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Singular anisotropic equations with a sign-changing perturbation*

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Abstract. We consider an anisotropic Dirichlet problem driven by the variable (p, q)-Laplacian (double phase problem). In the reaction, we have the competing effects of a singular term and of a superlinear perturbation. Contrary to most of the previous papers, we assume that the perturbation changes sign. We prove a multiplicity result producing two positive smooth solutions when the coefficient function in the singular term is small in the L^∞-norm.

Keywords: variable exponents, modular function, Luxemburg norm, regularity theory, maximum principle.

1 Introduction

In this paper, we study the following anisotropic Dirichlet problem:

$$\begin{aligned} &-\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = \theta(z)u(z)^{-\eta(z)} + f\bigl(z, u(z)\bigr) \quad \text{in } \Omega, \\ &u|_{\partial\Omega} = 0, \quad u > 0. \end{aligned} \tag{1}$$

In this problem, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. For $r \in C(\overline{\Omega})$ with $1 < r_- = \min_{\overline{\Omega}} r$, by $\Delta_{r(z)}$ we denote the anisotropic r-Laplace differential

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operator defined by

$$\Delta_{r(z)}u = \operatorname{div}\big(|Du|^{r(z)-2}Du\big) \quad \text{for all } u \in W_0^{1,r(\cdot)}(\varOmega).$$

This operator is not homogeneous (unless, of course, $r(\cdot)$ is constant), and this is a source of technical difficulties when we deal with anisotropic boundary value problems. In problem (1), we have the sum of two such operators with different variable exponents (anisotropic (p,q)-equation). In the right-hand side of (1), we have the combined effects of two different terms. One is the singular term $u \to \theta(z)u^{-\eta(z)}$ with $\theta \in L^\infty(\varOmega)_+ \setminus \{0\}$ and $\eta \in C(\overline{\varOmega})$ satisfying $0 < \eta(z) < 1$ for all $z \in \overline{\varOmega}$, and the other is a Carathéodory perturbation $f(z,x)$ (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable, and for a.e. $z \in \varOmega$, $x \to f(z,x)$ is continuous), which exhibits $(p_+ - 1)$-superlinear growth as $x \to +\infty$ with $p \in C^{0,1}(\overline{\varOmega})$ and $p_+ = \max_{\overline{\varOmega}} p$. The function $f(z,\cdot)$ need not satisfy the Ambrosetti–Rabinowitz condition (the AR-condition for short), which is common in the literature when studying superlinear problems. The function $f(z,\cdot)$ changes sign as we move from $x = 0$ to $+\infty$. This is in contrast to most previous anisotropic singular works in the literature, where the perturbation of the singular term is positive. We refer to the works of Byun and Ko [2], Saoudi and Ghanmi [17], Papageorgiou, Rădulescu, and Zhang [13], Papageorgiou and Winkert [16]. In all these works, $f \geqslant 0$. The fact that $f(z,\cdot)$ changes sign, leads to a different approach since now the unique solution of the purely singular problem cannot serve as a lower solution. We prove a multiplicity theorem producing two nontrivial smooth solutions when $\|\theta\|_\infty$ is small. We should also mention the related isotropic (constant exponents) works of Arora [1], Haddaoui et al. [8], Diaz and Giacomoni [4], Papageorgiou, Vetro, and Vetro [14] (balanced growth problems), Kumar, Rădulescu, and Sreenadh [10] (unbalanced growth problems) and the anisotropic work on systems of Leggat and Miri [11].

2 Mathematical background – hypotheses

Let $L^0(\varOmega)$ be the space of all measurable functions $u : \varOmega \to \mathbb{R}$. As usual, we identify two such functions, which differ only on a Lebesgue-null subset of $\varOmega$. Also, let

$$L_1^\infty(\varOmega) = \Big\{u \in L^\infty(\varOmega)\colon\ 1 \leqslant \operatorname*{ess\,inf}_{\varOmega} p\Big\}.$$

For every $p \in L_1^\infty(\varOmega)$, we set

$$p_- = \operatorname*{ess\,inf}_{\varOmega} p \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_{\varOmega} p.$$

Given $p \in L_1^\infty(\varOmega)$, we define the variable Lebesgue space $L^{p(\cdot)}(\varOmega)$ by

$$L^{p(\cdot)}(\varOmega) = \bigg\{u \in L^0(\varOmega)\colon\ \rho_p(u) = \int_{\varOmega} |u|^{p(z)}\,\mathrm{d}z < \infty\bigg\}.$$

We endow $L^{p(\cdot)}(\Omega)$ with the so-called Luxemburg norm defined by

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0\colon\ \rho_p\left(\frac{u}{\lambda}\right) \leqslant 1\right\}.$$

Normed this way, $L^{p(\cdot)}(\Omega)$ becomes a Banach space, which is separable if $p_+ < \infty$ and reflexive (in fact, uniformly convex) if $1 < p_- \leqslant p_+ < \infty$. The function $\rho_p(u) = \int_\Omega |u|^{p(z)}\,\mathrm{d}z$ is called the modular function, and it is continuous and convex.

If $p, q \in L_1^\infty(\Omega)$ and $q(z) \leqslant p(z)$ for a.e. $z \in \Omega$, then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ continuously.

Suppose that $p, p' \in L_1^\infty(\Omega)$ with $p_+ < \infty$ and for a.e. $z \in \Omega$, satisfy

$$\frac{1}{p(z)} + \frac{1}{p'(z)} = 1 \quad\Longrightarrow\quad p'(z) = \frac{p(z)}{p(z) - 1}.$$

Then we say that p, p' are conjugate variable exponents and $L^{p(\cdot)}(\Omega)^* = L^{p'(\cdot)}(\Omega)$. Also, we have the following version of Hölder's inequality:

$$\int_\Omega |uv|\,\mathrm{d}z \leqslant \left[\frac{1}{p_-} + \frac{1}{p'_-}\right]\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \quad \text{for all } u \in L^{p(\cdot)}(\Omega),\ v \in L^{p'(\cdot)}(\Omega).$$

Now let $p \in L_1^\infty(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$. Using the variable Lebesgue spaces, we can define the corresponding variable Sobolev spaces by

$$W^{1,p(\cdot)}(\Omega) = \bigl\{u \in L^{p(\cdot)}(\Omega)\colon\ |Du| \in L^{p(\cdot)}(\Omega)\bigr\}.$$

Here Du is the weak gradient of $u(\cdot)$. We endow with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|Du\|_{p(\cdot)} \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

With this norm, $W^{1,p(\cdot)}(\Omega)$ becomes a Banach space, which is separable and reflexive (recall that we have assumed $1 < p_- \leqslant p_+ < \infty$).

