21}\tau - \frac{1}{2}\tau^2\rho_{21}\right||\widetilde{M}_3|, \\
\widetilde{K}_6 &= \frac{1}{2}\tau^2|\rho_{21}||\widetilde{M}_5| + \tau^2|\gamma_2||c_1||\delta_{21}||\widetilde{M}_7|
\end{aligned}$$

at Eq. (25), we get

$$\begin{aligned}
&\left|u_{k+1}^h\right|_h^2 - \left|u_k^h\right|_h^2 + \left|u_{k+1}^h - u_k^h\right|_h^2 \\
&\quad + \frac{\beta_1}{4}\tau^2\mu_1\left\|u_{k+1}^h\right\|_h^2 + \beta_1\tau^2\mu_2\left\|u_k^h\right\|_h^2 + \frac{\beta_1}{2}\tau^2\mu_3\left\|u_{k-1}^h\right\|_h^2 \\
&\leqslant \tau^2\frac{4}{\beta_1\mu_1}c_1^2|f_k|_\infty^2 + K_1\left|u_{k+1}^h\right|_h^2 + K_2\left|u_{k-1}^h\right|_h^2 + K_3\left|u_k^h\right|_h^2 \\
&\quad + K_4\left|v_{k+1}^h\right|_h^2 + K_5\left|v_{k-1}^h\right|_h^2 + K_6\left|v_k^h\right|_h^2
\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2\\&\quad+\frac{\beta_2}{4}\tau^2\mu_1\left\|v_{k+1}^h\right\|^2+\beta_2\tau^2\mu_2\left\|v_k^h\right\|^2+\frac{\beta_2}{2}\tau^2\mu_3\left\|v_{k-1}^h\right\|^2\\&\quad\leqslant\tau^2\frac{4}{\beta_2\mu_1}c_2^2|g_k|_\infty^2+\widetilde{K}_1\left|v_{k+1}^h\right|_h^2+\widetilde{K}_2\left|v_{k-1}^h\right|_h^2+\widetilde{K}_3\left|v_k^h\right|_h^2\\&\qquad+\widetilde{K}_4\left|u_{k+1}^h\right|_h^2+\widetilde{K}_5\left|u_{k-1}^h\right|_h^2+\widetilde{K}_6\left|u_k^h\right|_h^2.\end{aligned}\tag{27}$$

Taking the sum of (26) and (27) and using the inequalities obtained so far, we get

$$\begin{aligned}&\left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2\\&\quad+\frac{\beta_1}{4}\tau^2\mu_1\left\|u_{k+1}^h\right\|_h^2+\beta_1\tau^2\mu_2\left\|u_k^h\right\|_h^2+\frac{\beta_1}{2}\tau^2\mu_3\left\|u_{k-1}^h\right\|_h^2\\&\quad+\frac{\beta_2}{4}\tau^2\mu_1\left\|v_{k+1}^h\right\|^2+\beta_2\tau^2\mu_2\left\|v_k^h\right\|^2+\frac{\beta_2}{2}\tau^2\mu_3\left\|v_{k-1}^h\right\|^2\\&\leqslant\tau^2\frac{4}{\beta_1\mu_1}c_1^2|f_k|_\infty^2+\tau^2\frac{4}{\beta_1\mu_1}c_2^2|g_k|_\infty^2+(K_1+\widetilde{K}_4)\left|u_{k+1}^h\right|_h^2\\&\quad+(K_4+\widetilde{K}_1)\left|v_{k+1}^h\right|_h^2+(K_2+\widetilde{K}_5)\left|u_{k-1}^h\right|_h^2+(K_5+\widetilde{K}_2)\left|v_{k-1}^h\right|_h^2\\&\quad+(K_3+\widetilde{K}_6)\left|u_k^h\right|_h^2+(K_6+\widetilde{K}_3)\left|v_k^h\right|_h^2.\end{aligned}\tag{28}$$

