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Bifurcation analysis of impulsive fractional-order Beddington–DeAngelis prey–predator model

Javad Alidousti[a], **Mojtaba Fardi**[a], **Shrideh Al-Omari**[b]

[a]Department of Mathematics,
Faculty of Mathematical Science,
Shahrekord University,
Shahrekord, P.O. Box 115, Iran

[b]Department of Mathematics,
Faculty of Science,
Al-Balqa Applied University,
Amman, Jordan
shridehalomari@bau.edu.jo

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Abstract. In this paper, a fractional density-dependent prey–predator model has been considered. Certain reading of local and global stabilities of an equilibrium point of a system was extracted and conducted by applying fractional systems' stability theorems along with Lyapunov functions. Meanwhile, the persistence of the aforementioned system has been discussed and claimed to imply a local asymptotic stability for the given positive equilibrium point. Moreover, the presented model was extended to a periodic impulsive model for the prey population. Such an expansion was implemented through the periodic catching of the prey species and the periodic releasing of the predator population. By studying the effect of changing some of the system's parameters and drawing their bifurcation diagram, it was observed that different periodic solutions appear in the system. However, the effect of an impulse on the system subjects the system to various dynamic changes and makes it experience behaviors including cycles, period-doubling bifurcation, chaos and coexistence as well. Finally, by comparing the fractional system with the classic one, it has been concluded that the fractional system is more stable than its classical one.

Keywords: prey–predator model, impulsive, stability, Lyapunov function, Caputo derivative, bifurcation, chaos, Beddington–DeAngelis functional response.

1 Introduction

The stability of the ecological systems is one of the essential topics in biological mathematics. In population models, the functional response of the predator to prey density indicates changes in prey density per unit time for each predator [7, 11]. This type of function response is similar to the Holing II function response with the additional assumption of predators' mutual interference [8]. The behavior of models with predator-dependent functional responses may be distinguishable from prey-dependent functional responses [16, 20]. We assume the prey–predator model with the Beddington–DeAngelis functional response [12, 18, 19] and density dependent predator as follows:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x\left(a - bx - \frac{\beta y}{\mu + \eta x + \gamma y}\right),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y\left(-\sigma + \frac{\xi x}{\mu + \eta x + \gamma y}\right),\end{aligned} \tag{1}$$

where all the parameters are positive, $x(t)$ and $y(t)$ indicate the population density of the prey and the predator. For biological description of the parameters, see [12, 18, 19].

Fractional calculus has attracted much attention due to its many applications in science and engineering. For example, the behavior of many physical and genetic phenomena is memory driven and therefore is described by fractional models [3–6, 9].

The Caputo derivative with a fractional order α of $x(t)$ is defined by

$$\frac{\mathrm{d}^\alpha}{ti^\alpha}x(t) := J^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}x(t), \quad 0 < \alpha \leqslant 1,$$

where J^α is the Riemann–Liouville integral operator defined by

$$J^\alpha x(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)\,\mathrm{d}\tau$$

in which $\Gamma(\cdot)$ is the Eulers gamma function. For $\alpha = 0$, we set $J^\alpha := Id$, the identity operator.

Consider the following system:

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = f\big(x(t)\big), \quad 0 < \alpha < 1, \tag{2}$$

with the initial condition $x(0) = x_0$, where $x_0, x(t) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$.

In this paper, by applying the fractional derivative on (1), we reach the following fractional model, which is equipped with the Caputo fractional derivative:

$$\begin{aligned}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) &= x\left(a - bx - \frac{\beta y}{\mu + \eta x + \gamma y}\right),\\ \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}y(t) &= y\left(-\sigma + \frac{\xi x}{\mu + \eta x + \gamma y}\right),\end{aligned} \tag{3}$$

where all of the parameters are positive and $\alpha \in (0, 1)$.

When $\eta = \gamma = 0$ and $\mu > 0$, (1) is reduced to Lotka–Volterra model.

When $\eta = 1$, $\gamma = 0$, system (1) will be the following Kolmogorov-type prey–predator model with Holling type II functional response:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) = x\left(a - bx - \frac{\beta y}{\mu + x}\right),$$
$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}y(t) = y\left(-\sigma + \frac{\xi x}{\mu + x}\right).$$

When $\mu = 0$, $\eta = 1$, system (1) turns into the following ratio-dependent prey–predator model:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) = x\left(a - bx - \frac{\beta y}{\gamma y + x}\right),$$
$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}y(t) = y\left(-\sigma + \frac{\xi x}{\gamma y + x}\right).$$

The purpose is to investigate the dynamics of (3), which is done through the Lyapunov method. In the following, we transform this model into a periodic impulsive model by periodically catching the prey population and releasing the predator population, and we examine the effect of the parameter changes on the model.

This paper is structured as follows. In Section 2, the local stability analysis of the equilibrium points of the system is discussed. In Section 3, the system persistence and global stability conditions of the system are presented using the Lyapunov method. In Section 4, by adding periodic impulse to the model, due to the control of the population of some species, we introduce a new population model and perform numerical analysis on this model. In Section 5, a conclusion is presented.

2 Stability analysis

We start our analysis by calling the next theorem.

Theorem 1. (*See* [22].) *Consider*

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t), \quad 0 < \alpha < 1,$$

where A is an arbitrary constant $n \times n$ matrix.

(i) *The solution $x(t) = 0$ of the system is asymptotically stable if and only if all eigenvalues λ_j ($j = 1, 2, \dots, n$) of A satisfy $|\arg(\lambda_j)| >$ $alpha\pi/2$.*

(ii) *The solution $x(t) = 0$ is stable if and only if the eigenvalues satisfy $|\arg(\lambda_j)| \geqslant \alpha\pi/2$ and all eigenvalues with $|\arg(\lambda_j)| = \alpha\pi/2$ have a geometric multiplicity that coincides with their algebraic multiplicity* (*i.e., an eigenvalue that is an l-fold zero of the characteristic polynomial has l linearly independent eigenvectors*).

