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Construction of the beta distributions using the random permutation divisors

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Abstract. A subset of cycles comprising a permutation σ in the symmetric group $\mathbb{S}_n$, $n \in \mathbb{N}$, is called a divisor of σ. Then the partial sums over divisors with sizes up to un, $0 \leqslant u \leqslant 1$, of values of a nonnegative multiplicative function on a random permutation define a stochastic process with nondecreasing trajectories. When normalized the latter is just a random distribution function supported by the unit interval. We establish that its expectations under various weighted probability measures defined on the subsets of $\mathbb{S}_n$ are quasihypergeometric distribution functions. Their limits as $n \to \infty$ cover the class of two-parameter beta distributions. It is shown that, under appropriate conditions, the convergence rate is of the negative power of n order. That opens a new possibility to model the beta distributions using divisors of permutations.

Keywords: random permutation, multiplicative function, Ewens distribution, quasihypergeometric distribution, arcsine law.

1 Introduction and result

We deal with random permutations and expose that the ubiquitous beta distribution law is also present in their statistical theory. By the definition, the cumulative beta distribution function is defined by

$$B(u; a, b) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^u x^{a-1}(1-x)^{b-1}\,\mathrm{d}x, \quad 0 \leqslant u \leqslant 1,$$

where Γ denotes Euler's gamma function. The parameters $a, b > 0$ make the density function $B'_u(u; a, b)$, $0 < u < 1$, particularly flexible at modeling different curves within the interval, including symmetrical, left- and right-skewed, concave and convex shapes, and straight lines. In particular, $B(u;1,1)=u$ and

$$B\left(u;\frac{1}{2},\frac{1}{2}\right)=\frac{2}{\pi}\arcsin\sqrt{u},$$

where $0\leqslant u\leqslant 1$, are just the uniform and the arcsine distributions. Application of the beta law is a versatile way to represent outcomes for proportions in vast fields of statistics (see [14]). It was not a surprise when it had been disclosed in number theory describing the proportion of the cardinality of divisors of a random natural number in a given interval compared to the total number of them (see [8]). The pioneering DDT theorem proved in this paper was extended in [6] and in the subsequent work [3], showing that, apart from the arcsine, other beta laws appear. We refer to [4] and [5] for the exhaustive historical account. In addition to the number theoretical value, these results gave simple arithmetical constructions approximating the function $B(u;a,b)$. As shown in [2], such a phenomenon is also common for the polynomials over a finite field and in a more general semigroup setting. This leads to a thought to examine permutations acting on the set $\mathbb{N}_n=:\{1,2,\ldots,n\}$ and comprising the symmetric group $\mathbb{S}_n$.

Recall that $\sigma\in\mathbb{S}_n$ is a one-to-one (bijective) mapping $\sigma:\mathbb{N}_n\to\mathbb{N}_n$. It can be represented by the table

$$\sigma=\begin{pmatrix}1 & 2 & \cdots & n\\ i_1 & i_2 & \cdots & i_n\end{pmatrix},$$

where $\sigma(r)=i_r$ for $1\leqslant r\leqslant n$, or by the digraph G_σ with vertex set $V(\sigma)=\mathbb{N}_n$. Its components are oriented cycles. A typical cycle has a vertex set $V(\varkappa)=\{k_1,\ldots,k_j\}\subset\mathbb{N}_n$ defined by

$$k_1\xrightarrow{\sigma}k_2\xrightarrow{\sigma}\cdots\xrightarrow{\sigma}k_j\xrightarrow{\sigma}k_1.$$

Here $1\leqslant j=\#V(\varkappa)\leqslant n$ is the cycle length. Such $\varkappa$, denoted later by the ordered j-tuple $(k_1,\ldots,k_j)$, can be considered the cyclic mapping acting on $V(\varkappa)$ as σ. Using mapping multiplication, we obtain the unique (up to the order of factors) decomposition of σ into cycles $\varkappa_i$ on pairwise disjoint subsets $V(\varkappa_i)$, namely,

$$\sigma=\varkappa_1\cdots\varkappa_w. \tag{1}$$

Here $w=w(\sigma)$ denotes the number of cycles, and $\mathbb{N}_n=V(\varkappa_1)\cup\cdots\cup V(\varkappa_w)$. Let us also introduce the empty permutation $\emptyset$ and $\mathbb{S}_0=\{\emptyset\}$.

We call a subset $\delta\subset\{\varkappa_1,\ldots,\varkappa_w\}$, including the empty one, *divisor* of σ. Being used to the product expression (1), we use the notation $\delta|\sigma$ rather than $\delta\subset\sigma$. The cycles from (1) not included into δ give another divisor, say, τ. Consequently, we obtain the ordered decomposition $\sigma=\delta\tau$ with $V(\delta)\cap V(\tau)=\emptyset$ and $V(\delta)\cup V(\tau)=\mathbb{N}_n$. Recall that a set $V\subset\mathbb{N}_n$ is a fixed set of the mapping σ if the image $\sigma(V):=\{\sigma(i),\, i\in V\}=V$. Such are the sets $V(\delta)$ and $V(\tau)$. Conversely, the disjoint partition $\mathbb{N}_n=V'\cup V''$ with the fixed sets $V',V''\subset\mathbb{N}_n$ of $\sigma\in\mathbb{S}_n$ define the product $\sigma=\delta\tau$ of divisors such that $V'=V(\delta)$ and $V''=V(\tau)$. For the definitions, we refer to [11, Sect. 1.9].

In contrast to σ, a divisor δ can be *weakly labelled*, that is, the vertexes are not necessarily numbered by numbers starting from 1 up to the *size* $|\delta|:=\#V(\delta)$. To overcome this inconvenience, we will use the methodology proposed in the book [10, Chap. II]. As suggested, to come to the *well-labelled* (or shortly, *standard*) δ, which uses only the numbers from $\mathbb{N}_{|\delta|}$, we do a *reduction* preserving the order relations among the labels.