If $p \in C^{0,1}(\overline{\Omega})$ (space of Lipschitz continuous functions on $\overline{\Omega}$), then $W^{1,p(\cdot)}(\Omega) \cap C^\infty(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$. For such an exponent $p(\cdot)$, we also define

$$W_0^{1,p(\cdot)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,p(\cdot)}}.$$

Then for $p \in C^{0,1}(\overline{\Omega})$ with $1 < p_- \leqslant p_+ < \infty$, we have that $W_0^{1,p(\cdot)}(\Omega)$ is a separable, reflexive (in fact, uniformly convex) Banach space and the Poincaré inequality holds, namely, we have

$$\|u\|_{p(\cdot)} \leqslant c\|Du\|_{p(\cdot)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega)$$

with $c = c(\Omega) > 0$ independent of u. So, on $W_0^{1,p(\cdot)}(\Omega)$, we consider the equivalent norm

$$\|u\| = \|Du\|_{p(\cdot)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

For a comprehensive treatment of variable Lebesgue and Sobolev spaces, we refer to the books of Cruz-Uribe and Fiorenza [3] and Diening, Harjulethto, Hästo, and Ruzicka [5].

Given $r \in L_1^\infty(\Omega)$, we define the critical Sobolev exponent $r^*(z)$ by

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N-r(z)} & \text{if } r(z) < N, \\ +\infty & \text{if } N \leqslant r(z). \end{cases}$$

We have the following embeddings.

Proposition 1. *If $r \in C^{0,1}(\overline{\Omega})$ with $1 < r_- \leqslant r_+ < \infty$ and $p \in C(\overline{\Omega})$, then*

(a) $1 \leqslant p(z) \leqslant r^*(z)$ *for all* $z \in \overline{\Omega} \Rightarrow W_0^{1,r(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *continuously*;
(b) $1 \leqslant p(z) < r^*(z)$ *for all* $z \in \overline{\Omega} \Rightarrow W_0^{1,r(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *compactly.*

There is a close relation between the Luxemburg norm and the modular function.

Proposition 2. *If $r \in L_1^\infty(\Omega)$ and $u \in L^{r(\cdot)}(\Omega)$, then*

(a) $\|u\|_{r(\cdot)} = \lambda \Leftrightarrow \rho_r(u/\lambda) = 1$;
(b) $\|u\|_{r(\cdot)} < 1$ (*resp.* > 1) $\Leftrightarrow \rho_r(u) < 1$ (*resp.* $\rho_r(u) > 1$);
(c) $\|u\|_{r(\cdot)} < 1 \Rightarrow \|u\|_{r(\cdot)}^{r_+} \leqslant \rho_r(u) \leqslant \|u\|_{r(\cdot)}^{r_-}$;
(d) $\|u\|_{r(\cdot)} > 1 \Rightarrow \|u\|_{r(\cdot)}^{r_-} \leqslant \rho_r(u) \leqslant \|u\|_{r(\cdot)}^{r_+}$;
(e) $\|u\|_{r(\cdot)} \to 0 \Leftrightarrow \rho_r(u) \to 0$;
(f) $\|u\|_{r(\cdot)} \to +\infty \Leftrightarrow \rho_r(u) \to +\infty$.

If $r \in C^{0,1}(\overline{\Omega}) \cap L_1^\infty(\Omega)$ and $r_+ < N$, then we have

$$W_0^{1,r(\cdot)}(\Omega) = W^{-1,r'(\cdot)}(\Omega) \quad \text{with } r'(z) = \frac{Nr(z)}{N - r(z)} \text{ for all } z \in \overline{\Omega}.$$

Then we can define the operator $A_r : W_0^{1,r(\cdot)}(\Omega) \to W^{-1,r'(\cdot)}(\Omega)$ by

$$\langle A_r(u), h \rangle = \int_\Omega |Du|^{r(z)-2}(Du, Dh)_{\mathbb{R}^N} \,\mathrm{d}z \quad \text{for all } u, h \in W_0^{1,r(\cdot)}(\Omega).$$

This operator is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too), and of type $(S)_+$, which means that it has the following property:

- $u_n \xrightarrow{\mathrm{w}} u$ in $W_0^{1,r(\cdot)}(\Omega)$ and $\limsup_{n\to\infty} \langle A_r(u_n),\, u_n - u\rangle \leqslant 0$ implies $u_n \to u$ in $W_0^{1,r(\cdot)}(\Omega)$.

If $p, q \in C^{0,1}(\overline{\Omega})$ with $1 < q(z) \leqslant p(z) < N$ for all $z \in \overline{\Omega}$, then we let $V : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ be defined by

$$V(u) = A_p(u) + A_q(u) \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

For this operator, we can state the following result (see [7, Thm. 3.1]).

Proposition 3. $V : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ *is bounded, continuous, strictly monotone (thus maximal monotone too), and of type* $(S)_+$.

Proof. We give an idea of the proof.

The strict monotonicity follows from elementary inequalities (see [7, Ineqs. (1), (2)]). Also, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \leqslant 0\\
&\quad\Longrightarrow\quad \limsup_{n\to\infty}\bigl[\langle A_p(u_n),\, u_n - u\rangle + \langle A_q(u),\, u_n - u\rangle\bigr] \leqslant 0\\
&\qquad\qquad\text{(by the monotonicity of } A_q(\cdot))\\
&\quad\Longrightarrow\quad \limsup_{n\to\infty}\langle A_p(u_n),\, u_n - u\rangle \leqslant 0.
\end{aligned}$$

So, the $(S)_+$-property of $V(\cdot)$ follows from that of $A_p(\cdot)$ (see [7, Thm. 3.1]). Indeed, from the last inequality and the weak lower semicontinuity of the modular function, we have

$$\rho_p(Du_n) \to \rho_p(Du).$$

Also, we know $Du_n \xrightarrow{\mathrm{w}} Du$ in $L^{p(\cdot)}(\Omega, \mathbb{R}^N)$. Invoking Lemma 2.4.17 (see also Remark 2.4.19) of [5], we conclude that

$$u_n \to u \quad \text{in } W_0^{1,p(\cdot)}(\Omega).$$

The continuity of $V(\cdot)$ is a consequence of Vitali's theorem. In fact, $V(\cdot)$ is a homeomorphism (see [7, Thm. 3.1]). □

Regularity theory will lead us to the space

$$C_0^1(\overline{\Omega}) = \bigl\{u \in C^1(\overline{\Omega}):\ u|_{\partial\Omega} = 0\bigr\}.$$

This is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}): u(z) \geqslant 0$ for all $z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+:\ u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0\right\}.$$

Here $n(\cdot)$ is the outward unit normed on $\partial\Omega$ and $\partial u/\partial n = (Du, n)_{\mathbb{R}^N}$.