If E is an equilibrium point of (2) and all of the eigenvalues $\lambda(Df(E))$ of the Jacobian matrix $Df(E)$ at the equilibrium point E satisfy $\lambda(Df(E)) \neq 0$ and $|\arg(\lambda(Df(E)))| \neq \alpha\pi/2$, then we call E a hyperbolic equilibrium point.

We pay attention of the following theorem that is the fractional type of the Hartman–Grobman theorem.

Theorem 2. (*See* [17].) *Suppose that E is a hyperbolic equilibrium point of* (2)*, then $f(x)$ is topologically equivalent with its linearization $Df(E)x$ in the neighborhood of the point E.*

Theorem 3. *If $\xi < \sigma\eta$, then the equilibrium point $E_1 = (a/b, 0)$ is locally asymptotically stable.*

Proof. The Jacobian evaluated at $(a/b, 0)$ has two eigenvalues $\lambda_1 = -a$, $\lambda_2 = -\sigma - a\xi/(a\eta + b\mu)$. Negative condition of the eigenvalue λ_2 implies that $\sigma b\mu + a(\sigma\eta - \xi) > 0$, i.e., $\xi < \sigma\eta$. This completes the proof. □

In the following, a positive equilibrium of (3) is denoted as $\overline{E}$. The equilibrium point $\overline{E}$ satisfies the following algebraic equation:

$$
\begin{aligned}
&(a - b\overline{x})(\mu + \eta\overline{x} + \gamma\overline{y}) - \beta\overline{y} = 0,\\
&(-\sigma)(\mu + \eta\overline{x} + \gamma\overline{y}) + \xi\overline{x} = 0.
\end{aligned}
$$

It is easy to check that if the condition

$$
(\xi - \sigma\eta)\frac{a}{b} > \sigma\mu \tag{4}
$$

holds, then system (3) has a positive equilibrium point.

Let $x(t) = \overline{x} + X(t)$, $y(t) = \overline{y} + Y(t)$, then linearized of (3) is

$$
\begin{aligned}
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) &= \overline{x}\big(a_{11}X(t) + a_{12}Y(t)\big),\\
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}y(t) &= \overline{y}\big(a_{21}X(t) + a_{22}Y(t)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{11} &= \frac{\beta\eta\overline{y}}{(\mu + \eta\overline{x} + \gamma\overline{y})^2} - b, \qquad & a_{12} &= -\frac{\beta(\mu + \eta\overline{x})}{(\mu + \eta\overline{x} + \gamma\overline{y})^2} < 0,\\
a_{21} &= \frac{\xi(\mu + \gamma\overline{y})}{(\mu + \eta\overline{x} + \gamma\overline{y})^2} > 0, \qquad & a_{22} &= -\frac{\xi\gamma\overline{x}}{(\mu + \eta\overline{x} + \gamma\overline{y})^2} < 0.
\end{aligned}
$$

Theorem 4. *If $\vartheta_2 > 0$, $\vartheta_1 + 2\sqrt{\vartheta_2}\cos(\alpha\pi/2) > 0$, then the positive equilibrium point $\overline{E} = (\overline{x}, \overline{y})$ of* (3) *is locally asymptotically stable, where*

$$
\begin{aligned}
\vartheta_1 &= \frac{b\xi^2\overline{x}^2 - \beta\overline{y}\eta\sigma^2 + \overline{y}\gamma\sigma^2\xi}{\overline{x}\xi^2},\\
\vartheta_2 &= \frac{\overline{y}\sigma^2(\overline{x}^2 b\xi\gamma - \overline{x}\beta\sigma\eta - \overline{y}\beta\sigma\gamma + \overline{x}\beta\xi)}{\xi^2(\overline{x})^2}.
\end{aligned}
$$

Proof. The Jacobian matrix of system (3) computed at $\overline{E}$ is given by

$$J|_{\overline{E}} = \begin{pmatrix} \overline{x}(-b + \frac{\beta\overline{y}\eta}{(\eta\overline{x}+\gamma\overline{y}+\mu)^2}) & -\frac{\overline{x}\beta(\eta\overline{x}+\mu)}{(\eta\overline{x}+\gamma\overline{y}+\mu)^2} \\ \frac{\overline{y}\xi(\gamma\overline{y}+\mu)}{(\eta\overline{x}+\gamma\overline{y}+\mu)^2} & -\frac{\overline{y}\xi\overline{x}\gamma}{(\eta\overline{x}+\gamma\overline{y}+\mu)^2} \end{pmatrix}.$$

Since $\mu + \eta\overline{x} + \gamma\overline{y} = \xi\overline{x}/\sigma$, we get

$$J|_{\overline{E}} = \begin{pmatrix} \overline{x}(-b + \frac{\beta\overline{y}\sigma^2\eta}{\xi^2(\overline{x})^2}) & \overline{x}(-\frac{\beta\sigma}{\xi\overline{x}} + \frac{\overline{y}\sigma^2\beta\gamma}{\xi^2(\overline{x})^2}) \\ \overline{y}(\frac{\sigma}{\overline{x}} - \frac{\sigma^2\eta}{\xi\overline{x}}) & -\frac{\overline{y}\sigma^2\gamma}{\xi\overline{x}} \end{pmatrix}.$$

The characteristic equation of $J|_{\overline{E}}$ is

$$P(\lambda;\vartheta_1,\vartheta_2) = \lambda^2 + \vartheta_1\lambda + \vartheta_2 = 0. \tag{5}$$