For example, the two-cycle divisor $(1,5)(3,4,7)$ of the permutation

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 5\,2\,4\,7\,1\,6\,3 \end{pmatrix} = (1\,5)(2)(3\,4\,7)(6) \in \mathbb{S}_7$$

is reduced to $(1,4)(2,3,5) \in \mathbb{S}_5$, as well as the divisor $(1,6)(2,3,7)$ of the permutation $(1,6)(2,3,7)(4)(5) \in \mathbb{S}_7$ is. We see that the same well-labelled divisor stems from several permutations of higher order. Conversely, the labels of $\delta \in \mathbb{S}_m$ from 1 to m, where $m \leqslant n$, can be substituted by an m-subset of $\mathbb{N}_n$ so that the former order is preserved. Any from $\binom{n}{m}$ strictly increasing functions $\psi : \mathbb{N}_m \to \mathbb{N}_n$ can be applied in the process, called *expansion*. In this way, we obtain such number of weakly labelled δ'.

An arbitrary pair of permutations $\delta \in \mathbb{S}_m$, $0 \leqslant m \leqslant n$, and $\tau \in \mathbb{S}_{n-m}$ can be expanded to some $\sigma \in \mathbb{S}_n$, namely, we firstly expand δ labels into the set $\mathbb{N}_n$. There are $\binom{n}{m}$ ways in doing this. Secondly, we relabel τ by the remaining numbers of $\mathbb{N}_n$ preserving the initial order of labels. The second step is performed in a unique way. Thus, we produce the ordered pair of sets of cycles (δ', τ') with $V(\delta') \cap V(\tau') = \emptyset$ and the decomposition into a product of two divisors $\sigma = \delta'\tau' \in \mathbb{S}_n$. The two steps give $\binom{n}{m}$ different standard permutations $\sigma \in \mathbb{S}_n$. We denote their set by

$$\delta \star \tau = \bigl\{\sigma = \delta'\tau' \colon \exists \text{ some expansion of } (\delta, \tau) \text{ to } (\delta', \tau')\bigr\}.$$

This is a subset of $\mathbb{S}_n$ of cardinality $\#\delta \star \tau = \binom{n}{m}$.

Having the above toolkit, we may explore proportions of divisors with specific sizes to their total number $2^{w(\sigma)}$. Actually, this has been started in the recent preprint by S.-K. Leung [15] with Theorem 1.3 concerning the sizes of fixed sets of a uniformly sampled random $\sigma \in \mathbb{S}_n$. His two-dimensional result reads as follows:

$$\frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} 2^{-w(\sigma)} \sum_{\substack{\mathbb{N}_n = A_1 \cup A_2 \\ \sigma(A_i) = A_i,\, i=1,2}} \mathbf{1}\{\#A_1 \leqslant nu\} = \frac{2}{\pi} \arcsin \sqrt{u} + O\bigl(n^{-1/2}\bigr).$$

As we have mentioned, the inner sum equals the number of decompositions into product of divisors $\sigma = \delta\tau$ with the size $|\delta| \leqslant un$. The presented asymptotic formula corresponds to the DDT theorem [8] mentioned at the beginning of the paper. By using the notion of a *divisor* instead of the notion of a *fixed set* of permutations, we can draw closer to the number theoretical investigations and leverage the ideas that have already been implemented.

In the present paper, we demonstrate that other beta distribution functions can also appear as the limits for ratios of sums over divisors. Avoiding too cumbersome calculations, we confine ourselves to specified classes of multiplicative functions whose definitions trace back to the papers [16] and [20].

Let $\mathcal{S}$ be the power set of cycles, that is, the set of all subsets of the cycle set on $\mathbb{N}$. The weakly labelled cycles are allowed. We call a function $q : \mathcal{S} \to \mathbb{R}$ *structure dependent* if its values depend on the lengths of cycles appearing in the argument including their multiplicities. Formally, if $k_j(\delta)$ is the number of cycles $\varkappa_i$ of length j involved in δ, then we require that $q(\delta)$ is a function depending on the cycle structure vector

$$\bar{k}(\delta) := \bigl(k_1(\delta), \dots, k_{|\delta|}(\delta)\bigr), \quad \delta \in \mathcal{S}.$$

Denote

$$J_\delta := \bigl\{j\colon\ k_j(\delta) \geqslant 1\bigr\}, \quad 1 \leqslant j \leqslant |\delta|.$$

A nonzero structure-dependent function q defined on $\mathcal{S}$ will be called *multiplicative* if, for every pair $\delta, \tau \in \mathcal{S}$ such that $J_\delta \cap J_\tau = \emptyset$, the following relation holds:

$$q(\delta\tau) = q(\delta)q(\tau). \tag{2}$$

Equivalently, a function having the decomposition

$$q(\delta) = \prod_{j \leqslant |\delta|} q_j\bigl(k_j(\delta)\bigr) \tag{3}$$

with some mappings $q_j : \mathbb{Z}_+ \to \mathbb{R}$ such that $q_j(0) = 1$, $j \geqslant 1$, is multiplicative. In the sequel, the multiplicative functions f, h, and q will have expressions as in (3) with $f_j(\cdot)$, $h_j(\cdot)$, and $q_j(\cdot)$, respectively. If (2) is satisfied for every pair δ and τ, then q will be called *completely* multiplicative. Then in (3), we have $q_j(k_j) = q_j(1)^{k_j}$. Set $\mathcal{M}$ and $\mathcal{M}_c$ for the classes of multiplicative and completely multiplicative functions. Observe that the function

$$f(\sigma) := \sum_{\delta|\sigma} g(\delta) \tag{4}$$

belongs to $\mathcal{M}$ and $\mathcal{M}_c$ if $g \in \mathcal{M}$ and $g \in \mathcal{M}_c$, respectively. In the second case, if $\vartheta_j := g_j(1)$, $\sigma \in \mathbb{S}_n$, then

$$f(\sigma) = \prod_{j \leqslant n} f_j\bigl(k_j(\sigma)\bigr), \qquad f_j(k) = \sum_{l=0}^{k} \binom{k}{l} \vartheta_j^l = (1 + \vartheta_j)^k.$$

In particular, the number-of-divisors function 2^w expressed by (4) with $g = 1$ belongs to $\in \mathcal{M}_c$.

