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Finite-time projective synchronization of fractional-order delayed quaternion-valued fuzzy memristive neural networks*

Yan He[a], **Weiwei Zhang**[a,b,c], **Hai Zhang**[a], **Jinde Cao**[d,e], **Fawaz E Alsaadi**[f]

[a]School of Mathematics and Physics, Anqing Normal University,
Anqing 246133, China
hyys110103@163.com; zhanghai0121@163.com

[b]Department of Mathematics,
Nanjing University of Aeronautics and Astronautics,
Nanjing 210093, China
wwzhahu@aliyun.com

[c]International Joint Research Center of Simulation and Control
for Population Ecology of Yangtze River in Anhui,
Anqing Normal University, Anqing 246011, China

[d]School of Mathematics, Southeast University,
Nanjing 210096, China
jdcao@seu.edu.cn

[e]Yonsei Frontier Lab, Yonsei University,
Seoul 03722, South Korea

[f]Department of Information Technology,
Faculty of Computing and Information Technology,
King Abdulaziz University, Jeddah, Saudi Arabia
fesalsaadi@kau.edu.sa

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Abstract. In this paper, the finite-time projective synchronization (FTPS) problem of fractional-order quaternion-valued fuzzy memristor neural networks (FOQVFMNNs) is studied. Through establishing a feedback controller with signed functions and an adaptive controller, sufficient conditions for FTPS for FOQVFMNNs are obtained. Furthermore, the synchronization establishment time is calculated. Finally, the practicability of the conclusions is verified by numerical simulations.

Keywords: quaternion-valued, adaptive control, fractional-order fuzzy memristive neural networks, finite-time projective synchronization.

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1 Introduction

The memristor was first proposed by Chua in 1971 [3] and was considered to be a component of the relationship between circuits. Although the resistance of this component changes with the amount of current flowing through it, when the current stops, its resistance remains unchanged, with the effect of memorizing charges. At that time, research on memristors did not attract much attention until 2008 [18], when researchers from the HP team first made nano memristors, which had the characteristics of low power consumption and high integration compared to traditional memories. In addition, scholars have found that using memristors to simulate synapses in neural networks (NNs) will significantly reduce the hardware implementation cost of NNs. Therefore, it is necessary to construct NNs with memristors. Based on the application prospects of memristors, more and more scholars have begun to study this device. For example, Yang et al. have studied a class of memristors with novel BST nanostructures in [26], and He et al. have explored chaos based on memristors in [7].

In the past few decades, there have been research achievements in fractional calculus in the fields of optics [12], mechanics [17], signal processing [1], NNs [19], and so on. Compared with integer-order calculus, it is more unique and irreplaceable. Firstly, fractional order can increase the degree of freedom of the system. Secondly, due to the two special properties of fractional-order inheritance and memory, fractional calculus is more accurately to calculate some mathematical models. Considering the above advantages, more and more scholars are now starting to study fractional-order NNs (FONNs), and have achieved significant results in [11, 15].

As far as the current research achievements are concerned, the research achievements on memristive FONNs (FOMNNs) are basically in the real or complex field. Different from them, a quaternion [20] includes one real part and three imaginary parts, and its imaginary units have no commutativity. Therefore, the exploration of quaternion-valued FOMNNs (FOQVMNNs) is more complex, there are very few results related to FOQVMNNs. In [21], Wang et al. discussed the FOQVMNNs about M-L stability analysis, what's more, in [2], Chen et al. went deeply into the FOQVMNNs on global exponential stability.

Fuzzy NNs (FNNs) is a system composed of fuzzy neurons, which is a combination of NNs and fuzzy theory. Therefore, it has the learning ability of NNs and the advantages of interpretability and uncertainty brought by fuzzy systems. FNNs can be seen as systems controlled by intelligent algorithms and trained through fuzzy systems. Therefore, the system formed by the combination of fuzzy theory and NNs is more efficient and superior. Furthermore, it is inevitable to encounter issues such as ambiguity and uncertainty in the process of establishing a model, and adding fuzzy terms to the model is the most advantageous solution. In recent years, many scholars have introduced $\vee$ and $\wedge$ into NNs to obtain FNNs, and achieved many excellent results [14, 24, 25]. For example, a memristive FNNs (FMNNs) system without time delay was constructed in [24], and a general approach was used to explore the synchronization problem of the system. In [14], the global asymptotic synchronization problem of fractional-order FNNs for second-order terms was explored without performing first-order and second-order transformations. What's more, the synchronization problem of complex valued FNNs was discussed in [25] and was not extended to the quaternion field.

Synchronization is considered to be an important phenomenon in NNs, which can be used to study the dynamic behavior in NNs. Therefore, it has received extensive attention from scholars in recent years. Generally speaking, synchronization can be divided into lag synchronization [28], projective synchronization [22], finite-time synchronization [33], and so on. Compared to general synchronization, FTPS achieves high convergence and good robustness in finite time (FT). In recent years, there have been relatively few achievements in researching the problem for FOQVFMNNs of FTPS, which requires us to further explore.

Based on the above research results, this paper considers the problems of FOQVMNN in FTPS and calculates the settling time of FTPS. The primary contributions are listed below:

(i) Passing through the differential inclusion technique and the measurable selection theory, the system in this paper is transformed into a system with uncertain parameters. What's more, one can make use of a unified method to study the FTPS problem in the quaternion field.

(ii) Unlike [21], this paper studies the problem of FTPS. The system can achieve high convergence and good robustness in a relatively short time.

(iii) In this paper, a feedback controller (3) with a symbolic function is considered, which can make the solution of a differential equation converge to zero or a neighborhood near zero in finite time. In addition, due to its fast convergence characteristics, it brings better robustness. What's more, the adaptive controller selected in (4) makes the results more universal and practical.

Notations. Quaternion algebra is a simple supercomplex number. $\mathbb{R}$ is the real domain, $\mathcal{Q}$ is the quaternion field. For arbitrary quaternion $Z \in \mathcal{Q}$, it can be defined by $Z = Z^R + \mathrm{i}Z^I + \mathrm{j}Z^J + \mathrm{k}Z^K$. Here i, j, and k are imaginary units that do not meet with commutativity.

2 Preliminaries

Definition 1. (See [13].) The Riemann–Liouville fractional-order integral of the function $u(\cdot)$ is represented as

$$ {}_aI_t^\epsilon u(t) = \frac{1}{\Gamma(\epsilon)} \int_a^t (t-\xi)^{\epsilon-1} u(\xi)\,\mathrm{d}\xi, \quad t > 0, $$

where $t \geqslant a$, $\epsilon > 0$, and $\Gamma(\cdot)$ is the gamma function defined as $\Gamma(\epsilon) = \int_0^\infty \mathrm{e}^{-t} t^{\epsilon-1}\,\mathrm{d}t$.

Definition 2. (See [13].) The Caputo fractional derivative of the function $u(\cdot) \in \mathcal{Q}$ is defined by

$$ {}_a^CD_t^\epsilon u(t) = \frac{1}{\Gamma(n-\epsilon)} \int_a^t (t-\xi)^{n-\epsilon-1} u'(\xi)\,\mathrm{d}\xi, $$

where $t \geqslant a$, $n = [\epsilon] + 1$, $n - 1 < \epsilon \leqslant n$.

Specifically, when $0 < \epsilon < 1$,

$$ {}_{a}^{C}D_{t}^{\epsilon}u(t) = \frac{1}{\Gamma(1-\epsilon)} \int_{a}^{t} (t-\xi)^{-\epsilon} u'(\xi)\,\mathrm{d}\xi. $$

Definition 3. (See [10].) The FONN drive system $x_\iota(t)$ and response system $\breve{x}_\iota(t)$ are FTPS for any initial value $x_\iota(t_0) = \phi_{\iota 0}$ and $\breve{x}_\iota(t_0) = \psi_{\iota 0}$ if there is a nonzero constant λ such that

$$ \lim_{t\to\hat{T}} \left|\breve{x}_\iota(t) - \lambda x_\iota(t)\right| = 0, \qquad \left|\breve{x}_\iota(t) - \lambda x_\iota(t)\right| = 0, \quad t \geqslant \hat{T}, $$

where $\lambda \in \mathbb{R} \setminus \{0\}$ is called the projection coefficient, $\hat{T}$ is called the settling time.

Definition 4. (See [6].) Let $h(\cdot)$ be a bounded set mapping of nonlinear function. For the following system

$$ {}_{t_0}^{C}D_{t}^{\epsilon}\phi(t) = h(t, \phi_t), \quad t > 0, \qquad \phi(t) = \hbar(t), \quad t \in [-\alpha, 0], $$

(i) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\hbar\| < \delta$, then the trivial solution of the above system is stable;
(ii) if the trivial solution of the system is stable and $\lim_{t\to\infty} \phi(t, \hbar) = 0$, then the trivial solution of the above system is asymptotically stable.

Here $h : R^+ \times C([-\alpha, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ and $h(t, 0) = 0$, $\hbar$ represents a set of continuous functions, and $\|\hbar\| = \max_{\eta\in[-\alpha,0]} |\hbar(\eta)|$.

Lemma 1. (*See* [29].) *As $\varrho(t)$ is a continuous differentiable function, one has*

$$ {}_{t_0}^{C}D_{t}^{\epsilon}\left|\varrho(t)\right| \leqslant \sum_{\iota=1}^{\varpi} \operatorname{sign}\bigl(\varrho(t)\bigr) {}_{t_0}^{C}D_{t}^{\epsilon}\varrho(t), \quad 0 < \epsilon < 1. $$

Remark 1. In Lemma 1, the function space is too strong. Generally, the function space can be an absolutely continuous space. Then one obtain a new lemma as below.

Lemma 2. *As $\varrho(t)$ is an absolutely continuous function, the following inequality holds almost everywhere*:

$$ {}_{t_0}^{C}D_{t}^{\epsilon}\left|\varrho(t)\right| \leqslant \sum_{\iota=1}^{\varpi} \operatorname{sign}\bigl(\varrho(t)\bigr) {}_{t_0}^{C}D_{t}^{\epsilon}\varrho(t), \quad 0 < \epsilon < 1. $$

Proof. Since $\varrho(t)$ is an absolutely continuous function, then $|\varrho(t)|$ is also an absolutely continuous function. According to the property of the absolutely continuous function, one has that $|\varrho(t)|$ is differentiable on $[0, +\infty)$ except for the set $\Omega = \{t | \varrho(0) = 0,\ \varrho'(t) \neq 0\}$ in which the Ω measure is 0.

