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Prešić-type fixed point results via Q-distance on quasimetric space and application to (p, q)-difference equations

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Abstract. In this paper, we introduce two new properties to the Q-function, called as the 0-property and the small self-distance property, which is frequently used in studies of fixed point theory in quasimetric spaces. Then, with the help of Q-functions having these properties, we present some fixed point theorems for Prešić-type mappings in quasimetric spaces. Finally, we state a theorem for the existence and uniqueness of the solution to a boundary value problem for (p, q)-difference equations to demonstrate the applicability of our theoretical results, which we support with an example.

Keywords: fixed point, quasimetric space, Prešić-type mapping, Q-function, (p, q)-difference equation.

1 Introduction and preliminaries

The theory of quantum calculus (q-calculus), which is known as the study of calculus without limit, and its applications have an important role in mathematical sciences, mechanics, physics, and other fields of real-world problems. For some papers, we refer the reader to [8, 9, 17, 21, 23].

Studies on q-difference equations arose at the beginning of the past century, particularly by Jackson [22], Carmichael [11], and Mason [28]. In this regard, studies on q-difference equations for both classic and some generalized versions have attracted the attention of several researchers, and their applications are discussed in the solutions of boundary value problems for q-difference equations (see [2–4, 38, 39]).

The (p, q)-calculus, a generalization of the classic q-calculus, was first introduced by Chakrabarti and Jagannathan [12] in quantum algebras, which contained two quantum numbers p and q. For some recent results, see [10, 20, 37] and the references cited therein.

The authors of the recent paper [24] initiated research on boundary value problems for the (p, q)-difference equation. In this paper, they considered the first-order quantum (p, q)-difference equation subject to a nonlocal condition of the form

$$D_{p,q}\xi(t) = f\bigl(t, \xi(pt)\bigr), \quad t \in \left[0, \frac{T}{p}\right],$$
$$\xi(0) = \alpha\xi(T) + \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} \xi(s)\,\mathrm{d}_{p_i q_i} s,$$

where $0 < q < p \leqslant 1$, $0 < q_i < p_i \leqslant 1$, $i = 1, 2, \dots, m$, are quantum numbers, $D_{p,q}$ is (p, q)-difference operator, $f : [0, T/p] \times \mathbb{R} \to \mathbb{R}$ is continuous, $T > 0$, α, β_i are given constants, and $\eta_i \in [0, p_i T]$. Then, taking into account some fundamental fixed point theorems such as the Banach contraction principle, the Boyd–Wong fixed point theorem, and the Leray–Schauder nonlinear alternative, they provided the existence (uniqueness within some cases) of solutions to this problem under some certain conditions on f and constants. A variety of new results on (p, q)-difference equations via fixed point theory can be found in [18, 19, 29–31, 34].

Now, let us review basic definitions and theorems about (p, q)-calculus, which are found in [36]. The (p, q)-derivative and (p, q)-integral of a function g are defined by the following formulas for constants $0 < q < p \leqslant 1$:

$$D_{p,q}g(t) = \begin{cases} \frac{g(pt)-g(qt)}{(p-q)t}, & t \neq 0, \\ \lim_{t\to 0} D_{p,q}g(t), & t = 0, \end{cases}$$

and

$$\int_0^t g(s)\,\mathrm{d}_{p,q}s = (p-q)t \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} g\left(\frac{q^n}{p^{n+1}} t\right),$$

provided that the right-hand side converges.

The (p, q)-integration by parts is given by

$$\int_a^b g(pt) D_{p,q}h(t)\,\mathrm{d}_{p,q}t = g(t)h(t)\big|_a^b - \int_a^b h(qt) D_{p,q}g(t)\,\mathrm{d}_{p,q}t,$$

and the following formulas hold:

$$D_{p,q}\left(\int_0^t g(s)\,\mathrm{d}_{p,q}s\right) = g(t),$$
$$\int_a^t D_{p,q}g(s)\,\mathrm{d}_{p,q}s = g(t) - g(a) \quad \text{for } a \in [0, t).$$

In the remainder of this section, we give fundamental notions of Prešić-type fixed point results, quasimetric spaces, and Q-functions. In Section 2, we set up the 0-property and small self-distance property of Q-functions and obtain some new fixed point theorems for Prešić-type mappings on quasimetric space via Q-functions. In the last section, using our theoretical results, we deal with an existence and uniqueness theorem for a second-order (p,q)-difference Langevin equation with boundary conditions of the form

$$\begin{aligned} &D_{p,q}(D_{p,q}+\gamma)\xi(t) = f\bigl(t,\xi(t)\bigr), \quad t\in[0,1],\\ &\xi(0)=\alpha, \qquad D_{p,q}\xi(0)=\beta, \end{aligned}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous, $0<q<p\leqslant 1$, and γ, α, β are given constants.

