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Study on the controllability of Hilfer fractional differential system with and without impulsive conditions via infinite delay

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Abstract. In this manuscript, we investigate the controllability of two different kinds of Hilfer fractional differential equations with an almost sectorial operator and infinite delay. First, we demonstrate the exact controllability of the Hilfer fractional system using the measure of noncompactness. Next, we develop the results for the controllability of the system under impulsive conditions. Finally, to show how the key findings may be utilised, applications are presented.

Keywords: Hilfer fractional derivative, measure of noncompactness, fixed point theorem, almost sectorial operators.

1 Introduction

Fractional calculus that include not only one but numerous fractional derivatives are highly concentrated in many physical processes. Because of its astounding uses in exhibiting the wonders of science and technology, fractional differential has recently received a lot of interest on its significance. Numerous problems in a number of domains, such

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as fluid flow, electrical systems, visco-elasticity, electro-chemistry, and others, can be managed through the use of fractional systems. There are many users and applications for the extension of differential equations and inequalities called differential inclusions, which may be thought of as an in optimal control theory. Dynamical systems having velocities that are not just governed by the system's state are simpler to investigate when one is skilled at using differential inclusions. Studies on boundary value issues have been conducted widely. Numerous studies have been done to determine whether there are solutions for fractional differential equations and inclusions and whether there are solutions for fractional differential inclusions. Consult research papers for further information [1, 2, 6, 7, 17–20, 25, 27, 32–34].

In both linear and nonlinear control systems, controllability is a key term that is at the foundation of mathematical control theory. In general, controllability pertains to the ability of a dynamical control system to be steered from a given beginning state to a given final state using a set of admissible control functions. Because controllability problems are important in so many domains, including engineering, biological systems, defence, controlling inflation rates in the economy, and control theory, a lot of scholars have recently dedicated themselves to studying them. The articles [3, 14, 15, 28, 31] give the excellent way to discuss the theory and applications related to controllability. In [9, 10], researchers studied the approximate controllability of fractional differential systems via fixed point approach. In [5, 13, 35], authors studied the existence of Hilfer fractional differential system using almost sectorial operators.

The impulsive effects may be seen in a variety of events and processes, including those in the sciences and engineering when the system state changes abruptly at specific times. Impulsive effects can significantly alter the behaviour of a system. It can introduce sudden changes, discontinuities, or jumps in the system's variables, leading to deviations from the expected or predicted behavior. This alteration can affect stability, convergence, and overall system dynamics. The impulsive effect can be intentionally applied to control or manipulate a system. By strategically introducing impulses, it is possible to drive the system towards desired states, induce specific behaviours, or stabilise unstable dynamics. Impulsive control strategies are employed in various fields, including engineering, physics, and biology. In the paper [8], author introduced the concept of the infinite delay on a functional impulsive differential equations. Nowadays, there are many researchers focusing on the theory and concepts related to fractional impulsive differential systems with infinite delay. We refer to [22–24, 29, 30, 38] and the references therein.

In [36], researcher studied existence of the mild solution for Hilfer fractional differential ($\mathrm{HFD_{tial}}$) equations given by

$$\begin{aligned} &{}^{H}D_{0}^{\lambda,\nu}\upsilon(\varrho) = A\upsilon(\varrho) + \varLambda\big(\varrho, \upsilon(\varrho)\big), \quad \varrho \in (0, T], \\ &I_{0+}^{(1-\lambda)(1-\nu)}\upsilon(0) = \upsilon_0. \end{aligned}$$

${}^{H}D_{0}^{\lambda,\nu}$ denotes the Hilfer fractional derivative ($\mathrm{HFD_{ve}}$) of order $0 < \lambda < 1$ and type $0 \leqslant \nu \leqslant 1$. Schauder fixed point theorem is used to prove the result, and here A is the almost sectorial operator.

In the research work [16], researchers focused on the controllability of $\mathrm{HFD}_{\mathrm{tial}}$ equations with infinite delay using MNC, the system defined as follows:

$$D_{0+}^{\gamma,\delta}\upsilon(\varrho) = A\upsilon(\varrho) + \Lambda(\varrho, y_\varrho) + Bu(\varrho), \quad \varrho \in J$$
$$I_{0+}^{(1-\gamma)(1-\delta)}\upsilon(\varrho) = h(\varrho) \in \mathcal{G}_h.$$

Here Mönch's fixed point theorem is used for the controllability of the equation.

Motivated from above works, we focus on the controllability of $\mathrm{HFD}_{\mathrm{tial}}$ equation using MNC. In this study, we will examine the following topic: $\mathrm{HFD}_{\mathrm{tial}}$ system

$$D_{0+}^{\mathsf{p},\sigma}\mathsf{q}(\mathsf{y}) = \mathsf{A}\mathsf{q}(\mathsf{y}) + \mathsf{H}(\mathsf{y}, \mathsf{q}_\mathsf{y}) + \mathsf{B}\mathsf{v}(\mathsf{y}), \quad \mathsf{y} \in \mathcal{J}' = (0, b], \tag{1}$$

$$I_{0+}^{(1-\mathsf{p})(1-\sigma)}\mathsf{q}(0) = \xi \in \mathsf{O}_\tau, \quad \mathsf{y} \in (-\infty, 0], \tag{2}$$

here A is the almost sectorial operator, $D_{0+}^{\mathsf{p},\sigma}$ denotes the $\mathrm{HFD}_{\mathrm{ve}}$ with order p, $0 < \mathsf{p} < 1$, and type σ, $0 \leqslant \sigma \leqslant 1$. Suppose $\mathsf{q}(\cdot)$ is the state in a Banach space Y with the norm $\|\cdot\|$. The histories $\mathsf{q}_\mathsf{y} : (-\infty, 0] \to \mathsf{O}_\tau$, $\mathsf{q}_\mathsf{y}(q) = \mathsf{q}(\mathsf{y} + q)$, $q \leqslant 0$, with phase space O_τ. Take $\mathcal{J} = [0, b]$. Let $\mathsf{H} : \mathcal{J} \times \mathsf{O}_\tau \to Y$ is the Y-valued function, and consider $\mathsf{v}(\cdot)$ in $L^2(\mathcal{J}, U)$, Banach space of admissible control function, the bounded linear operator $\mathsf{B} : U \to Y$.

The article consists of the following parts: Section 2 provides the theoretical concepts related to fractional differential, semigroups, phase spaces, almost sectorial operators, and measure of noncompatcness (MNC). Section 3 discussed the exact controllability of system (1)–(2). In Section 4, we continue our research to see whether we can extend to the controllability of an impulsive differential system (6). In Section 5, we provide two examples to explain our key points. Finally, some conclusions and some possible future directions for research are given.

2 Preliminaries

In this section, we present fundamental theorems, lemmas, and definitions that are used throughout the whole work.

Let us consider the set of all continuous function from $\mathcal{J}$ to Y, which is represented by Ω'' and $\mathcal{J} = [0, b]$ with $b > 0$. Take

$$\mathcal{X} = \Big\{\mathsf{q} \in \Omega'' \colon \lim_{\mathsf{y}\to 0} \mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}\mathsf{q}(\mathsf{y}) \text{ exists and finite}\Big\},$$

which is the Banach space with the norm $\|\cdot\|_\mathcal{X}$ defined by

$$\|\mathsf{q}\|_\mathcal{X} = \sup_{\mathsf{y}\in\mathcal{J}'}\big\{\mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}\big\|\mathsf{q}(\mathsf{y})\big\|\big\}.$$

Consider H with $\|\mathsf{H}\|_{L^p(\mathcal{J},\mathbb{R}^+)}$ through $\mathsf{H} \in L^p(\mathcal{J}, \mathbb{R}^+)$ for some p along with $p \in [1, \infty]$.

Definition 1. (See [37].) The RL fractional derivative of order $\mathsf{p} > 0$, $k - 1 \leqslant \mathsf{p} < k$, $k \in \mathbb{N}$, for the function $\mathsf{H} : [b, +\infty) \to \mathbb{R}$, is defined as

$${}^L D_{b+}^{\mathsf{p}}\mathsf{H}(\mathsf{y}) = \frac{1}{\Gamma(k-\mathsf{p})}\frac{\mathrm{d}^k}{\mathrm{d}\mathsf{y}^k}\int_b^{\mathsf{y}}\frac{\mathsf{H}(\lambda)}{(\mathsf{y}-\lambda)^{\mathsf{p}+1-k}}\,\mathrm{d}\lambda, \quad \mathsf{y} > b,\ \lambda \in \mathbb{R}^+.$$

Definition 2. (See [37].) The Caputo fractional derivative of order $\mathsf{p} > 0$, $k-1 \leqslant \mathsf{p} < k$, $k \in \mathbb{N}$, for the function $\mathsf{H} : [b, +\infty) \to \mathbb{R}$, is presented by

$$ {}^{C}D_{b+}^{\mathsf{p}}\mathsf{H}(\mathsf{y}) = \frac{1}{\Gamma(k-\mathsf{p})} \int_{b}^{\mathsf{y}} \frac{\mathsf{H}^{k}(\lambda)}{(\mathsf{y}-\lambda)^{\mathsf{p}+1-k}}\,\mathrm{d}\lambda = I_{b+}^{k-\mathsf{p}}\mathsf{H}^{k}(\mathsf{y}), \quad \mathsf{y} > b,\ \lambda \in \mathbb{R}^{+}. $$

Definition 3. (See [12].) The $\mathrm{HFD_{tial}}$ of order $0 < \mathsf{p} < 1$ and type $\sigma \in [0,1]$ for the function $\mathsf{H} : [b, +\infty) \to \mathbb{R}$, is presented by

$$ D_{b+}^{\mathsf{p},\sigma}\mathsf{H}(\mathsf{y}) = \left[I_{b+}^{(1-\mathsf{p})\sigma} D\left(I_{b+}^{(1-\mathsf{p})(1-\sigma)}\mathsf{H}\right)\right](\mathsf{y}). $$

In [8], we introduce the phase space, O_{τ}. Take the continuous function $\mathsf{g} : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^{0} \mathsf{g}(\mathsf{y})\,\mathrm{d}\mathsf{y} < +\infty$. Now, for every $m > 0$, define

$$ \mathsf{O} = \left\{\delta : [-m, 0] \to Y \;\middle|\; \delta(\mathsf{y}) \text{ is measurable and bounded}\right\} $$

with the norm

$$ \|\delta\|_{[-m,0]} = \sup_{\tau \in [-m,0]} \left\|\delta(\tau)\right\| \quad \forall \delta \in \mathsf{O}. $$

Next, take

$$ \mathsf{O}_{\tau} = \left\{\delta : (-\infty, 0] \to Y \;\middle|\; \forall m > 0,\ \exists \delta|_{[-m,0]} \in \mathsf{O} \text{ with } \int_{-\infty}^{0} \mathsf{g}(\tau)\|\delta\|_{[\tau,0]}\,\mathrm{d}\tau < +\infty\right\}. $$

Let O_{τ} be endowed with

$$ \|\delta\|_{\mathsf{g}} = \int_{-\infty}^{0} \mathsf{g}(\tau)\|\delta\|_{[\tau,0]}\,\mathrm{d}\tau \quad \forall \delta \in \mathsf{O}_{\tau}, $$

so $(\mathsf{O}_{\tau}, \|\cdot\|)$ is a Banach space.