Given a nonnegative $g \in \mathcal{M}$ and f defined by (4), we introduce

$$X(u) := X(u; g) = \frac{1}{f(\sigma)} \sum_{\substack{\delta|\sigma \\ |\delta| \leqslant un}} g(\delta), \quad \sigma \in \mathbb{S}_n,\ 0 \leqslant u \leqslant 1.$$

If $\sigma \in \mathbb{S}_n$ is taken at random, then $X(u)$ is a fairly mysterious random process with paths being cumulative distribution functions. With the present note, we start its theory, discussing the asymptotic behaviour of the expectation. Avoiding technical complications, but still getting the vast class of possible limits, we confine ourselves with nonnegative functions $g \in \mathcal{M}_c$. Apart from the uniform measure in $\mathbb{S}_n$, the weighted ones also raise an interest. One defines them in terms of nonnegative functions $p \in \mathcal{M}_c$ by setting the point probabilities

$$\mu_n^{(p)}\big(\{\sigma\}\big) := \frac{p(\sigma)}{P_n} := p(\sigma)\biggl(\sum_{\sigma\in\mathbb{S}_n} p(\sigma)\biggr)^{-1}, \quad \sigma \in \mathbb{S}_n,\ n \geqslant 1.$$

Let us agree on $\mu_0^{(p)}(\{\emptyset\}) = 1$. If $p = \theta^w$, where $\theta > 0$ is fixed, and, as above, $w = w(\cdot)$ stands for the number-of-cycles function, then $\widehat{\mu}_n^{(\theta)} := \mu_n^{(\theta^w)}$ is called the *Ewens* measure. The applications of the latter can be hardly overestimated (see [1, Chaps. 4 and 5] or [7]). Let $\widehat{\mathbf{E}}_n^{(\theta)}$ denote the expectation with respect to $\widehat{\mu}_n^{(\theta)}$. If $\theta = 1$, we return to the uniform measure. For brevity, then we put $\mathbf{E}_n = \widehat{\mathbf{E}}_n^{(1)}$.

The goal of the paper is to examine the expectation of the process $X(u)$ with respect to $\mu_n^{(p)}$, that is, the sequence

$$\mathbf{E}_n^{(p)} X(u; g) := \frac{1}{P_n} \sum_{\sigma\in\mathbb{S}_n} \frac{p(\sigma)}{f(\sigma)} \sum_{\substack{\delta|\sigma \\ |\delta|\leqslant un}} g(\delta). \tag{5}$$

At the very beginning, we discover that the distribution functions

$$B_n(u; a, b) := \binom{a+b+n-1}{n}^{-1} \sum_{0\leqslant m\leqslant un} \binom{a+m-1}{m}\binom{b+n-m-1}{n-m},$$

where $0 \leqslant u \leqslant 1$ and $a, b > 0$, called quasihypergeometric (see [13, formula (70)]) and appearing in some urn models, play the crucial role in our task. Secondly, we observe that, for large n, $B_n(u; a, b)$ lie close to the beta distribution function defined at the beginning of Section 1. Set $a \wedge b := \min\{a, b\}$ if $a, b \in \mathbb{R}$.

Theorem 1. *Let $\theta, \vartheta > 0$ be arbitrary fixed numbers, w be the number-of-cycles function, and $g = \vartheta^w$. Then*

$$\widehat{\mathbf{E}}_n^{(\theta)} X\big(u; \vartheta^w\big) = B_n(u; \alpha, \beta) = B(u; \alpha, \beta) + O\big(n^{-\gamma}\big),$$

uniformly in $0 \leqslant u \leqslant 1$ and $n \geqslant 1$. Here

$$\alpha := \frac{\vartheta\theta}{\vartheta + 1}, \qquad \beta := \frac{\theta}{\vartheta + 1}, \qquad \gamma := \alpha \wedge \beta \wedge 1. \tag{6}$$

The established approximation opens a new possibility to model the beta distribution $B(u; \alpha, \beta)$ with arbitrary $\alpha, \beta > 0$. For this, by (6), it suffices to use the functions in Theorem 1 with $\vartheta = \alpha/\beta$ and $\theta = \alpha + \beta$. The analogous models proposed in number theory [3] reach only the logarithmic approximation order. If $\vartheta = \theta = 1$, we obtain alternatively formulated Leung's result:

$$\mathbf{E}_n X(u;1) = \frac{2}{\pi}\arcsin(\sqrt{u}\,) + O\bigl(n^{-1/2}\bigr).$$

The uniform distribution appears in the limit when $\vartheta = 1$ and $\theta = 2$. In this case the direct calculations show that

$$\widehat{\mathbf{E}}_n^{(2)} X(u;1) = \frac{\lfloor un \rfloor + 1}{n+1} = u + O\bigl(n^{-1}\bigr). \tag{7}$$

This witnesses that, apart from the constants, the remainder term estimate given in Theorem 1 is optimal.

One can ask to describe the general class of possible multiplicative functions g, p available to model the beta distribution in this way. The next theorems give partial answers.

Theorem 2. *Let $n \geqslant 1$, $g, p \in \mathcal{M}_c$ be defined via $g_j(1) =: \vartheta_j \geqslant 0$ and $p_j(1) =: \theta_j \geqslant 0$. Assume that for positive constants ϑ, θ, C_1, C_2, c_1, c_2, the inequalities*

$$\bigl|\vartheta_j - \vartheta\bigr| \leqslant C_1 j^{-c_1}, \qquad \bigl|\theta_j - \theta\bigr| \leqslant C_2 j^{-c_2} \tag{8}$$

hold for every $j \geqslant 1$. Then

$$\mathbf{E}_n^{(p)} X(u;g) = B(u;\alpha,\beta) + O\bigl(n^{-c}\log^2(n+1) + n^{-\gamma}\bigr),$$

where α, β, γ have been defined in (6)*, and $c = c_1 \wedge c_2$. The constant in $O(\cdot)$ depends on that listed in condition* (8) *only.*

Dropping the interest to estimate the convergence rate, we can go even further.

Theorem 3. *Let $g, p \in \mathcal{M}_c$ be defined via nonnegative bounded $g_j(1) =: \vartheta_j$ and $p_j(1) =: \theta_j$, where $j \geqslant 1$ such that the series*

$$\sum_{j\geqslant 1} \frac{\vartheta_j - \vartheta}{j}, \qquad \sum_{j\geqslant 1} \frac{\theta_j - \theta}{j}$$

converge for some positive constants ϑ and θ. Then, uniformly in $0 \leqslant u \leqslant 1$,

$$\mathbf{E}_n^{(p)} X(u;g) = B(u;\alpha,\beta) + o(1), \quad n \to \infty,$$

where α, β have been defined in (6).