Furthermore, the process for proving that the above inequality holds true almost everywhere on $[0, +\infty)$ is the same as that in [32], which is omitted here. □

Lemma 3. (*See* [23].) *Let* $h, y \in \mathcal{Q}$, σ *is any positive parameter so that*

$$h\bar{y} + \bar{h}y \leqslant \sigma h\bar{h} + \frac{1}{\sigma} y\bar{y}.$$

Lemma 4. (*See* [34].) *Suppose* $h \in \mathcal{Q}$, *then*

$$h + \bar{h} = 2\,\mathrm{Re}(h) \leqslant 2|h|.$$

Proof. Take into account that $h = h^R + \mathrm{i}h^I + \mathrm{j}h^J + \mathrm{k}h^K = (h^R + \mathrm{i}h^I) + (h^J + \mathrm{i}h^K)\mathrm{j}$ is equivalent to $h = x + \mathrm{i}y$. Therefore, based on Lemma 7 in [34], it can be concluded that this conclusion is right. □

Lemma 5. (*See* [10].) *Suppose* $c(h) \in \mathcal{Q}$ *is a differentiable function, then*

$${}_{h_0}^{C}D_h^{\epsilon}\bigl(c(h)\overline{c(h)}\bigr) \leqslant c(h)\,{}_{h_0}^{C}D_h^{\epsilon}\overline{c(h)} + \bigl({}_{h_0}^{C}D_t^{\epsilon}c(h)\bigr)\overline{c(h)}$$

holds, where $0 < \epsilon < 1$.

Lemma 6. (*See* [8].) *Let* $g(t)$ *be the function that is continuous, at the same time, is also nonnegative such that*

$${}_{t_0}^{C}D_t^{\epsilon}g(t) \leqslant -\Omega g(t) + \Psi g(t - l) - \Theta g^{\varrho}(t),$$

where $0 < \epsilon < 1$, $0 < \varrho \leqslant 1$, $\Omega > \Psi > 0$, $\Theta > 0$.*Then the settling time can be obtained as below*:

(i) *if* $\varrho = 0$, $g(t) = 0$ *for all* $t > t_1$, *where*

$$t_1 = t_0 + \left[\frac{\Gamma(1+\epsilon)}{\Omega - \Psi} \ln \frac{(\Omega - \Psi)\Phi + \Theta}{\Theta}\right]^{1/\epsilon};$$

(ii) *if* $0 < \varrho < \epsilon < 1$, $g(t) = 0$ *for all* $t > t_2$, *where*

$$t_2 = t_0 + \left[\frac{\Gamma(1 + \frac{1}{1-\varrho})\Gamma(2-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1 + \frac{1}{1-\varrho} - \epsilon)(\Omega - \Psi)} \ln \frac{(\Omega - \Psi)\Phi^{1-\varrho} + \Theta}{\Theta}\right]^{1/\epsilon}$$

with $\Phi = \sup_{t_0 - l \leqslant s \leqslant t_0} g(s)$.

Assumption 1. If the activation function $f_\varsigma(\cdot)$ satisfies the Lipschitz condition on $\mathcal{Q}$, then for any $\varsigma \in \mathcal{N}$, there exists L_ς, which makes the following inequality hold:

$$\bigl|f_\varsigma(h_1) - f_\varsigma(h_2)\bigr| \leqslant L_\varsigma |h_1 - h_2|,$$

where $h_1 \neq h_2$ and $h_1, h_2 \in \mathcal{Q}$.

Assumption 2. Let $f_\varsigma(\cdot)$ be a bounded function, then for any ς, there exists M_ς such that

$$\bigl|f_\varsigma(y)\bigr| \leqslant M_\varsigma.$$

Assumption 3. Let x_ς, $\check{x}_\varsigma$ be two functions of the system so that

$$\left|\bigvee_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma(\check{x}_\varsigma)-\bigvee_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma(x_\varsigma)\right|\leqslant\sum_{\varsigma=1}^{\varpi}L_\varsigma|\alpha_{\iota\varsigma}||\check{x}_\varsigma-x_\varsigma|,$$
$$\left|\bigwedge_{\varsigma=1}^{n}\beta_{\iota\varsigma}f_\varsigma(\check{x}_\varsigma)-\bigwedge_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma(x_\varsigma)\right|\leqslant\sum_{\varsigma=1}^{\varpi}L_\varsigma|\beta_{\iota\varsigma}||\check{x}_\varsigma-x_\varsigma|.$$

3 Model description

In this section, we consider the master system of FOQVFMNNs with time delays as follows:

$$\begin{aligned}{}_{t_0}^{C}D_t^{\epsilon}x_\iota(t)={}&-d_\iota x_\iota(t)+\sum_{\varsigma=1}^{\varpi}a_{\iota\varsigma}\big(x_\varsigma(t)\big)f_\varsigma\big(x_\varsigma(t)\big)+\sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}\big(x_\varsigma(t)\big)f_\varsigma\big(x_\varsigma(t-l)\big)\\&+\sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}V_\varsigma+\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big)+\bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma+\bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma\\&+\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big)+I(t),\end{aligned}\tag{1}$$

where, $0<\epsilon<1$, $x_\iota(t)$ indicates the variable of state about the ιth neuron, $f_\varsigma(\cdot)$ stands for the active function, $a_{\iota\varsigma}(x_\iota(t))$ and $b_{\iota\varsigma}(x_\iota(t))$ mean the connection weights between memristors, l denotes the delay, $b_{\iota\varsigma}$, $T_{\iota\varsigma}$, $S_{\iota\varsigma}$ express the connection weights about fuzzy feedforward templates, $\bigwedge$ and $\bigvee$ mean fuzzy AND and OR, $\beta_{\iota\varsigma}$, $\alpha_{\iota\varsigma}$ show the connection weights of MAX and MIN templates for fuzzy feedback, $I(t)$ is a input from outside. The initial values of system (1) are recorded as $x_\iota(t_0)=\phi_\iota(t)$ with $\phi_\iota(0)=0$, where $\iota=1,2,3,\dots,\varpi$.

Assumption 4. For each ι, ς, the weights about $a_{\iota\varsigma}(x_\iota(t))$, $b_{\iota\varsigma}(x_\iota(t))$ in system (1) can be remembered as

$$a_{\iota\varsigma}\big(x_\iota(t)\big)=\begin{cases}\acute{a}_{\iota\varsigma}=a_{1\iota\varsigma}^R+\mathrm{i}a_{1\iota\varsigma}^I+\mathrm{j}a_{1\iota\varsigma}^J+\mathrm{k}a_{1\iota\varsigma}^K, & |x_\iota(t)|\leqslant \digamma_\iota,\\ \grave{a}_{\iota\varsigma}=a_{2\iota\varsigma}^R+\mathrm{i}a_{2\iota\varsigma}^I+\mathrm{j}a_{2\iota\varsigma}^J+\mathrm{k}a_{2\iota\varsigma}^K, & |x_\iota(t)|>\digamma_\iota,\end{cases}$$
$$b_{\iota\varsigma}\big(x_\iota(t)\big)=\begin{cases}\acute{b}_{\iota\varsigma}=b_{1\iota\varsigma}^R+\mathrm{i}b_{1\iota\varsigma}^I+\mathrm{j}b_{1\iota\varsigma}^J+\mathrm{k}b_{1\iota\varsigma}^K, & |x_\iota(t)|\leqslant \digamma_\iota,\\ \grave{b}_{\iota\varsigma}=b_{2\iota\varsigma}^R+\mathrm{i}b_{2\iota\varsigma}^I+\mathrm{j}b_{2\iota\varsigma}^J+\mathrm{k}b_{2\iota\varsigma}^K, & |x_\iota(t)|>\digamma_\iota,\end{cases}$$

where the $\digamma_\iota>0$ represents the switching jump, and $\acute{a}_{\iota\varsigma}$, $\grave{a}_{\iota\varsigma}$, $\acute{b}_{\iota\varsigma}$, $\grave{b}_{\iota\varsigma}$ are constants.

Let $a''_{\iota\varsigma}=\max\{\acute{a}_{\iota\varsigma},\grave{a}_{\iota\varsigma}\}$, $a'_{\iota\varsigma}=\min\{\acute{a}_{\iota\varsigma},\grave{a}_{\iota\varsigma}\}$, $b''_{\iota\varsigma}=\max\{\acute{b}_{\iota\varsigma},\grave{b}_{\iota\varsigma}\}$, $b'_{\iota\varsigma}=\min\{\acute{b}_{\iota\varsigma},\grave{b}_{\iota\varsigma}\}$, and $\hat{a}_{\iota\varsigma}=(a''_{\iota\varsigma}+a'_{\iota\varsigma})/2$, $\tilde{a}_{\iota\varsigma}=(a''_{\iota\varsigma}-a'_{\iota\varsigma})/2$, $\hat{b}_{\iota\varsigma}=(b''_{\iota\varsigma}+b'_{\iota\varsigma})/2$, $\tilde{b}_{\iota\varsigma}=(b''_{\iota\varsigma}-b'_{\iota\varsigma})/2$, $a^{\perp}_{\iota\varsigma}=\max\{|\acute{a}_{\iota\varsigma}|,|\grave{a}_{\iota\varsigma}|\}$, $b^{\perp}_{\iota\varsigma}=\max\{|\acute{b}_{\iota\varsigma}|,|\grave{b}_{\iota\varsigma}|\}$.