It is well known that the Banach contraction principle has been extended by many researchers in several different ways over the last few decades. In 1965, Prešić [33] generalized it as follows:

Theorem 1. *Let $(\mathcal{X},\rho)$ be a complete metric space, k be any positive integer, and let $\mathcal{F}:\mathcal{X}^k\to\mathcal{X}$ be a mapping satisfying the following contraction condition: for all $\xi_1,\xi_2,\dots,\xi_{k+1}\in\mathcal{X}$,*

$$\rho\bigl(\mathcal{F}(\xi_1,\xi_2,\dots,\xi_k),\mathcal{F}(\xi_2,\xi_3,\dots,\xi_{k+1})\bigr)\leqslant\sum_{i=1}^{k} q_i\rho(\xi_i,\xi_{i+1}),$$

where $q_1,q_2,\dots,q_k$ are positive constants such that $\sum_{i=1}^k q_i<1$. Then there exists a unique point $\xi\in\mathcal{X}$ such that $\xi=\mathcal{F}(\xi,\xi,\dots,\xi)$. Moreover, if $\xi_1,\xi_2,\dots,\xi_k$ are arbitrary points in $\mathcal{X}$ for $n\in\mathbb{N}$,

$$\xi_{n+k}=\mathcal{F}(\xi_n,\xi_{n+1},\dots,\xi_{n+k-1}),$$

then the sequence $\{\xi_n\}$ is convergent, and $\lim\xi_n=\mathcal{F}(\lim\xi_n,\lim\xi_n,\dots,\lim\xi_n)$.

Note that for $k=1$, Theorem 1 reduces to the Banach contraction principle.

Later on, in 2007, Ćirić and Prešić [16] further generalized Prešić-type contraction for complete metric space, which is stated as follows.

Theorem 2. *Let $(\mathcal{X},\rho)$ be a complete metric space, k be any positive integer, and let $\mathcal{F}:\mathcal{X}^k\to\mathcal{X}$ be a mapping satisfying the following contraction condition: for all $\xi_1,\xi_2,\dots,\xi_{k+1}\in\mathcal{X}$,*

$$\rho\bigl(\mathcal{F}(\xi_1,\xi_2,\dots,\xi_k),\mathcal{F}(\xi_2,\xi_3,\dots,\xi_{k+1})\bigr)\leqslant\lambda\max\bigl\{\rho(\xi_i,\xi_{i+1}),\ 1\leqslant i\leqslant k\bigr\},$$

where $\lambda\in(0,1)$. Then there exists a point $\xi\in\mathcal{X}$ such that $\xi=\mathcal{F}(\xi,\xi,\dots,\xi)$. Moreover, if $\xi_1,\xi_2,\dots,\xi_k$ are arbitrary points in $\mathcal{X}$ for $n\in\mathbb{N}$,

$$\xi_{n+k}=\mathcal{F}(\xi_n,\xi_{n+1},\dots,\xi_{n+k-1}),$$

then the sequence $\{\xi_n\}$ is convergent, and $\lim\xi_n=\mathcal{F}(\lim\xi_n,\lim\xi_n,\dots,\lim\xi_n)$. In addition, if for all $u,v\in\mathcal{X}$ with $u\neq v$, the condition

$$\rho\bigl(\mathcal{F}(u,u,\dots,u),\mathcal{F}(v,v,\dots,v)\bigr)<\rho(u,v)$$

holds, then ξ is the unique point in $\mathcal{X}$ such that $\xi=\mathcal{F}(\xi,\xi,\dots,\xi)$.

Some important applications of the above-stated results such as studying asymptotic stability of the equilibrium for the nonlinear difference equation and global attractivity of matrix difference equations can be found in [1, 13].

Consider the following properties of a function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, where $\mathcal{X}$ is a nonempty set:

(ρ1) $\rho(\xi, \xi) = 0$;
(ρ2) $\rho(\xi, \zeta) \leqslant \rho(\xi, \varsigma) + \rho(\varsigma, \zeta)$ for all $\xi, \zeta, \varsigma \in \mathcal{X}$;
(ρ3) $\rho(\xi, \zeta) = \rho(\zeta, \xi) = 0 \Rightarrow \xi = \zeta$,
(ρ4) $\rho(\xi, \zeta) = 0 \Rightarrow \xi = \zeta$.

- ρ is called a *quasi-pseudo metric* if (ρ1) and (ρ2) hold;
- ρ is called a *quasimetric* if (ρ1), (ρ2), and (ρ3) hold;
- ρ is called a T_1-quasimetric if (ρ1), (ρ2), (ρ3), and (ρ4) hold.

In this case, the space $(\mathcal{X}, \rho)$ is referred to by the name given to ρ.

Assume that $(\mathcal{X}, \rho)$ is a quasimetric space and $W : \mathcal{X} \to [0, \infty)$ is a function satisfying $\rho(\xi, \zeta) + W(\xi) = \rho(\zeta, \xi) + W(\zeta)$ for all $\xi, \zeta \in \mathcal{X}$. Then $(\mathcal{X}, \rho)$ is called weightable, and $(\mathcal{X}, \rho, W)$ is called a weighted quasimetric space.

Assume that $(\mathcal{X}, \rho)$ is a quasipseudometric space, $\xi_0 \in \mathcal{X}$, and $\varepsilon > 0$. Then the set

$$B_\rho(\xi_0, \varepsilon) = \{\zeta \in \mathcal{X}\colon \rho(\xi_0, \zeta) < \varepsilon\}$$

is called open sphere with center ξ_0 and radius ε. The family of all open spheres generates a topology τ_ρ on $\mathcal{X}$, which is T_0 whenever r is quasimetric. If ρ is a T_1-quasimetric, then τ_ρ is a T_1-topology on $\mathcal{X}$.