Given a measurable function $u : \Omega \to \mathbb{R}$, we define

$$u^{\pm}(z) = \max\bigl\{\pm u(z), 0\bigr\} \quad \text{for all } z \in \Omega.$$

Evidently, $u^{\pm}(\cdot)$ are both measurable and

$$u = u^+ - u^-, \qquad |u| = u^+ + u^-.$$

If $u \in W_0^{1,p(\cdot)}(\Omega)$, then $u^{\pm} \in W_0^{1,p(\cdot)}(\Omega)$.

Suppose $u, v : \Omega \to \mathbb{R}$ are measurable functions and $u(z) \leqslant v(z)$ for a.e. $z \in \Omega$. We define

$$[u,v] = \{h \in W_0^{1,p(\cdot)}(\Omega)\colon\ u(z) \leqslant h(z) \leqslant v(z) \ \text{ for a.e. } z \in \Omega\},$$
$$\operatorname{int}_{C_0^1(\overline{\Omega})}[u,v] \quad \text{the interior in } C_0^1(\overline{\Omega}) \text{ of } [u,v] \cap C_0^1(\overline{\Omega}),$$
$$[u) = \{h \in W_0^{1,p(\cdot)}(\Omega)\colon\ u(z) \leqslant h(z) \text{ for a.e. } z \in \Omega\}.$$

Now let X be a Banach space and $\varphi \in C^1(X)$. Then a critical set of φ

$$K_\varphi = \{u \in X\colon\ \varphi'(u) = 0\}.$$

We say that $\varphi(\cdot)$ satisfies the C-condition if the following is true:

- Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^*, admits a strongly convergent subsequence.

Finally, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

We introduce the hypotheses on the data of (1).

(H0) $p, q \in C^{0,1}(\overline{\Omega})$ with $1 < q(z) < p(z) < N$ for all $z \in \overline{\Omega}$, $\eta \in C(\overline{\Omega})$ with $0 < \eta(z) < 1$ for all $z \in \overline{\Omega}$, and $\theta \in L^\infty(\Omega) \setminus \{0\}$, $\theta(z) \geqslant 0$ for a.e. $z \in \Omega$.

(H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.e. $z \in \Omega$ and

(i) $|f(z,x)| \leqslant \hat{a}(z)[1 + x^{r(z)}]$ for a.e. $z \in \Omega$, all $x \geqslant 0$ with $\hat{a} \in L^\infty(\Omega)$, $r \in C(\overline{\Omega})$ with $p_+ < r(z) < p^*(z) = Np(z)/(N - p(z))$ for all $z \in \overline{\Omega}$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,\mathrm{d}s$, then $\lim_{x\to+\infty} F(z,x)x^{p_+} = +\infty$ uniformly for a.e. $z \in \Omega$, and if

$$\hat{e}(z,x) = \left(1 - \frac{p_+}{1-\eta(z)}\right)\theta(z)x^{1-\eta(z)} + f(z,x)x - p_+F(z,x),$$

then we can find $\gamma \in L^1(\Omega)$ such that

$$\hat{e}(z,x) \leqslant \hat{e}(z,y) + \gamma(z) \quad \text{for a.e. } z \in \Omega, \ \text{all } 0 \leqslant x \leqslant y;$$

(iii) there exist $k > 0$ and $\delta > 0$ such that

$$\theta(z)k^{-\eta(z)} + f(z,k) \leqslant -\beta < 0 \quad \text{for a.e. } z \in \Omega,$$
$$0 < c_s \leqslant f(z,x) \quad \text{for a.e. } z \in \Omega, \ \text{all } \delta \geqslant x \geqslant s > 0;$$

(iv) there exists $\hat{\xi} > 0$ such that for a.e. $z \in \Omega$, the function

$$x \to f(z,x) + \hat{\xi}x^{p(z)-1}$$

is nondecreasing on $[0,k]$.

Remark 1. Since we look for positive solutions and above hypotheses concern only the positive semiaxis, without any loss of generality, we may assume that $f(z,x) = 0$ for a.e. $z \in \Omega$, all $x \leqslant 0$. Hypothesis (H1)(ii) implies that for a.e. $z \in \Omega$, $f(z,\cdot)$ is $(p_+ - 1)$ superlinear but need not satisfy the well-known Ambrosetti–Rabinowitz condition, common when studying superlinear problems.

An example of a function, which satisfies hypotheses (H1) above, is the following:

$$f(z,x) = \begin{cases} (x^+)^{\tau(z)-1} - c(x^+)^{\lambda(z)-1} & \text{if } x \leqslant 1, \\ x^{p_+-1} \ln x - (c-1)(x^+)^{\mu(z)-1} & \text{if } 1 < x \end{cases}$$

with $\tau, \lambda, \mu \in C(\overline{\Omega})$, $\tau(z) < \lambda(z)$ and $\mu(z) < p(z)$ for all $z \in \overline{\Omega}$ and $c - 1 > \|\theta\|_\infty$.

By a solution of (1) we mean a function $u \in W_0^{1,p(\cdot)}(\Omega)$ such that $u(z) > 0$ for a.e. $z \in \Omega$, for all $h \in W_0^{1,p(\cdot)}(\Omega)$, $u^{-\eta(z)}h \in L^1(\Omega)$ and

$$\langle V(u), h \rangle = \int_\Omega \bigl[\theta(z)u^{-\eta(z)} + f\bigl(z, u(z)\bigr)\bigr] h \, \mathrm{d}z \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega).$$

3 Multiple positive solutions

In this section, we prove a multiplicity theorem producing two nontrivial smooth solutions for problem (1).