Adding these inequalities from $k=1,\dots,N-1$, we get

$$\begin{aligned}&\left|u_N^h\right|_h^2+\left|v_N^h\right|_h^2+\sum_{k=1}^{N-1}\left|u_{k+1}^h-u_k^h\right|_h^2+\sum_{k=1}^{N-1}\left|v_{k+1}^h-v_k^h\right|_h^2\\&\quad+\frac{\beta_1}{4}\tau^2\mu_1\sum_{k=1}^{N-1}\left\|u_{k+1}^h\right\|_h^2+\beta_1\tau^2\mu_2\sum_{k=1}^{N-1}\left\|u_k^h\right\|_h^2+\frac{\beta_1}{2}\tau^2\mu_3\sum_{k=1}^{N-1}\left\|u_{k-1}^h\right\|_h^2\\&\quad+\frac{\beta_2}{4}\tau^2\mu_1\sum_{k=1}^{N-1}\left\|v_{k+1}^h\right\|^2+\beta_2\tau^2\mu_2\sum_{k=1}^{N-1}\left\|v_k^h\right\|^2+\frac{\beta_2}{2}\tau^2\mu_3\sum_{k=1}^{N-1}\left\|v_{k-1}^h\right\|^2\\&\leqslant\left|u_0^h\right|_h^2+\left|v_0^h\right|_h^2+\frac{4\tau^2}{\beta_1\mu_1}c_1^2\sum_{k=1}^{N-1}|f_k|_\infty^2+\frac{4\tau^2}{\beta_1\mu_1}c_2^2\sum_{k=1}^{N-1}|g_k|_\infty^2\\&\quad+(K_1+\widetilde{K}_4)\sum_{k=1}^{N-1}\left|u_{k+1}^h\right|_h^2+(K_4+\widetilde{K}_1)\sum_{k=1}^{N-1}\left|v_{k+1}^h\right|_h^2\\&\quad+(K_2+\widetilde{K}_5)\sum_{k=1}^{N-1}\left|u_{k-1}^h\right|_h^2+(K_5+\widetilde{K}_2)\sum_{k=1}^{N-1}\left|v_{k-1}^h\right|_h^2\\&\quad+(K_3+\widetilde{K}_6)\sum_{k=1}^{N-1}\left|u_k^h\right|_h^2+(K_6+\widetilde{K}_3)\sum_{k=1}^{N-1}\left|v_k^h\right|_h^2.\end{aligned}\tag{29}$$

Using estimate (20), we obtain

$$\begin{aligned}&\left|u_N^h\right|_h^2+\left|v_N^h\right|_h^2+\sum_{k=1}^{N-1}\left|u_{k+1}^h-u_k^h\right|_h^2+\sum_{k=1}^{N-1}\left|v_{k+1}^h-v_k^h\right|_h^2\\&\quad+d_1\sum_{k=1}^{N-1}\left\|u_{k+1}^h\right\|_h^2+d_2\sum_{k=1}^{N-1}\left\|u_k^h\right\|_h^2+d_3\sum_{k=1}^{N-1}\left\|u_{k-1}^h\right\|_h^2\\&\quad+d_4\sum_{k=1}^{N-1}\left\|v_{k+1}^h\right\|^2+d_5\sum_{k=1}^{N-1}\left\|v_k^h\right\|^2+d_6\sum_{k=1}^{N-1}\left\|v_{k-1}^h\right\|^2\\&\quad\leqslant|u_0^h|_h^2+\left|v_0^h\right|_h^2+\tau^2d_7\sum_{k=1}^{N-1}|f_k|_\infty^2+\tau^2d_8\sum_{k=1}^{N-1}|g_k|_\infty^2.\end{aligned}$$

Here

$$\begin{aligned}d_1&=\frac{\beta_1}{4}\tau^2\mu_1-c_5(K_1+\widetilde{K}_4), & d_2&=\beta_1\tau^2\mu_2-c_6(K_3+\widetilde{K}_6),\\ d_3&=\frac{\beta_1}{2}\tau^2\mu-c_7(K_2+\widetilde{K}_5), & d_4&=\frac{\beta_2}{4}\tau^2\mu_1-c_7(K_4+\widetilde{K}_1),\\ d_5&=\beta_2\tau^2\mu_2-c_8(K_6+\widetilde{K}_3), & d_6&=\frac{\beta_2}{2}\tau^2\mu_3-c_9(K_5+\widetilde{K}_2),\\ d_7&=\frac{4}{\beta_1\mu_1}c_1^2, & d_8&=\frac{4}{\beta_2\mu_1}c_2^2.\end{aligned}$$

We refer to [24] for the useful inequality

$$\tau^2\sum_{k=1}^{N-1}|f_k|_\infty^2\leqslant\int_0^{\mathrm{T}}\left|f(s)\right|^2\mathrm{d}s. \tag{30}$$