Mean value theorems for multiplicative functions defined on $\mathbb{S}_m$, $1 \leqslant m \leqslant n$, play the main role in our problem. We collect them in the next section. The proofs of theorems will be given at the end of the paper.

2 Lemmata

Let us recall a few needed formulas. Afterwards, $[z^n]Q(z)$ will denote the coefficient at z^n in the formal power series expansion of $Q(z)$.

Lemma 1. *Let $q \in \mathcal{M}$ and $n \geqslant 0$, then*

$$\mathbf{E}_n q = [z^n] \prod_{j\geqslant 1}\biggl(1 + \sum_{r\geqslant 1} \frac{q_j(r) z^{rj}}{j^r r!}\biggr).$$

In particular, if $q \in \mathcal{M}_c$, then

$$\mathbf{E}_n q = [z^n] \exp\biggl\{\sum_{j\geqslant 1} \frac{q_j(1) z^j}{j}\biggr\}. \tag{9}$$

Proof. If $\mathbb{S}_n(\bar{k}) \subset \mathbb{S}_n$ is the subset of permutations σ with the cycle structure vector $\bar{k} = (k_1, \dots, k_n)$, then according to the Cauchy theorem [11, Thm. 1.2],

$$\#\mathbb{S}_n(\bar{k}) = n! \prod_{j\leqslant n} \frac{1}{j^{k_j} k_j!}.$$

By (3), the function q is constant on $\mathbb{S}_n(\bar{k})$. Hence summing over possible $\bar{k}$, we obtain

$$\mathbf{E}_n q = \sum_{\substack{1k_1+\cdots+nk_n=n\\ k_j\geqslant 0,\, j\leqslant n}} \prod_{j\leqslant n} \frac{q_j(k_j)}{j^{k_j} k_j!}.$$

The quantity on the right hand side is just the nth coefficient in the power series expansion of the product of functions as claimed in the lemma.

The second statement is a corollary of the previous one. The lemma is proved. □

Lemma 2. *If $\theta > 0$ and $n \geqslant 1$, then*

$$\mathbf{E}_n \theta^w = \binom{\theta + n - 1}{n} = \frac{n^{\theta-1}}{\Gamma(\theta)}\bigl(1 + O(n^{-1})\bigr).$$

The constant in $O(\cdot)$ depends on θ only.

Proof. The first equality follows from (9) when $q_j(1) = \theta$. The second estimate is a case of Theorem VI.1 in [10, p. 381]. □

The next lemma concerns a more general case. For short, we will also use $\ll$ as an analog of the symbol $O(\cdot)$.

Lemma 3. *Let a nonnegative completely multiplicative function q be defined via $q_j(1) = a_j$ satisfying the condition*

$$a_j - a \ll j^{-\varepsilon}, \quad j \geqslant 1, \tag{10}$$

with some $a, \varepsilon > 0$ and

$$A(z) := \sum_{j\geqslant 1} \frac{(a_j - a) z^j}{j}. \tag{11}$$

Then

$$\mathbf{E}_n q = \frac{n^{a-1}\mathrm{e}^{A(1)}}{\Gamma(a)}\bigl(1 + O\bigl(n^{-\varepsilon_1}\log(n+1)\bigr)\bigr), \quad \varepsilon_1 = 1 \wedge \varepsilon,\ n \geqslant 1.$$

Proof. By (9) in Lemma 1, we have the expression

$$\mathbf{E}_n q = \bigl[z^n\bigr](1-z)^{-a}\mathrm{e}^{A(z)}.$$

According to Lemma 4 in H.-K. Hwang's paper [12], condition (10) implies $[z^n]\mathrm{e}^{A(z)} \ll n^{-1-\varepsilon}$. Consequently, the desired asymptotic formula follows from Lemma 5 in [12].

The lemma is proved. □

Assuming an analytic continuation of the series $A(z)$ outside the unit disk, except, maybe, a sector $|\arg(z-1)| \leqslant \varepsilon_2 < \pi/2$, we could apply the so-called transfer theorems (see [9]) and get rid of the appearing logarithm in the remainder. Generalizing condition (8), we rather use the next lemma, which is based upon the information about the generating series only in the unit disk. We loose the remainder term estimate in this more general situation.

Lemma 4. *Let a nonnegative completely multiplicative function q be defined via bounded $q_j(1) = a_j$ such that the series $A(z)$ in* (11) *converges at the point $z = 1$. Then*

$$\mathbf{E}_n q = \frac{n^{a-1}\mathrm{e}^{A(1)}}{\Gamma(a)}\bigl(1 + o(1)\bigr), \quad n \to \infty.$$

Proof. See [18, Thm. 2]. □

Remark. For unbounded a_j, the asymptotic behaviour of $\mathbf{E}_n q$ raises more obstacles. The discussion including a counterexample is presented in the second author's paper [17].

Lemma 5. *If $v(x) := x^{a-1}(1-x)^{b-1}$, where $0 < x < 1$ and $a, b > 0$ are arbitrary fixed numbers, then*

$$n^{-1}\max\biggl\{v(x)\colon \frac{1}{n} \leqslant x \leqslant 1 - \frac{1}{n}\biggr\} \ll n^{-d}, \tag{12}$$

where $d := a \wedge b \wedge 1$.

If M, $1 < M \leqslant n-1$, is an integer, then

$$\frac{1}{n}\sum_{1 < m \leqslant M} v\biggl(\frac{m}{n}\biggr) = \int_{1/n}^{M/n} v(x)\,\mathrm{d}x + O\bigl(n^{-d}\bigr). \tag{13}$$

Furthermore, for $0 \leqslant u < 1$, we have

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)n}\sum_{1 \leqslant m \leqslant un} v\biggl(\frac{m}{n}\biggr) = B(u; a, b) + O\bigl(n^{-d}\bigr). \tag{14}$$

Proof. The first assertion (12) is a calculus exercise. To obtain (13), we apply the Euler–Maclaurin summation. Namely, if $x =: \lfloor x \rfloor + \langle x \rangle$ is the partition of $x \in \mathbb{R}$ into integer and fractional parts, then

$$\begin{aligned}\sum_{1<m\leqslant M} v\left(\frac{m}{n}\right) &= \int_1^M v\left(\frac{t}{n}\right) \mathrm{d}\big(\lfloor t \rfloor\big) \\ &= \int_1^M v\left(\frac{t}{n}\right) \mathrm{d}t + \frac{1}{2}\left(v\left(\frac{M}{n}\right) - v\left(\frac{1}{n}\right)\right) \\ &\quad + \int_1^M \left(\langle t \rangle - \frac{1}{2}\right) \mathrm{d}v\left(\frac{t}{n}\right).\end{aligned}$$