We start by examining the following auxiliary anisotropic Dirichlet problem:

$$-\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) = \theta(z) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2}$$

Proposition 4. *If hypotheses* (H0) *hold, then problem* (2) *has a unique solution* $\bar{u}_\theta \in \operatorname{int} C_+$*, and* $\bar{u}_\theta \to 0$ *in* $C_0^1(\overline{\Omega})$ *as* $\|\theta\|_\infty \to 0$.

Proof. From Proposition 3 we know that the operator $V : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ is continuous, maximal monotone, and strictly monotone. Also, for all $u \in W_0^{1,p(\cdot)}(\Omega)$, we have

$$\|Du\|_{p(\cdot)} \leqslant \langle V(u), u \rangle \quad \Longrightarrow \quad V(\cdot) \text{ is coercive (Poincaré's inequality).}$$

Therefore $V(\cdot)$ is surjective (see [12, p. 135]). So, we can find $\bar{u}_\theta \in W_0^{1,p(\cdot)}(\Omega)$, $\bar{u}_\theta \geqslant 0$, $\bar{u}_\theta \neq 0$ such that

$$V(\bar{u}_\theta) = \theta \quad \text{in } W^{-1,p'(\cdot)}(\Omega). \tag{3}$$

The strict monotonicity of $V(\cdot)$ implies that $\bar{u}_\theta$ is unique. From [13, Prop. A1] we know that $\bar{u}_\theta \in L^\infty(\Omega)$. Then the anisotropic regularity theory of [6] implies that $\bar{u}_\theta \in C_+ \setminus \{0\}$. From (3) we have (see [13, Prop. A2])

$$-\Delta_{p(z)} \bar{u}_\theta - \Delta_{q(z)} \bar{u}_\theta \geqslant 0 \quad \text{in } \Omega \quad \Longrightarrow \quad \bar{u}_\theta \in \operatorname{int} C_+.$$

On (3), we act with $\bar{u}_\theta \in \operatorname{int} C_+$ and obtain

$$\rho_p(D\bar{u}_\theta) \leqslant \int\limits_\Omega \theta(z)\bar{u}_\theta \,\mathrm{d}z \leqslant c_1\|\theta\|_\infty\|\bar{u}_\theta\| \quad \text{for some } c_1 > 0$$

(here we have used that $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$ continuously)

$$\implies \quad \min\bigl\{\|\bar{u}_\theta\|^{p_+}, \|\bar{u}_\theta\|^{p_-}\bigr\} \leqslant c_1\|\theta\|_\infty\|\bar{u}_\theta\| \quad \text{(see Proposition 2)},$$
$$\implies \quad \min\bigl\{\|\bar{u}_\theta\|^{p_+-1}, \|\bar{u}_\theta\|^{p_--1}\bigr\} \leqslant c_1\|\theta\|_\infty.$$

Since $1 < p_- \leqslant p_+$, we see that

$$\|\bar{u}_\theta\| \to 0 \quad \text{as } \|\theta\|_\infty \to 0. \tag{4}$$

Moreover, the anisotropic regularity theory (see [6]), implies that there exist $\alpha \in (0,1)$ and $c_2 > 0$ such that

$$\bar{u}_\theta \in C_0^{1,\alpha}(\overline{\Omega}) \text{ and } \|\bar{u}_\theta\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant 1 \quad \text{for all } \theta \in L^\infty(\Omega)_+,\ \|\theta\|_\infty \leqslant c_2.$$

The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (4) imply that

$$\bar{u}_\theta \to 0 \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } \|\theta\|_\infty \to 0.$$

□

Then for $\|\theta\|_\infty > 0$ small, we will have

$$\|\bar{u}_\theta\|_{C_0^1(\overline{\Omega})} \leqslant \min\{\delta, k\}, \qquad 1 < \frac{1}{\|\bar{u}_\theta\|_{C_0^1(\overline{\Omega})}}. \tag{5}$$

Now we can produce a positive solution for problem (1) when $\|\theta\|_\infty$ is small. Moreover, we can localize this solution.

Proposition 5. *If hypotheses* (H0)*,* (H1)*(i), (iii), (iv) hold and $\|\theta\|_\infty > 0$ is small, then problem* (1) *has a positive solution* (*see* (5))

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\bar{u}_\theta, k].$$

Proof. We already know that for $\|\theta\|_\infty > 0$ small, (5) holds. So, we can define the Carathéodory function $l(z,x)$ by

$$l(z,x) = \begin{cases} \theta(z)\bar{u}_\theta(z)^{-\eta(z)} + f(z,\bar{u}_\theta(z)) & \text{if } x < \bar{u}_\theta(z), \\ \theta(z)x^{-\eta(z)} + f(z,x) & \text{if } \bar{u}_\theta(z) \leqslant x \leqslant k, \\ \theta(z)k^{-\eta(z)} + f(z,k) & \text{if } k < x. \end{cases} \tag{6}$$

We set $L(z,x) = \int_0^x l(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\psi : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined for all $u \in W_0^{1,p(\cdot)}(\Omega)$ by

$$\psi(u) = \int\limits_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,\mathrm{d}z + \int\limits_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,\mathrm{d}z - \int\limits_\Omega L(z,u)\,\mathrm{d}z.$$

Note that $L(z,x)$ being a Carathéodory function (that is, measurable in $z \in \Omega$, continuous in $x \in \mathbb{R}$), it is jointly measurable. So, $L(\cdot, u(\cdot))$ is measurable and in $L^\infty(\Omega)$. Then $\psi(\cdot)$ is well defined, C^1, and

$$\langle \psi'(u), h\rangle = \langle A_p(u), h\rangle + \langle A_q(u), h\rangle - \int_\Omega l(z,u)h \,\mathrm{d}z \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega)$$

(see [7] and [9, p. 349]).