The initial u_0^h is the orthogonal projection of u_0 onto V^h in $L^2(\Omega)$. By this definition we have (see [24])

$$\left|u_0^h\right|\leqslant|u_0|\quad\forall h. \tag{31}$$

Making use of estimates (30) and (31), it follows that the right-hand side of (7) is bounded by

$$C=\left|u_0^h\right|_h^2+\left|v_0^h\right|_h^2+d_7\int_0^{\mathrm{T}}\left|f(s)\right|^2\mathrm{d}s+d_8\int_0^{\mathrm{T}}\left|g(s)\right|^2\mathrm{d}s.$$

This proves (12), (13), and (14). Next, adding inequalities (28) for $k=1,\dots r-1$ and dropping some positive terms, we get

$$\left|u_r^h\right|_h^2+\left|v_r^h\right|_h^2\leqslant\left|u_0^h\right|_h^2+\left|v_0^h\right|_h^2+\tau^2d_7\sum_{k=1}^{r-1}\left|f_k\right|_\infty^2+\tau^2d_8\sum_{k=1}^{r-1}\left|g_k\right|_\infty^2\leqslant C,$$

and this proves (11). Hence, Theorem 1 is proved. □

The next theorem states that the set of mesh functions $\{u_k^h\}$ and $\{v_k^h\}$ are compact in $L_{2h}(\Omega)$ topology.

Theorem 2. *Under the hypotheses of Theorem* 1*, there exist subsequences*

$$\{u_{k_m}^h\} \subset \{u_k^h\} \quad \textit{and} \quad \{v_{k_m}^h\} \subset \{v_k^h\},$$

which converge in V_h to bounded measurable functions u^h and v^h, respectively. Moreover, the limit functions u^h and v^h are unique weak solutions satisfying (7)*.*

Proof. Estimates (11)–(14) and discrete Gronwall lemma (see [8, 26, 27]) imply that

$$\{u_k^h\} \text{ and } \{v_k^h\} \text{ are bounded in } L^\infty(0,T;V).$$

Then by the Rellich theorem (see [7]) there exists a subsequence $\mathbf{w}_{k_m} = [u_{k_m}, v_{k_m}]^{\mathrm{T}}$ of $\mathbf{w}_k = [u_k, v_k]^{\mathrm{T}}$ and $\tilde{\mathbf{w}}_k \in L^\infty(0,T;\mathcal{V})$ such that

$$\tilde{\mathbf{w}}_k \in L^\infty(0,T;\mathcal{V}) \subset L^2(0,T;\mathcal{V})$$

and

$$\mathbf{w}_{k_m} \to \tilde{\mathbf{w}}_k \quad \text{weak-}{*} \text{ in } L^\infty(0,T;\mathcal{V}) \text{ and weakly in } L^2(0,T;\mathcal{V}).$$

By the Aubin theorem (see [5]) the above convergence results imply

$$\mathbf{w}_{k_m} \to \tilde{\mathbf{w}}_k \quad \text{strongly in } L^2(0,T;\mathcal{H}), \tag{32}$$

and by (32)

$$\sin\delta\mathbf{w}_{k_m} \to \sin\delta\tilde{\mathbf{w}}_k \quad \text{strongly in } L^2(0,T;\mathcal{H}),$$

which proves the existence of $\tilde{\mathbf{w}}_k$ a.e. in $\mathcal{H}$ and $\tilde{\mathbf{w}}_0 = \mathbf{w}_0$. Uniqueness follows from the results of Theorem 1 and convergence of difference scheme (7). Hence, Theorem 2 is proved. □