Suppose that $|\arg(\lambda)| = \alpha\pi/2$. By substituting $\lambda = re^{\text{ß}\alpha\pi/2}$ into (5), we deduce that

$$r^2\sin(\alpha\pi) + \vartheta_1 r\sin\frac{\alpha\pi}{2} = 0,$$
$$r^2\cos(\alpha\pi) + \vartheta_1 r\cos\frac{\alpha\pi}{2} + \vartheta_2 = 0.$$

Then we get

$$\vartheta_1 = -2r\cos\frac{\alpha\pi}{2} \leqslant 0, \tag{6}$$
$$\vartheta_2 = r^2 > 0. \tag{7}$$

From (6) we obtain $r = -\vartheta_1/(2\cos(\alpha\pi/2))$. By substituting r into (7), we get $\vartheta_2 = \vartheta_1^2/(4\cos^2(\alpha\pi/2))$, i.e.,

$$\vartheta_1 + 2\sqrt{\vartheta_2}\cos\frac{\alpha\pi}{2} = 0.$$

The equilibrium point $\overline{E}$ is locally asymptotically stable if all the zeros λ_i, $i = 1, 2$, of (5) satisfy $|\arg(\lambda_i)| > \alpha\pi/2$. Thus, $\overline{E}$ is locally asymptotically stable if and only if ϑ_2 and $\vartheta_1 + 2\sqrt{\vartheta_2}\cos(\alpha\pi/2) > 0$. □

Notice that due to the characteristic polynomial, if $\beta\eta\overline{y}/(\mu + \eta\overline{x} + \gamma\overline{y})^2 < b$, then for all $\alpha \in (0, 1]$, the equilibrium point $\overline{E}$ is locally asymptotically stable.

3 Fractional system persistence

In this section, the persistency of (3) is investigated, which means that all the solutions starting from an interior equilibrium point of the positive area $\mathbb{R}^2_+$ stay strictly positive and do not approach any boundary of the area as $t \to \infty$. This convinces us that all populations survive for all future times. First, we present the following lemma.

Lemma 1. *Suppose that* $x(t) \in C^1(\mathbb{R}_+)$ *and* $\alpha \in (0,1)$. *Then*

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x(t) - x(t)\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \ln x(t) \leqslant 0 \quad \forall t \geqslant 0. \tag{8}$$

Proof. We have

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \ln x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\dot{x}(\tau)}{x(\tau)}\, \mathrm{d}\tau.$$

It is enough to prove the following equation:

$$\int_0^t (t-\tau)^{-\alpha} \dot{x}(\tau) \frac{x(\tau)-x(t)}{x(\tau)}\, \mathrm{d}\tau \leqslant 0. \tag{9}$$

Equation (9) turn into

$$\int_0^t (t-\tau)^{-\alpha} \dot{w}(\tau) x(t) \left(1 - \frac{1}{w(\tau)+1}\right) \mathrm{d}\tau \leqslant 0,$$

where $w(\tau) = (x(\tau) - x(t))/x(t)$. Now by using the method reported in [26], (1) is archived. □

Theorem 5. *System* (3) *is persistent for all* $0 < \alpha < 1$ *if the equilibrium point* E_* *exists.*

Proof. Consider the following positive Lyapunov function:

$$V(x,y) = \lambda_1 \ln x + \lambda_2 \ln y, \quad (x,y) \in \mathbb{R}_+^2,\ \lambda_i > 0,\ i = 1,2.$$

By applying fractional Caputo differential operator along the solution of the model, we have

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} V(x,y) = \lambda_1 \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \ln x + \lambda_2 \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \ln y.$$

According to Lemma 1 and positivity of the solutions, we get

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} V(x,y) \geqslant \frac{\lambda_1}{x(t)} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x(t) + \frac{\lambda_2}{y(t)} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} y(t),$$

then

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} V(x,y) \geqslant \lambda_1 \left(a - bx - \frac{\beta y}{\mu + \eta x + \gamma y}\right) + \lambda_2 \left(-\sigma + \frac{\xi x}{\mu + \eta x + \gamma y}\right).$$

Following Lemma 4.6 in Huo et al. [15], to show that the system is persistent, we have to prove that $\mathrm{d}^\alpha V(x,y)/\mathrm{d}t^\alpha$ is positive at all the boundary points $E_0 = (0,0)$ and $E_1 = (a/b, 0)$ for appropriate choices of λ_i. Thus, the following conditions are extracted:

$$E_0\colon\ \lambda_1 > \lambda_2 \frac{\sigma}{a}, \qquad E_1\colon\ \lambda_2 \left(-\sigma + \frac{\xi a}{\mu b + \eta a}\right) > 0.$$

By choosing the suitable values of λ_i, the first condition is trivial, and the second condition is

$$(\xi - \sigma\eta)\frac{a}{b} > \sigma\mu.$$

But this condition is exactly the same as conditions for feasibility of this equilibrium point in (4). $\square$

By using the results obtained in Theorem 3, i.e. $\xi < \sigma\eta$, and persistence, we observe that all the boundary equilibrium points become unstable, and a unique interior equilibrium is revealed. Suppose that $\overline{x}$ and $\overline{y}$ are satisfied with $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$. Then we present the following theorem.