From (6) it is clear that $\psi(\cdot)$ is coercive. Also, Proposition 1 (the anisotropic Sobolev embedding theorem) implies that $\psi(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\psi(u_0) = \inf\left[\psi(u)\colon\ u \in W_0^{1,p(\cdot)}(\Omega)\right] \\ &\quad\implies \quad \langle \psi'(u_0),\ h\rangle = 0 \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega), \\ &\quad\implies \quad \langle V(u_0), h\rangle = \int_\Omega l(z,u_0)h\,\mathrm{d}z \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega). \end{aligned} \tag{7}$$

In (7), first, we choose the test function $(\bar{u}_\theta - u_0)^+ \in W_0^{1,p(\cdot)}(\Omega)$. We have

$$\begin{aligned} \langle V(u_0), (\bar{u}_\theta - u_0)^+\rangle &= \int_\Omega \left[\theta(z)\bar{u}_\theta^{-\eta(z)} + f(z,\bar{u}_\theta)\right](\bar{u}_\theta - u_0)^+\,\mathrm{d}z \quad \text{(see (6))} \\ &\geqslant \int_\Omega \theta(z)\bar{u}_\theta^{-\eta(z)}(\bar{u}_\theta - u_0)^+\,\mathrm{d}z \quad \text{(see (5) and (H1)(iii))} \\ &= \langle V(\bar{u}_\theta), (\bar{u}_\theta - u_0)^+\rangle \quad \text{(from Proposition 4)} \\ \implies \quad \bar{u}_\theta \leqslant u_0. \end{aligned}$$

Next, in (7), we use the test function $(u_0 - k)^+ \in W_0^{1,p(\cdot)}(\Omega)$. We obtain

$$\begin{aligned} \langle V(u_0),\ (u_0 - k)^+\rangle &= \int_\Omega \left[\theta(z)k^{-\eta(z)} + f(z,k)(u_0 - k)^+\right]\mathrm{d}z \\ &\leqslant 0 \quad \text{(see (H1)(iii))}. \end{aligned}$$

We have proved that

$$u_0 \in [\bar{u}_\theta, k]. \tag{8}$$

Then from (8), (6), and (7) it follows that u_0 is a positive solution of (1).

Invoking [17, Thm. B1], we have that $u_0 \in \operatorname{int} C_+$.

Let $\hat{\xi} > 0$ be as postulated by hypothesis (H1)(iv), we have

$$\begin{aligned} &-\Delta_{p(z)}\bar{u}_\theta - \Delta_{q(z)}\bar{u}_\theta + \hat{\xi}\bar{u}_\theta^{p(z)-1} - \theta(z)\bar{u}_\theta^{-\eta(z)} \\ &\quad\leqslant -\Delta_{p(z)}\bar{u}_\theta - \Delta_{q(z)}\bar{u}_\theta + \hat{\xi}\bar{u}_\theta^{p(z)-1} - \theta(z) \quad \text{(see (5))} \\ &\quad= \hat{\xi}\bar{u}_\theta^{p(z)-1} \quad \text{(see Proposition 4)} \end{aligned}$$

$$
\begin{aligned}
&\leqslant f(z,\bar u_\theta) + \hat\xi \bar u_\theta^{p(z)-1} \quad \text{(see (5) and (H1)(iii))}\\
&\leqslant f(z,u_0) + \hat\xi u_0^{p(z)-1} \quad \text{(see (8) and (H1)(iv))}\\
&= -\Delta_{p(z)}u_0 - \Delta_{q(z)}u_0 + \hat\xi u_0^{p(z)-1} - \theta(z)u_0^{-\eta(z)} \quad \text{in } \Omega.
\end{aligned} \tag{9}
$$

Note that for all $K \subseteq \Omega$ compact, we have

$$0 < c_K \leqslant f\big(z,\bar u_\theta(z)\big) \quad \text{for a.e. } z \in K$$

(recall $\bar u_\theta \in \operatorname{int} C_+$ and see (H1)(iii)). Then (9) and [15, Prop. 2.3] imply that

$$u_0 - \bar u_\theta \in \operatorname{int} C_+. \tag{10}$$

Also, we have

$$
\begin{aligned}
&-\Delta_{p(z)}u_0 - \Delta_{q(z)}u_0 + \hat\xi u_0^{p(z)-1} - \theta(z)u_0^{-\eta(z)}\\
&\quad= f(z,u_0) + \hat\xi u_0^{p(z)-1}\\
&\quad\leqslant f(z,k) + \hat\xi k^{p(z)-1} \quad \text{(see (8) and (H1)(iv))}\\
&\quad\leqslant -\Delta_{p(z)}k - \Delta_{q(z)}k + \hat\varepsilon k^{p(z)-1} - \theta(z)k^{-\eta(z)}.
\end{aligned}
$$

Note that $f(z,k) \leqslant -\beta - \theta(z)k^{-\eta(z)}$ for a.e. $z \in \Omega$ and $\beta > 0$. So, [13, Prop. A4] implies that

$$u_0(z) < k \quad \text{for all } z \in \overline{\Omega}. \tag{11}$$

From (10) and (11), it follows that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\bar u_\theta, k]. \qquad \square$$

Consider the Carathéodory function $g(z,x)$ defined by

$$
g(z,x) = \begin{cases} \theta(z)\bar u_\theta(z)^{-\eta(z)} + f(z,\bar u_\theta(z)) & \text{if } x \leqslant \bar u_\theta(z),\\ \theta(z)x^{-\eta(z)} + f(z,x) & \text{if } \bar u_\theta(z) < x. \end{cases} \tag{12}
$$

We see $G(z,x) = \int_0^x g(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\varphi : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined for all $u \in W_0^{1,p(\cdot)}(\Omega)$ by

$$\varphi(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,\mathrm{d}z + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,\mathrm{d}z - \int_\Omega G(z,u)\,\mathrm{d}z.$$

As for $\psi(\cdot)$, we see that $\varphi(\cdot)$ is well defined. In this case, from (12) and hypothesis (H1)(i) we see that $G(\cdot,u(\cdot)) \in L^1(\Omega)$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$.