If ρ is a quasimetric on $\mathcal{X}$, then ρ^{-1} and ρ^s are quasimetric and metric on $\mathcal{X}$, respectively, where

$$\rho^{-1}(\xi, \zeta) = \rho(\zeta, \xi) \quad \text{and} \quad \rho^s(\xi, \zeta) = \max\{\rho(\xi, \zeta), \rho^{-1}(\xi, \zeta)\}.$$

Example 1. The following are some examples of quasimetrics on $\mathbb{R}$.

(i) Consider $\rho(\xi, \zeta) = \max\{\zeta - \xi, 0\}$ for all $\xi, \zeta \in \mathbb{R}$. Then ρ is a quasimetric but not a T_1-quasimetric. Observe that τ_ρ is left-order topology and $\tau_{\rho^{-1}}$ is right-order topology on $\mathbb{R}$.
(ii) Consider $\rho(\xi, \zeta) = 0$ for $\xi = \zeta$ and $\rho(\xi, \zeta) = |\zeta|$ for $\xi \neq \zeta$. Then ρ is a weightable quasimetric with weighting function $W(\xi) = |\xi|$.
(iii) Consider $\rho(\xi, \zeta) = \zeta - \xi$ for $\xi \leqslant \zeta$ and $\rho(\xi, \zeta) = 1$ for $\xi > \zeta$. Then ρ is a T_1-quasimetric. Observe that τ_ρ is lower limit topology and $\tau_{\rho^{-1}}$ is upper limit topology on $\mathbb{R}$. Also, note that τ_{ρ^s} is a discrete topology on $\mathbb{R}$.

Let $\{\xi_n\}$ be a sequence in a quasimetric space $(\mathcal{X}, \rho)$ and $\xi \in \mathcal{X}$. In this case, if $\rho(\xi, \xi_n) \to 0$ as $n \to \infty$, then $\{\xi_n\}$ is said to be τ_ρ convergent to ξ. If for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\rho(\xi_k, \xi_n) < \varepsilon$ (resp. $\rho(\xi_n, \xi_k) < \varepsilon$) whenever $n \geqslant k \geqslant n_0$, then $\{\xi_n\}$ is said to left K-Cauchy (resp. right K-Cauchy) sequence. Finally, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\rho(\xi_n, \xi_k) < \varepsilon$ whenever $n, k \geqslant n_0$, then $\{\xi_n\}$ is said to be ρ^s-Cauchy sequence.

Although there are many approaches related to the completeness of quasimetric space in the literature (see [6, 7, 14, 15, 25, 27, 35]), the completeness concepts we will consider here are as follows. We call a quasimetric space $(\mathcal{X}, \rho)$ left K-complete (resp. left-M-complete) if every left K-Cauchy sequence is τ_ρ-convergent (resp. $\tau_{\rho^{-1}}$-convergent).

Let $\mathcal{X}$ be a nonempty set, ρ and σ be two quasimetrics on $\mathcal{X}$, $k \in \mathbb{N}$, and $\mathcal{F} : \mathcal{X}^k \to \mathcal{X}$ be a mapping. Then $\mathcal{F}$ is called k-sequentially ρ-σ-continuous at $\xi \in \mathcal{X}$ if for all sequence $\{\xi_n\}$ in $\mathcal{X}$ such that $\rho(\xi, \xi_n) \to 0$, implies

$$\sigma(\mathcal{F}(\xi, \xi, \dots, \xi), \mathcal{F}(\xi_n, \xi_{n+1}, \dots, \xi_{n+k-1})) \to 0.$$

If $\mathcal{F}$ is k-sequentially ρ-σ-continuous at all points of $\mathcal{X}$, then $\mathcal{F}$ is called k-sequentially ρ-σ-continuous on $\mathcal{X}$.

Now we recall the concept of Q-function, which is presented in [5] by Al-Hamidan et al.

Definition 1. Let $(\mathcal{X}, \rho)$ be a quasimetric space, and let $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function satisfying the following:

- (Q1) $q(\xi, \varsigma) \leqslant q(\xi, \zeta) + q(\zeta, \varsigma)$ for all $\xi, \zeta, \varsigma \in \mathcal{X}$;
- (Q2) if $\xi \in \mathcal{X}$, $M > 0$, and $\{\zeta_n\}$ is a sequence in $\mathcal{X}$ that ρ^{-1}-converges to a point $\zeta \in \mathcal{X}$ and satisfies $q(\xi, \zeta_n) \leqslant M$ for all $n \in \mathbb{N}$, then $q(\xi, \zeta) \leqslant M$;
- (Q3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(\xi, \zeta) \leqslant \delta$ and $q(\xi, \varsigma) \leqslant \delta$ imply $\rho(\zeta, \varsigma) \leqslant \varepsilon$.

Then q is called a Q-function on $(\mathcal{X}, \rho)$.

Let condition (Q2) is replaced by

(Q2′) $q(\xi, \cdot) : \mathcal{X} \to [0, \infty)$ is lower semicontinuous on $(\mathcal{X}, \tau_{\rho^{-1}})$ for all $\xi \in \mathcal{X}$.

Then q is called a w-distance on $(\mathcal{X}, \rho)$ [32]. Observe that if $q(\xi, \zeta) = 0$ and $q(\xi, \varsigma) = 0$, then $\zeta = \varsigma$. Obviously, if $(\mathcal{X}, \rho)$ is a metric space, then ρ is a Q-function on $(\mathcal{X}, \rho)$. Nevertheless, as it can be seen in [5], if ρ is a quasimetric, then ρ may not be a Q-function on $(\mathcal{X}, \rho)$.