Now, we consider the set

$$ \mathsf{O}_{\tau}' = \left\{\mathsf{q} : (-\infty, b] \to Y \;\middle|\; \mathsf{q} \in \Omega'',\ \xi \in \mathsf{O}_{\tau}\right\}. $$

Take the seminorm $\|\cdot\|_{\mathsf{g}}'$ in O_{τ}' defined as

$$ \|\mathsf{q}\|_{\mathsf{g}}' = \|\xi\|_{\mathsf{g}} + \sup\left\{\left\|\mathsf{q}(\tau)\right\|,\ \tau \in [0, b]\right\}, \quad \mathsf{q} \in \mathsf{O}_{\tau}'. $$

Definition 4. (See [26].) Suppose $\vartheta \in (0,1)$, $\varphi \in (0, \pi/2)$. Let $\Theta_{\varphi}^{-\vartheta}$ be the collections of closed linear operators, and let the sector C_{φ} be defined by $C_{\varphi} = \{\theta \in \mathbb{C} \setminus \{0\}: |\arg\theta| \leqslant \varphi\}$. Then the operator $\mathsf{A} : D(\mathsf{A}) \subset Y \to Y$ is said to be almost sectorial operator, which satisfies the given conditions:

(i) $\sigma'(\mathsf{A}) \subseteq C_{\varphi}$;
(ii) $\left\|(\theta I - \mathsf{A})^{-1}\right\| \leqslant L_{\delta}|\mathsf{y}|^{-\vartheta}$ for all $\varphi < \delta < \pi$, where L_{δ} is a constant,

i.e., $\mathsf{A} \in \Theta_{\varphi}^{-\vartheta}$ on Y.

Lemma 1. (*See* [26].) *Suppose* $\vartheta \in (0,1)$, $0 < \varphi \in (0, \pi/2)$, *and* $\mathsf{A} \in \Theta_{\varphi}^{-\vartheta}(Y)$. *Then*

(i) $\mathsf{T}(p_1 + p_2) = \mathsf{T}(p_1) + \mathsf{T}(p_2)$ *for any* $p_1, p_2 \in S^0_{\pi/2-\lambda}$;

(ii) *There exists the constant* $\kappa_0 > 0$ *such that* $\|\mathsf{T}(\mathsf{y})\|_{\Omega''} \leqslant \kappa_0 \mathsf{y}^{\vartheta-1}$ *for all* $\mathsf{y} > 0$;

(iii) *Let* $R(\mathsf{T}(\mathsf{y}))$ *be the range of* $\mathsf{T}(\mathsf{y})$, *and let* $z \in S^0_{\pi/2-\lambda}$ *in* $D(\mathsf{A}^\infty)$. *Mainly,* $R(\mathsf{T}(\mathsf{y})) \subset D(\mathsf{A}^\theta)$ *for all* $\theta \in \mathbb{C}$ *with* $\mathrm{Re}(\theta) > 0$,

$$\mathsf{A}^\theta \mathsf{T}(\mathsf{y})\mathsf{x} = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma_\gamma} \mathsf{y}^\theta \mathrm{e}^{-\mathsf{y}z} R(\mathsf{y}; \mathsf{A})\mathsf{x}\,\mathrm{d}z \quad \forall \mathsf{x} \in Y,$$

and there exists a constant $\kappa' = \kappa'(\beta, \theta) > 0$ *such that*

$$\left\|\mathsf{A}^\theta \mathsf{T}(\mathsf{y})\right\|_{B(Y)} \leqslant \kappa' \mathsf{y}^{-\beta-\mathrm{Re}(\theta)-1} \quad \forall \mathsf{y} > 0.$$

Let us consider the family of operators $\{\mathcal{S}_\mathsf{p}(\mathsf{y})\}_{\mathsf{y}\in S_{\pi/2-\varphi}}$ and $\{\mathcal{Q}_\mathsf{p}(\mathsf{y})\}_{\mathsf{y}\in S_{\pi/2-\varphi}}$ defined as

$$\mathcal{S}_\mathsf{p}(\mathsf{y}) = \int\limits_0^\infty \mathsf{W}_\mathsf{p}(\epsilon)\mathsf{T}\big(\mathsf{y}^\mathsf{p}\epsilon\big)\,\mathrm{d}\epsilon \quad \text{and} \quad \mathcal{Q}_\mathsf{p}(\mathsf{y}) = \int\limits_0^\infty \mathsf{p}\epsilon \mathsf{W}_\mathsf{p}(\epsilon)\mathsf{T}\big(\mathsf{y}^\mathsf{p}\epsilon\big)\,\mathrm{d}\epsilon,$$

where $\mathsf{W}_\mathsf{p}(\beta)$ is the following Wright-type function:

$$\mathsf{W}_\mathsf{p}(\beta) = \sum_{k\in\mathbb{N}} \frac{(-\beta)^{k-1}}{\Gamma(1-\mathsf{p}k)(k-1)!}, \quad \beta \in \mathbb{C}.$$

Lemma 2. *System* (1)–(2) *is equivalent to a integral equation stated by*

$$\mathsf{q}(\mathsf{y}) = \frac{\xi}{\Gamma(\sigma(1-\mathsf{p})+\mathsf{p})}\mathsf{y}^{(1-\mathsf{p})(\sigma-1)+\mathsf{p})} + \frac{1}{\Gamma(\mathsf{p})}\int\limits_0^\mathsf{y} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\big[\mathsf{A}\mathsf{q}_\lambda + \mathsf{H}\big(\lambda, \mathsf{q}_\lambda\big) + \mathsf{B}\mathsf{v}(\lambda)\big]\,\mathrm{d}\lambda.$$

Proof. The proof is similar to that of Lemma 2.12 in [11], so we omit it. □

Definition 5. The mild solution of system (1)–(2) is a function $\mathsf{q}(\mathsf{y}) \in C(\mathcal{J}', Y)$ such that

$$\mathsf{q}(\mathsf{y}) = \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})\xi + \int\limits_0^\mathsf{y} \mathcal{K}_\mathsf{p}(\mathsf{y}-\lambda)\mathsf{H}(\lambda, \mathsf{q}_\lambda)\,\mathrm{d}\lambda + \int\limits_0^\mathsf{y} \mathcal{K}_\mathsf{p}(\mathsf{y}-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda, \quad \mathsf{y} \in \mathcal{J}, \tag{3}$$

where

$$\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y}) = I_{0^+}^{\sigma(1-\mathsf{p})}\mathcal{K}_\mathsf{p}(\mathsf{y}) \quad \text{and} \quad \mathcal{K}_\mathsf{p}(\mathsf{y}) = \mathsf{y}^{\mathsf{p}-1}\mathcal{Q}_\mathsf{p}(\mathsf{y}),$$

i.e.,

$$\mathsf{q}(\mathsf{y}) = \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})\xi + \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{H}(\lambda, \mathsf{q}_\lambda)\,\mathrm{d}\lambda + \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda.$$

Lemma 3. (*See* [36].) *Assume that* $\mathsf{T}(\mathsf{y})$ *is equicontinuous, then* $\mathcal{Q}_{\mathsf{p}}(\mathsf{y})$, $\mathcal{K}_{\mathsf{p}}(\mathsf{y})$, *and* $\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})$ *are strongly continuous, that is, for all* $y \in Y$ *and* $\mathsf{y}_2 > \mathsf{y}_1 > 0$,

$$\big\|\mathcal{Q}_{\mathsf{p}}(\mathsf{a}_2)\mathsf{q} - \mathcal{Q}_{\mathsf{p}}(\mathsf{a}_1)\mathsf{q}\big\| \to 0, \qquad \big\|\mathcal{K}_{\mathsf{p}}(\mathsf{a}_2)\mathsf{q} - \mathcal{K}_{\mathsf{p}}(\mathsf{a}_1)\mathsf{q}\big\| \to 0,$$
$$\big\|\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{a}_2)\mathsf{q} - \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{a}_1)\mathsf{q}\big\| \to 0 \quad \textit{as } \mathsf{a}_2 \to \mathsf{a}_1.$$

Lemma 4. (*See* [36].) *For any fixed* $\mathsf{y} > 0$, $\mathcal{Q}_{\mathsf{p}}(\mathsf{a})$, $\mathcal{K}_{\mathsf{p}}(\mathsf{a})$, *and* $\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{a})$ *are linear operators, and for all* $\mathsf{q} \in Y$,

$$\big\|\mathcal{Q}_{\mathsf{p}}(\mathsf{a})\mathsf{q}\big\| \leqslant L'\mathsf{a}^{-\mathsf{p}+\mathsf{p}\vartheta}\|\mathsf{q}\|, \qquad \big\|\mathcal{K}_{\mathsf{p}}(\mathsf{a})\mathsf{q}\big\| \leqslant L'\mathsf{a}^{-1+\mathsf{p}\vartheta}\|\mathsf{q}\|,$$
$$\big\|\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{a})\mathsf{q}\big\| \leqslant L''\mathsf{a}^{-1+\sigma-\mathsf{p}\sigma+\mathsf{p}\vartheta}\|\mathsf{q}\|,$$

where

$$L' = \kappa_0\frac{\Gamma(\vartheta)}{\Gamma(\mathsf{p}\vartheta)}, \qquad L'' = \kappa_0\frac{\Gamma(\vartheta)}{\Gamma(\sigma(1-\mathsf{p})+\mathsf{p}\vartheta)}.$$

Lemma 5. (*See* [8].) *Suppose* $\mathsf{q} \in \mathsf{O}'_\tau$, *then for* $\mathsf{y} \in \mathcal{J}$, $\mathsf{q}_{\mathsf{y}} \in \mathsf{O}_\tau$. *Moreover,*

$$l\big|\mathsf{q}(\mathsf{y})\big| \leqslant \|\mathsf{q}_{\mathsf{y}}\|_{\mathsf{g}} \leqslant \|\xi\|_{\mathsf{g}} + l\sup_{r\in[0,\mathsf{y}]}\big|\mathsf{q}(r)\big|, \quad l = \int_{-\infty}^{0} \mathsf{g}(\mathsf{y})\,\mathrm{d}\mathsf{y} < \infty.$$

Definition 6. Suppose O is the bounded set in a Banach space Y, the Hausdorff MNC η is given by

$$\eta(\mathsf{O}) = \inf\big\{\epsilon > 0\colon\ \mathsf{O} \text{ can be covered by a finite number of balls with radii } \epsilon\big\}.$$

Lemma 6. (*See* [4].) *If* $E \subset C([p_1, p_2], Y)$ *is bounded and equicontinuous, then* $\eta(E(\mathsf{y}))$ *is continuous for* $p_1 \leqslant \mathsf{y} \leqslant p_2$, *and*

$$\eta(E) = \sup\big\{\eta\big(E(\mathsf{y})\big),\ p_1 \leqslant \mathsf{y} \leqslant p_2\big\},$$

where $E(\mathsf{y}) = \{y(\mathsf{y}),\ y \in E\} \subseteq Y$.