Theorem 6. *If $b > \beta\eta\sigma\overline{y}/(\xi\overline{x}(\mu + \eta\underline{x} + \gamma\underline{y}))$ holds, then the positive equilibrium $\overline{x}$ of system* (3) *is globally asymptotically stable.*

Proof. Rewrite (3) in the following form:

$$\begin{aligned}
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) &= x\left(-b(x - \overline{x}) + \frac{\beta\overline{y}}{\mu + \eta\overline{x} + \gamma\overline{y}} - \frac{\beta y}{\mu + \eta x + \gamma y}\right),\\
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}y(t) &= y\left(\frac{\xi x}{\mu + \eta x + \gamma y} - \frac{\xi\overline{x}}{\mu + \eta\overline{x} + \gamma\overline{y}}\right).
\end{aligned}$$

Due to the study of the global stability of $\overline{E} = (\overline{x}, \overline{y})$, we consider the Lyapunov function

$$V(t) = x - \overline{x} - \overline{x}\ln\frac{x}{\overline{x}} + L\left(y - \overline{y} - \overline{y}\ln\frac{y}{\overline{y}}\right),$$

where

$$L = \frac{\beta}{\xi}\frac{\mu + \eta\overline{x}}{\mu + \gamma\overline{y}}. \tag{10}$$

The function $V(x, y)$ satisfies

$$\frac{\partial V}{\partial x} = 1 - \frac{\overline{x}}{x}, \qquad \frac{\partial V}{\partial y} = L - L\frac{\overline{y}}{y}.$$

Hence the fixed point $(\overline{x}, \overline{y})$ is the only extremum of the function $V(x, y)$ in the positive quadrant. It is easy to see that $(\overline{x}, \overline{y})$ is a minimum. Since

$$\lim_{x\to 0} V(x, y) = \lim_{y\to 0} V(x, y) = \lim_{x\to\infty} V(x, y) = \lim_{y\to\infty} V(x, y) = \infty,$$

the point $(\overline{x}, \overline{y})$ is the global minimum, i.e.,

$$V(x, y) > V(\overline{x}, \overline{y}) = 0 \quad \forall x, y > 0,$$

thus

$$V(x, y) > 0 \quad \forall x, y > 0.$$

We compute the fractional derivative of $V(x,y)$ along with the solution of (3) as claimed by Lemma 3.1 [26]. Therefore,

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}V(x,y) \leqslant \left(1-\frac{\overline{x}}{x}\right)\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) + L\left(1-\frac{\overline{y}}{y}\right)\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}y(t).$$

Consequently,

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}V(x,y) \leqslant \frac{\chi}{(\mu+\eta\overline{x}+\gamma\overline{y})(\mu+\eta x+\gamma y)} - b(x-\overline{x})^2,$$

where

$$\chi = \beta\eta(x-\overline{x})(x\overline{y}-\overline{x}y) - (\beta\mu - L\xi\mu)(x-\overline{x})(y-\overline{y}) + L\xi\gamma(y-\overline{y})(x\overline{y}-\overline{x}y).$$

After that

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}V(x,y) \leqslant -\left(b - \frac{\beta\eta\overline{y}}{(\mu+\eta\overline{x}+\gamma\overline{y})(\mu+\eta x+\gamma y)}\right)(x-\overline{x})^2 - \frac{L\xi\gamma\overline{x}}{(\mu+\eta\overline{x}+\gamma\overline{y})(\mu+\eta x+\gamma y)}(y-\overline{y})^2.$$

From (10) and by setting $\varTheta = (\mu+\eta\overline{x}+\gamma\overline{y})(\mu+\eta\underline{x}+\gamma\underline{y}) = (\xi\overline{x}/\sigma)(\mu+\eta\underline{x}+\gamma\underline{y})$, we get

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}V(x,y) \leqslant -\left(b-\frac{\beta\eta\overline{y}}{\varTheta}\right)(x-\overline{x})^2 - \frac{L\xi\gamma\overline{x}}{\varTheta}(y-\overline{y})^2.$$

Therefore, if $b > \beta\eta\overline{y}/\varTheta$, the result is obtained. □

4 The impulsive system and its numerical analysis

The idea of impulsive differential equations was first introduced in 1960 by Milman and Myshkis [21] and then developed in the references [10, 23]. In real life, there are many phenomena that change state rapidly, which has led to the consideration of impulsive fractional differential equations (IFrDEs). Consider the following initial-value problem, which is equipped with the Caputo derivative for $0<\alpha<1$:

$$\begin{aligned}&\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}X(t) = f\big(t, X(t)\big), \quad t\neq t_i,\ i=1,2,3,\dots,\\ &X(t_i+0) = \varphi_i\big(X(t_i)\big), \qquad X(t_0)=X_0,\end{aligned} \tag{11}$$

where $X, X_0 \in \mathbb{R}^n$, $f:[0,+\infty)\times\mathbb{R}^n\to\mathbb{R}^n$, $\varphi_i=\mathbb{R}^n\to\mathbb{R}^n$, $t_1<t_2<t_3<\cdots$.

The second condition in the above equation is expressed as

$$\underbrace{X(t_i+0)-X(t_i)}_{:=\varDelta(X(t_i))} = \underbrace{\varphi_i\big(X(t_i)\big)-X(t_i)}_{:=I_i(X(t_i))}.$$

There are two views for calculating the solutions of the impulsive differential equation. Given that in each interval (t_i, t_{i+1}) between two consecutive impulses, the solution is calculated by the fractional differential equation. Besides, unlike the ordinary derivative, the Caputo fractional derivative depends significantly on the initial conditions, so that can have a different equation in each interval (t_i, t_{i+1}).