Example 2. (See [26].) The discrete metric q on every quasimetric space $(\mathcal{X}, \rho)$ is a Q-function.

Example 3. (See [26].) Let $(\mathcal{X}, \rho)$ be a weightable quasimetric space, and let $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function defined as $q(\xi, \zeta) = \rho(\xi, \zeta) + W(\xi)$, where W is the corresponding weighted function. Then q is a Q-function on $(\mathcal{X}, \rho)$.

Example 4. Let $\mathcal{X} = [0, \infty)$ and $\rho(\xi, \zeta) = \max\{\zeta - \xi, 0\}$ for all $\xi, \zeta \in \mathcal{X}$. Then $q_1(\xi, \zeta) = \max\{\xi, \zeta\}$, $q_2(\xi, \zeta) = \zeta$, and $q_3(\xi, \zeta) = (\xi + \zeta)/2$ are Q-functions on $(\mathcal{X}, \rho)$.

The following lemmas play important roles in our main results.

Lemma 1. (*See* [26].) *Assume that q is a q-function on a quasimetric space $(\mathcal{X}, \rho)$. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(\xi, \zeta) \leqslant \delta$, and $q(\xi, \varsigma) \leqslant \delta$ imply $\rho^s(\zeta, \varsigma) \leqslant \varepsilon$.*

Lemma 2. (*See* [5].) *Let $(\mathcal{X}, \rho)$ be a quasimetric space, $\{\xi_n\}$, $\{\zeta_n\}$ be sequences in $\mathcal{X}$, and $q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a Q-function. Assume that the sequences $\{\alpha_n\}, \{\beta_n\} \in \mathbb{R}^+$ are such that $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$. Then the following ones hold for all $\xi, \zeta, \varsigma \in \mathcal{X}$:*

(i) *If $q(\xi_n, \zeta) \leqslant \alpha_n$ and $q(\xi_n, \varsigma) \leqslant \beta_n$ for all $n \in \mathbb{N}$, then $\zeta = \varsigma$.*
(ii) *If $q(\xi_n, \zeta_n) \leqslant \alpha_n$ and $q(\xi_n, \zeta) \leqslant \beta_n$ for all $n \in \mathbb{N}$, then $\rho^s(\zeta_n, \zeta) \to 0$.*
(iii) *If $q(\xi_n, \xi_m) \leqslant \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{\xi_n\}$ is a ρ^s-Cauchy sequence.*

2 Fixed point results

Definition 2. Assume that $(\mathcal{X}, \rho)$ is a quasimetric space and q is a Q-function on $(\mathcal{X}, \rho)$. If the implication $q(\xi_n, \zeta) \to 0 \Rightarrow q(\zeta, \zeta) = 0$ is true for every sequence $\{\xi_n\}$ in $\mathcal{X}$ and $\zeta \in \mathcal{X}$, then the Q-function q is said to have 0-property.

It is clear that all Q-functions given in Examples 2, 3, and 4 have 0-property.

Definition 3. Assume that $(\mathcal{X}, \rho)$ and q are defined as in the above definition. If for all $\xi, \zeta \in \mathcal{X}$, the inequality $q(\zeta, \zeta) \leqslant q(\xi, \zeta)$ holds, then q is said to have small self-distance property.

Remark 1. Note that if a Q-function on a quasimetric space $(\mathcal{X}, \rho)$ has small self-distance property, then it has 0-property, but the converse may not be true. For example, the Q-function q_3 given in Example 4 has 0-property, but not small self-distance property.

Let $\mathcal{X}$ be a nonempty set, k be any positive integer, and $\mathcal{F} : \mathcal{X}^k \to \mathcal{X}$ be a mapping. In this case, for simplicity, we will use the following notation:

$$F_k(\mathcal{F}) = \bigl\{\xi \in \mathcal{X}\colon \xi = \mathcal{F}(\xi, \xi, \dots, \xi)\bigr\}.$$

If $k = 1$, then we will write $F(\mathcal{F})$ instead of $F_1(\mathcal{F})$, which is the set of fixed points of $\mathcal{F}$.

Now, we are ready to present our main result.

Theorem 3. *Let $(\mathcal{X}, \rho)$ be a left M-complete quasimetric space, q be a Q-function having 0-property, k be any positive integer, and $\mathcal{F} : \mathcal{X}^k \to \mathcal{X}$ be a mapping satisfying the following contraction condition: for all $\xi_1, \xi_2, \dots, \xi_{k+1} \in \mathcal{X}$,*

$$\begin{aligned} &q\bigl(\mathcal{F}(\xi_1, \xi_2, \dots, \xi_k), \mathcal{F}(\xi_2, \xi_3, \dots, \xi_{k+1})\bigr) \\ &\quad \leqslant \lambda \max\bigl\{q(\xi_i, \xi_{i+1}),\ 1 \leqslant i \leqslant k\bigr\}, \end{aligned} \tag{1}$$

where $\lambda \in (0, 1)$. Then there exists a point $\varsigma \in F_k(\mathcal{F})$ such that $q(\varsigma, \varsigma) = 0$. Moreover, if $\xi_1, \xi_2, \dots, \xi_k$ are arbitrary points in $\mathcal{X}$, then the sequence $\{\xi_n\}$ defined by

$$\xi_{n+k} = \mathcal{F}(\xi_n, \xi_{n+1}, \dots, \xi_{n+k-1}) \tag{2}$$

for $n \in \mathbb{N}$ is ρ^{-1}-convergent to some point in $F_k(\mathcal{F})$. In addition, if for all $u, v \in \mathcal{X}$ with $u \neq v$, the condition

$$q\bigl(\mathcal{F}(u, u, \dots, u), \mathcal{F}(v, v, \dots, v)\bigr) < q(u, v) \tag{3}$$

holds, then $F_k(\mathcal{F})$ is singleton.