Based on the above theory, the system can be looked upon as a switching system but has right-side discontinuity. Under the framework of Filippov's solution, one can use the theory about differential inclusion, the drive system can be converted into the system as follows:

$$\begin{aligned}{}_{t_0}^{C}D_t^{\epsilon}x_\iota(t) \in{}& -d_\iota x_\iota(t) + \sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma} + \overline{\mathrm{co}}[-\tilde{a}_{\iota\varsigma}, \tilde{a}_{\iota\varsigma}]\big)f_\varsigma\big(x_\varsigma(t)\big)\\ &+ \sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma} + \overline{\mathrm{co}}[-\tilde{b}_{\iota\varsigma}, \tilde{b}_{\iota\varsigma}]\big)f_\varsigma\big(x_\varsigma(t-l)\big) + \sum_{\varsigma=1}^{\varpi} b_{\iota\varsigma}V_\varsigma\\ &+ \bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big) + \bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma + \bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma\\ &+ \bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big) + I(t),\end{aligned}$$

where

$$\begin{aligned}\overline{\mathrm{co}}[-\tilde{a}_{\iota\varsigma}, \tilde{a}_{\iota\varsigma}] ={}& \overline{\mathrm{co}}\big([-1,1]\big)\big(a_{1\iota\varsigma}^R - a_{2\iota\varsigma}^R\big) + \mathrm{i}\overline{\mathrm{co}}\big([-1,1]\big)\big(a_{1\iota\varsigma}^I - a_{2\iota\varsigma}^I\big)\\ &+ \mathrm{j}\overline{\mathrm{co}}\big([-1,1]\big)\big(a_{1\iota\varsigma}^J - a_{2\iota\varsigma}^J\big) + \mathrm{k}\overline{\mathrm{co}}\big([-1,1]\big)\big(a_{1\iota\varsigma}^K - a_{2\iota\varsigma}^K\big),\\ \overline{\mathrm{co}}[-\tilde{b}_{\iota\varsigma}, \tilde{b}_{\iota\varsigma}] ={}& \overline{\mathrm{co}}\big([-1,1]\big)\big(b_{1\iota\varsigma}^R - b_{2\iota\varsigma}^R\big) + \mathrm{i}\overline{\mathrm{co}}\big([-1,1]\big)\big(b_{1\iota\varsigma}^I - b_{2\iota\varsigma}^I\big)\\ &+ \mathrm{j}\overline{\mathrm{co}}\big([-1,1]\big)\big(b_{1\iota\varsigma}^J - b_{2\iota\varsigma}^J\big) + \mathrm{k}\overline{\mathrm{co}}\big([-1,1]\big)\big(b_{1\iota\varsigma}^K - b_{2\iota\varsigma}^K\big).\end{aligned}$$

Based on the above theory, one can obtain the existence of measurable functions $\gamma_{\iota\varsigma}^{(1)}$ and $\gamma_{\iota\varsigma}^{(2)} \in \overline{\mathrm{co}}[-1,1]$ such that

$$\begin{aligned}{}_{t_0}^{C}D_t^{\epsilon}x_\iota(t) ={}& -d_\iota x_\iota(t) + \sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma} + \tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)f_\varsigma\big(x_\varsigma(t)\big)\\ &+ \sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma} + \tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)f_\varsigma\big(x_\varsigma(t-l)\big) + \sum_{\varsigma=1}^{\varpi} b_{\iota\varsigma}V_\varsigma\\ &+ \bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big) + \bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma + \bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma\\ &+ \bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big) + I(t).\end{aligned}$$

The following is the response system:

$$\begin{aligned}{}_{t_0}^{C}D_t^{\epsilon}\breve{x}_\iota(t) ={}& -d_\iota \breve{x}_\iota(t) + \sum_{\varsigma=1}^{\varpi}a_{\iota\varsigma}\big(\breve{x}_\varsigma(t)\big)f_\varsigma\big(\breve{x}_\varsigma(t)\big)\\ &+ \sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}\big(\breve{x}_\varsigma(t)\big)f_\varsigma\big(\breve{x}_\varsigma(t-l)\big) + \sum_{\varsigma=1}^{\varpi} b_{\iota\varsigma}V_\varsigma\end{aligned}$$

$$+\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(\breve{x}_\varsigma(t-l)\big)+\bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma+\bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma$$
$$+\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(\breve{x}_\varsigma(t-l)\big)+I(t)+u(t), \tag{2}$$

where $\breve{x}_\iota(t)$ indicates the state variable, which is related to system (2), $u(t)$ means the controller. The initial values of the response system are $\breve{x}_\iota(t_0)=\psi_{\iota 0}$ with $\psi_\iota(0)=0$, $\iota=1,2,3,\dots,\varpi$. Similarly, there exist the measurable functions $\xi_{\iota\varsigma}^{(1)}$, $\xi_{\iota\varsigma}^{(2)}\in\overline{\mathrm{co}}[-1,1]$ such that

$$\begin{aligned}{}_{t_0}^{C}D_t^{\epsilon}\breve{x}_\iota(t)&=-d_\iota\breve{x}_\iota(t)+\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)f_\varsigma\big(\breve{x}_\varsigma(t)\big)\\&\quad+\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\big)f_q\big(\breve{x}_\varsigma(t-l)\big)+\sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}V_\varsigma\\&\quad+\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(\breve{x}_\varsigma(t-l)\big)+\bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma+\bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma+\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(\breve{x}_\varsigma(t-l)\big)\\&\quad+I(t)+u(t).\end{aligned}$$

4 Main results

The projective error is represented as $e_\iota(t)=\breve{x}_\iota(t)-\lambda x_\iota(t)$, and $\lambda\in\mathbb{R}\setminus\{0\}$ denotes the projective coefficient, the projective error system is as below:

$$\begin{aligned}{}_{t_0}^{C}D_t^{\epsilon}e_\iota(t)&={}_{t_0}^{C}D_t^{\epsilon}\breve{x}_\iota(t)-\lambda\,{}_{t_0}^{C}D_t^{\epsilon}x_\iota(t)\\&=-d_\iota e_\iota(t)+(1-\lambda)\Bigg\{\sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}V_\varsigma+\bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma+\bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma+I(t)\Bigg\}\\&\quad+\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)f_\varsigma\big(\breve{x}_\varsigma(t)\big)+\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\big)f_\varsigma\big(\breve{x}_\varsigma(t-l)\big)\\&\quad-\lambda\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)f_\varsigma\big(x_\varsigma(t)\big)-\lambda\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)f_\varsigma\big(x_\varsigma(t-l)\big)\\&\quad+\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma(\breve{x}_\varsigma(t-l)\big)+\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(\breve{x}_\varsigma(t-l)\big),\end{aligned}$$

$$\begin{aligned}&-\lambda\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big)-\lambda\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big)+u(t)\\&\quad=-d_\iota e_\iota(t)+(1-\lambda)\Bigg\{\sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}V_\varsigma+\bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma+\bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma+I(t)\Bigg\}+u(t)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)\big[f_\varsigma\big(\check{x}_\varsigma(t)\big)-f_\varsigma\big(\lambda x_\varsigma(t)\big)\big]\\
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)\big[f_\varsigma\big(\lambda x_\varsigma(t)\big)-f_\varsigma\big(x_\varsigma(t)\big)\big]\\
&+\sum_{\varsigma=1}^{\varpi}\big[(1-\lambda)\hat{a}_{\iota\varsigma}+\big(\xi_{\iota\varsigma}^{(1)}-\lambda\gamma_{\iota\varsigma}^{(1)}\big)\tilde{a}_{\iota\varsigma}\big]f_\varsigma\big(x_\varsigma(t)\big)\\
&+\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(\lambda x_\varsigma(t-l)\big)-\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}\lambda f_\varsigma\big(x_\varsigma(t-l)\big)\\
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\big)\big[f_\varsigma\big(\check{x}_\varsigma(t-l)\big)-f_\varsigma\big(\lambda x_\varsigma(t-l)\big)\big]\\
&+\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(\check{x}_\varsigma(t-l)\big)-\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(\lambda x_\varsigma(t-l)\big)\\
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\big)\big[f_\varsigma\big(\lambda x_\varsigma(t-l)\big)-f_\varsigma\big(x_\varsigma(t-l)\big)\big]\\
&+\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(\check{x}_\varsigma(t-l)\big)-\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(\lambda x_\varsigma(t-l)\big)\\
&+\sum_{\varsigma=1}^{\varpi}\big[(1-\lambda)\hat{b}_{\iota\varsigma}+\big(\xi_{\iota\varsigma}^{(2)}-\lambda\gamma_{\iota\varsigma}^{(2)}\big)\tilde{b}_{\iota\varsigma}\big]f_\varsigma\big(x_\varsigma(t-l)\big)\\
&+\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(\lambda x_\varsigma(t-l)\big)-\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}\lambda f_\varsigma\big(x_\varsigma(t-l)\big).
\end{aligned}
$$

To achieve FTPS, the following controllers are set up:

$$
\begin{aligned}
u(t)&=u_h(t)+u_k(t),\\
u_h(t)&=-(1-\lambda)\Bigg\{\sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}V_\varsigma+\bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma+\bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma+I(t)\Bigg\},\\
u_k(t)&=-\omega_\iota e_\iota(t)-\frac{\mu_\iota e_\iota(t)}{[(\operatorname{sign}(e_\iota(t)))e_\iota(t)]^\theta}-\eta\operatorname{sign}\big(e_\iota(t)\big),
\end{aligned}
\tag{3}
$$

where $\omega_\iota,\mu_\iota>0$ and $1-\epsilon<\theta<1$ such that

$$
\begin{aligned}
\eta>\sum_{\varsigma=1}^{\varpi}\big[&2a_{\iota\varsigma}^{\perp}+2b_{\iota\varsigma}^{\perp}+\big|(1-\lambda)\hat{a}_{\iota\varsigma}+(1+\lambda)\tilde{a}_{\iota\varsigma}\big|+(1+\lambda)(\alpha_{\iota\varsigma}+\beta_{\iota\varsigma})\\
&+\big|(1-\lambda)\hat{b}_{\iota\varsigma}+(1+\lambda)\tilde{b}_{\iota\varsigma}\big|\big]M_\varsigma.
\end{aligned}
$$

Remark 2. In previous research [31, 32], one set up a delay-dependent controller to eliminate the delay term in the system, but it will increase the conservatism of system synchronization. In this paper, we set up a delay-independent feedback controller (3), which can still work without measuring the delay function.