The first approach leads to a change in the lower limit in order to calculate the Caputo derivative in $t_0 = 0$. To calculate the solution in the subsequent subintervals, this lower limit will be changed and moved to the beginning of the next interval [1, 2]. In this case, the solutions of IFrDE (11) would be

$$X(t) = \begin{cases} X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X(s))\,\mathrm{d}s, \quad t \in [0, t_1], \\ X_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f(s, X(s))\,\mathrm{d}s \\ \quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f(s, X(s))\,\mathrm{d}s \\ \quad + \sum_{i=1}^k I_i(X(t_i)), \quad t \in (t_k, t_{k+1}],\ k = 1, 2, 3, \dots . \end{cases}$$

Another approach is to keep the lower limit of t_0 in the Caputo derivative for all $t \geqslant t_0$, and only the initial conditions of the equation in each subinterval (t_i, t_{i+1}) will be changed [13, 14, 27–29]. This approach emphasizes that the restriction of the Caputo fractional derivative does not change in each subinterval, and only the initial conditions change between two consecutive impulses. In fact, in this approach, the solutions of IFrDE (11) are as follows:

$$X(t) = \begin{cases} X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X(s))\,\mathrm{d}s, \quad t \in [0, t_1], \\ X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X(s))\,\mathrm{d}s \\ \quad + \sum_{i=1}^k I_i(X(t_i)), \quad t \in (t_k, t_{k+1}],\ k = 1, 2, 3, \dots . \end{cases}$$

Minimizing losses caused by insect pests in agriculture is one of the major concerns. There are many ways to control agricultural insect pests. One of these methods is biological control in which the pests are destroyed through the release of their natural enemy. They use other insects such as parasites as pesticides [24,25]. Another method is chemical control and the use of insecticides, which have high effectiveness but are harmful to human health. The combination of the above two cases has the best effect in which insect pest reduction is applied at the lowest cost both to the grower and the environment. In this section, to control the population and by introducing periodic impulse, on the one hand, we were catching (poisoning) the prey population (pest), and on the other hand, we cause the periodic immigration of the predator population. We reduce the population of a particular species by catching or chemical poisoning in agriculture ($0 \leqslant p < 1$), and we increase the population of the particular species by artificial breeding or the release of other species ($\delta \geqslant 0$) [30]. We consider the following population model:

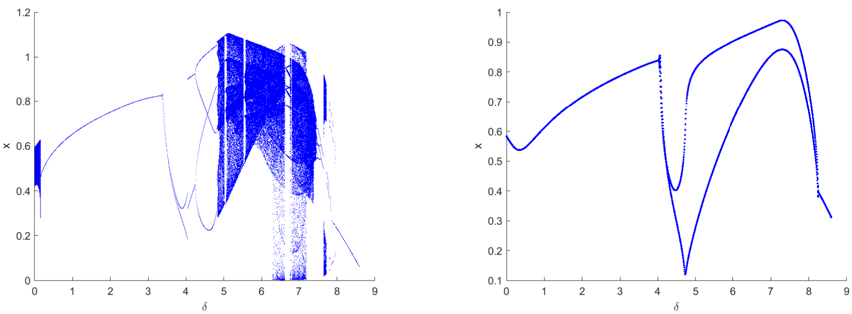
$$\begin{aligned} &\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x(t) = x\left(a - bx - \frac{\beta y}{\mu + \eta x + \gamma y}\right), \quad t \neq nT,\ n \in \mathbb{N}, \\ &\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} y(t) = y\left(-\sigma + \frac{\xi x}{\mu + \eta x + \gamma y}\right), \quad t \neq nT,\ n \in \mathbb{N}, \\ &x(t^+) = (1-p)x(t), \quad y(t^+) = y(t) + \delta, \quad t = nT,\ n \in \mathbb{N}, \end{aligned} \tag{12}$$

where $x(t^+) = \lim_{\tau\to 0^+} x(t+\tau)$, $y(t^+) = \lim_{\tau\to 0^+} y(t+\tau)$, $0 \leqslant p < 1$, $\delta \geqslant 0$, and T is the period of impulsive.

We put the parameter values as follows:

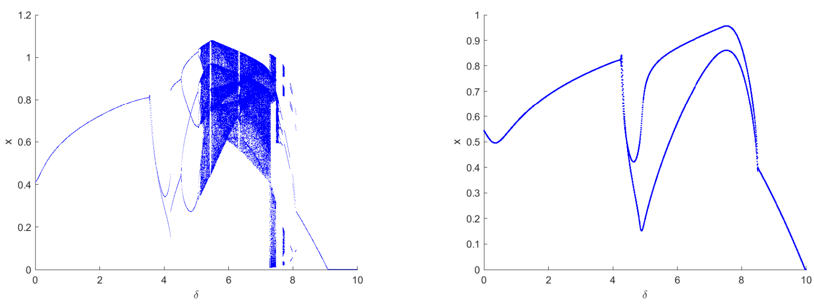
$$\alpha = 0.98, \quad \mu = \eta = 1, \quad a = 8, \quad b = 5, \quad \beta = 1.1, \quad \sigma = 0.2,$$
$$\xi = 1.045, \quad T = 6, \quad p = 0.1, \quad x(0) = 0.2, \quad y(0) = 7.5.$$

First, we consider the effect of the changes in the parameter δ. We will give bifurcation diagrams of different values $\gamma = 0.001, 0.005, 0.01, 0.02, 0.03$. When $\alpha = 1$ and $\gamma = 0.001$ as you can see in the Fig. 1(a), by increasing the parameter δ from 0.001 to 9, the system experiences a quasiperiodic oscillating, cycles (T-periodic solution), periodic doubling cascade leading to chaos, periodic halving cascade, nonunique dynamics, which means coexist of several attractors. Bifurcation diagrams illustrated in Figs. 2–5(a) show that system experiences process of periodic doubling cascade, cycles, periodic



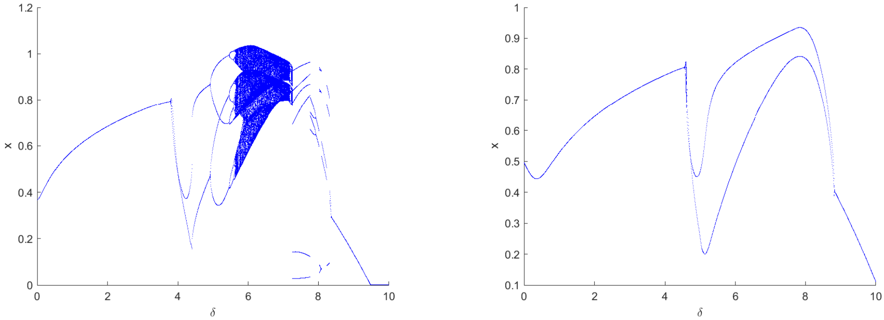
(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 1. Bifurcation diagrams of (12) showing the influence of δ with $\mu = \eta = 1$, $\gamma = 0.001$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$ when δ varies from 0.001 to 9.