Proposition 6. *If hypotheses* (H0)*,* (H1) *hold, then the functional* φ *satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(\cdot)}(\Omega)$ be a sequence such that

$$\left|\varphi(u_n)\right| \leqslant c_3 \quad \text{for some } c_3 > 0,\ \text{all } n \in \mathbb{N}, \tag{13}$$

$$\left(1 + \|u_n\|\right)\varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'(\cdot)}(\Omega). \tag{14}$$

From (14) we have for all $h \in W_0^{1,p(\cdot)}(\Omega)$,

$$\left|\left\langle V(u_n), h\right\rangle - \int_\Omega g(z,u_n)h\,\mathrm{d}z\right| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \text{with } \varepsilon_n \to 0^+. \tag{15}$$

In (15), we use the test function $h = -u_n^- \in W_0^{1,p(\cdot)}(\Omega)$. Using (12) and the fact that $f(z,\bar{u}_\theta(z)) \geqslant 0$ for a.e. $z \in \Omega$, we obtain

$$\rho_p(Du_n^-) \leqslant 0 \quad \text{for all } n \in \mathbb{N} \quad \Longrightarrow \quad u_n \geqslant 0 \quad \text{for all } n \in \mathbb{N} \tag{16}$$

(see Proposition 2).

From (16), (13), and (12) we have

$$\begin{aligned}
&\int_\Omega \frac{p_+}{p(z)}|Du_n|^{p(z)}\,\mathrm{d}z + \int_\Omega \frac{p_+}{q(z)}|Du_n|^{q(z)}\,\mathrm{d}z \\
&\quad - \int_{\{0\leqslant u_n\leqslant \bar{u}_\theta\}} p_+\left[\theta(z)\bar{u}_\theta^{-\eta(z)} + f(z,\bar{u}_\theta)\right]u_n\,\mathrm{d}z \\
&\quad - \int_{\{\bar{u}_\theta<u_n\}} \frac{p_+}{1-\eta(z)}\theta(z)\left[u_n^{1-\eta(z)} - \bar{u}_\theta^{1-\eta(z)}\right]\mathrm{d}z \\
&\quad - \int_{\{\bar{u}_\theta<u_n\}} p_+\left[F(z,u_n) - F(z,\bar{u}_\theta)\right]\mathrm{d}z \leqslant p_+c_1 \quad \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{17}$$

Inequality (17) implies that

$$\begin{aligned}
&\rho_p(Du_n) + \rho_q(Du_n) \\
&\quad - \int_{\{\bar{u}_\theta<u_n\}} \frac{p_+}{1-\eta(z)}\theta(z)u_n^{1-\eta(z)}\,\mathrm{d}z - \int_{\{\bar{u}_\theta<u_n\}} p_+F(z,u_n)\,\mathrm{d}z \\
&\quad - \int_{\{0\leqslant u_n\leqslant \bar{u}_\theta\}} p_+\left[\theta(z)\bar{u}_\theta^{1-\eta(z)} + f(z,\bar{u}_\theta)\right]u_n\,\mathrm{d}z \leqslant pc_1 \quad \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{18}$$

Also, if in (15), we choose the test function $h = u_n \in W_0^{1,p(\cdot)}(\Omega)$, we have

$$\begin{aligned}
&-\rho_p(Du_n) - \rho_q(Du_n) + \int_{\{0\leqslant u_n\leqslant \bar{u}_\theta\}} \left[\theta(z)\bar{u}_\theta^{-\eta(z)} + f(z,\bar{u}_\theta)\right]u_n\,\mathrm{d}z \\
&\quad + \int_{\{\bar{u}_\theta<u_n\}} \left[\theta(z)u_n^{1-\eta(z)} + f(z,u_n)u_n\right]\mathrm{d}z \leqslant \varepsilon_n \quad \text{for all } n \in \mathbb{N}
\end{aligned} \tag{19}$$

(see (12)). We add (18) and (19) and obtain

$$\begin{aligned}
&\int\limits_{\{1\leqslant u_n\leqslant \bar{u}_\theta\}} (1-p_+)\big[\theta(z)\bar{u}_\theta^{-\eta(z)}+f(z,\bar{u}_\theta)\big]u_n\,\mathrm{d}z\\
&\quad+\int\limits_{\{\bar{u}_\theta<u_n\}}\biggl(1-\frac{p_+}{1-\eta(z)}\biggr)\theta(z)u_n^{1-\eta(z)}\,\mathrm{d}z\\
&\quad+\int\limits_{\{\bar{u}_\theta<u_n\}}\big[f(z,u_n)u_n-p_+F(z,u_n)\big]\,\mathrm{d}z\leqslant c_4\quad\text{for some } c_4>0,\ \text{all } n\in\mathbb{N}\\
&\quad\Longrightarrow\quad\int\limits_{\Omega}\hat{e}(z,u_n)\,\mathrm{d}z\leqslant c_3\quad\text{for some } c_3>0,\ \text{all } n\in\mathbb{N}.
\end{aligned}\tag{20}$$

Claim. $\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,p(\cdot)}(\Omega)$ is bounded.

We argue by contradiction. So, suppose that at least for a subsequence, we have

$$\|u_n\|\to\infty.\tag{21}$$

We set $y_n=u_n/\|u_n\|$, $n\in\mathbb{N}$. Then $\|y_n\|=1$ and $y_n\geqslant 0$ for all $n\in\mathbb{N}$. So, we may assume that

$$y_n\xrightarrow{\mathrm{w}}y\quad\text{in } W_0^{1,p(\cdot)}(\Omega),\qquad y_n\to y\quad\text{in } L^{r(\cdot)}(\Omega),\qquad y\geqslant 0.\tag{22}$$

First, we suppose that $y\neq 0$ and let $\widehat{\Omega}=\{z\in\Omega\colon\ y(z)>0\}$. We have $|\widehat{\Omega}|_N>0$ (see (22)) and

$$u_n(z)\to+\infty\quad\text{for a.e. } z\in\widehat{\Omega}.\tag{23}$$

We may assume that $\|u_n\|\geqslant 1$ for all $n\in\mathbb{N}$ (see (21)). On account of hypothesis (H1)(ii) and (23), we have

$$\frac{F(z,u_n^+(z))}{\|u_n\|^{p_+}}=\frac{F(z,u_n^+(z))}{u_n(z)^{p_+}}y_n(z)^{p_+}\to+\infty\quad\text{for a.e. } z\in\widehat{\Omega}.$$

So, using Fatou's lemma, we have

$$\lim_{n\to\infty}\int\limits_{\Omega}\frac{F(z,u_n)}{\|u_n\|^{p_+}}\,\mathrm{d}z=+\infty.\tag{24}$$

