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Phase portraits on the unit sphere of the stretch-twist-fold flow

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Abstract. The so-called stretch-twist-fold flow consists in a Stokes flow depending on two parameters defined in a unit closed ball $\bar{B}$ that is associated with the motion of a fluid particle coming from the dynamo theory, and it models a mechanism for studying the magnetic field of the Earth and the Sun. Here for the first time, we classify all the local phase portraits of its equilibrium points, and we provide the global phase portraits on the 2-dimensional sphere of the boundary of the ball $\bar{B}$.

Keywords: stretch-twist-fold flow, equilibrium points, local phase portraits, global phase portraits.

1 Introduction and statement of main results

The stretch-twist-fold flow is a special case of the Stokes flow coming from the dynamo theory. More precisely, it is a two-parameter family of a three-dimensional incompressible flow defined in the unit closed ball that is associated with the fluid particle motion coming from the dynamo theory, and it was devised to represent the stretch-twist-fold action that is believed to be most conductive of the so-called "fast dynamo action" in magnetohydrodynamics; see for more details [11, 17, 18]. In other words, it is a model for studying the origin, maintenance and amplification of the magnetic field of the celestial bodies such as stars and planets.

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A prototype of the stretch-twist-fold mechanism of the magnetic field was introduced in [22–24] and is the following 3-dimensional differential system:

$$\begin{aligned} x' &= az - 8xy, \\ y' &= 11x^2 + 3y^2 + z^2 + bxz - 3, \\ z' &= -ax + 2yz - bxy, \end{aligned} \tag{1}$$

where $x, y, z \in ma$, a, b are positive real parameters related with the ratios of the intensities of the stretch, twist and fold components of the flow. It describes that an initially circular flux tube is subjected to stretch, twist and a fold sequence.

Note that system (1) is invariant under the symmetry $S(x, y, z) = (-x, y, -z)$, so the phase portrait of the differential system (1) is symmetric with respect to the y-axis. Moreover, its vector field

$$X = \bigl(az - 8xy,\ 11x^2 + 3y^2 + z^2 + bxz - 3,\ -ax + 2yz - bxy\bigr)$$

satisfies the incompressibility condition $\nabla X = 0$ on the open unit ball

$$B = \bigl\{(x, y, z) \in \mathbb{R}^3\colon\ x^2 + y^2 + z^2 < 1\bigr\},$$

which means that the system preserves the volume in its phase space, and so B is invariant by the flow of X. Moreover, $X \cdot n|_{\partial B} = 0$, that is, the vector field X is tangent to the boundary ∂B, which is the unit sphere

$$\mathbb{S}^2 = \bigl\{(x, y, z) \in \mathbb{R}^3\colon\ x^2 + y^2 + z^2 = 1\bigr\}.$$

Let $f = f(x, y, z) = x^2 + y^2 + z^2 - 1$. Since

$$\begin{aligned} &\frac{\partial f}{\partial x}(az - 8xy) + \frac{\partial f}{\partial y}\bigl(11x^2 + 3y^2 + z^2 + bxz - 3\bigr) \\ &\quad + \frac{\partial f}{\partial z}(-ax + 2yz - bxy) = 6yf, \end{aligned}$$

it follows that the sphere $\mathbb{S}^2$ is also invariant under the flow generated by the vector field X; for more details, see [13, Chap. 8]. In particular, the existence of this invariant sphere implies that the flow inside the open unit ball B remains always inside this ball. These facts prevent the existence of strange attractors. However, this differential system can still exhibits a rich variety of structures with chaotic and regular Lagrangian orbits intricately interspersed among each other inside the unit ball; see [4–8].

System (1) has been studied intensively from the analytical and numerical points of view such as the zero-Hopf bifurcation in [14], the integrability in [6, 15], the existence of Smale horseshoes in [1, 25], the existence of periodic solutions in [8, 15] as well as the existence of invariant tori in [8] to cite just a few. However, as far as the authors know, until now a complete study of the local phase portrait of the equilibrium points of the differential system (1) as well as the description of the flow of X on $\mathbb{S}^2$ have not been done. For generalized version of system (1), see [3, 9, 12].

The objective of this paper is double: first, to do a complete study of the local phase portraits of all equilibrium points of the differential system (1), and second, to describe the flow of the vector field X on the sphere $\mathbb{S}^2$ for all values of the positive real parameters a and b.