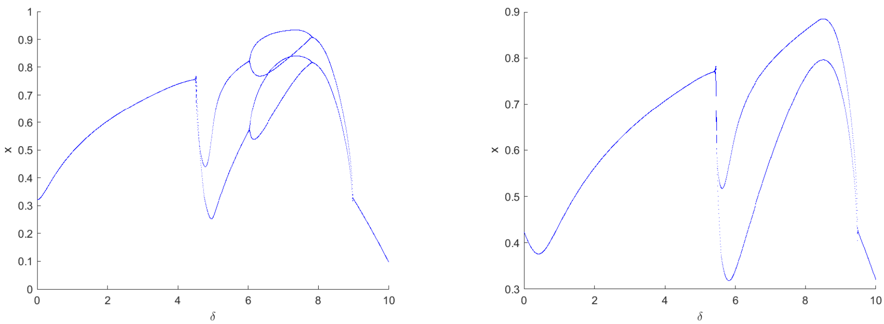
(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 2. Bifurcation diagrams of (12) showing the influence of δ with $\mu = \eta = 1$, $\gamma = 0.005$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$ when δ varies from 0.001 to 10.



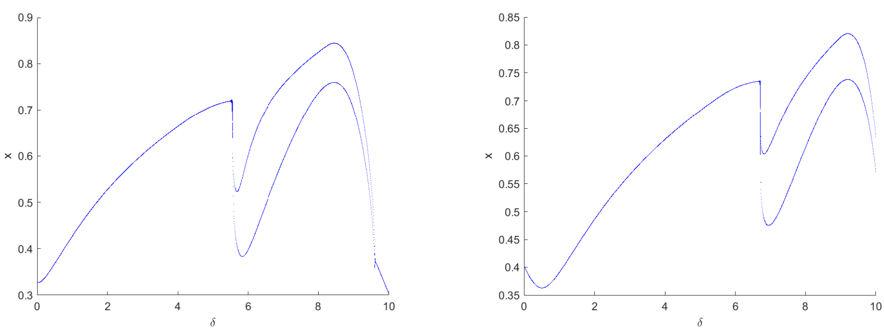
(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 3. Bifurcation diagrams of (12) showing the influence of δ with $\mu = \eta = 1$, $\gamma = 0.01$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$ when δ varies from 0.001 to 10.



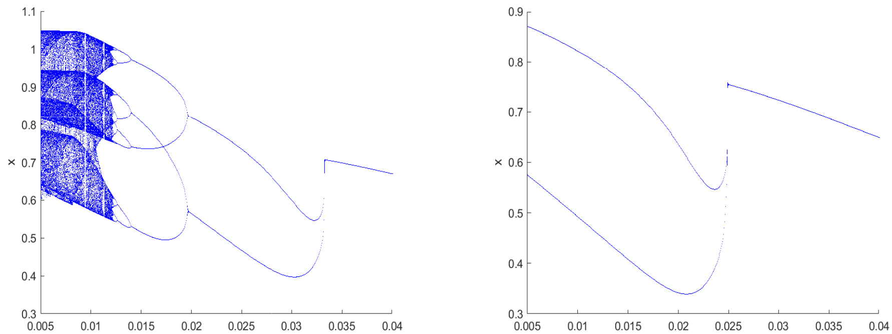
(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 4. Bifurcation diagrams of (12) showing the influence of δ with $\mu = \eta = 1$, $\gamma = 0.02$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$ when δ varies from 0.001 to 10.



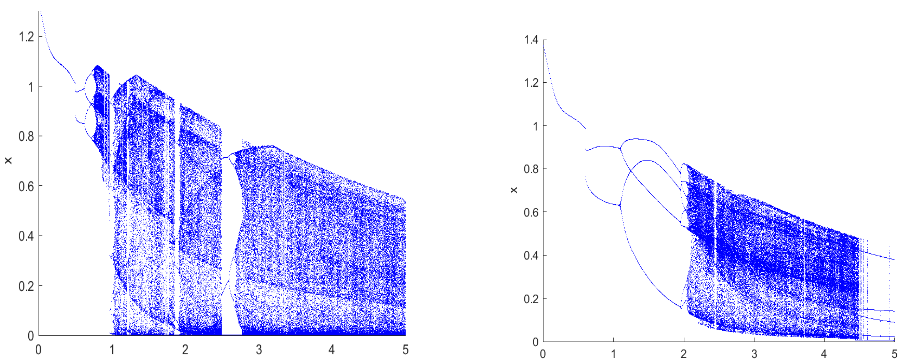
(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 5. Bifurcation diagrams of (12) showing the influence of δ with $\mu = \eta = 1$, $\gamma = 0.03$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$ when δ varies from 0.001 to 10.



(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 6. Bifurcation diagrams of (12) showing the influence of γ with $\mu = \eta = 1$, $\delta = 6$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$ when γ varies from 0.005 to 0.04.