Proof. Let $\xi_1, \xi_2, \ldots, \xi_k$ be arbitrary points in $\mathcal{X}$. Define a sequence $\{\xi_n\}$ by using these points as follows:

$$\xi_{n+k} = \mathcal{F}(\xi_n, \xi_{n+1}, \ldots, \xi_{n+k-1})$$

for $n \in \mathbb{N}$. For simplicity, define $q_n = q(\xi_n, \xi_{n+1})$ for all $n \in \mathbb{N}$. We will prove by induction that

$$q_n \leqslant M\theta^n, \tag{4}$$

where $\theta = \lambda^{1/k}$ and

$$M = \max\left\{\frac{q_i}{\theta^i},\ i \in \{1, 2, \ldots, k\}\right\}.$$

First, note that (4) is true for $n = 1, 2, \ldots, k$ because of the definition of M. Now let the k inequalities

$$q_{n+i} \leqslant M\theta^{n+i}$$

hold for $i \in \{1, 2, \ldots, k-1\}$. Then we have

$$\begin{aligned} q_{n+k} &= q(\xi_{n+k}, \xi_{n+k+1}) \\ &= q\big(\mathcal{F}(\xi_n, \xi_{n+1}, \ldots, \xi_{n+k-1}), \mathcal{F}(\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+k})\big) \\ &\leqslant \lambda \max\big\{q(\xi_{n+i-1}, \xi_{n+i}),\ i \in \{1, 2, \ldots, k\}\big\} \\ &= \lambda \max\big\{q_{n+i-1},\ i \in \{1, 2, \ldots, k\}\big\} \\ &\leqslant \lambda \max\big\{M\theta^{n+i-1},\ i \in \{1, 2, \ldots, k\}\big\} \\ &= \lambda M\theta^n = M\theta^{n+k}, \end{aligned}$$

and so (4) is true for all $n \in \mathbb{N}$. Using (4), we have for all $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} q(\xi_n, \xi_m) &\leqslant q(\xi_n, \xi_{n+1}) + q(\xi_{n+1}, \xi_{n+2}) + \cdots + q(\xi_{m-1}, \xi_m) \\ &\leqslant M\theta^n + M\theta^{n+1} + \cdots + M\theta^{m-1} \leqslant \frac{M\theta^n}{1-\theta}. \end{aligned}$$

Now, set $\varepsilon > 0$ and $0 < \delta < \varepsilon$ that satisfy (Q3). Hence, there exists $n_\delta \in \mathbb{N}$ such that $q(\xi_{n_\delta}, \xi_n) < \delta$ and $q(\xi_{n_\delta}, \xi_m) < \delta$ whenever $m, n > n_\delta$. Thereafter, by Lemma 1, we get $\rho^s(\xi_n, \xi_m) < \varepsilon$. Hence, $\{\xi_n\}$ is ρ^s-Cauchy sequence in $\mathcal{X}$, and thus it is left K-Cauchy sequence in $\mathcal{X}$. By using left M-completeness of $(\mathcal{X}, \rho)$, then we obtain that there exists $\varsigma \in \mathcal{X}$ such that $\{\xi_n\}$ is ρ^{-1}-convergent to ς, that is, $\rho(\xi_n, \varsigma) \to 0$ as $n \to \infty$. On the other hand, for $m > n \geqslant n_\delta$, we can deduce that $q(\xi_n, \xi_m) < \delta$. Hence, by (Q2), we get $q(\xi_n, \varsigma) < \delta < \varepsilon$, and so $q(\xi_n, \varsigma) \to 0$ as $n \to \infty$. By the 0-property of q, we have $q(\varsigma, \varsigma) = 0$. Now, using (Q1) and (1), we have

$$\begin{aligned} &q\big(\xi_{n+k}, \mathcal{F}(\varsigma, \varsigma, \ldots, \varsigma)\big) \\ &\quad= q\big(\mathcal{F}(\xi_n, \xi_{n+1}, \ldots, \xi_{n+k-1}), \mathcal{F}(\varsigma, \varsigma, \ldots, \varsigma)\big) \\ &\quad\leqslant q\big(\mathcal{F}(\xi_n, \xi_{n+1}, \ldots, \xi_{n+k-1}), \mathcal{F}(\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+k-1}, \varsigma)\big) \\ &\qquad+ q\big(\mathcal{F}(\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+k-1}, \varsigma), \mathcal{F}(\xi_{n+2}, \xi_{n+3}, \ldots, \varsigma, \varsigma)\big) \end{aligned}$$