The second summand on the right-hand side is of the order $O(n^{1-a} + n^{1-b})$. The last integral is majored by

$$(a-1)\int_{1/n}^{1/2} t^{a-2}\,\mathrm{d}t + (1-b)\int_{1/2}^{1-1/n} (1-t)^{b-2}\,\mathrm{d}t \ll n^{1-d}.$$

Proving (14), we note that in virtue of (12), the first summand in the sum on the left-hand side is negligible. Further, we apply (13) with $M = \lfloor un \rfloor$. The estimate

$$\max\left\{B(u;a,b) - B\left(\frac{M}{n}\right)\colon\ \frac{1}{n} \leqslant u \leqslant 1 - \frac{1}{n}\right\} \ll n^{-d},$$

stemming form (12), allows to complement the integration region up to the interval $[1/n, u]$, where $1/n \leqslant u \leqslant 1 - 1/n$. That already gives (14) for such u.

If $u < 1/n$, the sum is empty, thus, covered by the remainder term, and, if $1 - 1/n \leqslant u < 1$, the sum does not change. Extending the integration interval, we observe that the remainder $O(n^{-d})$ swallows $B(u;a,b)$ if $u \leqslant 1/n$, as well as $1 - B(u;a,b)$ if $1 - 1/n \leqslant u \leqslant 1$. Consequently, (14) remains valid for $0 \leqslant u < 1$.

The lemma is proved. □

3 Proofs of theorems

The idea is seen from the next observation. The σ's in the double sum (5) have the form $\sigma = \delta\tau$ with $|\delta| \leqslant un$, $|\tau| = n - |\delta|$. Stressing that the factors δ and τ can be weakly labelled, we temporarily return to the notation used in Section 1. Namely, we set $\delta' := \delta$ and $\tau' := \tau$, leaving the characters δ and τ to denote the well-labelled permutations from the symmetric groups $\mathbb{S}_m$ and $\mathbb{S}_{n-m}$ so that δ' and τ' are just the expansion outcomes for the pair (δ, τ). In this way, we gain the possibility to split the double sum into partial sums over the disjoint subsets $\delta \star \tau$ of $\mathbb{S}_n$ as defined in Section 1. Hence

$$\begin{aligned}\mathbf{E}_n^{(p)}X(u;g) &= \frac{1}{P_n}\sum_{0\leqslant m\leqslant un}\ \sum_{\substack{\delta\in\mathbb{S}_m\\ \tau\in\mathbb{S}_{n-m}}}\ \sum_{\sigma=\delta'\tau'\in\delta\star\tau}\frac{p(\delta'\tau')g(\delta')}{f(\delta'\tau')}\\ &= \frac{1}{P_n}\sum_{0\leqslant m\leqslant un}\ \sum_{\substack{\delta\in\mathbb{S}_m\\ \tau\in\mathbb{S}_{n-m}}}\frac{p(\delta\tau)g(\delta)}{f(\delta\tau)}\sum_{\sigma=\delta'\tau'\in\delta\star\tau}1\\ &= \frac{1}{P_n}\sum_{0\leqslant m\leqslant un}\binom{n}{m}\sum_{\delta\in\mathbb{S}_m}g(\delta)\sum_{\tau\in\mathbb{S}_{n-m}}\frac{p(\delta\tau)}{f(\delta\tau)}.\end{aligned}\tag{15}$$

The last equality stems from the fact that the structure of permutations or their divisors is reduction invariant.

Check that approximating $\mathbf{E}_n^{(p)}X(u;g)$ by $B(u;\alpha,\beta)$, we may confine ourselves to the interval $0\leqslant u<1$, excluding the point $u=1$ where these distribution functions equal one.

Proof of Theorem 1. If $g=\vartheta^w$, then $f=(\vartheta+1)^w$. By (15) and Lemma 2,

$$\begin{aligned}\widehat{\mathbf{E}}_n^{(\theta)}X(u;\vartheta^w) &= \frac{n!}{P_n}\sum_{0\leqslant m\leqslant un}\mathbf{E}_m\biggl(\frac{\vartheta\theta}{\vartheta+1}\biggr)^w\mathbf{E}_{n-m}\biggl(\frac{\theta}{\vartheta+1}\biggr)^w\\ &= \frac{n!}{P_n}\sum_{0\leqslant m\leqslant un}\binom{\alpha+m-1}{m}\binom{\beta+n-m-1}{n-m},\end{aligned}\tag{16}$$

where as we have denoted in (6), $\alpha=\vartheta\theta/(\vartheta+1)$ and $\beta=\theta/(\vartheta+1)$. Furthermore, the equality $\widehat{\mathbf{E}}_n^{(\theta)}X(1;\vartheta^w)=1$ reveals that

$$\begin{aligned}\frac{P_n}{n!} &= \sum_{0\leqslant m\leqslant n}\bigl[z^m\bigr](1-z)^{-\alpha}\bigl[z^{n-m}\bigr](1-z)^{-\beta}\\ &= \bigl[z^n\bigr](1-z)^{-\theta} = \binom{\theta+n-1}{n} = \frac{n^{\theta-1}}{\Gamma(\theta)}\bigl(1+O\bigl(n^{-1}\bigr)\bigr)\end{aligned}\tag{17}$$

by Lemma 2. Together the latter two equalities yield the claimed formula with the sequence of discrete distribution functions $B_n(u;\alpha,\beta)$.

Analysing the asymptotical behaviour of $\widehat{\mathbf{E}}_n^{(\theta)}X(u;\vartheta^w)$ as $n\to\infty$, we firstly reckon the case $\alpha=\beta=1$. Evidently, formula (16) yields the desired expression (7) given in Section 1.