Computing the equilibrium points of system (1), they are (whenever they are real)

$$
\begin{aligned}
p_1 &= (0,1,0), & p_2 &= (0,-1,0),\\
p_3 &= \left(\frac{\sqrt{P}}{8C}, \frac{Aa}{32}, \frac{A\sqrt{P}}{32C}\right), & p_4 &= \left(-\frac{\sqrt{P}}{8C}, \frac{Aa}{32}, -\frac{A\sqrt{P}}{32C}\right) = S(p_3),\\
p_5 &= \left(\frac{\sqrt{Q}}{8C}, \frac{Ba}{32}, \frac{B\sqrt{Q}}{32C}\right), & p_6 &= \left(-\frac{\sqrt{Q}}{8C}, \frac{Ba}{32}, -\frac{B\sqrt{Q}}{32C}\right) = S(p_5),
\end{aligned}
$$

where

$$
\begin{gathered}
A = b - \sqrt{64+b^2}, \qquad B = b + \sqrt{64+b^2}, \qquad C = \sqrt{2(100+b^2)},\\
P = -a^2(160+bA) + 64(40+bB),\\
Q = -a^2(160+bB) + 64(40+bA).
\end{gathered}
$$

Note that all these points when they exists, i.e., when they are real, are on the sphere $\mathbb{S}^2$.

Note that since $p_4 = S(p_3)$ and $p_6 = S(p_5)$ and the phase portrait of the differential system (1) is invariant with respect to the symmetry S, it follows that the local phase portraits of the equilibrium points p_3 and p_4 are equal, and the local phase portraits of the equilibrium points p_5 and p_6 are equal, of course, when they exist.

We define

$$
\begin{gathered}
b_1 = \sqrt{\frac{4096 - 2256a^2 + 25a^4 + (256+25a^2)\sqrt{256+68a^2+a^4}}{50a^2}},\\
b_2 = \frac{16-a^2}{a}, \qquad b_3 = \frac{25-a^2}{a}, \qquad b_4 = \frac{a^2-16}{a}, \qquad b_5 = \frac{a^2-25}{a},
\end{gathered}
$$

and we consider the sets of parameters

$$
\begin{aligned}
R_1 &= \{(a,b)\colon\ a \in (0,4),\ 0 < b < b_2\},\\
L_1 &= \{(a,b)\colon\ a \in (0,4),\ b = b_2,\}\\
R_2 &= \{(a,b)\colon\ a > 0,\ b > b_2,\ b > b_4\},\\
L_2 &= \{(a,b)\colon\ a > 4,\ b = b_4\},\\
R_3 &= \{(a,b)\colon\ a > 4,\ b < b_4\},
\end{aligned}
$$

where R_i denotes regions, L_i denotes lines, and all together form a partition of the plane formed by the points (a,b) with a and b positive; see Fig. 1(a).

The equilibrium points of the differential system (1) are described in the next proposition.

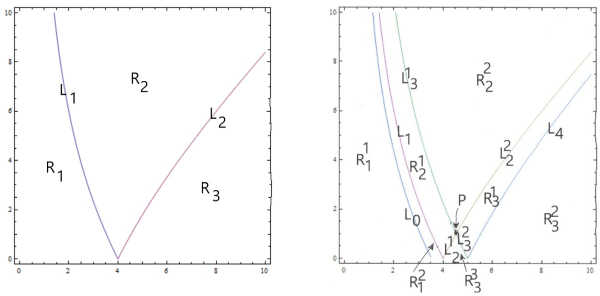
Theorem 1. *The differential system* (1) *has the following equilibrium points*:

(i) p_1, p_2, p_3, p_4, p_5 *and* p_6 *if* $(a,b) \in R_1$;
(ii) $p_1 = p_5 = p_6$, p_2, p_3 *and* p_4 *if* $(a,b) \in L_1$;
(iii) p_1, p_2, p_3 *and* p_4 *if* $(a,b) \in R_2$;
(iv) p_1, $p_2 = p_3 = p_4$ *if* $(a,b) \in L_2$;
(v) p_1 *and* p_2 *if* $(a,b) \in R_3$.

The straight line formed by the y-axis is invariant under the flow of the differential system (1) containing a heteroclinic orbit, which travels inside the ball B from the equilibrium point p_1 to the equilibrium point p_2. Indeed, when $x = z = 0$, we get that $(\dot{x}, \dot{y}, \dot{z}) = (0, 3(y^2 - 1), 0)$.