Lemma 7. (*See* [4].) *Suppose* Y *is a Banach space and* $G_1, G_2 \subseteq Y$ *are bounded. Then the following statements hold*:

(i) G_1 *is precompact iff* $\eta(G_1) = 0$;
(ii) $\eta(G_1) = \eta(\overline{G}_1) = \eta(\mathrm{conv}(G_1))$, *where* $\mathrm{conv}(G_1)$ *and* $\overline{G}_1$ *denote the convex hull and closure of* G_1, *respectively*;

(iii) *If* $G_1 \subseteq G_2$, *then* $\eta(G_1) \leqslant \eta(G_2)$;
(iv) $\eta(G_1+G_2) \leqslant \eta(G_1)+\eta(G_2)$, *where* $G_1+G_2 = \{b_1+b_2\colon b_1 \in G_1, b_2 \in G_2\}$;
(v) $\eta(G_1 \cup G_2) \leqslant \max\{\eta(G_1), \eta(G_2)\}$;
(vi) $\eta(\lambda G_1) = |\lambda|\eta(G_1$ *for every* $\lambda \in \mathbb{R}$ *when* Y *is a real Banach space*;
(vii) *If the operator* $\Psi : D(\Psi) \subseteq Y \to Y_1$ *is Lipschitz continuous and* Λ_1 *is the constant, then we know that* $\rho(\Psi(G_1)) \leqslant \Lambda_1\eta(G_1)$ *for any bounded subset* $G_1 \subset D(\Psi)$, *where* ρ *represents the Hausdorff MNC in the Banach space* Y_1.

Theorem 1. (*See* [32]. *If* $\{\mathsf{q}_k\}_{k=1}^{\infty}$ *is a set of Bochner-integrable functions from* $\mathcal{J}$ *to* Y *with the estimate property* $\|\mathsf{q}_k(\mathsf{y})\| \leqslant \mu_1(\mathsf{y})$ *for almost all* $\mathsf{y} \in \mathcal{J}$ *and every* $k \geqslant 1$, *where* $\mu_1 \in L^1(\mathcal{J}, \mathbb{R})$, *then the function* $\varphi(\mathsf{y}) = \mu(\{\mathsf{q}_k(\mathsf{y}), k \geqslant 1\})$ *is in* $L^1(\mathcal{J}, \mathbb{R})$ *and satisfies*

$$\mu\left(\left\{\int_0^{\mathsf{y}} \mathsf{q}_k(\omega)\,\mathrm{d}\omega,\ k \geqslant 1\right\}\right) \leqslant 2\int_0^{\mathsf{y}} \varphi(\omega)\,\mathrm{d}\omega.$$

Lemma 8. (*See* [4].) *If* $G \subset C([a,b], Y)$ *is bounded and equicontinuous, then* $\eta(G(\mathsf{y}))$ *is continuous for* $a \leqslant \mathsf{y} \leqslant b$, *and*

$$\eta(G) = \sup\bigl\{\eta\bigl(G(\mathsf{y})\bigr),\ a \leqslant \mathsf{y} \leqslant b\bigr\},$$

where $G(\mathsf{y}) = \{\mathsf{q}(\mathsf{y}), \mathsf{q} \in G\} \subseteq Y$.

Lemma 9. (*See* [21].) *Let* E *be a closed convex subset of a Banach space* Y, *and let* $0 \in E$. *Assume that* $\mathsf{H} : E \to Y$ *is continuous map satisfying Mönch's condition, i.e.,* $E_1 \subset E$ *is countable, and* $E_1 \subset \operatorname{conv}(\{0\} \cup \mathsf{H}(E_1))$ *implies that* $\overline{E_1}$ *is compact. Then* H *has a fixed point in* E.

3 Controllability

We require the following hypotheses:

(R1) The operator A generated semigroup $\mathsf{T}(\mathsf{y})$ satisfies $\|\mathsf{T}(\mathsf{y})\| \leqslant \mathcal{K}_1$, where $\mathcal{K}_1 \geqslant 0$ is the constant.

(R2) $\mathsf{H} : \mathcal{J} \times \mathsf{O}_\tau \to Y$ is the function with:
- (i) $\mathsf{H}(\cdot, \mathsf{q})$ is strongly measurable for all $\mathsf{q} \in \mathsf{O}_\tau$, $\mathsf{H}(\mathsf{y}, \cdot)$ is continuous for a.e. $\mathsf{y} \in \mathcal{J}$, and $\mathsf{H}(\cdot,\cdot) : [0,b] \to Y$ is strongly measurable;
- (ii) There exist $0 < \mathsf{p}_1 < \mathsf{p}$, $m \in L^{1/\mathsf{p}_1}(\mathcal{J}, \mathbb{R}^+)$, and nondecreasing continuous function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\mathsf{H}(\mathsf{y}, \mathsf{q})\| \leqslant m(\mathsf{y})h(\|\mathsf{q}\|)$, $\mathsf{q} \in Y$, $\mathsf{y} \in \mathcal{J}$, where h satisfies $\liminf_{k\to\infty} h(k)/k = 0$;
- (iii) There exists a constant $0 < \mathsf{p}_2 < \mathsf{p}$ and $h \in L^{1/\mathsf{p}_2}(\mathcal{J}, \mathbb{R}^+)$ such that, for all bounded subsets $\mathsf{O} \subset Y$, $\eta(\mathsf{H}(\mathsf{y}, \mathsf{O})) \leqslant h(\mathsf{y})\eta(\mathsf{O})$ for a.e. $\mathsf{y} \in \mathcal{J}$.

(R3) (i) Let $\mathsf{B} : L^2(\mathcal{J}, U) \to L^1(\mathcal{J}, Y)$ be the bounded linear operator, $\mathsf{L} : L^2(\mathcal{J}, U) \to Y$ defined by $\mathsf{L}\mathsf{v} = \int_0^b (b-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(b-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda$ has an inverse operator L^{-1}, which take the values in $L^2(\mathcal{J}, U)/\ker\mathsf{L}$, and $\mathcal{K}_2, \mathcal{K}_3 > 0$ with $\|\mathsf{B}\|_{L^2(U,Y)} \leqslant \mathcal{K}_2$, $\|\mathsf{L}^{-1}\|_{L^2(Y,U/\ker\mathsf{L})} \leqslant \mathcal{K}_3$;

(ii) There exists a constant $\mathsf{p}_0 \in (0, \mathsf{p})$ and $K_{\mathsf{J}} \in L^{1/\mathsf{p}_0}(\mathcal{J}, \mathbb{R}^+)$ such that, for all bounded sets $Q \subset Y$, $\eta((\mathsf{L}^{-1}Q)(\mathsf{y})) \leqslant K_{\mathsf{L}}(\mathsf{y})\eta(Q)$.

To make things easier for us, we introduce

$$K_{\mathsf{p}_j} = \frac{\mathsf{p}_j - 1}{\mathsf{p}\vartheta - 1} b^{(\mathsf{p}\vartheta-1)/(\mathsf{p}_j-1)}, \quad j = 1, 2,$$
$$\mathcal{K}_4 = K_{\mathsf{p}_1}\|h\|_{L^{1/\mathsf{p}_1}(\mathcal{J},\mathbb{R}^+)} \quad \text{and} \quad \mathcal{K}_5 = K_{\mathsf{p}_2}^2\|h\|_{L^{1/\mathsf{p}_2}(\mathcal{J},\mathbb{R}^+)}.$$

Theorem 2. *Suppose* (R1)–(R3) *hold, then the HF system* (1)–(2) *has a solution on* $\mathcal{J}$ *with* $\widehat{\mathcal{K}} = b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}(L'\mathcal{K}_4 + L'^2\mathcal{K}_2\mathcal{K}_5K_{\mathsf{L}}) < 1$ *and* $\theta > 1 + \vartheta$.

Proof. Consider the operator $\Psi : \mathsf{O}'_\tau \to \mathsf{O}'_\tau$ given by

$$\Psi\big(\mathsf{q}(\mathsf{y})\big) = \begin{cases} \Psi_1(\mathsf{y}), \quad (-\infty, 0], \\ \mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}[\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})\xi \\ \quad + \int_0^{\mathsf{y}}(\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{H}\big(\lambda, \mathsf{q}_\lambda\big)\,\mathrm{d}\lambda \\ \quad + \int_0^{\mathsf{y}}(\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda], \quad \mathsf{y} \in \mathcal{J}. \end{cases}$$

For $\Psi_1 \in \mathsf{O}_\tau$, we define $\widehat{\Psi}$ by

$$\widehat{\Psi}(\mathsf{y}) = \begin{cases} \Psi_1(\mathsf{y}), & \mathsf{y} \in (-\infty, 0], \\ \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})\xi, & \mathsf{y} \in \mathcal{J}, \end{cases}$$

then $\widehat{\Psi} \in \mathsf{O}'_\tau$. Let $\mathsf{q}(\mathsf{y}) = y(\mathsf{y}) + \widehat{\Psi}(\mathsf{y})$, $-\infty < \mathsf{y} \leqslant b$, q fulfill from a simple standpoint from (3) iff y satisfies $y_0 = 0$ and

$$y(\mathsf{y}) = \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{H}\big(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda\big)\,\mathrm{d}\lambda$$
$$+ \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda,$$

where

$$\mathsf{v}(\mathsf{y}) = \mathsf{L}^{-1}\Bigg[\mathsf{q}_1 - \mathcal{S}_{\mathsf{p},\sigma}(b)\xi - \int_0^b (b-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(b-\lambda)\mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda\Bigg].$$

Let $\mathsf{O}''_\tau = \{y \in \mathsf{O}'_\tau\colon\ y_0 \in \mathsf{O}_\tau\}$. For any $y \in \mathsf{O}'_\tau$,

$$\|y\|'_{\mathsf{g}} = \|y_0\|_{\mathsf{g}} + \sup\big\{\|y(\lambda)\|,\, 0 \leqslant \lambda \leqslant b\big\} = \sup\big\{\big\|y(\lambda)\big\|,\, 0 \leqslant \lambda \leqslant b\big\}.$$

Thus, $(\mathsf{O}''_\tau, \|\cdot\|'_{\mathsf{g}})$ is a Banach space.