4 Numerical analysis

In this section, the theoretical statements are verified by numerical implementations. A unified numerical method based on the fixed point iterations and finite-difference schemes are presented. We aid the nonlinear part by the fixed point iterations. We introduce a composite numerical method to obtain accurate results for the solution of a system of PDEs with initial and boundary conditions for one-dimensional coupled sine-Gordon equations. We generate an exact solution at random

$$w(t,x) = \{u(t,x), v(t,x)\},$$

where

$$u(t,x) = \sin\pi x \cos|4t|, \qquad v(t,x) = \sin\pi x \cos|9t|,$$

and we formulate a boundary value problem that leads to this solution. Let us consider the following boundary value problem for the system of sine-Gordon equations:

$$\begin{aligned}
&u_{tt}+u_t-u_{xx}+u\\
&\quad=\sin(u-v)+\bigl(\bigl(\pi^2-15\bigr)\cos|4t|-4\sin|4t|\bigr)\sin\pi x\\
&\qquad-\sin\bigl(\cos|4t|\sin\pi x-\cos|9t|\sin\pi x\bigr),\quad 0<t<1,\ 0<x<1,\\
&v_{tt}+v_t-v_{xx}+v\\
&\quad=-\sin(u-v)+\bigl(\bigl(\pi^2-80\bigr)\cos|9t|-9\sin|9t|\bigr)\sin\pi x\\
&\qquad+\sin\bigl(\cos|4t|\sin\pi x-\cos|9t|\sin\pi x\bigr),\quad 0<t<1,\ 0<x<1,\\
&u(0,x)=\sin\pi x,\quad u_t(0,x)=0,\quad 0\leqslant x\leqslant 1,\\
&v(0,x)=\sin\pi x,\quad v_t(0,x)=0,\quad 0\leqslant x\leqslant 1,\\
&u(t,0)=u(t,1)=0,\quad 0\leqslant t\leqslant 1,\\
&v(t,0)=v(t,1)=0,\quad 0\leqslant t\leqslant 1,
\end{aligned}\tag{33}$$

and the corresponding difference problem

$$\begin{aligned}
&\frac{{}_mu_n^{k+1}-2{}_mu_n^k+{}_mu_n^{k-1}}{\tau^2}+\frac{{}_mu_n^{k+1}-{}_mu_n^{k-1}}{2\tau}\\
&\quad-\frac{{}_mu_{n+1}^k-2{}_mu_n^k+{}_mu_{n-1}^k}{2h^2}+\frac{1}{2}{}_mu_n^k-\frac{{}_mu_{n+1}^{k+1}-2{}_mu_n^{k+1}+{}_mu_{n-1}^{k+1}}{4h^2}\\
&\quad-\frac{{}_mu_{n+1}^{k-1}-2{}_mu_n^{k-1}+{}_mu_{n-1}^{k-1}}{4h^2}+\frac{1}{4}{}_mu_n^{k+1}+\frac{1}{4}{}_mu_n^{k-1}\\
&\quad=\bigl(\bigl(\pi^2-15\bigr)\cos|4t_k|-4\sin|4t_k|\bigr)\sin\bigl(\pi x_n\bigr)\\
&\qquad-\sin\bigl(\cos|4t_k|\sin\pi x_n-\cos|9t_k|\sin\pi x_n\bigr)+\sin\bigl({}_mu_n^k-{}_mv_n^k\bigr),\\
&\quad\frac{{}_mv_n^{k+1}-2{}_mv_n^k+{}_mv_n^{k-1}}{\tau^2}+\frac{{}_mv_n^{k+1}-{}_mv_n^{k-1}}{2\tau}-\frac{{}_mv_{n+1}^k-2{}_mv_n^k+{}_mv_{n-1}^k}{2h^2}\\
&\quad+\frac{1}{2}{}_mv_n^k-\frac{{}_mv_{n+1}^{k+1}-2{}_mv_n^{k+1}+{}_mv_{n-1}^{k+1}}{4h^2}-\frac{{}_mv_{n+1}^{k-1}-2{}_mv_n^{k-1}+{}_mv_{n-1}^{k-1}}{4h^2}\\
&\quad+\frac{1}{4}{}_mv_n^{k+1}+\frac{1}{4}{}_mv_n^{k-1}\\
&\quad=\bigl(\bigl(\pi^2-80\bigr)\cos|9t_k|-9\sin|9t_k|\bigr)\sin(\pi x_n)\\
&\qquad-\sin\bigl(\cos|3t_k|\sin\pi x_n-\cos|2t_k|\sin\pi x_n\bigr)+\sin\bigl({}_mu_n^k-{}_mv_n^k\bigr),\\
&t_k=k\tau,\quad 1\leqslant k\leqslant N-1,\ N\tau=1,\\
&x_n=nh,\quad 1\leqslant n\leqslant M-1,\ Mh=1,\\
&{}_mu_n^0=\sin(\pi x_n),\quad {}_mv_n^0=\sin(\pi x_n),\quad 0\leqslant n\leqslant M,\\
&(2\tau)^{-1}\bigl(-3{}_mu_n^0(x_n)+4{}_mu_n^1(x_n)-{}_mu_n^2(x_n)\bigr)=0,\quad 1\leqslant n\leqslant M-1,\\
&(2\tau)^{-1}\bigl(-3{}_mv_n^0(x_n)+4{}_mv_n^1(x_n)-{}_mv_n^2(x_n)\bigr)=0,\quad 1\leqslant n\leqslant M-1,\\
&{}_mu_0^k={}_mu_M^k=0,\quad {}_mv_0^k={}_mv_M^k=0,\quad 0\leqslant k\leqslant N.
\end{aligned}\tag{34}$$