Now we define the sets of parameters (see Fig. 1(b))

$$
\begin{aligned}
R_1^1 &= \left\{(a,b)\colon\ a \in \left(0, 16\sqrt{\frac{2}{41}}\right),\ 0 < b < b_1\right\},\\
L_0 &= \left\{(a,b)\colon\ a \in \left(0, 16\sqrt{\frac{2}{41}}\right),\ b = b_1,\right\}\\
R_1^2 &= \big\{(a,b)\colon\ a \in (0,4),\ b_1 < b < b_2\big\},\\
L_1 &= \big\{(a,b)\colon\ a \in (0,4),\ b = b_2\big\},\\
R_2^1 &= \left\{(a,b)\colon\ a \in \left(0, \sqrt{\frac{41}{2}}\right),\ b_2 < b < b_3,\ b > b_4\right\},\\
L_3^1 &= \left\{(a,b)\colon\ a \in \left(0, \sqrt{\frac{41}{2}}\right),\ b = b_3\right\},\\
P &= \left(\sqrt{\frac{41}{2}}, \frac{9}{\sqrt{82}}\right),\\
L_2^1 &= \left\{(a,b)\colon\ a \in \left(4, \sqrt{\frac{41}{2}}\right),\ b = b_4\right\},\\
R_3^1 &= \big\{(a,b)\colon\ a \in (4,5),\ b_4 < b < b_3\big\},\\
L_3^2 &= \left\{(a,b)\colon\ a \in \left(\sqrt{\frac{41}{2}}, 5\right),\ b = b_3,\ b > b_4\right\},\\
R_2^2 &= \big\{(a,b)\colon\ a > 0,\ b > b_3,\ b > b_4\big\},\\
L_2^2 &= \left\{(a,b)\colon\ a > \sqrt{\frac{41}{2}},\ b = b_4\right\},\\
R_3^2 &= \left\{(a,b)\colon\ a > \sqrt{\frac{41}{2}},\ b_5 < b < b_4,\ b > b_3\right\},\\
L_4 &= \big\{(a,b)\colon\ a > 5,\ b = b_5\big\},\\
R_3^3 &= \big\{(a,b)\colon\ a > 5,\ b < b_5\big\}.
\end{aligned}
$$



(a) (b)

Figure 1. Bifurcation diagram on the number of equilibrium points (a); on the local phase portraits at the equilibrium points (b) in the parameter plane (a, b) with a and b positive.

The local phase portraits of the equilibrium points of the differential system (1) are described in the next theorem. For the definitions of hyperbolic, semihyperbolic equilibrium points, saddle, focus, node, saddle-node, see [13] and Section 2. We recall that a *nondiagonalizable node* is a node with equal eigenvalues whose Jordan normal form does not diagonalize.

Theorem 2. *The local phase portraits of the differential system* (1) *in its equilibrium points are*:

(i) *In the region* R_1^1:
p_1 *is a hyperbolic saddle on* $\mathbb{S}^2$, *and its eigenvalue in the direction inside the ball* B *is positive*;
p_2 *is a hyperbolic saddle on* $\mathbb{S}^2$, *and its eigenvalue in the direction inside the ball* B *is negative*;
p_3 *and* p_4 *are hyperbolic unstable foci on* $\mathbb{S}^2$, *and their eigenvalue in the direction inside the ball* B *is negative*;
p_5 *and* p_6 *are hyperbolic stable foci on* $\mathbb{S}^2$, *and their eigenvalue in the direction inside the ball* B *is positive.*

(ii) *In the line* L_0:
p_1 *is a hyperbolic saddle on* $\mathbb{S}^2$, *and its eigenvalue in the direction inside the ball* B *is positive*;
p_2 *is a hyperbolic saddle on* $\mathbb{S}^2$, *and its eigenvalue in the direction inside the ball* B *is negative*;
p_3 *and* p_4 *are hyperbolic unstable foci on* $\mathbb{S}^2$, *and their eigenvalue in the direction inside the ball* B *is negative*;
p_5 *and* p_6 *are hyperbolic stable nondiagonalizable nodes on* $\mathbb{S}^2$, *and their eigenvalue in the direction inside the ball* B *is positive.*

(iii) *In the region* R_1^2:
p_1 *is a hyperbolic saddle on* $\mathbb{S}^2$, *and its eigenvalue in the direction inside the ball* B *is positive*;
p_2 *is a hyperbolic saddle on* $\mathbb{S}^2$, *and its eigenvalue in the direction inside the ball* B *is negative*;

p_3 and p_4 are hyperbolic unstable foci on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is negative;
p_5 and p_6 are hyperbolic stable nodes on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is positive.

(iv) *In the line L_1*:
p_1 is a semihyperbolic stable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative;
p_3 and p_4 are hyperbolic unstable foci on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is negative.

(v) *In the region R_2^1*:
p_1 is a hyperbolic stable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative;
p_3 and p_4 are hyperbolic unstable foci on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is negative.

(vi) *In the line L_2^1*:
p_1 is a hyperbolic stable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a semihyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative.

(vii) *In the region R_3^3*:
p_1 is a hyperbolic stable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a hyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative.

(viii) *In the line L_3^1*:
p_1 is a hyperbolic stable nondiagonalizable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative;
p_3 and p_4 are hyperbolic unstable foci on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is negative.