Let $\mathsf{P} > 0$, take $\mathsf{O}_{\mathsf{P}} = \{y \in \mathsf{O}''_\tau\colon\ \|y\|_{\mathsf{g}} \leqslant \mathsf{P}\}$, then $\mathsf{O}_{\mathsf{P}} \subset \mathsf{O}''_\iota$ is uniformly bounded, and for $y \in \mathsf{O}_P$, from Lemma 5

$$\|y_{\mathsf{y}} + \widehat{\Psi}_{\mathsf{y}}\|_{\mathsf{g}} \leqslant \|y_{\mathsf{y}}\|_{\mathsf{g}} + \|\widehat{\Psi}\|_{\mathsf{g}} \leqslant l\big(\mathsf{P} + L''\mathsf{y}^{-1+\sigma-\mathsf{p}\sigma+\mathsf{p}\vartheta}\big) + \|\Psi_1\|_{\mathsf{g}} = \mathsf{P}'.$$

Consider the operator $\Phi : \mathsf{O}''_\tau \to \mathsf{O}''_\tau$ defined as

$$\Phi y(\mathsf{y}) = \begin{cases} 0, & \mathsf{y} \in (-\infty, 0], \\ \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\, \mathrm{d}\lambda \\ \quad + \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{B}\mathsf{v}(\lambda)\, \mathrm{d}\lambda, & \mathsf{y} \in \mathcal{J}. \end{cases}$$

Now, we prove that Φ has a fixed point.

Step 1. Show that $\Phi(\mathsf{O}_\mathsf{P}) \subseteq \mathsf{O}_\mathsf{P}$.

Suppose that the condition is not true, so, for all $\mathsf{P} > 0$, there exists $y^\mathsf{P} \in \mathsf{O}_\mathsf{P}$, but $\Phi(y^\mathsf{P}) \notin \mathsf{O}_\mathsf{P}$,

$$\begin{aligned} \mathsf{P} &< \sup \mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \big\| \Phi\big(y^\mathsf{P}(\mathsf{y})\big) \big\| \\ &\leqslant b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \big\| \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{H}\big(\lambda,\, y_\lambda^\mathsf{P} + \widehat{\Psi}_\lambda\big) \big\|\, \mathrm{d}\lambda \\ &\quad + b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \big\| \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{B}\mathsf{v}^\mathsf{P}(\lambda) \big\|\, \mathrm{d}\lambda \\ &\leqslant b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \Bigg[\int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \big\| \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \big\| \big\| \mathsf{H}\big(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda\big) \big\| \\ &\quad + \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \big\| \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \big\| \\ &\qquad \times \Bigg\| \mathsf{B} W^{-1} \Bigg(\mathsf{q}_1 - \mathfrak{S}_{\mathsf{p},\sigma}(b)\xi - \int_0^b (b-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(b-\lambda) \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda) \Bigg)\, \mathrm{d}\lambda \Bigg\| \Bigg] \\ &\leqslant b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \bigg[M_1^* + L' \mathcal{K}_2 \mathcal{K}_3 \frac{b^{\mathsf{p}\vartheta}}{\mathsf{p}\vartheta} \big(\mathsf{q}_1 - L'' b^{-1+\sigma-\mathsf{p}\sigma+\mathsf{p}\vartheta} - M_1^* \big) \bigg]. \end{aligned}$$

Dividing P in both side, from the limit value we get a contradiction to our assumption (here $M_1^* = L'(b^{\mathsf{p}\vartheta}/\mathsf{p}\vartheta) m(b) h(\mathsf{P}'))$. So, $\Phi(\mathsf{O}_\mathsf{P}) \subseteq \mathsf{O}_\mathsf{P}$.

Step 2. Prove that Φ is continuous on O_P.

Φ maps O_P into O_P. For any $y^m, y \in \mathsf{O}_\mathsf{P}$, $m = 0, 1, 2, \dots$, with $\lim_{m\to\infty} y^m = y$, we know $\lim_{m\to\infty} y^m(\mathsf{y}) = y(\mathsf{y})$ and $\lim_{m\to\infty} \mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} y^m(\mathsf{y}) = \mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} y(\mathsf{y})$.

By (R2),

$$\mathsf{H}\big(\mathsf{y}, \mathsf{q}_m(\mathsf{y})\big) = \mathsf{H}\big(\mathsf{y},\, y_\mathsf{y}^m + \widehat{\Psi}_\mathsf{y}\big) \to \mathsf{H}(\mathsf{y},\, y_\mathsf{y} + \widehat{\Psi}_\mathsf{y}) = \mathsf{H}(\mathsf{y}, \mathsf{q}_\mathsf{y}) \quad \text{as } m \to \infty.$$

Take

$$F_k(\lambda) = \mathsf{H}\big(\lambda,\, y_\lambda^k + \widehat{\Psi}_\lambda\big) \quad \text{and} \quad F(\lambda) = \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda).$$

Then from hypotheses (R2) and Lebesgue's dominated convergence theorem we can obtain

$$\int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \| F_k(\lambda) - F(\lambda) \| \,\mathrm{d}\lambda \to 0 \quad \text{as } m \to \infty,\ \mathsf{y} \in \mathcal{J}. \tag{4}$$

Next,

$$\mathsf{v}^k(\mathsf{y}) = \mathsf{L}^{-1}\left[\mathsf{q}_1 - \mathcal{S}_{\mathsf{p},\sigma}(b)\xi - \int_0^b (b-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(b-\lambda)\mathsf{H}\big(\lambda,\, y_\lambda^k + \widehat{\Psi}_\lambda\big)\,\mathrm{d}\lambda\right]$$

$$\begin{aligned}&\|\mathsf{v}^k(\mathsf{y}) - \mathsf{v}(\mathsf{y})\| \\ &\quad = \mathsf{L}^{-1}\left[\int_0^b (b-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(b-\lambda)\big\|\mathsf{H}\big(\lambda, y_\lambda^k + \widehat{\Psi}_\lambda\big) - \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\big\|\right].\end{aligned} \tag{5}$$

From (4), Eq. (5) converges to zero as $k \to \infty$.

Now,

$$\begin{aligned}&\big\|\Phi y^k(\mathsf{y}) - \Phi y(\mathsf{y})\big\|_{\mathsf{g}} \\ &\quad \leqslant \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\big(\|F_k(\lambda) - F(\lambda)\|\,\mathrm{d}\lambda + \mathsf{B}\|\mathsf{v}^k(\mathsf{y}) - \mathsf{v}(\mathsf{y})\|\big)\,\mathrm{d}\lambda.\end{aligned}$$

Using (4) and (5), we obtain $\|\Phi y^k - \Phi y\|_{\mathsf{g}} \to 0$ as $k \to \infty$. So, Φ is continuous on O_{P}.

Step 3. Next, we need to prove that Φ is equicontinuous.

For $\mathsf{q} \in \mathsf{O}_{\mathsf{P}}$ and $0 \leqslant \mathsf{y}_1 < \mathsf{y}_2 \leqslant b$, we have

$$\begin{aligned}&\big\|\Phi\mathsf{q}(\mathsf{y}_2) - \Phi\mathsf{q}(\mathsf{y}_1)\big\| \\ &\quad = \Bigg\|\mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}\Bigg(\int_0^{\mathsf{y}_2} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}_2-\lambda)\mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda \\ &\qquad + \int_0^{\mathsf{y}_2} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}_2-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda\Bigg) \\ &\qquad - \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}\Bigg(\int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}_1-\lambda)\mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda \\ &\qquad - \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}_1-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda\Bigg)\Bigg\| \\ &\quad \leqslant \Bigg\|\mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}\int_0^{\mathsf{y}_1} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}_2-\lambda)\mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda\end{aligned}$$

$$
\begin{aligned}
&- \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda \Bigg\| \\
&+ \Bigg\| \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda \\
&- \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_1-\lambda) \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda \Bigg\| \\
&+ \Bigg\| \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_{\mathsf{y}_1}^{\mathsf{y}_2} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda \Bigg\| \\
&+ \Bigg\| \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \\
&- \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \Bigg\| \\
&+ \Bigg\| \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \\
&- \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_1-\lambda) \mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \Bigg\| \\
&+ \Bigg\| \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_{\mathsf{y}_1}^{\mathsf{y}_2} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \Bigg\| \\
&\leqslant \sum_{i=1}^{6} I_i.
\end{aligned}
$$

$$
\begin{aligned}
I_1 &\leqslant \Bigg\| \Bigg(\int_0^{\mathsf{y}_1} \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} - \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \Bigg) \\
&\qquad \times \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda \Bigg\| \\
&\leqslant L' \Bigg\| \Bigg(\int_0^{\mathsf{y}_1} \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} - \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \Bigg) \\
&\qquad \times (\mathsf{y}_2-\lambda)^{-\mathsf{p}+\mathsf{p}\vartheta}\,\mathrm{d}\lambda \Bigg\| m(b) f(\mathsf{P}')
\end{aligned}
$$

implies $I_1 \to 0$ as $\mathsf{y}_2 \to \mathsf{y}_1$.

$$I_2 \leqslant \left\| \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \big[\mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) - \mathcal{Q}_\mathsf{p}(\mathsf{y}_1-\lambda)\big] \mathsf{H}\big(\lambda, y_\lambda + \widehat{\Psi}_\lambda\big)\, \mathrm{d}\lambda \right\|$$
$$\leqslant \left\| \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \big[\mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) - \mathcal{Q}_\mathsf{p}(\mathsf{y}_1-\lambda)\big]\, \mathrm{d}\lambda \right\| m(b) f(\mathsf{P}').$$

From Lemma 3 we obtain that $I_2 \to 0$ as $\mathsf{y}_2 \to \mathsf{y}_1$.

$$I_3 = \left\| \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_{\mathsf{y}_1}^{\mathsf{y}_2} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{H}(\lambda,\, y_\lambda + \widehat{\Psi}_\lambda)\, \mathrm{d}\lambda \right\|$$
$$\leqslant L' \left\| \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_{\mathsf{y}_1}^{\mathsf{y}_2} (\mathsf{y}_2-\lambda)^{\mathsf{p}\vartheta-1}\, \mathrm{d}\lambda \right\| m(b) f(\mathsf{P}').$$

Integrating and $\mathsf{y}_2 \to \mathsf{y}_1$ imply $I_3 = 0$.

$$I_4 = \left\| \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{B}\mathsf{v}(\lambda)\, \mathrm{d}\lambda \right.$$
$$\left. - \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{B}\mathsf{v}(\lambda)\, \mathrm{d}\lambda \right\|$$
$$\leqslant \left\| \int_0^{\mathsf{y}_1} \big(\mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} - \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1}\big) \right.$$
$$\left. \times\, \mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) \mathsf{B}\mathsf{v}(\lambda)\, \mathrm{d}\lambda \right\|$$
$$\leqslant L'\mathcal{K}_2 \left\| \int_0^{\mathsf{y}_1} \big(\mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} - \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1}\big) \right.$$
$$\left. \times\, (\mathsf{y}_2-\lambda)^{-\mathsf{p}+\mathsf{p}\vartheta} \mathsf{v}(\lambda)\, \mathrm{d}\lambda \right\|$$

implies $I_4 \to 0$ as $\mathsf{y}_2 \to \mathsf{y}_1$.