System (33) is proposed as the model of wave propagation on an infinite chain of elastically bound atoms lying over a fixed lower chain of similar atoms. The higher-order derivatives describe the elastic interaction energy between neighboring atoms and their kinetic energy, respectively. The nonlinear terms containing the sine trigonometric function stand for the potential energy due to the fixed lower chain. The rest of the terms are damping terms and source functions.

Difference scheme (34) corresponds to the approximate solution of problem (33). The modified Gauss elimination method is applied in solving system (34). The set of a family of grid points

$$\begin{aligned}\Omega_h &= [0,1]_\tau \times [0,1]_h \\ &= \bigl\{(t_k, x_n)\colon t_k = k\tau,\ 0 \leqslant k \leqslant N,\ N\tau = 1, \\ &\qquad x_n = nh,\ 0 \leqslant n \leqslant M,\ Mh = 1\bigr\}\end{aligned}$$

is considered. We present the numerical results of errors, the number of iterations, and the related CPU times in the following tables at different N and M values. The Matlab implementations are carried out by MATLAB R2023a software package, by a PC System of 64-bit, Core i5 CPU, 1.80 GHz, 8 GB of RAM. We use the following formula:

$$\max_{\substack{1\leqslant k\leqslant N-1\\ 1\leqslant n\leqslant M-1}} \bigl|w(t_k, x_n) - w_n^k\bigr|$$

to compute errors. The algorithm is performed for $m = 1, 2, \dots, p$, where p depends on a given error tolerance ε such that

$$\bigl|{}_p u_n - {}_{p-1}u_n\bigr| < \varepsilon \quad \text{and} \quad \bigl|{}_p v_n - {}_{p-1}v_n\bigr| < \varepsilon.$$

Here m index represents the number of fixed point iterations. The exact solution is denoted by $w(t_k, x_n) = [u(t_k, x_n), v(t_k, x_n)]^{\mathrm{T}}$, and the numerical solution is denoted by $w_n^k = [u_n^k, v_n^k]^{\mathrm{T}}$ for the approximate solution of problem (33) at (t_k, x_n).

The numerical results are presented in the tables below. Table 1 gives the errors for the solution of (34) with a terminating criteria $\varepsilon = 10^{-15}$ and initials in the matrix form (35). Table 2 presents the errors for the solution of (34) with a terminating criteria $\varepsilon = 10^{-20}$ and initials (36). Table 3 gives the errors for the solution of (34) with $\varepsilon = 10^{-20}$. In the iteration, the initials are taken as the identity matrices of the form (37).

$$_0u_n^k = \mathrm{rand}(N+1,\ 1), \qquad {}_0v_n^k = 0(N+1,\ 1), \tag{35}$$

$$_0u_n^k = \mathrm{rand}(N+1,\ 1), \qquad {}_0v_n^k = 0(N+1,\ 1), \tag{36}$$

$$_0u_n^k = I(N+1,\ M+1), \qquad {}_0v_n^k = I(N+1,\ M+1). \tag{37}$$

Table 1. Numerical results of problem (34) with $\varepsilon = 10^{-15}$ and initials (35).

$N = M$	Error of w	m	CPU times
20	0.0508	9	0.268
40	0.0116	10	0.633
80	0.0056	10	2.711
160	0.0028	10	16.636

Table 2. Numerical results of problem (34) with $\varepsilon = 10^{-20}$ and initials (36).

$N = M$	Error of w	m	CPU times
20	0.0508	10	0.292
40	0.0116	11	0.713
80	0.0056	12	3.260
160	0.0028	12	19.916

Table 3. Numerical results of problem (34) with $\varepsilon = 10^{-20}$ and initials (37).