(ix) *In the point P*:
p_1 is a hyperbolic stable nondiagonalizable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a semihyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative.

(x) *In the line L_3^2*:
p_1 is a hyperbolic stable nondiagonalizable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;

p_2 is a hyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative.

(xi) *In the region R_2^2:*
p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative;
p_3 and p_4 are hyperbolic unstable foci on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is negative.

(xii) *In the line L_2^2:*
p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a semihyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative.

(xiii) *In the region R_3^1:*
p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a hyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative.

(xiv) *In the line L_4:*
p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a hyperbolic unstable nondiagonalizable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative.

(xv) *In the region R_3^2:*
p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive;
p_2 is a hyperbolic unstable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative.

From Theorem 2 and by the Hartman–Grobman theorem (see, for instance, [10]) we note that the segment of the invariant y-axis with endpoints p_1 and p_2 is contained in the unstable manifold of the equilibrium point p_1 and in the stable manifold of the equilibrium point p_2.

Again, from Theorem 2 and by the Hartman–Grobman theorem it follows that at each equilibrium point on the sphere $\mathbb{S}^2$ of the differential system (1), there is either a stable or an unstable manifold of at most dimension two contained inside the open ball B.

In the next three theorems, we describe the dynamics on the invariant sphere $\mathbb{S}^2$ of the flow of the differential system (1) in function of the positive parameters a and b. We have numerical evidences that the differential system (1) has no periodic orbits on the sphere $\mathbb{S}^2$ (see the Appendix). So we do the next conjecture.

Conjecture 1. For all positive values of the parameters a and b, the differential system (1) on the sphere $\mathbb{S}^2$ has no periodic orbits.

It is known that the stretch-twist-fold flow has periodic orbits in the open ball B; see, for instance, [8, 15].

For definitions of separatrix, canonical region, strip flow and spiral or nodal flow, see, for instance, [13, Sect. 1.9] or Section 3.

Let $\varphi(t, p)$ be an orbit of a vector field X on the sphere $\mathbb{S}^2$ such that $\varphi(0, p) = p$. We define the set

$$\omega(p) = \{q \in \mathbb{S}^2\colon \text{ there exist } \{t_n\} \text{ with } t_n \to \infty \text{ and } \varphi(t_n) \to q \text{ when } n \to \infty\}.$$

In a similar way, we define the set

$$\alpha(p) = \{q \in \mathbb{S}^2\colon \text{ there exist } \{t_n\} \text{ with } t_n \to -\infty \text{ and } \varphi(t_n) \to q \text{ when } n \to \infty\}.$$

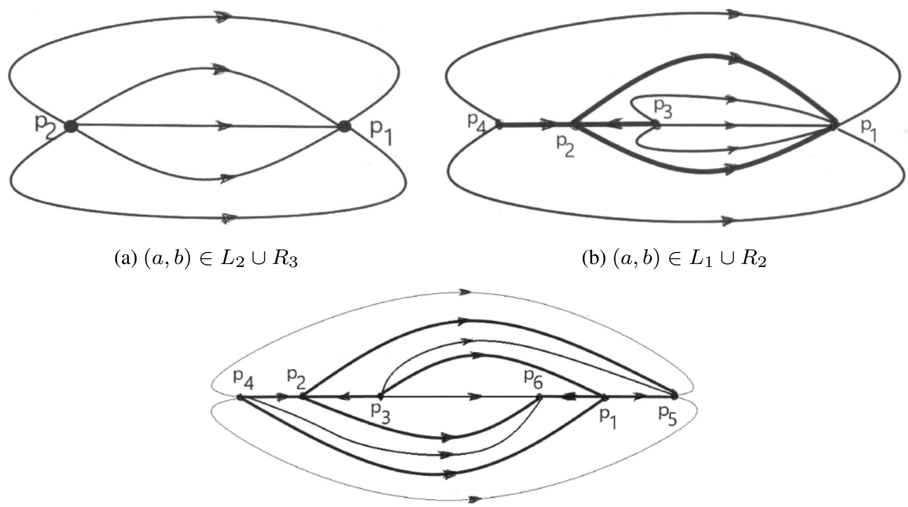
The sets $\omega(p)$ and $\alpha(p)$ are called the *ω-limit set* and the *α-limit set* of p, respectively.

Theorem 3. *Assume that the differential system* (1) *has no periodic orbits on the invariant sphere $\mathbb{S}^2$, and that $(a, b) \in L_2 \cup R_3$. Then every orbit on $\mathbb{S}^2$ different from the equilibrium points p_1 and p_2 has α-limit in p_2 and ω-limit in p_1. Removing the two equilibria, we obtain one canonical region with a spiral or nodal flow*; *see Fig.* 2(a).