$$I_5 \leqslant \left\| \mathsf{y}_1^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}_1} (\mathsf{y}_1-\lambda)^{\mathsf{p}-1} \big(\mathcal{Q}_\mathsf{p}(\mathsf{y}_2-\lambda) - \mathcal{Q}_\mathsf{p}(\mathsf{y}_1-\lambda)\big) \mathsf{B}\mathsf{v}(\lambda)\, \mathrm{d}\lambda \right\|.$$

Since $\mathcal{Q}_{\mathsf{p}}(\mathsf{y})$ is uniform continuous, we get $I_5 \to 0$ as $\mathsf{y}_2 \to \mathsf{y}_1$.

$$\begin{aligned} I_6 &= \left\| \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int\limits_{\mathsf{y}_1}^{\mathsf{y}_2} (\mathsf{y}_2-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}_2-\lambda) \mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \right\| \\ &\leqslant L'\mathcal{K}_2 \left\| \mathsf{y}_2^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int\limits_{\mathsf{y}_1}^{\mathsf{y}_2} (\mathsf{y}_2-\lambda)^{\mathsf{p}\vartheta-1} \mathsf{v}(\lambda)\,\mathrm{d}\lambda \right\|. \end{aligned}$$

Integrating and applying limit imply $I_6 = 0$. Therefore, Φ is equicontinuous on $\mathcal{J}$.

Step 4. Prove Mönch's conditions.

Let $\mathsf{O}_0 \subset \mathsf{O}_P$ by countable, and let $\mathsf{O}_0 \subset \operatorname{conv}(\{0\} \cup \Phi(\mathsf{O}_0))$. We prove $\eta(\mathsf{O}_0) = 0$. Suppose that $\mathsf{O}_0 = \{y^k\}_{k=1}^{\infty}$. We have to show that $\Phi(\mathsf{O}_0)(\mathsf{y})$ is relatively compact in Y for each $\mathsf{y} \in \mathcal{J}$.

$$\begin{aligned} \eta\big(\Phi(\mathsf{O}_0)\big) &= \eta\big(\Phi\big(\{y^k\}_{k=1}^{\infty}\big)\big) \\ &\leqslant \eta\Bigg(\mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int\limits_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{H}\big(\lambda, \{y_\lambda^k + \widehat{\Psi}_\lambda\}_{k=1}^{\infty}\big)\,\mathrm{d}\lambda \\ &\quad + \int\limits_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \Bigg) \\ &\leqslant \eta\Bigg(\mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int\limits_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{H}\big(\lambda, \{y_\lambda^k + \widehat{\Psi}_\lambda\}_{k=1}^{\infty}\big)\,\mathrm{d}\lambda \Bigg) \\ &\quad + \eta\Bigg(\mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int\limits_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \Bigg) \\ &= J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \eta\Bigg(\mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int\limits_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{H}\big(\lambda, \{y_\lambda^k + \widehat{\Psi}_\lambda\}_{k=1}^{\infty}\big)\,\mathrm{d}\lambda \Bigg) \\ &\leqslant b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int\limits_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \eta\big(\mathsf{H}\big(\lambda, \{y_\lambda^k + \widehat{\Psi}_\lambda\}_{k=1}^{\infty}\big)\,\mathrm{d}\lambda\big) \\ &\leqslant L' b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int\limits_0^{\mathsf{y}} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)^{\mathsf{p}\vartheta-1} h(\lambda) \eta\big(\mathsf{O}_0\big)\,\mathrm{d}\lambda \end{aligned}$$

$$
\begin{aligned}
&\leqslant L' b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \left(\int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{(\mathsf{p}\vartheta-1)/(1-\mathsf{p}_1)} \,\mathrm{d}\lambda \right)^{\mathsf{p}_1-1} \left(\int_0^{\mathsf{y}} \big\|h(\lambda)\big\|^{\mathsf{p}_1} \,\mathrm{d}\lambda \right)^{1/\mathsf{p}_1} \eta(\mathsf{O}_0) \\
&\leqslant L' b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} K_{\mathsf{p}_1} \|h\|_{L^{(1/\mathsf{p}_1)}(\mathcal{J},\mathbb{R}^+)} \eta(\mathsf{O}_0),
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \eta\left(\mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{B} v(\lambda) \,\mathrm{d}\lambda \right) \\
&\leqslant b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \eta\left(\int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \right. \\
&\qquad \left. \times \mathsf{B}\mathsf{L}^{-1} \left(\mathsf{q}_1 - \mathfrak{S}_{\mathsf{p},\sigma}(b)\xi - \int_0^b (b-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(b-\lambda) \mathsf{H}\big(b, \big\{y_\lambda^k + \widehat{\Psi}_\lambda\big\}_{k=1}^{\infty}\big) \right) \mathrm{d}\lambda \right) \\
&\leqslant L'^2 \mathcal{K}_2 b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^b (b-\lambda)^{2\mathsf{p}\vartheta-2} \eta\big(\mathsf{L}^{-1}\mathsf{H}\big(\lambda, \big\{y_\lambda^k + \widehat{\Psi}_\lambda\big\}_{k=1}^{\infty}\big)\big) \,\mathrm{d}\lambda \\
&\leqslant L'^2 \mathcal{K}_2 K_{\mathsf{L}} b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \int_0^b (b-\lambda)^{2\mathsf{p}\vartheta-2} \eta\big(\mathsf{H}\big(b, \big\{y_\lambda^k + \widehat{\Psi}_\lambda\big\}_{k=1}^{\infty}\big)\big) \,\mathrm{d}\lambda \\
&\leqslant L'^2 \mathcal{K}_2 K_{\mathsf{L}} b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \left(\int_0^b (b-\lambda)^{(\mathsf{p}\vartheta-1)/(\mathsf{p}_2-1)} \,\mathrm{d}\lambda \right)^{2\mathsf{p}_2-2} \left(\int_0^b \|h\|^{\mathsf{p}_2} \,\mathrm{d}\lambda \right)^{\mathsf{p}_2} \eta(\mathsf{O}_0) \\
&\leqslant L'^2 \mathcal{K}_2 K_{\mathsf{L}} b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} K_{\mathsf{p}_2}^2 \|h\|_{L^{(1/\mathsf{p}_1)}(\mathcal{J},\mathbb{R}^+)} \eta(\mathsf{O}_0).
\end{aligned}
$$

Now,

$$
J_1 + J_2 \leqslant b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \big[L'\mathcal{K}_4 + L'^2 \mathcal{K}_2 \mathcal{K}_5 K_{\mathsf{L}} \big] \eta(\mathsf{O}_0).
$$

Therefore,

$$
\eta\big(\Phi(\mathsf{O}_0)\big) \leqslant \widehat{\mathcal{K}} \eta\big(\mathsf{O}_0\big),
$$

where $\widehat{\mathcal{K}} = b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}(L'\mathcal{K}_4 + L'^2\mathcal{K}_2\mathcal{K}_5 K_{\mathsf{L}})$.

Therefore, using Mönch's condition, we get

$$
\begin{aligned}
&\eta(\mathsf{O}_0) \leqslant \eta\big(\operatorname{conv}\big(\{0\} \cup \Phi(\mathsf{O}_0)\big)\big) = \eta\big(\Phi(\mathsf{O}_0)\big) \leqslant \widehat{\mathcal{K}} \eta(\mathsf{O}_0) \\
&\quad \Longrightarrow \quad \eta(\mathsf{O}_0) = 0.
\end{aligned}
$$

Then from Lemma 9 Φ has a fixed point q in O_P. So $\mathsf{q} = y + \widehat{\Psi}$ is the mild solution of Eqs. (1)–(2) such that $\mathsf{q}(b) = \mathsf{q}_1$. Therefore, system (1)–(2) is controllable in Y. □

4 Impulsive conditions

Consider the HF impulsive differential equations

$$\begin{aligned} &D_{0+}^{\mathsf{p},\sigma}\mathsf{q}(\mathsf{y}) = \mathsf{A}\mathsf{q}(\mathsf{y}) + \mathsf{H}(\mathsf{y},\mathsf{q}_\mathsf{y}) + \mathsf{B}\mathsf{v}(\mathsf{y}), \quad \mathsf{y} \in \mathcal{J}' = (0,b],\ \mathsf{y} \neq \mathsf{y}_m, \\ &\Delta\mathsf{q}(\mathsf{y}_m) = I_m\big(\mathsf{q}(\mathsf{y}_m^+)\big), \quad m = 1,2,\ldots,n, \\ &I_{0+}^{(1-\mathsf{p})(1-\sigma)}\mathsf{q}(0) = \xi \in \mathsf{O}_\tau, \quad \mathsf{y} \in (-\infty,0], \end{aligned} \tag{6}$$

where $I_m : Y \to Y$, $0 < \mathsf{y}_1 < \mathsf{y}_2 < \mathsf{y}_3 < \cdots < \mathsf{y}_m < \mathsf{y}_{m+1} = b$, and

$$\begin{aligned} \Delta\mathsf{q}|_{\mathsf{y}=\mathsf{q}_{\mathsf{y}_m}} &= \mathsf{q}\big(\mathsf{y}_m^+\big) - \mathsf{q}\big(\mathsf{y}_m^-\big), \\ \mathsf{q}\big(\mathsf{y}_m^+\big) = \lim_{\delta\to 0^+}\mathsf{q}(\mathsf{y}_m+\delta), &\qquad \mathsf{q}\big(\mathsf{y}_m^-\big) = \lim_{\delta\to 0^-}\mathsf{q}(\mathsf{y}_m+\delta). \end{aligned}$$

Here we introduce the set

$$\begin{aligned} \mathsf{O}_\tau''' = \big\{\mathsf{q} : (-\infty,b] \to Y\colon\ \big|\ &\mathsf{q}|_{J_m} \in C(J_m,Y), \\ &\exists\mathsf{u}\big(\mathsf{y}_m^+\big),\mathsf{u}\big(\mathsf{y}_\mathsf{m}^-\big)\colon\ \mathsf{u}\big(\mathsf{y}_m^+\big) = \mathsf{u}\big(\mathsf{y}_m^-\big),\ \mathsf{u}_0 = \xi,\ \xi \in \mathsf{O}_\tau\big\}, \end{aligned}$$

where $\mathsf{u}|_{J_m}$ is the restriction of u to $J_m = (\mathsf{y}_m, \mathsf{y}_{m+1}]$, $m = 0,1,2,\ldots,n$. The function $\|\cdot\|_\mathsf{g}'$ is the seminorm in O_τ''' defined as

$$\|\mathsf{u}\|_\mathsf{g}' = \sup\big\{\big|\mathsf{u}(\tau)\big|,\ \tau \in [0,b]\big\} + \|\xi\|_\mathsf{g}, \quad \mathsf{u} \in \mathsf{O}_\tau'''.$$

(R4) (i) Let $I_m : Y \to Y$ be the continuous function, and there exists a constant $\mathcal{K}_6$ such that, for all $\mathsf{y} \in I$, we have

$$\big\|I_m(\mathsf{u}_1) - I_m(\mathsf{u}_2)\big\| \leqslant \mathcal{K}_6\|\mathsf{u}_1 - \mathsf{u}_2\|.$$

(ii) There exists $\mathcal{K}_6^* > 0$ such that

$$\big\|I_m(\mathsf{u})\big\| \leqslant \mathcal{K}_6^* \quad \forall \mathsf{u} \in Y, m = 1,2,\ldots,n.$$

(iii) There exist $\mathsf{p}_3 \in (0,\mathsf{p})$ and $\mathcal{K}_7 \in L^{(1/\mathsf{p}_3)}(J_m,Y)$ such that, for every bounded sets $\mathsf{O}_m \subset Y$, $m = 1,2,\ldots,n$, $\eta(I_m(\mathsf{O}_m)) \leqslant \mathcal{K}_7 \times \eta(\mathsf{O}_m)$.