Theorem 1. *In the light of Assumptions* 1–2 *and controller* (3)*, if there is a positive scalar quaternion, systems* (1) *and* (2) *can reach FTPS when* $\delta_1 > \delta_2$ *defined as*

$$\delta_1 = \min_{1\leqslant\iota\leqslant\varpi}\left\{d_\iota + \omega_\iota - \sum_{\varsigma=1}^{\varpi} a_{\iota\varsigma}^{\perp} L_\varsigma\right\} > 0,$$
$$\delta_2 = \max_{1\leqslant\iota\leqslant\varpi}\left\{\sum_{\varsigma=1}^{\varpi}(b_{\iota\varsigma}^{\perp} L_\varsigma + \left(|\alpha_{\iota\varsigma}| + |\alpha_{\iota\varsigma}|L_\varsigma\right))\right\} > 0,$$

where the settling time t_2 *is as follows:*

$$t_2 = t_0 + \left[\frac{\Gamma(1+\frac{1}{1-\theta})\Gamma(2-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+\frac{1}{1-\theta}-\epsilon)(\delta_1-\delta_2)}\ln\frac{(\delta_1-\delta_2)\Phi^{1-\theta}+\delta_3}{\delta_3}\right]^{1/\epsilon}.$$

Proof. One can establish the auxiliary function by

$$V(t) = \sum_{\iota=1}^{\varpi}\left|e_\iota(t)\right|.$$

On the basis of Lemma 2, after adding controller (3), we can get

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\epsilon} V(t) &\leqslant \sum_{\iota=1}^{\varpi}\operatorname{sign}\big(e_\iota(t)\big)\,{}_{t_0}^{C}D_t^{\epsilon} e_\iota(t)\\
&\leqslant \sum_{\iota=1}^{\varpi}\operatorname{sign}\big(e_\iota(t)\big)\Bigg\{-(d_\iota+\omega_\iota)e_\iota(t) - \frac{\mu_\iota e_\iota(t)}{[(\operatorname{sign}(e_\iota(t)))e_\iota(t)]^\theta}\\
&\quad - \eta\operatorname{sign}\big(e_\iota(t)\big) + \bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}\big(f_\varsigma\big(\check{x}_\varsigma(t-l)\big) - f_\varsigma\big(\lambda x_\varsigma(t-l)\big)\big)\\
&\quad + \bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}\big(f_\varsigma\big(\check{x}_\varsigma(t-l)\big) - f_\varsigma\big(\lambda x_\varsigma(t-l)\big)\big)\\
&\quad + \bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}\big(f_\varsigma\big(\lambda x_\varsigma(t-l)\big) - \lambda f_\varsigma\big(x_\varsigma(t-l)\big)\big)\\
&\quad + \bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}\big(f_\varsigma\big(\lambda x_\varsigma(t-l)\big) - \lambda f_\varsigma\big(x_\varsigma(t-l)\big)\big)\\
&\quad + \sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma} + \tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)\big[f_\varsigma\big(\check{x}_\varsigma(t)\big) - f_\varsigma\big(\lambda x_\varsigma(t)\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)\big[f_{\varsigma}\big(\lambda x_{\varsigma}(t)\big)-f_{\varsigma}\big(x_{\varsigma}(t)\big)\big]\\
&+\sum_{\varsigma=1}^{\varpi}\big[(1-\lambda)\hat{a}_{\iota\varsigma}+\big(\xi_{\iota\varsigma}^{(1)}-\lambda\gamma_{\iota\varsigma}^{(1)}\big)\tilde{a}_{\iota\varsigma}\big]f_{\varsigma}\big(x_{\varsigma}(t)\big)\\
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\big)\big[f_{\varsigma}(\check{x}_{\varsigma}(t-l))-f_{\varsigma}\big(\lambda x_{\varsigma}(t-l)\big)\big]\\
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\big)\big[f_{\varsigma}\big(\lambda x_{\varsigma}(t-l)\big)-f_{\varsigma}\big(x_{\varsigma}(t-l)\big)\big]\\
&+\sum_{\varsigma=1}^{\varpi}\big[(1-\lambda)\hat{b}_{\iota\varsigma}+\big(\xi_{\iota\varsigma}^{(2)}-\lambda\gamma_{\iota\varsigma}^{(2)}\big)\tilde{b}_{\iota\varsigma}\big]f_{\varsigma}\big(x_{\varsigma}(t-l)\big)\Bigg\}.
\end{aligned}
$$

Based on Assumptions 1 and 4, one has

$$
\begin{aligned}
&\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\operatorname{sign}\big(e_{\iota}(t)\big)\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)\big[f_{\varsigma}\big(\check{x}_{\varsigma}(t)\big)-f_{\varsigma}\big(\lambda x_{\varsigma}(t)\big)\big]\\
&\quad\leqslant\sum_{\iota=1}^{\varpi}\operatorname{sign}\big(e_{\iota}(t)\big)\sum_{\varsigma=1}^{\varpi}\big[a_{\iota\varsigma}^{\perp}L_{\varsigma}\big|\check{x}_{\varsigma}(t)-\lambda x_{\varsigma}(t)\big|\big]\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}a_{\iota\varsigma}^{\perp}L_{\varsigma}\big|e_{\iota}(t)\big|.
\end{aligned}
$$

On the basis of Assumptions 2 and 4, we have

$$
\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\operatorname{sign}\big(e_{\iota}(t)\big)\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)\big[f_{\varsigma}\big(\lambda x_{\varsigma}(t)\big)-f_{\varsigma}\big(x_{\varsigma}(t)\big)\big]\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}2a_{\iota\varsigma}^{\perp}M_{\varsigma}.
$$

Under Assumptions 2 and 3, then

$$
\begin{aligned}
&\sum_{\iota=1}^{\varpi}\operatorname{sign}\big(e_{\iota}(t)\big)\left[\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_{\varsigma}\big(\lambda x_{\varsigma}(t-l)\big)-\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}\lambda f_{\varsigma}\big(x_{\varsigma}(t-l)\big)\right]\\
&\quad\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}|\alpha_{\iota\varsigma}|(1+\lambda)M_{\varsigma},\\
&\sum_{\iota=1}^{\varpi}\operatorname{sign}\big(e_{\iota}(t)\big)\left[\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_{\varsigma}\big(\check{x}_{\varsigma}(t-l)\big)-\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_{\varsigma}\big(\lambda x_{\varsigma}(t-l)\big)\right]\\
&\quad\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_{\varsigma}|\alpha_{\iota\varsigma}|\big|e_{\iota}(t-l)\big|.
\end{aligned}
$$

Moreover,

$$\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\operatorname{sign}\bigl(e_\iota(t)\bigr)\bigl[(1-\lambda)\hat{a}_{\iota\varsigma}+(\xi_{\iota\varsigma}^{(1)}-\lambda\gamma_{\iota\varsigma}^{(1)})\tilde{a}_{\iota\varsigma}\bigr]f_\varsigma\bigl(x_\varsigma(t)\bigr)$$
$$\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\bigl|(1-\lambda)\hat{a}_{\iota\varsigma}+(1+\lambda)\tilde{a}_{\iota\varsigma}\bigr|M_\varsigma,$$
$$\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\operatorname{sign}\bigl(e_\iota(t)\bigr)\bigl[(1-\lambda)\hat{b}_{\iota\varsigma}+(\xi_{\iota\varsigma}^{(2)}-\lambda\gamma_{\iota\varsigma}^{(2)})\tilde{b}_{\iota\varsigma}\bigr]f_\varsigma(x_\varsigma(t-l))$$
$$\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\bigl|(1-\lambda)\hat{b}_{\iota\varsigma}+(1+\lambda)\tilde{b}_{\iota\varsigma}\bigr|M_\varsigma.$$

Combining the above inequalities, one gets the similar results:

$$\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\operatorname{sign}\bigl(e_\iota(t)\bigr)\bigl(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\bigr)\bigl[f_\varsigma\bigl(\check{x}_\varsigma(t-l)\bigr)-f_\varsigma\bigl(\lambda x_\varsigma(t-l)\bigr)\bigr]$$
$$\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}^{\perp}L_\varsigma\bigl|e_\iota(t-l)\bigr|,$$
$$\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\operatorname{sign}\bigl(e_\iota(t)\bigr)\bigl(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\bigr)\bigl[f_\varsigma\bigl(\lambda x_\varsigma(t-l)\bigr)-f_\varsigma\bigl(x_\varsigma(t-l)\bigr)\bigr]$$
$$\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}2b_{\iota\varsigma}^{\perp}M_\varsigma,$$
$$\sum_{\iota=1}^{\varpi}\operatorname{sign}\bigl(e_\iota(t)\bigr)\left[\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\bigl(\check{x}_\varsigma(t-l)\bigr)-\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\bigl(\lambda x_\varsigma(t-l)\bigr)\right]$$
$$\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\varsigma|\beta_{\iota\varsigma}|\bigl|e_\iota(t-l)\bigr|,$$
$$\sum_{\iota=1}^{\varpi}\operatorname{sign}\bigl(e_\iota(t)\bigr)\left[\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\bigl(\lambda x_\varsigma(t-l)\bigr)-\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}\lambda f_\varsigma\bigl(x_\varsigma(t-l)\bigr)\right]$$
$$\leqslant\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}|\beta_{\iota\varsigma}|(1+\lambda)M_\varsigma.$$

Combined with the above inequalities, because $\operatorname{sign}^2(e_\iota(t)) = 0$ or 1 such that $\sum_{\iota=1}^{\varpi}\operatorname{sign}^2(e_\iota(t))\geqslant 1$, then

$$ {}_{t_0}^{C}D_t^{\epsilon}V(t)$$
$$\leqslant-\sum_{\iota=1}^{\varpi}\left[\left(d_\iota+\omega_\iota-\sum_{\varsigma=1}^{\varpi}a_{\iota\varsigma}^{\perp}L_\varsigma\right)\bigl|e_\iota(t)\bigr|\right]-\sum_{\iota=1}^{\varpi}\mu_\iota\bigl[\operatorname{sign}\bigl(e_\iota(t)\bigr)e_\iota(t)\bigr]^{1-\theta}$$

$$
\begin{aligned}
&+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\Big[\big(b_{\iota\varsigma}^{\perp}L_\varsigma+\big(|\alpha_{\iota\varsigma}|+|\beta_{\iota\varsigma}|\big)L_\varsigma\big)\big|e_\varsigma(t-l)\big| \\
&\qquad +\Big(\big((1+\lambda)(\alpha_{\iota\varsigma}+\beta_{\iota\varsigma})+\big|(1-\lambda)\hat{b}_{\iota\varsigma}+(1+\lambda)\tilde{b}_{\iota\varsigma}\big| \\
&\qquad +2a_{\iota\varsigma}^{\perp}+2b_{\iota\varsigma}^{\perp}+\big|(1-\lambda)\hat{a}_{\iota\varsigma}+(1+\lambda)\tilde{a}_{\iota\varsigma}\big|\big)M_\varsigma-\eta\Big)\Big] \\
\leqslant{}& -\sum_{\iota=1}^{\varpi}\Bigg[\Bigg(d_\iota+\omega_\iota-\sum_{\varsigma=1}^{\varpi}a_{\iota\varsigma}^{\perp}L_\varsigma\Bigg)\big|e_\iota(t)\big|\Bigg]-\sum_{\iota=1}^{\varpi}\mu_\iota\big[\operatorname{sign}(e_\iota(t))e_\iota(t)\big]^{1-\theta} \\
&+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\big(b_{\iota\varsigma}^{\perp}L_\varsigma+\big(|\alpha_{\iota\varsigma}|+|\beta_{\iota\varsigma}|\big)L_\varsigma\big)\big|e_\varsigma(t-l)\big| \\
\leqslant{}& -\delta_1\sum_{\iota=1}^{\varpi}|e_\iota(t)|+\delta_2\sum_{\iota=1}^{\varpi}\big|e_\iota(t-l)\big|-\delta_3\sum_{\iota=1}^{\varpi}\big|e_\iota(t)\big|^{1-\theta} \\
\leqslant{}& -\delta_1V(t)+\delta_2V(t-l)-\delta_3V(t)^{1-\theta}.
\end{aligned}
$$

Therefore, according to Lemma 6, system(1) can reach FTPS with system (2).