Let $\alpha=1$, $\beta\neq1$, and $0\leqslant u<1$. Then $\theta=1+\beta$, $\Gamma(1+\beta)=\beta\Gamma(\beta)$, and, by Lemma 2,

$$\begin{aligned}\widehat{\mathbf{E}}_n^{(\theta)}X\bigl(u;\vartheta^w\bigr) &= \beta n^{-\beta}\bigl(\bigl(1+O\bigl(n^{-1}\bigr)\bigr)\sum_{1\leqslant m\leqslant nu}(n-m)^{\beta-1}\\ &\quad+O\biggl(n^{-\beta}\sum_{1\leqslant m\leqslant n-1}(n-m)^{\beta-2}\biggr)+O\bigl(n^{-\gamma}\bigr).\end{aligned}$$

Further, we apply relation (14) to approximate the main sum and the estimate

$$\sum_{1\leqslant k\leqslant n} k^{\beta-2} \ll n^{\beta-1}+1, \tag{18}$$

valid for $\beta>0$ and $\beta\neq 1$, to bound the sums in the error terms. So we deduce that

$$\widehat{\mathbf{E}}_n^{(\theta)} X\bigl(u;\vartheta^w\bigr) = \beta\int\limits_0^u (1-x)^{\beta-1}\,\mathrm{d}x + O\bigl(n^{-\gamma}\bigr) = B(u;1,\beta)+O\bigl(n^{-\gamma}\bigr).$$

The case $\alpha\neq 1$ and $\beta=1$ is treated similarly. Then

$$\widehat{\mathbf{E}}_n^{(\theta)} X\bigl(u;\vartheta^w\bigr) = \alpha\int\limits_0^u x^{\alpha-1}\,\mathrm{d}x + O\bigl(n^{-\gamma}\bigr) = B(u;\alpha,1)+O\bigl(n^{-\gamma}\bigr).$$

Let $\alpha\neq 1$, $\beta\neq 1$, $\alpha+\beta=\theta$, and $0\leqslant u<1$. Using the definition of $v(x)$ introduced in Lemmas 5 and 2, from (16), (17), and (14), we deduce that

$$\begin{aligned}
&\widehat{\mathbf{E}}_n^{(\theta)} X\bigl(u;\vartheta^w\bigr)\\
&\quad= \frac{\Gamma(\theta)}{\Gamma(\alpha)\Gamma(\beta)}\biggl(1+O\biggl(\frac{1}{n}\biggr)\biggr)\\
&\qquad\times\biggl(\frac{1}{n}\sum_{1\leqslant m\leqslant nu} v\biggl(\frac{m}{n}\biggr)\biggl(1+O\biggl(\frac{1}{m}\biggr)\biggr)\biggl(1+O\biggl(\frac{1}{n-m}\biggr)\biggr)+O\bigl(n^{-\gamma}\bigr)\biggr)\\
&\quad= B(u;\alpha,\beta)+O(R)+O\bigl(n^{-\gamma}\bigr),
\end{aligned}$$

where

$$\begin{aligned}
R &= n^{1-\theta}\biggl(\sum_{1\leqslant m\leqslant n/2}+\sum_{n/2<m\leqslant n-1}\biggr)\bigl(m^{\alpha-2}(n-m)^{\beta-1}+m^{\alpha-1}(n-m)^{\beta-2}\bigr)\\
&\ll n^{-\alpha}\sum_{1\leqslant m\leqslant n/2} m^{\alpha-2} + n^{-\beta}\sum_{n/2\leqslant m\leqslant n-1}(n-m)^{\beta-2}\\
&\ll n^{-\alpha}\bigl(n^{\alpha-1}+1\bigr)+n^{-\beta}\bigl(n^{\beta-1}+1\bigr)\ll n^{-\gamma},
\end{aligned}$$

according to (18).

The theorem is proved. $\square$

Proof of Theorem 2. For brevity, let us introduce the completely multiplicative functions:

$$\begin{aligned}
G(\delta) &= \frac{g(\delta)p(\delta)}{f(\delta)} = \prod_{j\leqslant m}\biggl(\frac{\vartheta_j\theta_j}{1+\vartheta_j}\biggr)^{k_j(\delta)} =: \prod_{j\leqslant m}\alpha_j^{k_j(\delta)}, \quad \delta\in\mathbb{S}_m;\\
F(\tau) &= \frac{p(\tau)}{f(\tau)} = \prod_{j\leqslant n-m}\biggl(\frac{\theta_j}{1+\vartheta_j}\biggr)^{k_j(\tau)} =: \prod_{j\leqslant n-m}\beta_j^{k_j(\tau)}, \quad \tau\in\mathbb{S}_{n-m}.
\end{aligned}$$

Formula (15) yields the expression

$$\mathbf{E}_n^{(p)} X(u;g) = \bigl(\mathbf{E}_n p\bigr)^{-1} \sum_{0\leqslant m\leqslant un} \mathbf{E}_m G\, \mathbf{E}_{n-m} F. \tag{19}$$

Under the conditions listed in Theorem 2, we have

$$\alpha_j - \alpha \ll j^{-c}, \quad \beta_j - \beta \ll j^{-c}, \quad c = c_1 \wedge c_2,$$

for $j \geqslant 1$ and α, β introduced in (6). As earlier, let $\alpha_j + \beta_j = \theta_j$ and $\alpha + \beta = \theta$. Hence, by Lemma 3,

$$\mathbf{E}_n G = \frac{n^{\alpha-1}\mathrm{e}^{K_1}}{\Gamma(\alpha)}(1+r_n), \qquad \mathbf{E}_n F = \frac{n^{\beta-1}\mathrm{e}^{K_2}}{\Gamma(\beta)}(1+r_n), \tag{20}$$

where r_n is a remainder term, not the same in different places but having the order $n^{-c}\log(n+1)$, and

$$K_1 = \sum_{j\geqslant 1} \frac{\alpha_j - \alpha}{j}, \qquad K_2 = \sum_{j\geqslant 1} \frac{\beta_j - \beta}{j}.$$