Definition 7. A continuous function $\mathsf{q} : (-\infty,b] \to Y$ is known as the mild solution of Eq. (6) if there exists $\mathsf{q}_0 = \xi \in \mathsf{O}_\tau$ on $(-\infty,0]$ such that

$$\begin{aligned} \mathsf{q}(\mathsf{y}) = \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})\xi &+ \int_0^\mathsf{y} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_\mathsf{p}(\mathsf{y}-\lambda)\mathsf{H}\big(\mathsf{y},\mathsf{q}_\mathsf{y}\big)\,\mathrm{d}\lambda \\ &+ \int_0^\mathsf{y} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_\mathsf{p}(\mathsf{y}-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda + \sum_{0<\mathsf{y}_m<\mathsf{y}} \mathfrak{S}_{\mathsf{p},\sigma}(\mathsf{y}-\mathsf{y}_m)I_m\big(\mathsf{q}(\mathsf{y}_m)\big), \end{aligned}$$

is satisfied.

Theorem 3. *Suppose* (R1)–(R4) *hold, then* (6) *is controllable on* $[0,b]$ *with* $\xi \in D(\mathsf{A}^{\theta})$, $\theta > 1 + \vartheta$, *and*

$$\widehat{\mathcal{K}}^{*} = b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}\big(L'\mathcal{K}_4 + L'^2\mathcal{K}_2\mathcal{K}_5K_{\mathsf{L}}\big) + L''n\mathcal{K}_7\big(\big(L'\mathcal{K}_2K_{\mathsf{L}}b^{\mathsf{p}\vartheta/(\mathsf{p}\vartheta)}\big) + 1\big) < 1.$$

Proof. Consider the operator $\Psi^{*} : \mathsf{O}_{\tau}''' \to \mathsf{O}_{\tau}'''$ given by

$$\Psi^{*}(\mathsf{q}(\mathsf{y})) = \begin{cases} \Psi_1^{*}(\mathsf{y}), \quad (-\infty, 0], \\ \mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta}[\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})\xi + \int_0^{\mathsf{y}}(\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{H}(\lambda, \mathsf{q}_{\lambda})\,\mathrm{d}\lambda \\ + \int_0^{\mathsf{y}}(\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda \\ \quad + \sum_{0<\mathsf{y}_m<\mathsf{y}}\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y}-\mathsf{y}_m)I_m(\mathsf{q}(\mathsf{y}_m))], \quad \mathsf{y} \in \mathcal{J}. \end{cases}$$

For $\Psi_1^{*} \in \mathsf{O}_{\tau}$, we define $\widehat{\Psi}^{*}$ by

$$\widehat{\Psi}^{*}(\mathsf{y}) = \begin{cases} \Psi_1^{*}(\mathsf{y}), & \mathsf{y} \in (-\infty, 0], \\ \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})\xi, & \mathsf{y} \in \mathcal{J}, \end{cases}$$

then $\widehat{\Psi}^{*} \in \mathsf{O}_{\tau}'$. Let $\mathsf{q}(\mathsf{y}) = y(\mathsf{y}) + \widehat{\Psi}^{*}(\mathsf{y})$, $-\infty < \mathsf{y} \leqslant b$, q fulfill from a simple standpoint from (3) iff y satisfies $y_0 = 0$ and

$$\begin{aligned} y(\mathsf{y}) = {} & \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{H}\big(\lambda, y_{\lambda} + \widehat{\Psi}_{\lambda}^{*}\big)\,\mathrm{d}\lambda \\ & + \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda)\mathsf{B}\mathsf{v}(\lambda)\,\mathrm{d}\lambda + \sum_{0<\mathsf{y}_m<\mathsf{y}} \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y}-\mathsf{y}_m)I_m\big(y_m + \widehat{\Psi}_m^{*}\big), \end{aligned}$$

where

$$\begin{aligned} \mathsf{v}(\mathsf{y}) = \mathsf{L}^{-1}\Bigg[& \mathsf{q}_1 - \mathcal{S}_{\mathsf{p},\sigma}(b)\xi - \int_0^{b}(b-\lambda)^{\mathsf{p}-1}\mathcal{Q}_{\mathsf{p}}(b-\lambda)\mathsf{H}\big(\lambda, y_{\lambda} + \widehat{\Psi}_{\lambda}^{*}\big)\,\mathrm{d}\lambda \\ & - \sum_{0<\mathsf{y}_m<b} \mathcal{S}_{\mathsf{p},\sigma}(b-\mathsf{y}_m)I_m\big(y_m + \widehat{\Psi}_m^{*}\big)\Bigg]. \end{aligned}$$

Let $\mathsf{O}_{\tau}'''^{*} = \{y \in \mathsf{O}_{\tau}''' \colon y_0 \in \mathsf{O}_{\tau}\}$. For any $y \in \mathsf{O}_{\tau}'$,

$$\|y\|_{\mathsf{g}} = \|y_0\|_{\mathsf{g}} + \sup\big\{\big\|y(\lambda)\big\|, 0 \leqslant \lambda \leqslant b\big\} = \sup\big\{\big\|y(\lambda)\big\|, 0 \leqslant \lambda \leqslant b\big\}.$$

Thus, $(\mathsf{O}_{\tau}'''^{*}, \|\cdot\|_b')$ is a Banach space.

Consider $\mathsf{P}^{*} > 0$ and take $\mathsf{O}_{\mathsf{P}^{*}} = \{y \in \mathsf{O}_{\tau}'' \colon \|y\|_{\mathsf{g}} \leqslant \mathsf{P}^{*}\}$. Then $\mathsf{O}_{\mathsf{P}^{*}} \subset \mathsf{O}_{\tau}'''^{*}$ is uniformly bounded, and for $y \in \mathsf{O}_{P}$, from Lemma 5

$$\|y_{\mathsf{y}} + \widehat{\Psi}_{\mathsf{y}}\|_{\mathsf{g}} \leqslant \|y_{\mathsf{y}}\|_{\mathsf{g}} + \|\widehat{\Psi}\|_{\mathsf{g}} \leqslant l\big(\mathsf{P}^{*} + L''\mathsf{y}^{-1+\sigma-\mathsf{p}\sigma+\mathsf{p}\vartheta}\big) + \|\Psi_1\|_{\mathsf{g}} = \mathsf{P}'^{*}.$$

Let $\Phi^*: \mathsf{O}_\tau'''^* \to \mathsf{O}_\tau'''^*$ be the operator defined by

$$\Phi^* y(\mathsf{y}) = \begin{cases} 0, \quad \mathsf{y} \in (-\infty, 0], \\ \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{H}(\lambda, y_\lambda + \widehat{\Psi}_\lambda)\,\mathrm{d}\lambda \\ \quad + \int_0^{\mathsf{y}} (\mathsf{y}-\lambda)^{\mathsf{p}-1} \mathcal{Q}_{\mathsf{p}}(\mathsf{y}-\lambda) \mathsf{Bv}(\lambda)\,\mathrm{d}\lambda \\ \quad + \sum_{0<\mathsf{y}_m<\mathsf{y}} \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y}-\mathsf{y}_m) I_m(y_m + \widehat{\Psi}_m^*), \quad \mathsf{y} \in \mathcal{J}. \end{cases}$$

Obviously, the statement that the operator Ψ^* has a fixed point is equivalent to that Φ^* has one. So it turns out to prove that Φ^* has a fixed point.

Step 1. We prove that $\Phi^*(\mathsf{O}_{\mathsf{P}^*}) \subseteq \mathsf{O}_{\mathsf{P}^*}$.
From Step 1 of the previous part and (R4) we can easily prove it.

Step 2. Prove that Φ^* is continuous.
Φ^* maps $\mathsf{O}_{\mathsf{P}*}$ into $\mathsf{O}_{\mathsf{P}^*}$. Take $y^k, y \in \mathsf{O}_{\mathsf{P}^*}$, $k = 0, 1, 2, \dots$, such that $\lim_{k\to\infty} y^k = y$. Then we have $\lim_{k\to\infty} y^m(\mathsf{y}) = y(\mathsf{y})$ and $\lim_{k\to\infty} \mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} y^k(\mathsf{y}) = \mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} y(\mathsf{y})$. From hypotheses (R4)

$$\begin{aligned} I_m(\mathsf{y}_m^k) &= I_m\big(\mathsf{y}_m^k + \widehat{\Psi}_m^*\big) \to I_m(\mathsf{y}_m + \widehat{\Psi}_m^*) \\ \implies \quad & \sum_{0<\mathsf{y}_m<\mathsf{y}} \big\| \mathfrak{S}_{\mathsf{p},\sigma}(\mathsf{y}^k - \mathsf{y}_m^k) I_m\big(\mathsf{y}_m^k + \widehat{\Psi}_m^*\big) \\ &\quad - \mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y}-\mathsf{y}_m) I_m\big(\mathsf{y}_m + \widehat{\Psi}_m^*\big)\big\| \to 0 \end{aligned} \tag{7}$$

since $\mathcal{S}_{\mathsf{p},\sigma}(\mathsf{y})$ is strongly continuous. By (4), (5), and (7) we get that Φ^* is continuous.

Step 3. Next, we have to show Φ^* is equicontinuous.
Using hypotheses (R4) and Step 3 in previous section, we can verify the equicontinuity of Φ^*.