In the above discussion, we established a feedback controller. To obtain more general results, we will set up an adaptive controller to achieve FTPS:

$$
\begin{aligned}
u(t)&=u_h(t)+u_j(t)+u_k(t) \\
u_h(t)&=-(1-\lambda)\Bigg\{\sum_{\varsigma=1}^{\varpi}b_{\iota\varsigma}V_\varsigma+\bigvee_{\varsigma=1}^{\varpi}S_{\iota\varsigma}V_\varsigma+\bigwedge_{\varsigma=1}^{\varpi}T_{\iota\varsigma}V_\varsigma+I(t)\Bigg\} \\
u_j(t)&=-\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}\big)\big[f_\varsigma\big(\lambda x_\varsigma(t)\big)-\lambda f_\varsigma\big(x_\varsigma(t)\big)\big] \\
&\quad-\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}\big)\big[f_\varsigma\big(\lambda x_\varsigma(t-l)\big)-\lambda f_\varsigma\big(x_\varsigma(t-l)\big)\big] \\
&\quad+\sum_{\varsigma=1}^{\varpi}\lambda\tilde{a}_{\iota\varsigma}\big[\gamma_{\iota\varsigma}^{(1)}f_\varsigma\big(\breve{x}_\varsigma(t)\big)-\xi_{\iota\varsigma}^{(1)}f_\varsigma\big(x_\varsigma(t)\big)\big] \\
&\quad+\sum_{\varsigma=1}^{\varpi}\lambda\tilde{b}_{\iota\varsigma}\big[\gamma_{\iota\varsigma}^{(2)}f_\varsigma\big(\breve{x}_\varsigma(t-l)\big)-\xi_{\iota\varsigma}^{(2)}f_\varsigma\big(x_\varsigma(t-l)\big)\big] \\
&\quad-\sum_{\varsigma=1}^{\varpi}\tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big[f_\varsigma\big(\lambda x_\varsigma(t)\big)-\lambda f_\varsigma\big(x_\varsigma(t)\big)\big] \\
&\quad-\sum_{\varsigma=1}^{\varpi}\tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big[f_\varsigma\big(\lambda x_\varsigma(t-l)\big)-\lambda f_\varsigma\big(x_\varsigma(t-l)\big)\big] \\
&\quad-\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(\lambda x_\varsigma(t-l)\big)+\lambda\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big) \\
&\quad-\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(\lambda x_\varsigma(t-l)\big)+\lambda\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_\varsigma\big(x_\varsigma(t-l)\big)
\end{aligned}
$$

$$u_k(t) = -\omega_\iota(t)e_\iota(t) - \frac{\mu_\iota e_\iota(t)}{[e_\iota(t)\overline{e_\iota(t)}]^\theta} \tag{4}$$

in which ${}_{t_0}^{C}D_t^\epsilon \omega_\iota(t) = \zeta_\iota e_\iota(t)\overline{e_\iota(t)}$, $\mu_\iota > 0$, $\zeta_\iota > 0$. □

Theorem 2. *In view of Assumptions* 1, 3 *and controller* (4)*, systems* (1) *and* (2) *can reach FTPS when there exists* $\kappa > \upsilon$ *such that*

$$\kappa = \min_{1\leqslant\iota\leqslant\varpi} 2d_\iota > 0, \qquad \mu = \min_{1\leqslant\iota\leqslant\varpi} 2\mu_\iota > 0,$$

$$\upsilon = \max_{1\leqslant\iota\leqslant\varpi}\left\{\sum_{\varsigma=1}^{\varpi} L_\iota\left(\left|\hat{b}_{\varsigma\iota} + \tilde{b}_{\varsigma\iota}\xi_{\varsigma\iota}^{(2)} + \tilde{b}_{\varsigma\iota}\gamma_{\varsigma\iota}^{(2)}\right| + |\alpha_{\varsigma\iota} + \beta_{\varsigma\iota}|\right)\right\} > 0$$

in which the settling time t_2^* *is expressed as*

$$t_2^* = t_0 + \left[\frac{\Gamma(1+\frac{1}{1-\theta})\Gamma(2-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+\frac{1}{1-\theta}-\epsilon)(\kappa-\upsilon)} \ln\frac{(\kappa-\upsilon)\Phi^{1-\theta}+\mu}{\mu}\right]^{1/\epsilon}.$$

Proof. Let us construct the auxiliary function:

$$V(t) = \sum_{\varsigma=1}^{\varpi} e_\iota(t)\overline{e_\iota(t)} + \sum_{\varsigma=1}^{\varpi}\frac{1}{\zeta_\iota}\left(\omega_\iota(t) - \omega_\iota^*\right)^2,$$

where

$$\begin{aligned}
2\omega_\iota^* = {}& \sum_{\varsigma=1}^{\varpi} L_\varsigma\left(\left|\hat{a}_{\iota\varsigma} + \tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)} + \tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\right| + \left|\hat{b}_{\iota\varsigma} + \tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)} + \tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\right| + |\alpha_{\iota\varsigma} + \beta_{\iota\varsigma}|\right) \\
& + \sum_{\varsigma=1}^{\varpi} L_\iota\left|\hat{a}_{\varsigma\iota} + \tilde{a}_{\varsigma\iota}\xi_{\varsigma\iota}^{(1)} + \tilde{a}_{\varsigma\iota}\gamma_{\varsigma\iota}^{(1)}\right|,
\end{aligned}$$

$$\begin{aligned}
{}_{t_0}^{C}D_t^\epsilon V(t) \leqslant {}& \sum_{\iota=1}^{\varpi}\left[e_\varsigma(t)\,{}_{t_0}^{C}D_t^\epsilon\overline{e_\iota(t)} + {}_{t_0}^{C}D_t^\epsilon e_\iota(t)\,\overline{e_\iota(t)}\right] + \sum_{\iota=1}^{\varpi}\frac{2}{\zeta_\iota}\left(\omega_\iota(t) - \omega_\iota^*\right){}_{t_0}^{C}D_t^\epsilon\omega_\iota(t) \\
\leqslant {}& \sum_{\iota=1}^{\varpi}\Bigg\{e_\iota(t)\Bigg[-\overline{\left(d_\iota + \omega_\iota(t)\right)}\overline{e_\iota(t)} - \frac{\overline{\mu_\iota}\overline{e_\iota(t)}}{[e_\iota(t)\overline{e_\iota(t)}]^\theta} \\
& + \sum_{\varsigma=1}^{\varpi}\overline{\left(\hat{a}_{\iota\varsigma} + \tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)} + \tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\right)}\left(\overline{f_\varsigma\left(\check{x}_\varsigma(t)\right)} - \overline{f_\varsigma\left(\lambda x_\varsigma(t)\right)}\right) \\
& + \sum_{\varsigma=1}^{\varpi}\overline{\left(\hat{b}_{\iota\varsigma} + \tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)} + \tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\right)}\left(\overline{f_\varsigma\left(\check{x}_\varsigma(t-l)\right)} - \overline{f_\varsigma\left(\lambda x_\varsigma(t-l)\right)}\right) \\
& + \bigwedge_{\varsigma=1}^{\varpi}\overline{\alpha_{\iota\varsigma}}\overline{f_\varsigma\left(\check{x}_\varsigma(t-l)\right)} - \bigwedge_{\varsigma=1}^{\varpi}\overline{\alpha_{\iota\varsigma}}\overline{f_\varsigma\left(\lambda x_\varsigma(t-l)\right)} + \bigvee_{\varsigma=1}^{\varpi}\overline{\beta_{\iota\varsigma}}\overline{f_\varsigma\left(\check{x}_\varsigma(t-l)\right)} \\
& - \bigvee_{\varsigma=1}^{\varpi}\overline{\beta_{\iota\varsigma}}\overline{f_\varsigma\left(\lambda x_\varsigma(t-l)\right)}\Bigg] + \Bigg[-\left(d_\iota + \omega_\iota(t)\right)e_\iota(t) - \frac{\mu_\iota e_\iota(t)}{[e_\iota(t)\overline{e_\iota(t)}]^\theta}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)\big(f_{\varsigma}\big(\check{x}_{\varsigma}(t)\big)-f_{\varsigma}\big(\lambda x_{\varsigma}(t)\big)\big)\\
&+\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_{\varsigma}\big(\check{x}_{\varsigma}(t-l)\big)-\bigwedge_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}f_{\varsigma}\big(\lambda x_{\varsigma}(t-l)\big)\\
&+\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)\big(f_{\varsigma}\big(\check{x}_{\varsigma}(t-l)\big)-f_{\varsigma}\big(\lambda x_{\varsigma}(t-l)\big)\big)\\
&+\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_{\varsigma}\big(\check{x}_{\varsigma}(t-l)\big)-\bigvee_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}f_{\varsigma}\big(\lambda x_{\varsigma}(t-l)\big)\Bigg]\overline{e_{\varsigma}(t)}\Bigg\}\\
&+2\sum_{\iota=1}^{\varpi}\big(\omega_{\iota}(t)-\omega_{\iota}^{*}\big)e_{\iota}(t)\overline{e_{\iota}(t)}.
\end{aligned}
$$