$$
\begin{aligned}
&\quad + q\big(\mathcal{F}(\xi_{n+2}, \xi_{n+3}, \dots, \varsigma, \varsigma), \mathcal{F}(\xi_{n+3}, \xi_{n+4}, \dots, \varsigma, \varsigma, \varsigma)\big) \\
&\quad + \dots + q\big(\mathcal{F}(\xi_{n+k-1}, \varsigma, \dots, \varsigma), \mathcal{F}(\varsigma, \varsigma, \dots, \varsigma)\big) \\
&\leqslant \lambda \max\big\{q(\xi_{n+i-1}, \xi_{n+i}), q(\xi_{n+k-1}, \varsigma),\ 1 \leqslant i \leqslant k-1\big\} \\
&\quad + \lambda \max\big\{q(\xi_{n+i-1}, \xi_{n+i}), q(\xi_{n+k-1}, \varsigma),\ 2 \leqslant i \leqslant k-1\big\} \\
&\quad + \dots + \lambda \max\big\{q(\xi_{n+k-2}, \xi_{n+k-1}), q(\xi_{n+k-1}, \varsigma)\big\} + \lambda q(\xi_{n+k-1}, \varsigma).
\end{aligned}
$$

Taking limit $n \to \infty$, we have $q(\xi_n, \mathcal{F}(\varsigma, \varsigma, \dots, \varsigma)) \to 0$. Therefore, since $q(\xi_n, \varsigma) \to 0$ and $q(\xi_n, \mathcal{F}(\varsigma, \varsigma, \dots, \varsigma)) \to 0$ as $n \to \infty$, by Lemma 2(i), we get $\varsigma = \mathcal{F}(\varsigma, \varsigma, \dots, \varsigma)$, and so $\varsigma \in F_k(\mathcal{F})$.

Now suppose that (3) holds. To prove $F_k(\mathcal{F}) = \{\varsigma\}$, let $w \neq \varsigma$ and $w \in F_k(\mathcal{F})$. Then, by (3), we have

$$q(\varsigma, w) = q\big(\mathcal{F}(\varsigma, \varsigma, \dots, \varsigma), \mathcal{F}(w, w, \dots, w)\big) < q(\varsigma, w).$$

This contradicts our assumption. So, $F_k(\mathcal{F}) = \{\varsigma\}$. □

By Remark 1, we can present the following theorem, which its proof is clear.

Theorem 4. *Assume that $(\mathcal{X}, \rho)$ is a left M-complete quasimetric space, q is a Q-function having small self-distance property, k is any positive integer, and $\mathcal{F}\colon \mathcal{X}^k \to \mathcal{X}$ is a mapping satisfying the contraction condition* (1). *Then there exists a point $\varsigma \in F_k(\mathcal{F})$ such that $q(\varsigma, \varsigma) = 0$.*

Moreover, the sequence $\{\xi_n\}$ given in (2) *for arbitrary initial points $\xi_1, \xi_2, \dots, \xi_k \in \mathcal{X}$ is ρ^{-1}-convergent to some point in $F_k(\mathcal{F})$. In addition, if* (3) *holds for all $u, v \in \mathcal{X}$ with $u \neq v$, then $F_k(\mathcal{F})$ is singleton.*

If we take $k = 1$ in Theorem 3 (and also in Theorem 4), then it may be concluded the following fixed point result. Note that neither small self-distance nor 0-property for Q-function is required in this result.

Corollary 1. *Let $(\mathcal{X}, \rho)$ be a left M-complete quasimetric space, q be a Q-function on $\mathcal{X}$, and $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ be a mapping. Assume that there exists $\lambda \in (0, 1)$ satisfying*

$$q(\mathcal{F}\xi, \mathcal{F}\zeta) \leqslant \lambda q(\xi, \zeta)$$

for all $\xi, \zeta \in \mathcal{X}$. Then $\mathcal{F}$ has a unique fixed point $\varsigma \in \mathcal{X}$. Furthermore, $q(\varsigma, \varsigma) = 0$.

Now, an example is demonstrated to illustrate our main theorem.

Example 5. Set $\mathcal{X} = [0, \infty)$ and $\rho(\xi, \zeta) = \max\{\zeta - \xi, 0\}$ for all $\xi, \zeta \in \mathcal{X}$. Because $\rho(\xi, 0) = 0$ for all $\xi \in \mathcal{X}$, it follows that every sequence ρ^{-1}-converges to 0, and so $(\mathcal{X}, \rho)$ is a left M-complete quasimetric space. Define $q(\xi, \zeta) = \zeta$, then q is a Q-function on $(\mathcal{X}, \rho)$, which have both small self-distance and 0-property. Consider a mapping $\mathcal{F}$: $\mathcal{X}^2 \to \mathcal{X}$ defined by

$$\mathcal{F}(\xi, \zeta) = \begin{cases} 0, & \max\{\xi, \zeta\} < 1, \\ \frac{\ln(1+\xi+\zeta)}{1+\sqrt{\xi^2+\zeta^2}}, & \max\{\xi, \zeta\} \geqslant 1. \end{cases}$$

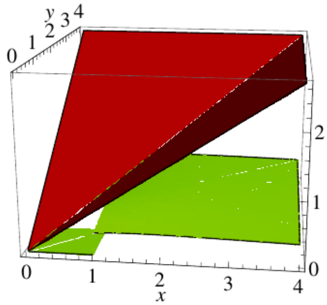

Figure 1. Graphical representation of the LHS (green) and the RHS (red) of inequality (5)