$N = M$	Error of w	m	CPU times
20	0.0508	10	0.292
40	0.0116	11	0.923
80	0.0056	12	3.191
160	0.0028	12	20.150

Difference scheme (7) is used together with fixed point iteration to obtain numerical solutions. By the numerical results it is observed that the difference scheme (34) converges to a solution for different $N = M$ values, initial vectors ${}_0u_n^k$, ${}_0v_n^k$, termination criteria ε at different iteration numbers m. When reaching the maximum difference value at specific grid points of two successive results gets less than ε, the iterative process stops. We noticed that for several different ε and initial values ${}_0u_n^k$, ${}_0v_n^k$ presented at (35)–(37), the errors decrease. On the other hand, the number of iterations and the CPU times increase for certain $N = M$ values. The results of numerical experiments support the theoretical results and verify the efficiency of the numerical method.

5 Conclusion

This study presents the unique solvability of the system of finite-difference schemes for coupled sine-Gordon equations. The existence and uniqueness of the solutions are proved by the variational formulation. A useful unified numerical method that combines the second-order accuracy unconditionally stable difference scheme with the fixed point iteration is presented. Numerical implementations verify the theoretical results, which supports the efficiency of the unified method. In future studies, unconditionally stable difference schemes corresponding to multidimensional nonlinear systems of PDEs, which are derived from biological and phase-field models, will be obtained. The weak and global solutions of these systems of difference schemes will be studied. Perturbation problems corresponding to these problems will also be studied.

Conflicts of interest. The author declares no conflicts of interest.

Acknowledgment. I want to thank Prof. Dr. Allaberen Ashyralyev for all his support so far.

References

1. M. Ashyraliyev, On Gronwall's type integral inequalities with singular kernels, *Filomat*, **31**(4):1041–1049, 2017, `https://doi.org/10.2298/FIL1704041A`.
2. A. Ashyralyev, P.E. Sobolevskii, A note on the difference schemes for hyperbolic equations, *Abstr. Appl. Anal.*, **6**(2):63–70, 2001, `https://doi.org/10.1155/S1085337501000501`.
3. A. Ashyralyev, P.E. Sobolevskii, *New Difference Schemes for Partial Differential Equations, Operator Theory: Advances and Applications*, Birkhäuser, Berlin, 2004.
4. A. Ashyralyev, O. Yildirim, On multipoint nonlocal boundary value problems for hyperbolic differential and difference equations, *Taiwanese J. Math.*, **14**(1):165–194, 2010, `https://doi.org/10.11650/twjm/1500405734`.
5. J.P. Aubin, Un théoreme de compacité, *C. R. Math. Acad. Sci. Paris*, **256**:5042–5044, 1963.
6. R. Baronas, J. Kulys, L. Petkevičius, Modelling the enzyme catalysed substrate conversion in a microbioreactor acting in continuous flow mode, *Nonlinear Anal. Model. Control*, **23**(3): 437–456, 2018, `https://doi.org/10.15388/NA.2018.3.9`.
7. R. Dautray, J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 2, Functional and Variational Methods*, Springer, Berlin, 1988.
8. E. Emmrich, *Discrete Versions of Gronwall's Lemma and Their application to the Numerical Analysis of Parabolic Problems*, Technische Universität Berlin, 1999, `https://doi.org/10.14279/depositonce-14714`.
9. L.C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, CBMS Reg. Conf. Ser. Math., Vol. 74, AMS, Providence, RI, 1990.
10. L.C. Evans, *Partial Differential Equations,* 2nd ed., Grad. Stud. Math., Vol. 19, AMS, Providence, RI, 1998, `https://doi.org/10.1090/gsm/019`.
11. E. Gaidamauskaitė, R. Baronas, A comparison of finite difference schemes for computational modelling of biosensors, *Nonlinear Anal. Model. Control*, **12**(3):359–369, 2007, `https://doi.org/10.15388/NA.2007.12.3.14697`.
12. H.W. Hethcote, The mathematics of infectious diseases, *SIAM Rev.*, **42**(4):599–653, 2000, `https://doi.org/10.1137/S0036144500371907`.
13. A. Hocquet, M. Hofmanova, An energy method for rough partial differential equations, *J. Differ. Equations*, **265**(4):1407–1466, 2018, `https://doi.org/10.1016/j.jde.2018.04.006`.
14. C. Huang, B. Liu, Traveling wave fronts for a diffusive Nicholson's blowflies equation accompanying mature delay and feedback delay, *Appl. Math. Lett.*, **134**:108321, 2022, `https://doi.org/10.1016/j.aml.2022.108321`.
15. L. Huang, Z. Xu, Traveling wave fronts of a diffusive Nicholson's blowflies equation with two delays, *Appl. Math. Lett.*, **139**:108545, 2023, `https://doi.org/10.1016/j.aml.2022.108545`.
16. T. Kato, *Perturbation Theory for Linear Operators*, Class. Math., Vol. 132, Springer, Berlin, 1995, `https://doi.org/10.1007/978-3-642-66282-9`.
17. K.R. Khusnutdinova, D.E. Pelinovsky, On the exchange of energy in coupled Klein–Gordon equations, *Wave Motion*, **38**(1):1–10, 2003, `https://doi.org/10.1016/S0165-2125(03)00022-2`.