According to Assumptions 1 and 3,

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\epsilon}V(t)\leqslant{}&-\sum_{\iota=1}^{\varpi}(2d_{\iota}+2\omega_{\iota}^{*})e_{\iota}(t)\overline{e_{\iota}(t)}-2\mu_{\iota}\big[e_{\iota}(t)\overline{e_{\iota}(t)}\big]^{1-\theta}\\
&+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}e_{\iota}(t)\overline{\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)}\overline{L_{\varsigma}}\overline{e_{\varsigma}(t)}\\
&+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)L_{\varsigma}e_{\varsigma}(t)\overline{e_{\iota}(t)}\\
&+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}e_{\iota}(t)\overline{\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)}\overline{L_{\varsigma}}\overline{e_{\varsigma}(t-l)}\\
&+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\big(\hat{b}_{\iota\varsigma}+\tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)L_{\varsigma}e_{\varsigma}(t-l)\overline{e_{\iota}(t)}\\
&+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}e_{\iota}(t)\overline{\alpha_{\iota\varsigma}}\overline{L_{\varsigma}}\overline{e_{\varsigma}(t-l)}+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}L_{\varsigma}e_{\varsigma}(t-l)\overline{e_{\iota}(t)}\\
&+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}e_{\iota}(t)\overline{\beta_{\iota\varsigma}}\overline{L_{\varsigma}}\overline{e_{\varsigma}(t-l)}+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\beta_{\iota\varsigma}L_{\varsigma}e_{\varsigma}(t-l)\overline{e_{\iota}(t)}.
\end{aligned}
$$

According to Lemma 4, we can get

$$
\begin{aligned}
&\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}e_{\iota}(t)\overline{\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)}\overline{L_{\varsigma}}\overline{e_{\varsigma}(t)}\\
&\quad+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\big(\hat{a}_{\iota\varsigma}+\tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)L_{\varsigma}e_{\varsigma}(t)\overline{e_{\iota}(t)}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 2\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\big(e_\iota(t)\overline{\big(\hat a_{\iota\varsigma}+\tilde a_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde a_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)}\,\overline{L_\varsigma}\,\overline{e_\varsigma(t)}\\
&\qquad\qquad\times\big(\hat a_{\iota\varsigma}+\tilde a_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde a_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big)L_\varsigma e_\varsigma(t)\overline{e_\iota(t)}\big)^{1/2}\\
&\leqslant 2\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\varsigma\big|\hat a_{\iota\varsigma}+\tilde a_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde a_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big|\big|e_\iota(t)\big|\big|e_\varsigma(t)\big|\\
&\leqslant \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\varsigma\big|\hat a_{\iota\varsigma}+\tilde a_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde a_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big|\big(e_\iota(t)\overline{e_\iota(t)}+e_\varsigma(t)\overline{e_\varsigma(t)}\big)\\
&= \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\varsigma\big|\hat a_{\iota\varsigma}+\tilde a_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)}+\tilde a_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big|e_\iota(t)\overline{e_\iota(t)}\\
&\quad+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\iota\big|\hat a_{\varsigma\iota}+\tilde a_{\varsigma\iota}\xi_{\varsigma\iota}^{(1)}+\tilde a_{\varsigma\iota}\gamma_{\varsigma\iota}^{(1)}\big|e_\iota(t)\overline{e_\iota(t)}.
\end{aligned}
$$

What's more,

$$
\begin{aligned}
&\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}e_\iota(t)\overline{\big(\hat b_{\iota\varsigma}+\tilde b_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde b_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)}\,\overline{L_\varsigma}\,\overline{e_\varsigma(t-l)}\\
&\qquad+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\big(\hat b_{\iota\varsigma}+\tilde b_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde b_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)L_\varsigma e_\varsigma(t-l)\overline{e_\iota(t)}\\
&\quad\leqslant 2\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\big(e_\iota(t)\overline{\big(\hat b_{\iota\varsigma}+\tilde b_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde b_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)}\,\overline{L_\varsigma}\,\overline{e_\varsigma(t-l)}\\
&\qquad\qquad\times\big(\hat b_{\iota\varsigma}+\tilde b_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde b_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big)L_\varsigma e_\varsigma(t-l)\overline{e_\iota(t)}\big)^{1/2}\\
&\quad\leqslant 2\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\varsigma\big|\hat b_{\iota\varsigma}+\tilde b_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde b_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big|\big|e_\iota(t)\big|\big|e_\varsigma(t-l)\big|\\
&\quad\leqslant \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\varsigma\big|\hat b_{\iota\varsigma}+\tilde b_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde b_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big|\big(e_\iota(t)\overline{e_\iota(t)}+e_\varsigma(t-l)\overline{e_\varsigma(t-l)}\big)\\
&\quad= \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\varsigma\big|\hat b_{\iota\varsigma}+\tilde b_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)}+\tilde b_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big|e_\iota(t)\overline{e_\iota(t)}\\
&\qquad+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\iota\big|\hat b_{\varsigma\iota}+\tilde b_{\varsigma\iota}\xi_{\varsigma\iota}^{(2)}+\tilde b_{\varsigma\iota}\gamma_{\varsigma\iota}^{(2)}\big|e_\iota(t-l)\overline{e_\iota(t-l)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}e_\iota(t)\overline{\alpha_{\iota\varsigma}}\,\overline{L_\varsigma}\,\overline{e_\varsigma(t-l)}+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}\alpha_{\iota\varsigma}L_\varsigma e_\varsigma(t-l)\overline{e_\iota(t)}\\
&\quad\leqslant \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\varsigma|\alpha_{\iota\varsigma}|e_\iota(t)\overline{e_\iota(t)}+\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi}L_\iota|\alpha_{\varsigma\iota}|e_\iota(t-l)\overline{e_\iota(t-l)},
\end{aligned}
$$

$$\sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi} e_\iota(t)\overline{\beta_{\iota\varsigma}L_\varsigma e_\varsigma(t-l)} + \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi} \beta_{\iota\varsigma}L_\varsigma e_\varsigma(t-l)\overline{e_\iota(t)}$$
$$\leqslant \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi} L_\varsigma|\beta_{\iota\varsigma}|e_\iota(t)\overline{e_\iota(t)} + \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi} L_\iota|\beta_{\iota\varsigma}|e_\iota(t-l)\overline{e_\iota(t-l)},$$

$$\begin{aligned}
&{}_{t_0}^{C}D_t^{\epsilon}V(t) \\
&\leqslant -\sum_{\iota=1}^{\varpi}\Bigg[2d_\iota + 2\omega_\iota^* \\
&\quad - \sum_{\varsigma=1}^{\varpi} L_\varsigma\big(\big|\hat{a}_{\iota\varsigma} + \tilde{a}_{\iota\varsigma}\xi_{\iota\varsigma}^{(1)} + \tilde{a}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(1)}\big| + \big|\hat{b}_{\iota\varsigma} + \tilde{b}_{\iota\varsigma}\xi_{\iota\varsigma}^{(2)} + \tilde{b}_{\iota\varsigma}\gamma_{\iota\varsigma}^{(2)}\big| + |\alpha_{\iota\varsigma} + \beta_{\iota\varsigma}|\big) \\
&\quad - \sum_{\varsigma=1}^{\varpi} L_\iota\big|\hat{a}_{\varsigma\iota} + \tilde{a}_{\varsigma\iota}\xi_{\varsigma\iota}^{(1)} + \tilde{a}_{\varsigma\iota}\gamma_{\varsigma\iota}^{(1)}\big|\Bigg] e_\iota(t)\overline{e_\iota(t)} - 2\mu_\iota\big[e_\iota(t)\overline{e_\iota(t)}\big]^{1-\theta} \\
&\quad + \sum_{\iota=1}^{\varpi}\sum_{\varsigma=1}^{\varpi} L_\iota\big(\big|\hat{b}_{\varsigma\iota} + \tilde{b}_{\varsigma\iota}\xi_{\varsigma\iota}^{(2)} + \tilde{b}_{\varsigma\iota}\gamma_{\varsigma\iota}^{(2)}\big| + |\alpha_{\varsigma\iota} + \beta_{\varsigma\iota}|\big)e_\iota(t-l)\overline{e_\iota(t-l)} \\
&\leqslant -\kappa V(t) + \upsilon V(t-l) - \mu V(t)^{1-\theta}.
\end{aligned}$$

In view of Lemma 6, system (1) can achieve FTPS under controller (4) with system (2). □

Remark 3. Compared to linear controllers, the adaptive controller is more versatile, which is designed in this paper. In addition to the control gain $\omega_\iota(t)$ being updated adaptively and converging to some constants when achieving FTPS, its control cost will also be significantly reduced.

Remark 4. Among the existing research results, there are relatively few research results on QV memristors. In [9], one considered the FOQVFMNNs about its estimation and synchronization problems, and generalized inequalities were used to compare two quaternion-values, thereby exploring the stability of FOQVFMNNs.

Remark 5. In this paper, the system is considered as a whole instead of being divided into several subsystems, which reduces the complexity of the system compared to [32]. In addition, considering the inevitable ambiguity problem in the system, this article adds fuzzy terms to the system, which can be seen as a generalization of [8] to a certain extent.

Remark 6. There have been relatively few studies on FTPS of QVFOMNNs in the available results. The projective synchronization problem in NNs was studied in [22, 30], and the FTS problem was explored in [4, 33]. In comparison, the exploration process for FTPS is the prolongation of the above achievements, which can enable the system to achieve high convergence and good robustness in a short time, and the resulting projective synchronization results are more common.

Remark 7. Compared with existing literature on FTS, it can be found that most papers use $D^{\epsilon}V(t) \leqslant -\Theta V^{\varrho}(t)$ to prove that master-slave systems can achieve FTS. For example, in [31], this inequality was used to prove the FTS problem of the system. However, in [27], it was pointed out that this inequality can only ensure the synchronization of the system and cannot estimate the settling time of the system. Although some papers provide the conditions that settling time should meet, its display upper bound is difficult to achieve. As obtained in [16], the inequality $\mathrm{e}^{-\gamma t}(1+t)E_{\alpha}(J(t)\Gamma(\alpha)t^{\alpha}) < \epsilon/\delta$. The inequality ${}_{t_0}^{C}D_t^{\epsilon}g(t) \leqslant -\Omega g(t)+\Psi g(t-l)-\Theta g^{\varrho}(t)$ mentioned in Lemma 6 of this paper not only improves the universality and flexibility of the system, but also results in a quick calculation of settling time, saving computational costs.