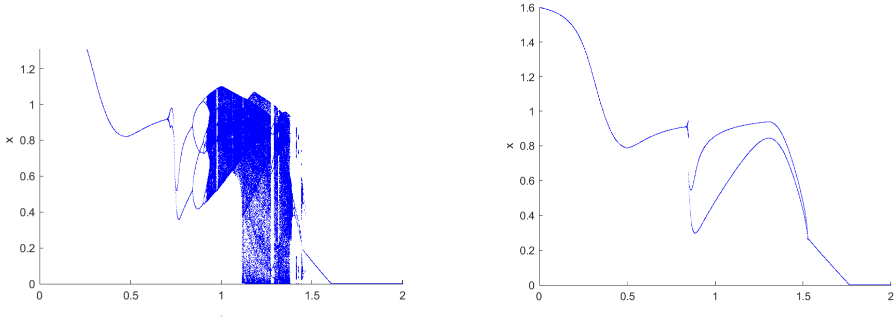


(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 7. Bifurcation diagrams of (12) showing the influence of ξ with $\mu = \eta = 1$, $\gamma = 0.001$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $T = 6$, $p = 0.1$, $\delta = 6$ when ξ varies from 0.001 to 5.

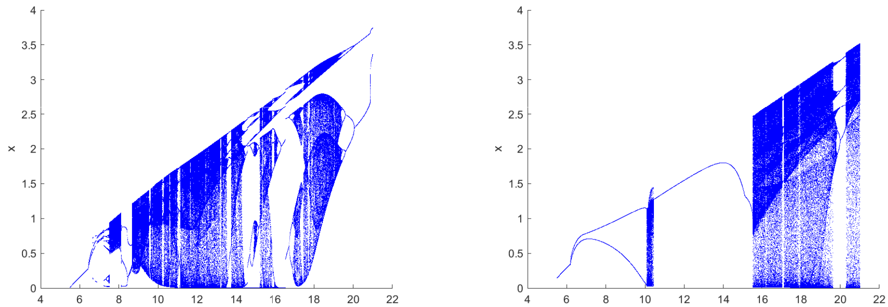
halving cascade and cycles, whereas in the fractional case, Figs. 2–5(b) such behavior is not observed, and only periodic solutions with period 2 are available. In classical model when $\gamma = 0.01$, the system has chaotic behavior, but when $\gamma = 0.02$, only the periodic solution exists. So γ plays an important role in discussing complexity of the classical system, furthermore, impulsive immigration of the predator makes sense when densities of the predator are low or the parameter γ is small. However, in the fractional system, the behavior is more stable, and there is no chaotic behavior.

On the other hand, in order to further investigate the dynamical behaviors of system (12), the bifurcation diagrams with respect to the parameters γ, ξ, β and a are also carried out in Figs. 6–9(a) and Figs. 6–9(b) in classical and fractional cases, respectively. Phase space for some values of the bifurcation diagrams, comparing with the classical cases, can be seen in Figs. 10–13.



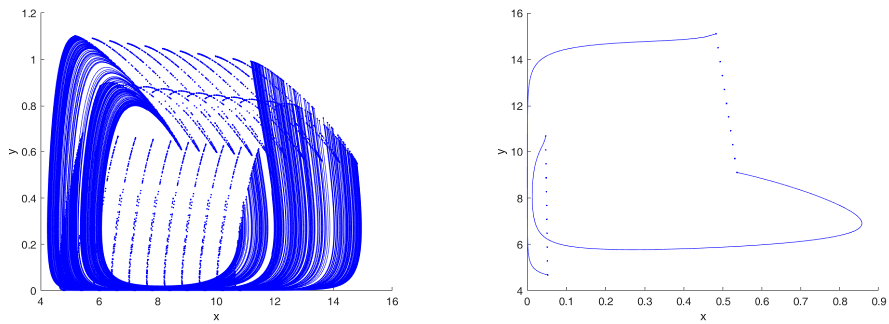
(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 8. Bifurcation diagrams of (12) showing the influence of β with $\mu = \eta = 1$, $\gamma = 0.001$, $a = 8$, $b = 5$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$, $\delta = 6$ when β varies from 0.001 to 2.



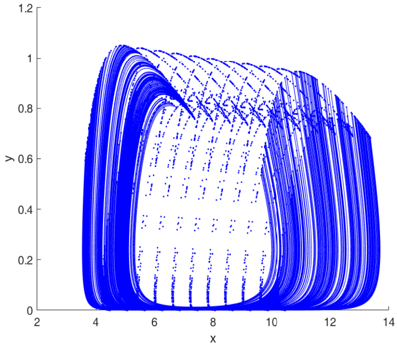
(a) $\alpha = 1$ (b) $\alpha = 0.98$

Figure 9. Bifurcation diagrams of (12) showing the influence of a with $\mu = \eta = 1$, $\gamma = 0.001$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$, $\delta = 6$ when a varies from 5.5 to 25.

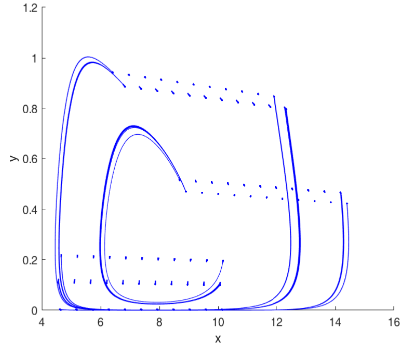


(a) $\alpha = 1$ (b) $\alpha = 0.98$

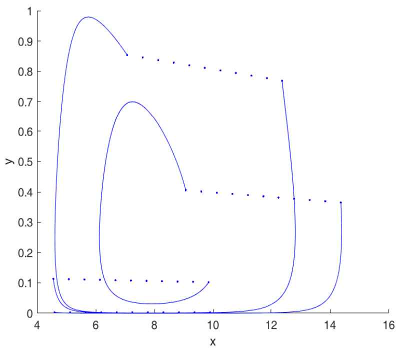
Figure 10. Phase portrait of (12) under condition $\mu = \eta = 1$, $\gamma = 0.001$, $a = 8$, $b = 5$, $\sigma = 0.2$, $\xi = 1.045$, $T = 6$, $p = 0.1$, $\delta = 6$, $\beta = 1$.