Theorem 4. *Assume that the differential system* (1) *has no periodic orbits on the invariant sphere $\mathbb{S}^2$, and that $(a, b) \in L_1 \cup R_2$. Then one of the two stable separatrices of the saddle p_2 comes from the unstable equilibrium p_3, and the other from the unstable equilibrium p_4, and the two unstable separatrices of p_2 go to the stable equilibrium p_1. Removing the four separatrices of the saddle p_2 and all the equilibria, we obtain two canonical regions with strip flows. In one canonical region, every orbit has α-limit at p_3 and ω-limit at p_1. In the other canonical region, every orbit has α-limit at p_4 and ω-limit at p_1*; *see Fig.* 2(b).

Theorem 5. *Assume that the differential system* (1) *has no periodic orbits on the invariant sphere $\mathbb{S}^2$. Then for the flow on the sphere $\mathbb{S}^2$ and for some values $(a, b) \in R_1$, one of the two stable separatrices of the saddle p_2 comes from the unstable equilibrium p_3, and the other from the unstable equilibrium p_4, and one unstable separatrix of p_2 goes to the stable equilibrium p_5 and the other goes to the stable equilibrium p_6. One of the two unstable separatrices of the saddle p_1 goes to the stable equilibrium p_5, and the other goes to the stable equilibrium p_6, and one of the stable separatrix of p_1 comes from the unstable equilibrium p_3, and the other comes from the unstable equilibrium p_4. Removing the separatrices of the two saddles p_1 and p_2 and all the equilibria, we obtain four canonical regions with strip flows. In a canonical region, every orbit has α-limit at p_3 and ω-limit at p_5. In other canonical region, every orbit has α-limit at p_3 and ω-limit at p_6. In another canonical region, every orbit has α-limit at p_4 and ω-limit at p_5. Finally, in the fourth canonical region, every orbit has α-limit at p_4 and ω-limit at p_6*; *see Fig.* 2(c).

We note that Theorems 3 and 4 characterize completely the topological phase portraits of the differential system (1) when the parameters $(a, b) \in L_1 \cup R_2 \cup L_2 \cup \mathbb{R}_3$, but when the parameters $(a, b) \in R_1$, we have proved in Theorem 5 that for some values of such parameters, its phase portrait is topologically equivalent to the phase portrait of Fig. 2(c). We have numerical evidence that the following conjecture must hold.



(a) $(a,b) \in L_2 \cup R_3$

(b) $(a,b) \in L_1 \cup R_2$

(c) $(a,b) \in R_1$

Figure 2. The phase portrait of system (1). The thick lines are formed by the separatrices of the saddles, and the thin lines are some orbits, which are not separatrices.

Conjecture 2. For all values of the parameters $(a,b) \in R_1$, the phase portrait of the differential system (1) on the sphere $\mathbb{S}^2$ is topologically equivalent to the one of Fig. 2(c).

We recall the stereographic projection from the south pole. We identify $\mathbb{R}^2$ as the tangent plane to the sphere $\mathbb{S}^2$ at the point $(0,0,-1)$, and we denote the points of $\mathbb{R}^2$ as $(u,v) = (u,v,-1)$. Let $\pi : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{(0,0,1)\}$ be the diffeomorphism given by

$$\pi(u,v) = \left(x = \frac{2u}{1+u^2+v^2},\ y = \frac{2v}{1+u^2+v^2},\ z = \frac{u^2+v^2-1}{1+u^2+v^2}\right).$$

That is, π is the inverse map of the stereographic projection $\pi^{-1} : \mathbb{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ defined by

$$\pi^{-1}(x,y,z) = \left(u = \frac{x}{1-z},\ v = \frac{y}{1-z}\right).$$

2 The equilibria of system (1)

In this section, we prove Theorems 1 and 2.

Proof of Theorem 1. First, we note that $P = 0$ if and only if $b = (a^2 - 16)/a$ (i.e., P vanishes on L_2), and that $Q = 0$ if and only if $b = (16 - a^2)/a$ (i.e., Q vanishes on L_1).

Since P and Q are positive in the region R_1, for the values of the parameters (a,b) in this region, the differential system (1) has the six equilibria p_1, p_2, p_3, p_4, p_5 and p_6. Therefore, statement (i) of Theorem 1 is proved.

Since $P > 0$ and $Q = 0$ on the line L_1, it follows that $p_1 = p_5 = p_6$, p_2, p_3 and p_4. This completes the proof of statement (ii) of Theorem 1.

Since $P > 0$ and $Q < 0$ in the region R_2, for the values of the parameters (a, b) in this region, the differential system (1) has the four equilibria p_1, p_2, p_3 and p_4, and consequently, statement (iii) of Theorem 1 follows.