Now let $\xi, \zeta, \varsigma \in \mathcal{X}$ be arbitrary points, then we have (except for the obvious case)

$$q\big(\mathcal{F}(\xi,\zeta),\mathcal{F}(\zeta,\varsigma)\big)=\mathcal{F}(\zeta,\varsigma)=\frac{\ln(1+\zeta+\varsigma)}{1+\sqrt{\zeta^2+\varsigma^2}}\leqslant\frac{2}{3}\max\{\zeta,\varsigma\}$$
$$=\frac{2}{3}\max\big\{q(\xi,\zeta),q(\zeta,\varsigma)\big\}. \tag{5}$$

Therefore, all conditions of Theorem 3 hold with $k = 2$. Then there exists $\varsigma \in F_2(\mathcal{F})$ such that $q(\varsigma,\varsigma) = 0$. Figure 1 confirms inequality (5).

Now, considering the sequential continuity of $\mathcal{F}$, we can state the following theorem.

Theorem 5. *Assume that $(\mathcal{X},\rho)$ is a left M-complete quasimetric space, q is a Q-function on $\mathcal{X}$, and $\mathcal{F}:\mathcal{X}^k\to\mathcal{X}$ is a mapping satisfying contraction condition* (1)*. Then $F_k(\mathcal{F})\neq\emptyset$, provided that one of the following conditions holds*:

(C1) *ρ is T_1-quasimetric and $\mathcal{F}$ is sequentially ρ^{-1}-ρ-continuous*;
(C2) *$(\mathcal{X},\tau_{\rho^{-1}})$ is Hausdorff and $\mathcal{F}$ is sequentially ρ^{-1}-ρ^{-1}-continuous.*

Proof. Let $\xi_1,\xi_2,\dots,\xi_k$ be arbitrary points in $\mathcal{X}$. Define a sequence $\{\xi_n\}$ by using these points as follows:

$$\xi_{n+k}=\mathcal{F}(\xi_n,\xi_{n+1},\dots,\xi_{n+k-1})$$

for $n\in\mathbb{N}$. As in the proof of Theorem 3, we may assert that $\{\xi_n\}$ is ρ^s-Cauchy sequence, and consequently, it is left K-Cauchy sequence in $\mathcal{X}$. Since $(\mathcal{X},\rho)$ is left M-complete, it follows that there exists $\varsigma\in\mathcal{X}$ such that $\{\xi_n\}$ is ρ^{-1}-convergent to ς, that is, $\rho(\xi_n,\varsigma)\to 0$ as $n\to\infty$.

Now, if (C1) holds, then we obtain $\rho(\mathcal{F}(\varsigma,\varsigma,\dots,\varsigma),\mathcal{F}(\xi_n,\xi_{n+1},\dots,\xi_{n+k-1}))\to 0$ as $n\to\infty$. Therefore, we get

$$\begin{aligned}&\rho\big(\mathcal{F}(\varsigma,\varsigma,\dots,\varsigma),\varsigma\big)\\&\quad\leqslant\rho\big(\mathcal{F}(\varsigma,\varsigma,\dots,\varsigma),\xi_{n+k}\big)+\rho(\xi_{n+k},\varsigma)\\&\quad=\rho\big(\mathcal{F}(\varsigma,\varsigma,\dots\varsigma),\mathcal{F}(\xi_n,\xi_{n+1},\dots,\xi_{n+k-1})\big)+\rho(\xi_{n+k},\varsigma)\to 0\end{aligned}$$

as $n\to\infty$. Since ρ is T_1-quasimetric, we get $\varsigma=\mathcal{F}(\varsigma,\varsigma,\dots\varsigma)$.

If (C2) holds, then we have

$$\rho\big(\mathcal{F}(\xi_n, \xi_{n+1}, \dots, \xi_{n+k-1}), \mathcal{F}(\varsigma, \varsigma, \dots \varsigma)\big) = \rho\big(\xi_{n+k}, \mathcal{F}(\varsigma, \varsigma, \dots \varsigma)\big) \to 0$$

as $n \to \infty$. Since $(\mathcal{X}, \tau_{\rho^{-1}})$ is Hausdorff, we have $\varsigma = \mathcal{F}(\varsigma, \varsigma, \dots \varsigma)$. □

3 Existence and uniqueness result

This section is devoted to presenting a novel application with the aid of Theorem 3. In this section, we will study the existence and uniqueness of the solution of second-order (p, q)-difference Langevin equation with boundary conditions of the form

$$\begin{aligned} &D_{p,q}(D_{p,q} + \gamma)\xi(t) = f\big(t, \xi(t)\big), \quad t \in [0, 1], \\ &\xi(0) = \alpha, \qquad D_{p,q}\xi(0) = \beta, \end{aligned} \tag{6}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $0 < q < p \leqslant 1$, and γ, α, β are given constants.