Remark 8. In Theorem 1, we designed a 1-norm Lyapunov function, where $V(t) = \sum_{\iota=1}^{\varpi}|e_{\iota}(t)|$, which is equivalent to the sum of the absolute values of all error variables $e_{\iota}(t)$, thus ensuring the nonnegativity of $V(t)$. The Lyapunov function designed in Theorem 2 is composed of 2-norm error variable $\sum_{\varsigma=1}^{\varpi} e_{\iota}(t)\overline{e_{\iota}(t)}$ and the adaptive part with control gain $\sum_{\varsigma=1}^{\varpi}(\omega_{\iota}(t)-\omega_{\iota}^{*})^2/\zeta_{\iota}$, which can ensure the nonnegative of $V(t)$.

Remark 9. Finite-time stability refers to the fact that for a given time T related to the initial value of the system, the fractional-order system is stable on intervals $[t, T]$, but its stability on $[T, +\infty]$ cannot be determined. In [5], the authors discussed the synchronization issue of fractional differential equations based on fractional inequality ${}_{t_0}^{C}D_t^{\epsilon}g(t) \leqslant -\Omega g(t) + \Psi g(t-l)$, then $\lim_{t\to\infty} g(t) = 0$. However, above mentioned fractional inequality cannot ensure finite-time stability.

5 Numerical simulations

In this part, one can draw upon numerical simulation to checking the correctness of the theoretical analysis results in this paper.

Example. Firstly, we choose the two-dimensional FOQVFMNNs system. One has

$$\begin{aligned}
&{}_{0}^{C}D_t^{0.98}x_{\iota}(t)\\
&\quad= -d_{\iota}x_{\iota}(t) + \sum_{\varsigma=1}^{2} a_{\iota\varsigma}\big(x_{\varsigma}(t)\big)f_{\varsigma}\big(x_{\varsigma}(t)\big) + \sum_{\varsigma=1}^{2} b_{\iota\varsigma}\big(x_{\varsigma}(t)\big)f_{\varsigma}\big(x_{\varsigma}(t-l)\big) + \sum_{\varsigma=1}^{2} b_{\iota\varsigma}V_{\varsigma}\\
&\qquad+ \bigwedge_{\varsigma=1}^{2} \alpha_{\iota\varsigma}f_{\varsigma}\big(x_{\varsigma}(t-l)\big) + \bigwedge_{\varsigma=1}^{2} T_{\iota\varsigma}V_{\varsigma} + \bigvee_{\varsigma=1}^{2} S_{\iota\varsigma}V_{\varsigma} + \bigvee_{\varsigma=1}^{2} \beta_{\iota\varsigma}f_{\varsigma}\big(x_{\varsigma}(t-l)\big) + I_{\iota}(t),
\end{aligned}$$

$$\begin{aligned}
&{}_{0}^{C}D_t^{0.98}\breve{x}_{\iota}(t)\\
&\quad= -d_{\iota}\breve{x}_{\iota}(t) + \sum_{\varsigma=1}^{2} a_{\iota\varsigma}\big(\breve{x}_{\varsigma}(t)\big)f_{\varsigma}\big(\breve{x}_{\varsigma}(t)\big) + \sum_{\varsigma=1}^{2} b_{\iota\varsigma}\big(\breve{x}_{\varsigma}(t)\big)f_{\varsigma}\big(\breve{x}_{\varsigma}(t-l)\big) + \sum_{\varsigma=1}^{2} b_{\iota\varsigma}V_{\varsigma}\\
&\qquad+ \bigwedge_{\varsigma=1}^{2} \alpha_{\iota\varsigma}f_{\varsigma}\big(\breve{x}_{\varsigma}(t-l)\big) + \bigwedge_{\varsigma=1}^{2} T_{\iota\varsigma}V_{\varsigma} + \bigvee_{\varsigma=1}^{2} S_{\iota\varsigma}V_{\varsigma} + \bigvee_{\varsigma=1}^{2} \beta_{\iota\varsigma}f_{\varsigma}\big(\breve{x}_{\varsigma}(t-l)\big)\\
&\qquad+ I_{\iota}(t) + u_{\iota}(t)
\end{aligned}$$

in which $\iota = 1,2$, $x_\iota(t) = x_\iota^R(t) + \mathrm{i}x_\iota^I(t) + \mathrm{j}x_\iota^J(t) + \mathrm{k}x_\iota^K(t)$ with $x_\iota^R(t)$, $x_\iota^I(t)$, $x_\iota^J(t)$, $x_\iota^K(t) \in \mathbb{R}$, $\breve{x}_\iota(t) = \breve{x}_\iota^R(t) + \mathrm{i}\breve{x}_\iota^I(t) + \mathrm{j}\breve{x}_\iota^J(t) + \mathrm{k}\breve{x}_\iota^K(t)$ with $\breve{x}_\iota^R(t)$, $\breve{x}_\iota^I(t)$, $\breve{x}_\iota^J(t)$, $\breve{x}_\iota^K(t) \in \mathbb{R}$, $f_\iota(x_\iota(t)) = \tanh(x_\iota^I(t)) + \mathrm{i}\tanh(x_\iota^I(t)) + \mathrm{j}\tanh(x_\iota^J(t)) + \mathrm{k}\tanh(x_\iota^K(t))$, $\lambda = 2.3$, $l = 1$, $I_1 = I_2 = 0$, $d_1 = d_2 = 3$, it can be calculated that $L_\varsigma = M_\varsigma = 1$.

$$a_{11}\bigl(x_1(t)\bigr) = \begin{cases} 0.23 - 0.15\mathrm{i} + 0.24\mathrm{j} - 0.26\mathrm{k}, & |x_1(t)| > 0.46, \\ 0.1 - 0.1\mathrm{i} - 0.2\mathrm{j} - 0.2\mathrm{k}, & |x_1(t)| \leqslant 0.46, \end{cases}$$

$$a_{12}\bigl(x_1(t)\bigr) = \begin{cases} -0.23 + 0.1\mathrm{i} - 0.15\mathrm{j} + 0.13\mathrm{k}, & |x_1(t)| > 0.46, \\ -0.3 - 0.18\mathrm{i} + 0.1\mathrm{j} + 0.1\mathrm{k}, & |x_1(t)| \leqslant 0.46, \end{cases}$$

$$a_{21}\bigl(x_2(t)\bigr) = \begin{cases} 0.1 + 0.13\mathrm{i} - 0.16\mathrm{j} + 0.14\mathrm{k}, & |x_2(t)| > 0.46, \\ -0.2 - 0.2\mathrm{i} + 0.2\mathrm{j} + 0.18\mathrm{k}, & |x_2(t)| \leqslant 0.46, \end{cases}$$

$$a_{22}\bigl(x_2(t)\bigr) = \begin{cases} -0.15 - 0.23\mathrm{i} + 0.24\mathrm{j} - 0.24\mathrm{k}, & |x_2(t)| > 0.46, \\ -0.2 + 0.2\mathrm{i} - 0.1\mathrm{j} - 0.2\mathrm{k}, & |x_2(t)| \leqslant 0.46, \end{cases}$$

$$b_{11}\bigl(x_1(t)\bigr) = \begin{cases} -0.16 - 0.17\mathrm{i} - 0.14\mathrm{k} + 0.14\mathrm{k}, & |x_1(t)| > 0.46, \\ -0.2 + 0.13\mathrm{i} - 0.12\mathrm{j} - 0.11\mathrm{k}, & |x_1(t)| \leqslant 0.46, \end{cases}$$

$$b_{12}\bigl(x_1(t)\bigr) = \begin{cases} -0.15 + 0.13\mathrm{i} - 0.2\mathrm{j} + 0.24\mathrm{k}, & |x_1(t)| > 0.46, \\ -0.3 - 0.12\mathrm{i} + 0.13\mathrm{j} - 0.14\mathrm{k}, & |x_1(t)| \leqslant 0.46, \end{cases}$$

$$b_{21}\bigl(x_2(t)\bigr) = \begin{cases} -0.16 + 0.14\mathrm{i} + 0.15\mathrm{j} - 0.15\mathrm{k}, & |x_2(t)| > 0.46, \\ -0.2 - 0.2\mathrm{i} - 0.13\mathrm{j} - 0.1\mathrm{k}, & |x_2(t)| \leqslant 0.46, \end{cases}$$

$$b_{22}\bigl(x_2(t)\bigr) = \begin{cases} -0.4 - 0.8\mathrm{i} + 0.6\mathrm{j} + 0.3\mathrm{k}, & |x_2(t)| > 0.46, \\ 0.2 - 0.3\mathrm{i} + 0.2\mathrm{j} + 0.1\mathrm{k}, & |x_2(t)| \leqslant 0.46. \end{cases}$$

What's more,

$$B = \mathrm{diag}[b_1, b_2] = \begin{bmatrix} -0.18 + 0.28\mathrm{i} + 0.1\mathrm{j} - 0.27\mathrm{k} & 0 \\ 0 & 0.23 - 0.23\mathrm{i} - 0.15\mathrm{j} + 0.25\mathrm{k} \end{bmatrix},$$

$$S = \mathrm{diag}[s_1, s_2] = \begin{bmatrix} -0.28 + 0.2\mathrm{i} + 0.1\mathrm{j} + 0.2\mathrm{k} & 0 \\ 0 & -0.15 - 0.13\mathrm{i} + 0.24\mathrm{j} - 0.34\mathrm{k} \end{bmatrix},$$

$$T = \mathrm{diag}[t_1, t_2] = \begin{bmatrix} -0.15 + 0.15\mathrm{i} - 0.25\mathrm{j} - 0.19\mathrm{k} & 0 \\ 0 & -0.15 + 0.2\mathrm{i} + 0.2\mathrm{j} + 0.28\mathrm{k} \end{bmatrix},$$

$$V = \mathrm{diag}[v_1, v_2] = \begin{bmatrix} -0.14 - 0.12\mathrm{i} + 0.4\mathrm{j} - 0.5\mathrm{k} & 0 \\ 0 & 0.12 - 0.12\mathrm{i} - 0.1\mathrm{j} - 0.16\mathrm{k} \end{bmatrix},$$

$$\alpha_{pq} = \begin{bmatrix} 0.12 + 0.15\mathrm{i} - 0.12\mathrm{j} - 0.12\mathrm{k} & 0.2 + 0.17\mathrm{i} - 0.2\mathrm{j} - 0.15\mathrm{k} \\ -0.25 + 0.15\mathrm{i} - 0.25\mathrm{j} - 0.15\mathrm{k} & 0.14 + 0.18\mathrm{i} - 0.14\mathrm{j} - 0.14\mathrm{k} \end{bmatrix},$$

$$\beta_{pq} = \begin{bmatrix} 0.2 + 0.15\mathrm{i} - 0.27\mathrm{j} + 0.34\mathrm{k} & -0.12 - 0.18\mathrm{i} + 0.24\mathrm{j} + 0.2\mathrm{k} \\ -0.4 - 0.2\mathrm{i} + 0.12\mathrm{j} + 0.15\mathrm{k} & -0.18 - 0.25\mathrm{i} - 0.25\mathrm{j} + 0.25\mathrm{k} \end{bmatrix}.$$