Since $P = 0$ and $Q < 0$ on the line L_2, it follows that p_1, $p_2 = p_3 = p_4$. This completes the proof of statement (iv) of Theorem 1.

Finally, since $P < 0$ and $Q < 0$ in the region R_2, for the values of the parameters (a, b) in this region, the differential system (1) has the two equilibria p_1 and p_2. This proves statement (v) of Theorem 1.

In summary, Theorem 1 is proved. □

In what follows, we recall some basic definitions and results that we shall need for proving Theorem 2.

An equilibrium point of a differential system or vector field in a 2-dimensional manifold is *hyperbolic* if the real part of its two eigenvalues are nonzero. The local phase portraits of the hyperbolic equilibrium points in dimension two are classified; see, for instance, [13, Thm. 2.15].

An equilibrium point of a differential system or vector field in a 2-dimensional manifold is *semihyperbolic* if it has only one eigenvalue equal to zero. The semihyperbolic equilibrium points only can be saddles, nodes or saddle-nodes; see, for instance, [13, Thm. 2.19].

We recall that each isolated equilibrium point of a continuous differential system or vector field in a 2-dimensional manifold has associated a unique integer number called its (*topological*) *index*. The nodes and foci have index 1, the saddles have index -1, and the saddle-nodes have index 0; for more details, see [13, Chap. 6].

The next theorem is well known, and for a proof, see, for instance, [2, Sect. 36] or [13, p. 179].

Theorem 6 [Poincaré–Hopf theorem]. *For every continuous vector field on the sphere $\mathbb{S}^2$ with a finite number of equillibrium points, the sum of the indices of its equilibrium points is* 2.

Proof of Theorem 2. The Jacobian matrix of the differential system (1) is

$$\begin{pmatrix} -8y & -8x & a \\ 22x + bz & 6y & 2z + bx \\ -a - by & 2z - bx & 2y \end{pmatrix}.$$

The equilibrium points p_3 and p_4 exist in the regions $R_1 \cup L_1 \cup R_2$. The Jacobian matrix evaluated at these two equilibrium points has the same characteristic polynomial

$$\begin{aligned} P_{34}(\lambda) = {} & \frac{3}{256}\bigl(a\bigl(a^2\bigl(b^2+32\bigr) - 512\bigr)\sqrt{b^2+64} - a^3 b\bigl(b^2+64\bigr)\bigr) \\ & + \frac{1}{128}\bigl(a^2\bigl(13b^2+544\bigr) - 64\bigl(b^2+64\bigr) - b\bigl(13a^2+64\bigr)\sqrt{b^2+64}\bigr)\lambda - \lambda^3. \end{aligned}$$

Therefore, the Jacobian matrix evaluated at both equilibrium points have the same eigenvalues. It is easy to check that the independent term of this characteristic polynomial does not vanish if $(a,b) \in R_1 \cup L_1 \cup R_2$. The discriminant of this cubic characteristic polynomial is negative if $(a,b) \in R_1 \cup L_1 \cup R_2$, hence, this polynomial has only one real root and two complex roots. Since the independent term of the polynomial $P_{34}(\lambda)$ does not vanish, in order to see that the real root of this polynomial is always negative when $(a,b) \in R_1 \cup L_1 \cup R_2$, it is sufficient to compute the roots of this polynomial in a point $(a,b) \in R_1 \cup L_1 \cup R_2$. Since the system formed by the coefficients of the polynomial $P_{34}(\lambda) + (\lambda - r)(\lambda^2 + \omega^2)$ has no solution in the real variables r, ω and $(a,b) \in R_1 \cup L_1 \cup R_2$, it follows that the real part of the two complex eigenvalues of the polynomial $P_{34}(\lambda)$ never vanish. So, in order to see that the real part of the two complex eigenvalues is always positive, it is sufficient to see that for a particular value of $(a,b) \in R_1 \cup L_1 \cup R_2$. Hence, the equilibrium points p_3 and p_4 are always hyperbolic. Moreover, by [13, Thm. 2.15] and taking into account that p_3 and p_4 are on the invariant sphere $\mathbb{S}^2$, they are always hyperbolic unstable foci on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is negative.