18. V. Kozlov, S. Radosavljevic, V. Tkachev, U. Wennergren, Global stability of an age-structured population model on several temporally variable patches, *Math. Biol.*, **83**(68), 2021, https://doi.org/10.1007/s00285-021-01701-3.
19. B. Liu, Finite-time stability of CNNs with neutral proportional delays and time-varying leakage delays, *Math. Methods Appl. Sci.*, **40**(1):167–174, 2017, https://doi.org/10.1002/mma.3976.
20. S. Nakagiri, J. Ha, Coupled sine-Gordon equations as nonlinear second order evolution equations, *Taiwanese J. Math.*, **5**(2):297–315, 2001, https://doi.org/10.11650/twjm/1500407338.
21. D. Pham, R. Temam, Weak solutions of the Shigesaka–Kawasaki–Teramoto equations and their attractors, *Nonlinear Anal.*, **159**(2):339–364, 2017, https://doi.org/10.1016/j.na.2017.01.017.
22. F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar, New York, 1955.
23. S. Roman, A. Štikonas, Green's function for discrete second-order problems with nonlocal boundary conditions, *Bound. Value Probl.*, **2011**:767024, 2011, https://doi.org/10.1155/2011/767024.
24. R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis*, Stud. Math. Appl., Vol. 2, North-Holland, Amsterdam, 1977, https://doi.org/10.1016/S0168-2024(09)X7004-9.
25. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Appl. Math. Sci., Vol. 68, Springer, New York, 1997, https://doi.org/10.1007/978-1-4612-0645-3.
26. F. Tone, On the long-time stability of the implicit Euler scheme for the 2D space-periodic Navier-Stokes equations, *Asymptotic Anal.*, **51**(3–4):231–245, 2007, https://api.semanticscholar.org/CorpusID:30080067.
27. F. Tone, D. Wirosoetisno, On the long-time stability of the implicit euler scheme for the two-dimensional navier–stokes equations, *SIAM J. Numer. Anal.*, **44**(1):29–40, 2006, https://doi.org/10.1137/040618527.
28. L.V. Yakushevich, *Nonlinear Physics of DNA*, Wiley, Weinheim, 2004, https://doi.org/10.1002/3527603700.
29. O. Yildirim, M. Uzun, Weak solvability of the unconditionally stable difference scheme for the coupled sine-Gordon system, *Nonlinear Anal. Model. Control*, **25**(6):997–1014, 2020, https://doi.org/10.15388/namc.2020.25.20558.
30. H.M. Yin, J. Zou, Asymptotic analysis for a nonlinear reaction diffusion system modeling an infectious disease, *Nonlinear Anal., Real World Appl.*, **75**:103984, 2024, https://doi.org/10.1016/j.nonrwa.2023.103984.
31. K. Yosida, *Operational Calculus: A Theory of Hyperfunctions*, Appl. Math. Sci., Vol. 55, Springer, New York, 1984.
32. A. Žilinskas, R. Baronas, L. Litvinas, L. Petkevičius, Multi-objective optimization and decision visualization of batch stirred tank reactor based on spherical catalyst particles, *Nonlinear Anal. Model. Control*, **24**(6):1019–1033, 2019, https://doi.org/10.15388/NA.2019.6.10.