From (13) and (12) we have

$$\begin{aligned}
&-\frac{1}{q_-}\big[\rho_p(Du_n)+\rho_q(Du_n)\big]\\
&\quad+\int\limits_{\Omega}\biggl[\frac{1}{1-\eta(z)}\theta(z)u_n^{1-\eta(z)}+F(z,u_n)\biggr]\,\mathrm{d}z\leqslant c_5\quad\text{for some } c_5>0,\ \text{all } n\in\mathbb{N},
\end{aligned}$$

$$
\begin{aligned}
\Longrightarrow\quad & \int_\Omega \left[\frac{1}{1-\eta(z)}\theta(z)u_n^{1-\eta(z)} + F(z,u_n)\right] \mathrm{d}z \\
& \leqslant \frac{1}{q_-}\left[\rho_p(Du_n) + \rho_q(Du_n)\right] + c_5 \\
& \leqslant \frac{2}{q_-}\rho_p(Du_n) + c_6 \quad \text{for some } c_6 > 0, \text{ all } n \in \mathbb{N}, \\
& \left(q(z) < p(z)\ \forall z \in \widehat{\Omega} \ \Rightarrow\ \rho_q(Du_n) \leqslant |\Omega|_N + \rho_p(Du_n)\ \forall n \in \mathbb{N}\right) \\
& \leqslant \frac{2}{q_-}\|u_n\|^{p_+} + c_6 \quad \text{for all } n \in \mathbb{N}, \\
\Longrightarrow\quad & \int_\Omega \frac{F(z,u_n)}{\|u_n\|^{p_+}}\,\mathrm{d}z \leqslant \frac{2}{q_-} + \frac{c_6}{\|u_n\|^{p_+}} \quad \text{for all } n \in \mathbb{N} (\text{recall } \|y_n\| = 1).
\end{aligned} \tag{25}
$$

We compare (25) and (24), and we have a contradiction.

Next, suppose $y = 0$. We consider the C^1-functional $\sigma : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\sigma(u) = \frac{1}{p_+}\rho_p(Du_n) - \int_\Omega G(z,u)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

Clearly, we have

$$\sigma \leqslant \varphi. \tag{26}$$

Consider the function $[0,1] \ni t \to \sigma(tu_n)$, $n \in \mathbb{N}$. This function is continuous, and so we can find $t_n \in [0,1]$ such that

$$\sigma(t_n u_n) = \max_{0\leqslant t\leqslant 1} \sigma(tu_n). \tag{27}$$

Let $\mu > 1$ and set $v_n = (2\mu)^{1/p_+} y_n$. From (22) and since we have assumed that $y = 0$, we have

$$v_n \to 0 \quad \text{in } L^{r(z)}(\Omega) \quad \Longrightarrow \quad \int_\Omega G(z,v_n)\,\mathrm{d}z \to 0. \tag{28}$$

From (21) we see that we can find $n_0 \in \mathbb{N}$ such that

$$\frac{(2\mu)^{1/p_-}}{\|u_n\|} \in (0,1] \quad \text{for all } n \geqslant n_0. \tag{29}$$

Then (27) and (29) imply that

$$
\begin{aligned}
\sigma(t_n u_n) &\geqslant \sigma\left(\frac{(2\mu)^{1/p_-}}{\|u_n\|}u_n\right) = \frac{1}{p_+}\int_\Omega \frac{(2\mu)^{p(z)/p_-}}{\|u_n\|^{p(z)}}|Du_n|^{p(z)}\,\mathrm{d}z - \int_\Omega G(z,v_n)\,\mathrm{d}z \\
&\geqslant \frac{2\mu}{p_+}\rho_p(Dy_n) - \int_\Omega G(z,v_n)\,\mathrm{d}z \quad (\text{since } \mu > 1)
\end{aligned}
$$

$$= \frac{2\mu}{p_+} - \int_\Omega G(z, v_n)\,\mathrm{d}z \quad \text{for all } n \geqslant n_0 \text{ (recall } \|y_n\| = 1 \text{ for all } n \in \mathbb{N})$$

$$\geqslant \frac{\mu}{p_+} \quad \text{for all } n \geqslant n_1 \geqslant n_0 \text{ (see (28))}.$$

But $\mu > 1$ is arbitrary. Hence we infer that

$$\sigma(t_n u_n) \to +\infty \quad \text{as } n \to \infty. \tag{30}$$

We have

$$0 \leqslant t_n u_n \leqslant u_n \quad \text{for all } n \in \mathbb{N}.$$

Hypothesis (H1)(iii) implies that

$$\int_\Omega \hat{e}(z, t_n u_n)\,\mathrm{d}z \leqslant \int_\Omega \hat{e}(z, u_n)\,\mathrm{d}z + \|\gamma\|_1 \quad \text{for all } n \in \mathbb{N}. \tag{31}$$

Let $e_0(z,x) = g(z,x)x - p_+ G(z,x)$, then from (12) and (31) it follows that

$$\int_\Omega e_0(z, t_n u_n)\,\mathrm{d}z \leqslant \int_\Omega e_0(z, u_n)\,\mathrm{d}z + c_7 \quad \text{for some } c_7 > 0, \text{ all } n \in \mathbb{N}. \tag{32}$$

Note that

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(u_n) \leqslant \varphi(u_n) \leqslant c_3 \quad \text{for all } n \in \mathbb{N}$$

(see (26)).

From (30) we see that there exists $n_\alpha \in \mathbb{N}$ such that

$$t_n \in (0,1) \quad \text{for all } n \geqslant n_\alpha. \tag{33}$$

Then from (27) we see that for $n \geqslant n_\alpha$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sigma(t u_n)\bigg|_{t=t_n} = 0,\\
&\quad\Longrightarrow\quad \langle \sigma'(t_n u_n), u_n \rangle = 0 \quad \text{(by the chain rule)}\\
&\quad\Longrightarrow\quad \langle \sigma'(t_n u_n), t_n u_n \rangle = 0 \quad \text{(see (33))},\\
&\quad\Longrightarrow\quad \langle A_p(t_n u_n), t_n u_n \rangle = \int_\Omega g(z, t_n u_n)(t_n u_n)\,\mathrm{d}z,\\
&\quad\Longrightarrow\quad \rho_p\big(D(t_n u_n)\big) = \int_\Omega g(z, t_n u_n)(t_n u_n)\,\mathrm{d}z.
\end{aligned} \tag{34}$$

From (12) and (20) we have

$$\int_\Omega e_0(z, u_n)\,\mathrm{d}z \leqslant c_8 \quad \text{for some } c_8 > 0, \text{ all } n \in \mathbb{N}. \tag{35}$$

We return to (32) and use (34) and (35). We obtain

$$\rho_p\big(D(t_n u_n)\big) - \int_\Omega p_+ G(z, t_n u_n)\,\mathrm{d}z \leqslant c_9 \quad \text{for some } c_9 > 0,\ \text{all } n \geqslant n_\alpha,$$
$$\Longrightarrow \quad p_+\sigma(t_n u_n) \leqslant c_9 \quad \text{for all } n \geqslant n_\alpha. \tag{36}$$

We compare (36) and (30), and we reach a contradiction. Therefore $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(\cdot)}(\Omega)$ is bounded. This proves the claim.