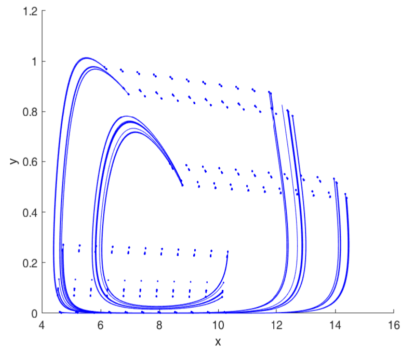
(a) $\delta = 6$



(b) $\delta = 5.5$

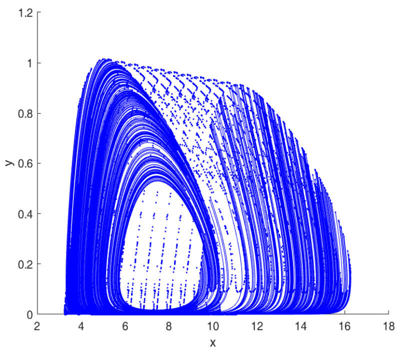


(c) $\delta = 5.3$

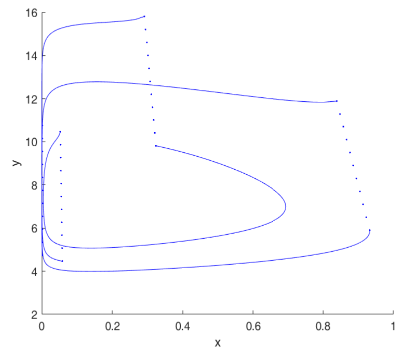


(d) $\delta = 5.6$

Figure 11. Phase portrait of system (12) under condition $\mu = \eta = 1$, $\gamma = 0.001$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\alpha = 1$, $\xi = 1.045$, $T = 6$, $p = 0.1$.

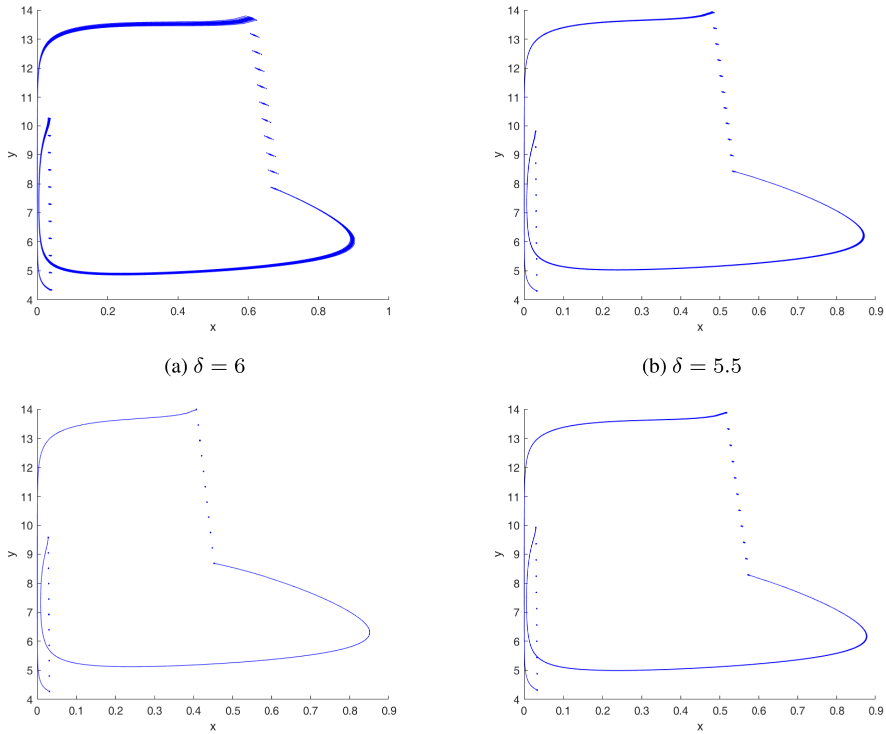


(a) $\alpha = 1$



(b) $\alpha = 0.98$

Figure 12. Phase portrait of system (12) under condition $\mu = \eta = 1$, $\gamma = 0.001$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $T = 6$, $p = 0.1$, $\delta = 6$, $\xi = 1.5$.



(a) $\delta = 6$ (b) $\delta = 5.5$ (c) $\delta = 5.3$ (d) $\delta = 5.6$

Figure 13. Phase portrait of system (12) under condition $\mu = \eta = 1$, $\gamma = 0.001$, $a = 8$, $b = 5$, $\beta = 1.1$, $\sigma = 0.2$, $\alpha = 0.98$, $\xi = 1.045$, $T = 6$, $p = 0.1$.

5 Conclusion

We have considered a fractional density-dependent prey–predator model and examined the stability and persistence of the system by using the Lyapunov method. The results obtained in this paper are based on the Beddington–DeAnglis function response that leads us to the Kolmogorov prey–predator model by placing $\eta = 1$ and $\gamma = 0$, and will be the ratio-dependent prey–predator model when $\mu = 0$ and $\eta = 1$. We extend this model to a periodic impulsive model for the prey population. This expansion is done through the periodic catching of the prey population and the periodic releasing of the predator population. This model has the potential to protect the predator from extinction. However, under some circumstances, it can also lead to the extinction of the prey. It can be seen that the fractional order α plays a vital role in the complexity of the system. Our results show that there is a significant difference between the classical and fractional model, and its fractional cases. We have more stable system compared with the classical mode.

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