The equilibrium points p_5 and p_6 exist in the region R_1. The Jacobian matrix evaluated at these two equilibrium points have the same characteristic polynomial

$$\begin{aligned} P_{56}(\lambda) = &-\frac{3}{256}\bigl(a\bigl(a^2(b^2+32) - 512\bigr)\sqrt{b^2+64} + a^3 b\bigl(b^2+64\bigr)\bigr) \\ &+ \frac{1}{128}\bigl(a^2\bigl(13b^2+544\bigr) - 64\bigl(b^2+64\bigr) + b\bigl(13a^2+64\bigr)\sqrt{b^2+64}\bigr)\lambda - \lambda^3. \end{aligned}$$

Therefore, the Jacobian matrix evaluated at both equilibrium points have the same eigenvalues. It is easy to check that the independent term of this characteristic polynomial does not vanish if $(a,b) \in R_1$. So these points are always hyperbolic. Moreover, it is easy to check that the discriminant D of this cubic characteristic polynomial is negative if $(a,b) \in R_1^1$, zero if $(a,b) \in L_0$, and positive if $(a,b) \in R_1^2$.

Using for the polynomial $P_{56}(\lambda)$ the same kind of arguments than the ones used in the study of the roots of the polynomial $P_{34}(\lambda)$, we obtain that when $D < 0$, the polynomial $P_{56}(\lambda)$ has only one positive real root and two complex roots, and the real part of the complex roots is negative. Hence, the equilibrium points p_5 and p_6 are always hyperbolic. Furthermore, by [13, Thm. 2.15] and taking into account that p_5 and p_6 are on the invariant sphere $\mathbb{S}^2$, they are always hyperbolic stable foci on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is positive.

When $D = 0$, the two complex roots become a negative double real root, and the remaining real root continues being positive. Therefore, by [13, Thm. 2.15] taking into account that p_5 and p_6 are on the invariant sphere $\mathbb{S}^2$, they are always hyperbolic stable nondiagonalizable nodes on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is positive.

When $D > 0$, the previous negative double real root splits into two distinct negative real roots, and the remaining real root continues being positive. Hence, by [13, Thm. 2.15] taking into account that p_5 and p_6 are on the invariant sphere $\mathbb{S}^2$, they are always hyperbolic stable nodes on $\mathbb{S}^2$, and their eigenvalue in the direction inside the ball B is positive.

Computing the eigenvalues of the Jacobian matrix at p_1 and p_2, we get that they are

$$\lambda_1 = 6, \qquad \lambda_2 = -3 - \sqrt{25 - a^2 - ab}, \qquad \lambda_3 = -3 + \sqrt{25 - a^2 - ab}$$

and

$$\lambda_1 = -6, \qquad \lambda_2 = 3 - \sqrt{25 - a^2 + ab}, \qquad \lambda_3 = 3 + \sqrt{25 - a^2 + ab},$$

respectively. Then in the region $R_1 = R_1^1 \cup L_0 \cup R_1^2$, it is easy to check that p_1 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and that p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. This completes the proof of statements (i), (ii) and (iii) of Theorem 2.

In the line L_1 the equilibrium p_1 is semihyperbolic on $\mathbb{S}^2$ having an eigenvalue positive and the other zero, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. On this line, we only have the four equilibrium points p_i for $i = 1, 2, 3, 4$. We know that p_3 and p_4 are foci on $\mathbb{S}^2$, and that p_2 is a saddle, so the sum of the indices of these three equilibria is 1. Therefore, by the Poincaré–Hopf theorem the index of the semihyperbolic equilibrium p_1 must be 1, and consequently, p_1 must be a semihyperbolic stable node. This completes the proof of statements (iv) of Theorem 2.

In the region R_2^1 the equilibrium p_1 becomes a hyperbolic stable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. This completes the proof of statements (v) of Theorem 2.

In the line L_2^1 the equilibrium p_1 continues being a hyperbolic stable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; but p_2 becomes a semihyperbolic equilibrium on $\mathbb{S}^2$ with a positive real eigenvalue and a zero eigenvalue, and its eigenvalue in the direction inside the ball B is negative. Since in the line L_2^1 the unique equilibrium points are p_1 and p_2 and the index of p_1 is 1, by the Poincaré–Hopf theorem the index of the semihyperbolic equilibrium p_2 is also 1. So p_2 is a semihyperbolic unstable node on $\mathbb{S}^2$. This completes the proof of statements (vi) of Theorem 2.

In the region R_3^3 the equilibrium p_1 continues being a hyperbolic stable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 becomes a hyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. So statement (vii) of Theorem 2 is proved.

In the line L_3^1 the equilibrium p_1 becomes a hyperbolic stable nondiagonalizable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. Therefore, statement (viii) of Theorem 2 is proved.

In the point P the equilibrium p_1 is a hyperbolic stable nondiagonalizable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a semihyperbolic equilibrium on $\mathbb{S}^2$ with a positive eigenvalue and a zero eigenvalue, and its eigenvalue in the direction inside the ball B is negative. By the Poincaré–Hopf theorem p_2 is a semihyperbolic unstable node on $\mathbb{S}^2$. This completes the proof of statement (ix) of Theorem 2.