Furthermore, by (19) and Lemma 1,

$$\begin{aligned}
\mathbf{E}_n p &= \bigl[z^n\bigr]\biggl(1 + \sum_{m\geqslant 1} \mathbf{E}_m G z^m\biggr)\biggl(1 + \sum_{k\geqslant 1} \mathbf{E}_k F z^k\biggr) \\
&= \bigl[z^n\bigr] \exp\biggl\{\sum_{j\geqslant 1} \frac{\theta_j z^j}{j}\biggr\} = \frac{n^{\theta-1}\mathrm{e}^{K}}{\Gamma(\theta)}(1+r_n)
\end{aligned} \tag{21}$$

with $K = K_1 + K_2$. It remains to insert the latter formulas into (19) and to complete a standard asymptotic analysis. The arguments applied proving Theorem 1 suffice. Now we have

$$\begin{aligned}
\mathbf{E}_n^{(p)} X(u;g) &= \frac{\Gamma(\theta)}{\Gamma(\alpha)\Gamma(\beta)}(1+r_n) \\
&\quad \times \biggl(\frac{1}{n} \sum_{1\leqslant m\leqslant nu} v\biggl(\frac{m}{n}\biggr)(1+r_m)(1+r_{n-m}) + O\bigl(n^{-\gamma}\bigr)\biggr) \\
&= B(u;\alpha,\beta) + O(R_1) + O\bigl(n^{-\gamma}\bigr).
\end{aligned} \tag{22}$$

Here

$$\begin{aligned}
R_1 &\ll n^{-\alpha} \sum_{1\leqslant m\leqslant n/2} m^{\alpha-c-1}\log(m+1) + n^{-\beta} \sum_{1\leqslant m\leqslant n/2} m^{\beta-c-1}\log(m+1) \\
&\ll n^{-c}\log^2(n+1) + n^{-\gamma}.
\end{aligned}$$

This estimate is valid for every fixed $c, \alpha, \beta > 0$.

The theorem is proved. $\square$

Proof of Theorem 3. We may use the notation introduced in the proof of Theorem 2 and formula (19). Our agreement $\mu_0^{(p)}(\{\emptyset\}) = 1$ and $F(\emptyset) = G(\emptyset) = 1$ yield $\mathbf{E}_0 G = 1$. According to Lemma 4, the asymptotic expressions (20) and (21) hold with the errors $r_n = o(1)$ if $n \to \infty$.

Since $\mathbf{E}_n F = O(n^{\beta-1})$, having in mind (22), we have

$$\mathbf{E}_n^{(p)} X(u; g) = B(u; \alpha, \beta) + O\bigl(R_2(n)\bigr) + O\bigl(n^{-\gamma}\bigr), \tag{23}$$

where

$$R_2(n) \ll \frac{1}{n} \sum_{1 \leqslant m \leqslant n-1} v\left(\frac{m}{n}\right)\bigl(|r_m| + |r_{n-m}|\bigr).$$

Here $r_n = o(1)$ as $n \to \infty$; therefore, for an arbitrary $\eta > 0$, there exists $N := N(\eta)$ such that $|r_n| \leqslant \eta$ if $n > N > 1$. Hence using Lemma 5,

$$R_2(n) \ll \frac{1}{n} \sum_{1 \leqslant m \leqslant N} \left(\left(\frac{m}{n}\right)^{\alpha-1} + \left(\frac{m}{n}\right)^{\beta-1}\right) + \frac{\eta}{n} \sum_{1 \leqslant m \leqslant n-1} v\left(\frac{m}{n}\right).$$

Bearing in mind (13),

$$R_2(n) \ll \left(\frac{N}{n}\right)^{\alpha} + \left(\frac{N}{n}\right)^{\beta} + \eta.$$

Hence $\overline{\lim}_{n\to\infty} R_2(n) \leqslant \eta$ giving $R_2(n) = o(1)$ as $n \to \infty$. Inserting the last estimate into (23), we complete the proof of Theorem 3. □

4 Concluding remarks

(i) The graphs below were made using k samples of trajectories $X(u; 1)$, where $g(\delta) \equiv 1$ and permutation σ is generated uniformly at random from $\mathbb{S}_n$, and cycles in the permutation were found using Python package SymPy [19]. The graphs on the left and right were generated using parameter sets $n = 100$, $k = 50$ and $n = 1000$, $k = 500$, respectively.

The bold curves in the center of Figs. 3 and 4 depict the arcsine distribution function and the empirical mean values of the curves shown in Figs. 1 and 2. In Figs. 3 and 4, we also see a band around the mean curve depicting the change in standard error, which reduces to zero at the value of 0.5. It can be seen that as the number of samples is increased, the expectation of the combinatorial process distribution approaches the arcsine distribution function. Further numerical analysis witnesses regular asymptotic behaviour as $n \to \infty$ of higher power moments of $X(u; 1)$ and of the variance. A future search of the theoretical results would be reasonable.

(ii) We began this paper studying problems formulated in terms of the associative and commutative convolution $f := g \odot h$ defined by the equality

$$f(\sigma) = \sum_{\delta\tau=\sigma} g(\delta)h(\tau), \quad \sigma \in \mathcal{S}.$$

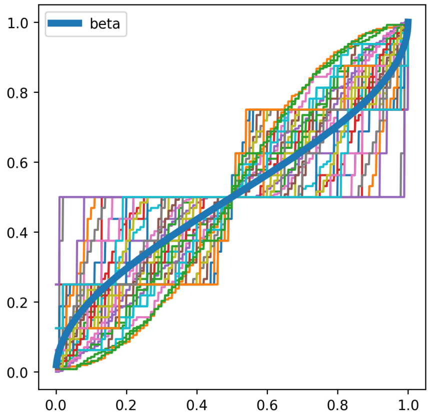

Figure 1. $k = 50$.

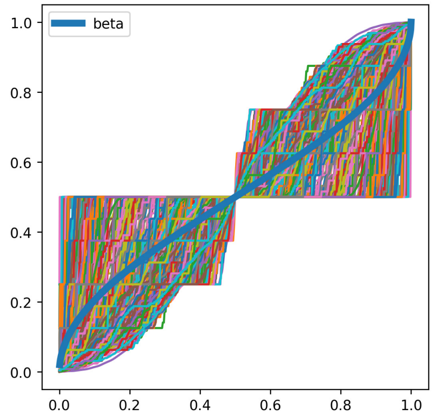

Figure 2. $k = 500$.

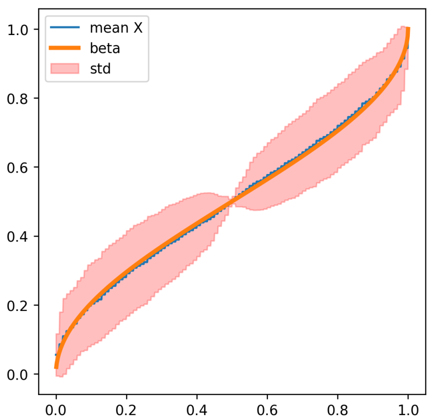

Figure 3. $k = 50$.