On account of the claim and by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{\mathrm{w}} u \quad \text{in } W_0^{1,p(\cdot)}(\Omega), \qquad u_n \to u \quad \text{in } L^{r(\cdot)}(\Omega). \tag{37}$$

We return to (15) and use the test function $h = u_n - u \in W_0^{1,p(\cdot)}(\Omega)$. Passing to the limit as $n \to \infty$ and using (37), we obtain

$$\lim_{n\to\infty} \big\langle V(u_n),\, u_n - u\big\rangle = 0, \quad \Longrightarrow \quad u_n \to u \quad \text{in } W_0^{1,p(\cdot)}(\Omega)$$

(see Proposition 3). This proves that $\varphi(\cdot)$ satisfies the C-condition. □

Proposition 7. *If hypotheses* (H0), (H1) *hold, then* $K_\varphi \subseteq [\bar{u}_\theta) \cap \operatorname{int} C_+$.

Proof. Let $u \in K_\varphi$. We have

$$\big\langle \varphi'(u), h\big\rangle = 0 \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega),$$
$$\Longrightarrow \quad \big\langle V(u), h\big\rangle = \int_\Omega g(z,u) h\,\mathrm{d}z \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega).$$

We use the test function $[\bar{u}_\theta - u]^+ \in W_0^{1,p(\cdot)}(\Omega)$. Then

$$\begin{aligned}
\big\langle V(u), (\bar{u}_\theta - u)^+\big\rangle &= \int_\Omega \big[\theta(z)\bar{u}_\theta^{-\eta(z)} + f(z,\bar{u}_\theta)\big](\bar{u}_\theta - u)^+\,\mathrm{d}z \quad \text{(see (12))}\\
&\geqslant \int_\Omega \theta(z)(\bar{u}_\theta - u)^+\,\mathrm{d}z \quad \text{(see (5) and (H1)(iii))}\\
&= \big\langle V(\bar{u}_\theta), (\bar{u}_\theta - u)^+\big\rangle \quad \text{(see Proposition 4)}\\
\Longrightarrow \quad \bar{u}_\theta &\leqslant u.
\end{aligned}$$

The anisotropic regularity theory (see [6]) implies that $u \in \operatorname{int} C+$. Therefore $K_\varphi \subseteq [\bar{u}_\theta) \cap \operatorname{int} C_+$. □

On account of Proposition 7, we see that we may assume that

$$K_\varphi \text{ is finite.} \tag{38}$$

Otherwise, Proposition 7 and (12) imply that we have a whole sequence of distinct positive smooth solutions of (1), and so, we are done.

Proposition 8. *If hypotheses* (H0)*,* (H1) *hold and* $\|\theta\|_\infty$ *is small, then problem* (1) *has a second positive solution*

$$\hat{u} \in \operatorname{int} C_+, \quad \hat{u} \neq u_0.$$

Proof. From (6) and (12) we see that

$$\psi|_{[\bar{u}_\theta, k]} = \varphi|_{[\bar{u}_\theta, k]}. \tag{39}$$

Let u_0 be the first positive solution of problem (1) produced in Proposition 2. From Proposition 5 we know that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\bar{u}_\theta, k]. \tag{40}$$

From the proof of Proposition 5 we know that

$$u_0 \text{ is a minimizer of } \psi(\cdot). \tag{41}$$

From (39), (40), and (41) we see that

$$\begin{aligned} &u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi(\cdot) \\ &\quad \Longrightarrow \quad u_0 \text{ is a local } W_0^{1,p(\cdot)}(\Omega)\text{-minimizer of } \varphi(\cdot) \end{aligned} \tag{42}$$

(see [13, Prop. A2]).

Then (38), (42), and Theorem 5.7.6 of [12, p. 449] imply that we can find $\rho \in (0,1)$ small such that

$$\varphi(u_0) < \inf\bigl[\varphi(u)\colon \|u - u_0\| = \rho\bigr] = \hat{m}. \tag{43}$$

Suppose that $u \in \operatorname{int} C_+$. Then hypothesis (H1)(ii) implies that

$$\varphi(tu) \to -\infty \quad \text{as } t \to \infty. \tag{44}$$

Finally, from Proposition 6 we know that

$$\varphi(\cdot) \text{ satisfies the C-condition.} \tag{45}$$

From (43), (44), and (45) we see that we can use the mountain pass theorem and find $\hat{u} \in W_0^{1,p(\cdot)}(\Omega)$ such that

$$\begin{aligned} &\hat{u} \in K_\varphi \subseteq [\bar{u}_\theta) \cap \operatorname{int} C_+ \quad \text{(see Proposition 7)}, \\ &\hat{m} \leqslant \varphi(\hat{u}) \quad \text{(see (43))} \\ &\quad \Longrightarrow \quad \hat{u} \neq u_0, \hat{u} \in \operatorname{int} C_+, \text{is the second positive solution of problem (1).} \end{aligned}$$ ☐

Summarizing, we can state the following multiplicity theorem for problem (1).

Theorem. *If hypotheses* (H0)*,* (H1) *hold and* $\|\theta\|_\infty$ *is small, then problem* (1) *has at least two positive solutions:* u_0*,* $\hat{u} \in \operatorname{int} C_+$*,* $u_0 \neq \hat{u}$*, and* $u_0(z) < k$ *for all* $z \in \overline{\Omega}$*.*

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