Step 4. Show Mönch's conditions.
Using Step 4 in previous part and hypotheses (R4)(ii), take

$$\begin{aligned} J_3 &= \eta\bigg(\mathsf{y}^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \sum_{0<\mathsf{y}_m<\mathsf{y}} \mathfrak{S}_{\mathsf{p},\sigma}(\mathsf{y}-\mathsf{y}_m) I_m\big(\mathsf{q}\big\{\big(y_m^k + \widehat{\Psi}_m^*\big)\big\}_{k=1}^\infty\big)\bigg) \\ &\leqslant b^{1-\sigma+\mathsf{p}\sigma-\mathsf{p}\vartheta} \sum_{0<\mathsf{y}_m<\mathsf{y}} \mathfrak{S}_{\mathsf{p},\sigma}(\mathsf{y}-\mathsf{y}_m) \eta\big(I_m\big(\mathsf{q}\big\{\big(y_m^k + \widehat{\Psi}_m^*\big)\big\}_{k=1}^\infty\big)\big) \\ &\leqslant L'' n \mathcal{K}_7 \eta(\mathsf{O}_o). \end{aligned}$$

Thus, by expressions of J_1, J_2, J_3 and (R4) we can verify Mönch's conditions. Hence, by Lemma 9 system (6) is controllable in Y. □

5 Applications

Application 1

Suppose the $\mathrm{HFD}_{\mathrm{tial}}$ equation is given by

$$\begin{aligned} &D_{0+}^{2/3,\sigma}\mathsf{q}(\mathsf{y},\tau) = \mathsf{q}_{\tau\tau}(\mathsf{y},\tau) + \mathsf{W}\widehat{\varphi}(\mathsf{y},\tau) + \chi\left(\mathsf{y}, \int_{-\infty}^{\mathsf{y}} \chi_1(\lambda-\mathsf{y})\mathsf{q}(\mathsf{y},\tau)\,\mathrm{d}\lambda\right), \\ &I_{0+}^{1/3(1-\sigma)}\mathsf{q}(0,\tau) = \mathsf{q}_0(\tau), \quad \tau \in [0,\pi], \\ &\mathsf{q}(\mathsf{y},0) = \mathsf{q}(\mathsf{y},\pi) = 0, \quad \mathsf{y} \in \mathcal{J}, \\ &\mathsf{q}(\mathsf{y},\tau) = \xi(\mathsf{y},\tau), \quad \mathsf{y} \in (-\infty, 0], \end{aligned} \tag{8}$$

where $D_{0+}^{2/3,\sigma}$ is the HFD$_{\text{tial}}$ of order $\mathsf{p} = 2/3$ and type $\sigma \in [0,1]$, $\chi : \mathcal{J} \times \mathsf{O}_\tau \to Y$ and ξ are continuous, which satisfies some smoothness conditions.

Let $Y = U = L^2[0,\pi]$, and let $\mathsf{A} : D(\mathsf{A}) \subset Y \to Y$ be defined as $\mathsf{Ax} = \mathsf{x}'$ such that

$$D(A) = \{\mathsf{x} \in Y\colon\ \mathsf{x}, \mathsf{x}' \text{ are absolutely continuous},\ \mathsf{x}'' \in Y,\ \mathsf{x}(0) = \mathsf{x}(\pi) = 0\},$$

where A is the almost sectorial operator of the semigroup T defined as $\mathsf{T}(\mathsf{y})\varrho(\lambda) = \varrho(\mathsf{y}+\lambda)$ for $\varrho \in Y$. The semigroup $\mathsf{T}(\mathsf{y})$ is noncompact semigroup in Y with $\eta(\mathsf{T}(\mathsf{y})\mathsf{O}) \leqslant \eta(\mathsf{O})$, where η denotes the Hausdorff MNC, and there exists a constant $\mathcal{K}_1 \geqslant 1$ such that $\sup_{\mathsf{y}\in\mathcal{J}} \|\mathsf{T}(\mathsf{y})\| \leqslant \mathcal{K}_1$. Furthermore, $\mathsf{y} \to \varrho(\mathsf{y}^{2/3}\theta + \lambda)\mathsf{q}$ is equicontinuous for $\mathsf{y} \geqslant 0$ and $0 < \theta < \infty$.

Define

$$\mathsf{q}(\mathsf{y})(\tau) = \mathsf{q}(\mathsf{y},\tau), \qquad \mathsf{H}(\mathsf{y},\mathsf{q}_\mathsf{y})(\tau) = \chi\left(\mathsf{y}, \int_{-\infty}^{\mathsf{y}} \chi_1(\lambda-\mathsf{y})\mathsf{q}(\mathsf{y},\tau)\,\mathrm{d}\lambda\right).$$

Let us consider $\mathsf{B} : U \to Y$ defined as

$$(\mathsf{Bv})(\tau) = \mathsf{W}\eta(\mathsf{y},\tau), \quad \tau \in (0,\pi).$$

Taking into account the entries A, B, and H, system (8) looks like this:

$$\begin{aligned} &D_{0+}^{\mathsf{p},\sigma}\mathsf{q}(\mathsf{y}) = \mathsf{Aq}(\mathsf{y}) + \mathsf{H}(\mathsf{y},\mathsf{q}_\mathsf{y}) + \mathsf{Bv}(\mathsf{y}), \quad \mathsf{p} = \frac{2}{3} \in (0,1),\ \mathsf{y} \in \mathcal{J}, \\ &I_{0+}^{(1-\mathsf{p})(1-\sigma)}(\mathsf{q}_0) = \xi \in \mathsf{O}_\tau. \end{aligned}$$

We set

$$\chi\left(\mathsf{y}, \int_{-\infty}^{\mathsf{y}} \chi_1(\lambda-\mathsf{y})\mathsf{q}(\mathsf{y},\tau)\,\mathrm{d}\lambda\right) = \mathsf{C}_0 \sin(\mathsf{q}(\mathsf{y}))$$

with a constant C_0. Then the required system (8) satisfies the assumption, so we conclude that the HFD$_{\text{tial}}$ system (1)–(2) is controllable.

Application 2

Consider the Hilfer fractional impulsive differential system

$$\begin{aligned} &D_{0+}^{2/3,\sigma}\mathsf{q}(\mathsf{y},\tau) = \frac{\partial^2}{\partial\tau^2}(\mathsf{y},\tau) + \mathsf{Z}\widehat{\varphi}(\mathsf{y},\tau) + \int_{-\infty}^{\mathsf{y}} \mathsf{F}(\mathsf{y},\ \tau,\ \lambda-\mathsf{y})\Lambda(\mathsf{q}(\lambda,\tau))\,\mathrm{d}\lambda, \\ &\quad \mathsf{y} \in [0,b],\ \tau \in [0,\pi],\ \mathsf{y} \neq \mathsf{y}_m, \end{aligned}$$

$$\begin{aligned}
&\mathsf{q}\big(\mathsf{y}_m^+,\tau\big)-\mathsf{q}\big(\mathsf{y}_m^-,\tau\big)=I_m\big(\mathsf{q}\big(\mathsf{y}_m^+,\tau\big)\big),\quad m=0,1,\dots,n,\\
&I_{0^+}^{1/3(1-\sigma)}\mathsf{q}(0,\tau)=\mathsf{q}_0(\tau),\quad \tau\in[0,\pi],\\
&\mathsf{q}(\mathsf{y},0)=\mathsf{q}(\mathsf{y},\pi)=0,\quad \mathsf{y}\in\mathcal{J},\\
&\mathsf{q}(\mathsf{y},\tau)=\xi(\mathsf{y},\tau),\quad \mathsf{y}\in(-\infty,0].
\end{aligned}$$

Here $D_{0^+}^{2/3,\sigma}$ specify the Hilfer fractional derivative of order $2/3$ and type σ. Also, $\mathsf{F}:\mathcal{J}\times[0,\pi]\times(-\infty,0]\to Y$ is continuous function such that $\mathsf{F}(\mathsf{y},\tau,\omega)>0$, $\int_{-\infty}^0\mathsf{F}(\mathsf{y},\tau,\omega)\,\mathrm{d}\omega<\infty$, and $\varLambda:(-\infty,0]\times[0,\pi]$, $I_m:Y\to Y$ are the continuous functions.

Take $Y=U=L^2[0,\pi]$, and let $\mathsf{A}:D(\mathsf{A})\subset Y\to Y$ be defined as $\mathsf{A}\mathsf{x}=\mathsf{x}'$ such that

$$D(A)=\big\{\mathsf{x}\in Y\colon \mathsf{x},\mathsf{x}' \text{ are absolutely continuous},\ \mathsf{x}''\in Y,\ \mathsf{x}(0)=\mathsf{x}(\pi)=0\big\}$$

and

$$\mathsf{A}\mathsf{x}=\sum_{k=1}^{\infty}k^2\langle\mathsf{x},\mathsf{x}_k\rangle\mathsf{x}_k,\quad \mathsf{x}\in D(\mathsf{A}),$$

where $\mathsf{x}_k(x)$ is the orthogonal set of eigenvectors of A.

Suppose we took the function $\mathsf{g}=\mathrm{e}^{4\lambda}$, $\lambda<0$. Then $l=\int_{-\infty}^0\mathsf{g}(\lambda)\,\mathrm{d}\lambda=1/4$, and we define

$$\|\xi\|_{\mathsf{g}}=\int_{-\infty}^{0}\mathsf{g}(\lambda)\sup_{\omega\in[\lambda,0]}\big|\xi(\omega)\big|_{L^2}\,\mathrm{d}\lambda.$$

Thus, if $(\mathsf{y},\xi)\in\mathcal{J}\times\mathsf{O}_\tau$, then $\xi(\omega)(\tau)=\xi(\omega,\tau)$, $(\omega,\tau)\in(-\infty,0]\times[0,\pi]$. Here we take

$$\mathsf{q}(\mathsf{y})(\tau)=\mathsf{q}(\mathsf{y},\tau),\qquad \mathsf{H}(\mathsf{y},\omega)(\tau)=\int_{-\infty}^{0}\mathsf{F}(\mathsf{y},\tau,\omega)\varLambda\big(\xi(\omega)(\tau)\big)\,\mathrm{d}\omega.$$