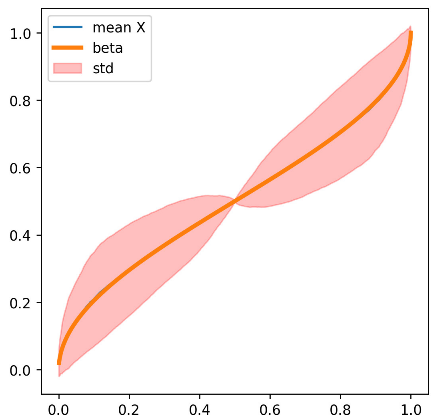

Figure 4. $k = 500$.

The resulting algebra of functions has many parallels with the algebra of number-theoretic functions on $\mathbb{N}$. Furthermore, restricted to the class $\mathcal{M}$, the identities for the summed functions

$$\sum_{\sigma\in\mathbb{S}_n} f(\sigma) = \sum_{\delta\in\mathbb{S}_m} \binom{n}{m} \sum_{\delta\in\mathbb{S}_m} g(\delta) \sum_{\tau\in\mathbb{S}_{n-m}} h(\tau), \quad n \geqslant 0,$$

lead to the product formula for the corresponding exponential generating functions. Namely, we have

$$\sum_{n\geqslant 0} \frac{1}{n!} \sum_{\sigma\in\mathbb{S}_n} f(\sigma) z^n = \left(\sum_{m\geqslant 0} \frac{1}{m!} \sum_{\delta\in\mathbb{S}_m} g(\delta) z^m\right) \left(\sum_{k\geqslant 0} \frac{1}{k!} \sum_{\tau\in\mathbb{S}_k} h(\tau) z^k\right).$$

This recalls the relation between the Dirichlet convolution and the product formula for the corresponding generating series of number-theoretic functions. Consequently, taking into account the experience when proving Theorems 1–3, one can go much further exploiting the parallelism with number theory.

(iii) Permutations are just a particular case of the so-called labelled combinatorial structures (see [10, Chap. II]). The notion of a divisor can be easily extended to other structures, together with our results.

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References

1. R. Arratia, A.D. Barbour, S. Tavaré, *Logarithmic Combinatorial Structures: A Probabilistic Approach*, EMS Publishing House, Zürich, Swiss, 2003.
2. G. Bareikis, A. Mačiulis, On the numbers of divisors in arithmetical semigroup, *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Comput.*, **39**:35–44, 2013.
3. G. Bareikis, A. Mačiulis, Modeling the beta distribution using multiplicative functions, *Lith. Math. J.*, **57**:171–182, 2017, https://doi.org/10.15388/namc.2020.25.16518.
4. G. Bareikis, A. Mačiulis, Modeling the Dirichlet distribution using multiplicative functions, *Nonlinear Anal. Model. Control*, **25**(2):282–300, 2020, https://doi.org/10.1007/s10986-017-9351-6.
5. G. Bareikis, A. Mačiulis, Bivariate Beta distribution and multiplicative functions, *Eur. J. Math.*, **7**:1668–1688, 2021, https://doi.org/10.1007/s40879-021-00492-7.
6. G. Bareikis, E. Manstavičius, On the DDT theorem, *Acta Arith.*, **126**:155–168, 2007, https://doi.org/10.4064/aa126-2-5.
7. H. Crane, The ubiquitous Ewens sampling formula, *Stat. Sci.*, **31**:1–19, 2016, https://doi.org/10.1214/15-STS529.
8. J.-M. Deshouillers, F. Dress, G. Tenenbaum, Lois de répartition des diviseurs 1, *Acta Arith.*, **34**: 273–285, 1979, https://doi.org/10.4064/aa-34-4-273-285.
9. Ph. Flajolet, A. Odlyzko, Singularity analysis of generating functions, *SIAM J. Discrete Math.*, **3**:216–240, 1990, https://doi.org/10.1137/0403019.
10. Ph. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, Cambridge, 2008, https://doi.org/10.1017/CBO9780511801655.
11. K. Ford, Cycle type of random permutations: A toolkit, *Discrete Anal.*, **9**:36, 2022, https://doi.org/10.48550/arXiv.2104.12019.
12. H.-K. Hwang, Asymptotic of Poisson approximation to random discrete distributions, *Adv. Appl. Probab.*, **31**:448–491, 1999, https://doi.org/10.1239/aap/1029955143.
13. N.L. Johnson, S. Kotz, Developments in discrete distributions, 1969–1980, *Int. Stat. Rev.*, **50**: 71–101, 1982, https://doi.org/10.2307/1402460.
14. N.L. Johnson, S. Kotz, N. Balakrishnan, Beta distributions, in *Continuous Univariate Distributions, Vol. 2*, Wiley, New York, 1995, pp. 210–275.

15. S.K. Leung, Dirichlet law for factorization of integers, polynomials and permutations, 2022, arXiv:2206.14728v2.
16. E. Manstavičius, Additive and multiplicative functions on random permutations, *Lith. Math. J.*, **36**:400–408, 1996.
17. E. Manstavičius, A Tauber theorem and multiplicative functions on permutations, in K. Győry et al. (Eds.), in *Proceedings of the International Conference Organized by the Stefan Banach International Mathematical Center in Honor of the 60th Birthday of Andrzej Schinzel, Zakopane, Poland, June 30–July 9, 1997, Vol. 2: Elementary and Analytic Number Theory*, Walter de Gruyter, Berlin, 1999, pp. 1025–1038.
18. E. Manstavičius, An asymptotic enumeration method, in A. Dubickas et al. (Eds.), in *Analytic and Probabilistic Methods in Number Theory. Proceedings of the Sixth International Conference, Palanga, Lithuania, 11–17 September 2016*, Vilnius Univ. Press, Vilnius, 2017, pp. 109–121.
19. A. Meurer, et al., Sympy: Symbolic computing in Python, *PeerJ Comput. Sci.*, **3**:e103, 2017, 10.7717/peerj-cs.103.
20. A.M. Vershik, Statistical mechanics of combinatorial structures and their limit shapes, *Funct. Anal. Appl.*, **30**:90–105, 1996, https://doi.org/10.1007/BF02509449.