Assuming that $f(t, \xi(t)) = 0$ for each $t \in [p, 1]$, we can see that (6) is equivalent to the integral equation defined by

$$\xi(t) = \alpha + (\beta + \gamma\alpha)t - \gamma \int_0^t \xi(s)\, \mathrm{d}_{p,q}s + \int_0^{t/p} (t - pqs) f\big(s, \xi(s)\big)\, \mathrm{d}_{p,q}s. \tag{7}$$

Assume that $C[0, 1]$ is the space of all real-valued continuous functions defined on $[0, 1]$. Define an operator $\mathcal{F} : C[0, 1] \to C[0, 1]$ by

$$\mathcal{F}u(t) = \alpha + (\beta + \gamma\alpha)t - \gamma \int_0^t u(s)\, \mathrm{d}_{p,q}s + \int_0^{t/p} (t - pqs) f\big(s, u(s)\big)\, \mathrm{d}_{p,q}s.$$

Hence, if u is a fixed point of $\mathcal{F}$, then it is a solution of the integral equation (7), and so, identically, we can say that it is a solution of (p, q)-difference Langevin equation (6).

To show the existence of fixed point of $\mathcal{F}$ by using Corollary 1, we will consider the space $\mathcal{X}$ as the positive cone of $C[0, 1]$, that is,

$$\mathcal{X} = \big\{u \in C[0, 1]\colon\ u(t) \geqslant 0\ \text{for}\ t \in [0, 1]\big\}.$$

Define a quasimetric on $\mathcal{X}$ as

$$\rho(u, v) = \sup_{t \in [0,1]} \max\big\{v(t) - u(t),\ 0\big\}.$$

In this case, it is clear that the function

$$q(u, v) = \sup_{t \in [0,1]} v(t)$$

is a Q-function on $\mathcal{X}$. Also, for all $u \in \mathcal{X}$, we have $\rho(u, 0) = 0$, then every sequence in $\mathcal{X}$ ρ^{-1}-converges to zero function. Therefore, $(\mathcal{X}, \rho)$ is a left M-complete quasimetric space.

Now consider the following assumptions:

(A1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, and $f(t,\xi) = 0$ for each $t \in [p,1]$;
(A2) $\alpha, \beta \geqslant 0$ and $\alpha(1+\gamma) + \beta = 0$;
(A3) There exists $L \geqslant 0$ such that $f(t,\xi) \leqslant L\xi$ for all $\xi \in [0,\infty)$.

Theorem 6. *In addition to* (A1)–(A3)*, suppose that*

$$|\gamma| + L\frac{p+q-qp^2}{p^2(p+q)} < 1.$$

Then the (p,q)*-difference Langevin equation* (6) *has a unique positive solution.*

Proof. Consider the quasimetric space $(\mathcal{X}, \rho)$, which is mentioned above. Then $\mathcal{F}$ is a self-mapping of $\mathcal{X}$ because of (A1) and (A2). Also, from (A2) and (A3) we have, for all $u, v \in \mathcal{X}$,

$$\begin{aligned}
q(\mathcal{F}u, \mathcal{F}v) &= \sup_{t\in[0,1]} \mathcal{F}v(t) \\
&= \sup_{t\in[0,1]} \left\{ \alpha + (\beta+\gamma\alpha)t - \gamma \int_0^t v(s)\,\mathrm{d}_{p,q}s + \int_0^{t/p} (t-pqs) f\bigl(s, v(s)\bigr)\,\mathrm{d}_{p,q}s \right\} \\
&\leqslant \sup_{t\in[0,1]} \left| \alpha + (\beta+\gamma\alpha)t - \gamma \int_0^t v(s)\,\mathrm{d}_{p,q}s + \int_0^{t/p} (t-pqs) f\bigl(s, v(s)\bigr)\,\mathrm{d}_{p,q}s \right| \\
&\leqslant \alpha + (\beta+\gamma\alpha) + |\gamma| \sup_{t\in[0,1]} v(t) + \sup_{t\in[0,1]} \int_0^{t/p} (t-pqs) f\bigl(s, v(s)\bigr)\,\mathrm{d}_{p,q}s \\
&\leqslant |\gamma| \sup_{t\in[0,1]} v(t) + \sup_{t\in[0,1]} \int_0^{t/p} (t-pqs) L v(s)\,\mathrm{d}_{p,q}s \\
&\leqslant |\gamma| \sup_{t\in[0,1]} v(t) + L \sup_{t\in[0,1]} v(t) \sup_{t\in[0,1]} \int_0^{t/p} (t-pqs)\,\mathrm{d}_{p,q}s \\
&= \left(|\gamma| + L\frac{p+q-qp^2}{p^2(p+q)} \right) \sup_{t\in[0,1]} v(t) \\
&\leqslant \lambda \sup_{t\in[0,1]} v(t) = \lambda q(u,v),
\end{aligned}$$

where

$$\lambda = |\gamma| + L\frac{p+q-qp^2}{p^2(p+q)} < 1.$$

Therefore, by Corollary 1, $\mathcal{F}$ has a unique fixed point in $\mathcal{X}$. That is, the (p,q)-difference Langevin equation (6) has a unique positive solution. □

4 Conclusions

In this paper, two new properties of Q-functions on quasimetric spaces named 0-property and small self-distance property were introduced. Then, taking into account these properties, some fixed point results for Prešić-type mappings were presented. To support the main theorem, an example was provided. Finally, an existence and uniqueness theorem for (p,q)-difference equations having boundary conditions was presented. The properties of the Q-function introduced in this study will be used to derive fixed point theorems for Prešić-type mappings satisfying various contractive inequalities.

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