Then we can analyze

$$\begin{aligned}
&\big\|\mathsf{H}(\mathsf{y},\xi)\big\|_{L^2}\\
&\quad=\left[\int_0^{\pi}\left(\int_{-\infty}^{0}\mathsf{F}(\mathsf{y},\tau,\omega)\varLambda\big(\xi(\omega)(\tau)\big)\,\mathrm{d}\omega\right)^2\mathrm{d}\tau\right]^{1/2}\\
&\quad\leqslant\left[\int_0^{\pi}\left(\int_{-\infty}^{0}\mathsf{F}\big(\mathsf{y},\tau,\omega\big)\,\mathrm{d}\omega\cdot\varXi\left(\int_{-\infty}^{0}\mathrm{e}^{4\lambda}\big\|\xi\big(\lambda(\cdot)\big)\big\|_{L^2}\,\mathrm{d}\lambda\right)\mathrm{d}\omega\right)^2\mathrm{d}\tau\right]^{1/2}\\
&\quad\leqslant\left[\int_0^{\pi}\left(\int_{-\infty}^{0}\mathsf{F}\big(\mathsf{y},\tau,\omega\big)\,\mathrm{d}\omega\cdot\varXi\left(\int_{-\infty}^{0}\mathrm{e}^{4\lambda}\sup_{\lambda\in\omega,0}\big\|\xi(\lambda)\big\|_{L^2}\,\mathrm{d}\lambda\right)\mathrm{d}\omega\right)^2\mathrm{d}\tau\right]^{1/2}\\
&\quad\leqslant\left[\int_0^{\pi}\left(\int_{-\infty}^{0}\mathsf{F}\big(\mathsf{y},\tau,\omega\big)\,\mathrm{d}\omega\right)^2\mathrm{d}\tau\right]^{1/2}\varXi\big(\|\xi\|_{\mathsf{g}}\big)=\left[\int_0^{\pi}\big(\mathsf{K}(\mathsf{y},\tau)\big)^2\,\mathrm{d}\tau\right]^{1/2}\varXi\big(\|\xi\|_{\mathsf{g}}\big),
\end{aligned}$$

where $\int_{-\infty}^{0} \mathsf{F}(\mathsf{y},\tau,\omega)\,\mathrm{d}\omega = \mathsf{K}(\mathsf{y},\tau)$ is continuous nondecreasing function. So the required hypotheses and Theorem 3 are satisfied, hence the system is controllable.

6 Conclusion

In this article, we discussed the controllability of the Hilfer fractional differential equation with and without impulsive conditions via infinite delay. The abstract Cauchy problem was proved using fractional calculus, almost sectorial operators, measure of noncompactness, and the fixed point technique. Firstly, we proved the exact controllability of the fractional system, then extended the system to an impulsive condition and proved the controllability. Finally, we provided two applications to illustrate the theory. In the future, we will try to focus on the controllability of the Ψ-Hilfer fractional impulsive differential systems with infinite delay. Also, we will study some real-life problems related to fractional differential systems via semigroup theory.

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References

1. R.P. Agarwal, V. Lakshmikanthan, J.J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal., Theory Methods Appl.*, **72**(6):2859–2862, 2010, `https://doi.org/10.1016/j.na.2009.11.029`.
2. B. Ahmad, A. Alsaedi, S.K. Ntouyas, J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, 2017, `https://doi.org/10.1007/978-3-319-52141-1`.
3. K. Balachandran, R. Sakthivel, Controllability of integro-differential systems in Banach spaces, *Appl. Math. Comput.*, **118**(1):63–71, 2001, `https://doi.org/10.1016/S0096-3003(00)00040-0`.
4. J. Banaś, Measure of noncompactness in Banach spaces, *Commentat. Math. Univ. Carol.*, **21**(1):131–143, 1980.
5. P. Bedi, A. Kumar, T. Abdeljawad, Z.A. Khan, A. Khan, Existence and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators, *Adv.*

Differ. Equ., **2020**:615, 2020, https://doi.org/10.1186/s13662-020-03074-1.

6. C. S. Varun Bose, R. Udhayakumar, A note on the existence of Hilfer fractional differential inclusions with almost sectorial operators, *Math. Methods Appl. Sci.*, **45**(5):2530–2541, 2021, https://doi.org/10.1002/mma.7938.
7. C. S. Varun Bose, R. Udhayakumar, Analysis on the controllability of Hilfer fractional neutral differential equations with almost sectorial operators and infinite delay via measure of noncompactness, *Qual. Theory Dyn. Syst.*, **22**(1):22, 2023, https://doi.org/10.1007/s12346-022-00719-2.
8. Y.K. Chang, Controllability of impulsive differential systems with infinite delay in Banach spaces, *Chaos Solitons Fractals*, **33**(5):1601–1609, 2007, https://doi.org/10.1016/j.chaos.2006.03.006.
9. C. Dineshkumar, R. Udhayakumar, New results concerning to approximate controllability of Hilfer fractional neutral stochastic delay integro-differential system, *Numer. Methods Partial Differ. Equations*, **37**(2):1072–1090, 2020, https://doi.org/10.1002/num.22567.
10. C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, K. S. Nisar, Results on approximate controllability of neutral integro-differential stochastic system with state-dependent delay, *Numer. Methods Partial Differ. Equations*, **40**(1):e22698, 2020, https://doi.org/10.1002/num.22698.
11. H. Gu, J.J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Appl. Math. Comput.*, **257**:344–354, 2015, https://doi.org/10.1016/j.amc.2014.10.083.
12. R. Hilfer, *Application of fractional calculus in physics*, World Scientific, Singapore, 2000, https://doi.org/10.1142/3779.
13. A. Jaiswal, D. Bahuguna, Hilfer fractional differential equations with almost sectorial operators, *Differ. Equ. Dyn. Syst.*, **31**:301–317, 2023, https://doi.org/10.1007/s12591-020-00514-y.
14. S. Ji, G. Li, M. Wang, Controllability of impulsive differential systems with nonlocal conditions, *Appl. Math. Comput.*, **217**(16):6981–6989, 2011, https://doi.org/10.1016/j.amc.2011.01.107.
15. K. Kavitha, V. Vijayakumar, R. Udhayakumar, Results on controllability on Hilfer fractional neutral differential equations with infinite delay via measure of noncompactness, *Chaos Solitons Fractals*, **139**:110035, 2020, https://doi.org/10.1016/j.chaos.2020.110035.
16. K. Kavitha, V. Vijayakumar, R. Udhayakumar, C. Ravichandran, Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness, *Asian J. Control*, **24**(3):1406–1415, 2022, https://doi.org/10.1002/asjc.2549.
17. B. Khaminsou, C. Thaiprayoon, W. Sudsutad, S. A. Jose, Qualitative analysis of a proportional Caputo fractional Pantograph differential equation with mixed nonlocal conditions, *Nonlinear Funct. Anal. Appl.*, **26**(1):197–223, 2021, https://doi.org/10.22771/nfaa.2021.26.01.14.
18. V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal., Theory Methods Appl.*, **69**(8):2677–2682, 2008, https://doi.org/10.1016/j.na.2007.08.042.

19. F. Li, Mild solutions for abstract differential equations with almost sectorial operators and infinite delay, *Adv. Differ. Equ.*, **2013**:327, 2013, `https://doi.org/10.1186/1687-1847-2013-327`.
20. K.S. Miller, B. Ross, *it An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
21. H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal., Theory Methods Appl.*, **4**(5):985–999, 1980.
22. K. Munusamy, C. Ravichandran, K.S. Nisar, R. Jagatheeshwari, N. Valliammal, Results on neutral integrodifferential system using Krasnoselskii-Schaefer theorem with initial conditions, in *5th International Conference on Current Scenario in Pure and Applied Mathematics (ICCSPAM-2022)*, AIP Conf. Proc., Vol. 2718, American Institute of Physics, New York, 2023, p. 040001, `https://doi.org/10.1063/5.0137023`.
23. K.S. Nisar, R. Jagatheeshwari, C. Ravichandran, P. Veeresha, High performance computational method for fractional model of solid tumour invasion, *Ain Shams Eng. J.*, p. 102226, 2023, `https://doi.org/10.1016/j.asej.2023.102226`.
24. K.S. Nissar, K. Manusamy, C. Ravichandran, Results on existence of solutions in nonlocal partial functional integrodifferential equations with finite delay in nondense domain, *Alexandria Eng. J.*, **73**:377–384, 2023, ISSN 1110-0168, `https://doi.org/10.1016/j.aej.2023.04.050`.
25. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., Vol. 44, Springer, New York, 1983, `https://doi.org/10.1007/978-1-4612-5561-1`.
26. F. Periago, B. Straub, A functional calculus for almost sectorial operators and applications to abstract evolution equations, *J. Evol. Equ.*, **2**:41–68, 2002, `https://doi.org/10.1007/s00028-002-8079-9`.
27. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.
28. C. Ravichandran, D. Baleanu, On the controllability of fractional functional integro-differential systems with an infinite delay in Banach spaces, *Adv. Differ. Equ.*, **2013**:291, 2013, `https://doi.org/10.1186/1687-1847-2013-291`.
29. C. Ravichandran, K. Logeswari, A. Khan, T. Abdeljawad, J.F. Gómez-Aguilar, An epidemiological model for computer virus with Atangana–Baleanu fractional derivative, *Results Phys.*, **51**:106601, 2023, `https://doi.org/10.1016/j.rinp.2023.106601`.
30. P. Sawantong, K. Logeswari, K.S. Nisar, R. Jagatheeshwari, C. Ravichandran, P. Veeresha, Fractional order geminivirus impression in capsicum annuum model with Mittag-Leffler kernal, *Fractals*, **31**(4):23040049, 2023, `https://doi.org/10.1142/S0218348X23400492`.
31. V. Singh, Controllability of Hilfer fractional differential systems with non-dense domain, *Numer. Funct. Anal. Optim.*, **40**(13):1572–1592, 2019, `https://doi.org/10.1080/01630563.2019.1615947`.
32. J. Wang, Z. Fan, Y. Zhou, Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces, *J. Optim. Theory Appl.*, **154**(1):292–302, 2012, `https://doi.org/10.1007/s10957-012-9999-3`.

33. M. Yang, Q. Wang, Existence of mild solutions for a class of Hilfer fractional evolution equations with nonlocal conditions, *Fract. Calc. Appl. Anal.*, **20**(3):679–705, 2017, https://doi.org/10.1515/fca-2017-0036.

34. L. Zhang, Y. Zhou, Fractional Cauchy problems with almost sectorial operators, *Appl. Math. Comput.*, **257**:145–157, 2014, https://doi.org/10.1016/j.amc.2014.07.024.

35. M. Zhou, C. Li, Y. Zhou, Existence of mild solutions for Hilfer fractional evolution equations with almost sectorial operators, *Axioms*, **11**(4):144, 2022, https://doi.org/10.3390/axioms11040144.

36. M. Zhou, C. Li, Y. Zhou, Existence of mild solutions for Hilfer fractional evolution equations with almost sectorial operators, *Axioms*, **11**(4):144, 2022, https://doi.org/10.3390/axioms11040144.

37. Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014, https://doi.org/10.1142/9789814579902.

38. Y. Zhou, H.J. Wei, A Cauchy problem for fractional evolution equations with Hilfer's fractional derivative on semi-infinite interval, *Fract. Calc. Appl. Anal.*, **25**(3):924–961, 2022, https://doi.org/10.1007/s13540-022-00057-9.