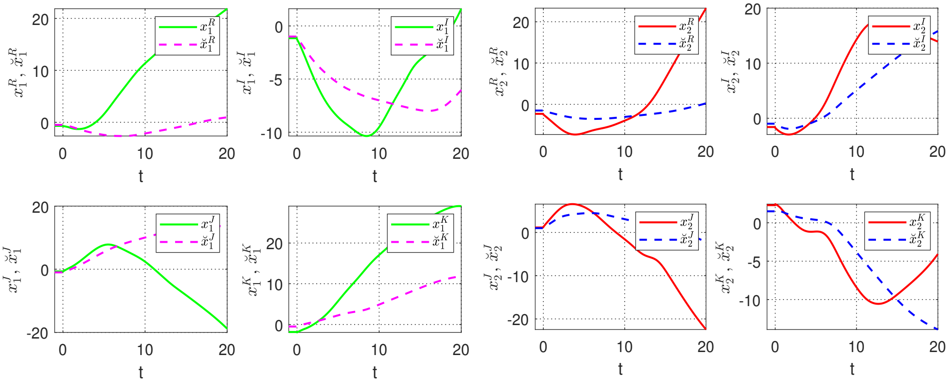

Figure 1. The state trajectories of system (1) and (2) without controller (3).

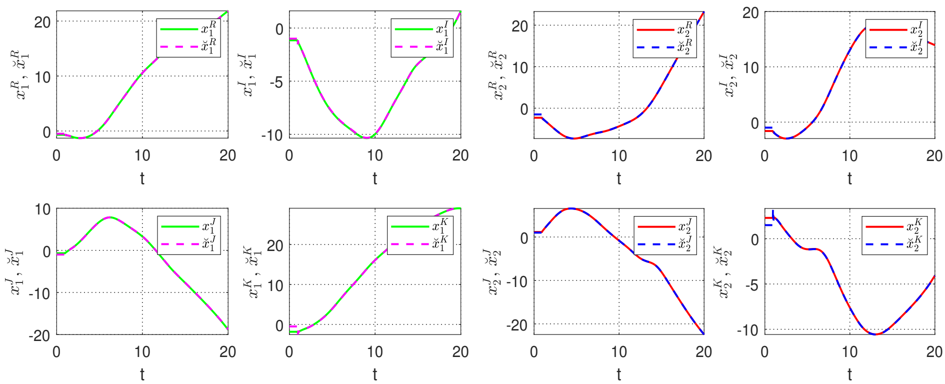

Figure 2. The state trajectories of system (1) and (2) with controller (3).

Under controller (3), one can set the values

$$x(0) = (-0.3 - 0.5\mathrm{i} - \mathrm{j} - 0.7\mathrm{k},\ -0.3 - 0.8\mathrm{i} + 0.5\mathrm{j} + \mathrm{k}),$$
$$\breve{x}(0) = (-0.5 - \mathrm{i} - 1.5\mathrm{j} - \mathrm{k},\ -1 - 0.5\mathrm{i} + \mathrm{j} + 1.5\mathrm{k}).$$

The state curves without controller (3) are showing in Fig. 1, and we set the values of controller (3) such as $\omega_1 = \omega_2 = 32$, $\mu_1 = 45$, $\mu_2 = 55$, $\theta = 0.94$, $\eta = 30$. Based on the above values, one can calculate the following values in Theorem 1: $\delta_1 = 33.36$, $\delta_2 = 4.11$. So that $\delta_1 > \delta_2$, which meets the condition of FTPS. Therefore, system (1) can achieve FTPS with system (2) under controller (3), the settling time $t_2 \approx 0.4348$. Figure 2 means the changing trajectory of state variables. What's more, the error trajectory is shown in Fig. 3.

Under controller (4), we set the values

$$x(0) = (1.4 + 1.4\mathrm{i} - 1.2\mathrm{j} - 1.7\mathrm{k},\ -1.5 - 0.4\mathrm{i} - 0.6\mathrm{j} - 0.4\mathrm{k}),$$
$$\breve{x}(0) = (1.2 + \mathrm{i} - \mathrm{j} - 0.8\mathrm{k},\ -2 + 0.5\mathrm{i} + 0.3\mathrm{j} + 0.5\mathrm{k}).$$

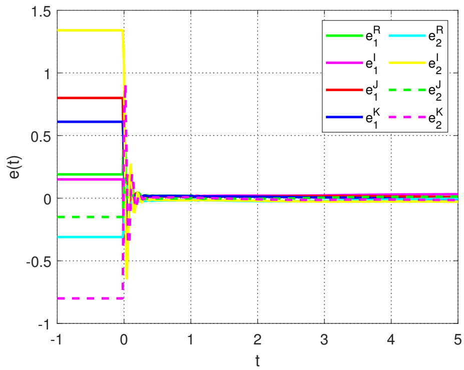

Figure 3. The synchronization error with controller (3).

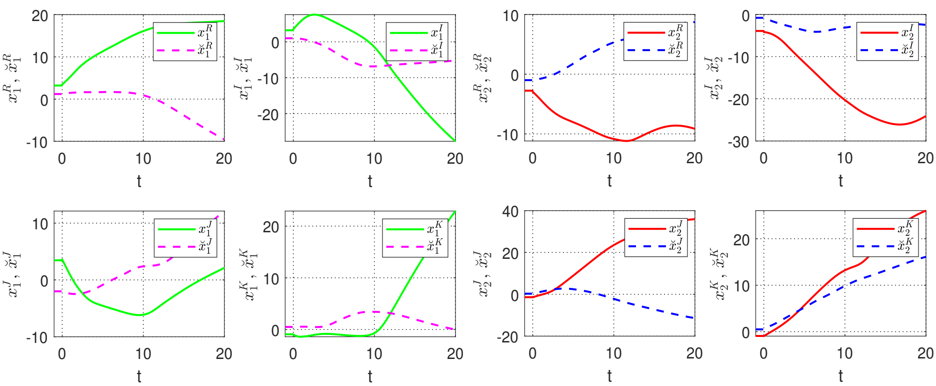

Figure 4. The state trajectories of system (1) and (2) without controller (4).

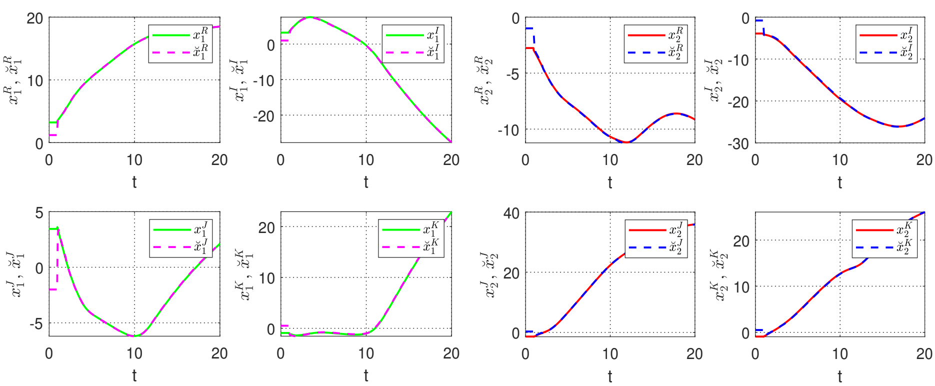

Figure 5. The state trajectories of system (1) and (2) with controller (4).

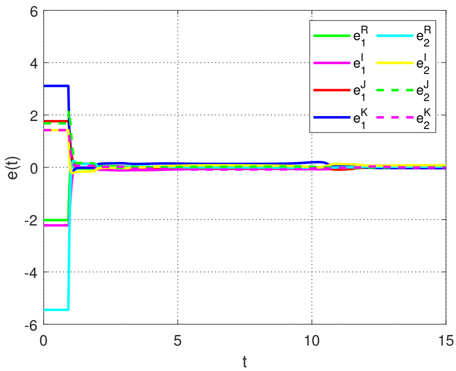

Figure 6. The synchronization error with controller (4).

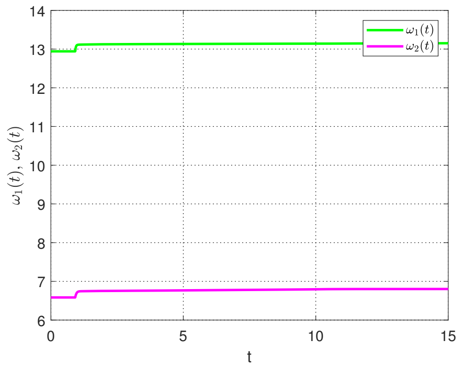

Figure 7. The trajectory of evolutions with controller (4).

The state curves without controller (4) are showing in Fig. 4, and we set the values of controller (4) as $\omega_1(0) \approx 12.94$, $\omega_2(0) \approx 6.59$, $\mu_1 = \mu_2 = 0.22$, $\theta = 0.87$, we can calculate that $\kappa = 6$, $\upsilon = 1.1512$, $\mu = 0.44$, so that $\kappa > \upsilon$, which meets the condition of FTPS. Therefore, system (1) can achieve FTPS with system (2) under controller (4). Furtherly, one can have $t_2^* \approx 3.8196$. Figures 5 and 6 show the track of state variables and the error trajectory, respectively. What's more, Fig. 7 represents the evolutions of $\omega_1(t)$, $\omega_2(t)$.

6 Conclusion

This paper recommends the issue related to FTPS in FOQVMNNs. Theorem 1 establishes a 1-norm auxiliary function, and Theorem 2 establishes an adaptive auxiliary function. Sufficient conditions for FTPS are obtained, then one can calculate the settling time of FTPS. In the end, the simulation results demonstrate the effectiveness of the theoretical results. In practical problems, external disturbance is inevitable. In the next step of research, uncertain parameters will be considered and its dynamic behaviors will be studied.

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