In the line L_3^2 the equilibrium p_1 is a hyperbolic stable nondiagonalizable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a hyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. This completes the proof of statement (x) of Theorem 2.

In the region R_2^2 the equilibrium p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a hyperbolic saddle on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. Hence, statement (xi) of Theorem 2 is proved.

In the line L_2^2 the equilibrium p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a semihyperbolic equilibrium on $\mathbb{S}^2$ with a positive eigenvalue and a zero eigenvalue, and its eigenvalue in the direction inside the ball B is negative. Again, by the Poincaré–Hopf theorem p_2 is a semihyperbolic unstable node on $\mathbb{S}^2$. This completes the proof of statement (xii) of Theorem 2.

In the region R_3^1 the equilibrium p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a hyperbolic unstable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. Therefore, statement (xiii) of Theorem 2 is proved.

In the line L_4 the equilibrium p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a hyperbolic unstable nondiagonalizable node on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. This completes the proof of statement (xiv) of Theorem 2.

In the region R_3^2 the equilibrium p_1 is a hyperbolic stable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is positive; and p_2 is a hyperbolic unstable focus on $\mathbb{S}^2$, and its eigenvalue in the direction inside the ball B is negative. So statement (xv) of Theorem 2 is proved. This completes the proof of Theorem 2. □

3 Proofs of Theorems 3, 4 and 5

We recall the Poincaré–Bendixson theorem on the sphere $\mathbb{S}^2$. For a proof, see the more general proof of this theorem for a compact region of the plane provided in [13, Sect. 1.7] or see [20].

Theorem 7 [Poincaré–Bendixson theorem I]. *Let $\varphi(t,p)$ be an orbit of a C^1 vector field X on the sphere $\mathbb{S}^2$. Assume that X has finitely many equilibrium points. Then one of the following statements holds*:

(i) *If $\omega(p)$ does not contains equilibrium points, then $\omega(p)$ is a periodic orbit.*
(ii) *If $\omega(p)$ contains both regular and equilibrium points, then $\omega(p)$ is formed by a set of orbits, every one of which tends to one of the equilibrium points in $\omega(p)$ as $t \to \pm\infty$.*
(iii) *If $\omega(p)$ does not contain regular points, then $\omega(p)$ is a unique equilibrium point.*

A *separatrix* of a vector field on the sphere $\mathbb{S}^2$ is an equilibrium point, or a limit cycle, or an orbit on the boundary of a hyperbolic sector at an equilibrium point. The set of all separatrices is closed (see [19]), and we denote it by Σ_X. An open connected component of $\mathbb{S}^2 \setminus \Sigma_X$ is a *canonical region* of X. It is known that the flow on a canonical region is topologically equivalent to one of the following three flows (see [16, 19, 21]):

(i) The flow defined on $\mathbb{R}^2$ by the differential system $\dot{x} = 1$, $\dot{y} = 0$, which we denote by *strip flow*.

(ii) The flow defined on $\mathbb{R}^2 \setminus \{0\}$ by the differential system given in polar coordinates $r' = 0$, $\theta' = 1$, which we denote by *annulus flow*.

(iii) The flow defined on $\mathbb{R}^2 \setminus \{0\}$ by the differential system given in polar coordinates $r' = r$, $\theta' = 0$, which we denote by *spiral* or *nodal flow*.

Proof of Theorem 3. By assumptions the differential system (1) has no periodic orbits on the invariant sphere $\mathbb{S}^2$, and from Theorem 2 if $(a, b) \in L_2 \cup R_3$, then the unique separatrices of the system are the two equilibrium points p_1 and p_2, being p_1 a stable equilibrium and p_2 an unstable equilibrium. Therefore, the flow on the canonical region $\mathbb{S}^2 \setminus \{p_1, p_2\}$ is a spiral or nodal flow. This completes the proof of the theorem. □

Proof of Theorem 4. By hypotheses the differential system (1) has no periodic orbits on the invariant sphere $\mathbb{S}^2$, and from Theorem 2 if $(a, b) \in L_1 \cup R_2$, the system has the equilibrium points p_i for $i = 1, 2, 3, 4$, being p_1 a stable equilibrium, p_2 a saddle, and p_3 and p_4 are unstable equilibria. By the Poincaré–Bendixson theorem the two stable separatrices of the saddle p_2 come from the unstable equilibrium p_3, and the other from the unstable equilibrium p_4, and the two unstable separatrices of p_2 go to the stable equilibrium p_1. Removing the four separatrices of the saddle p_2 and the four equilibria, we obtain two canonical regions with strip flows. In one canonical region, every orbit distinct from the equilibrium points p_3 and p_1 has α-limit in p_3 and ω-limit in p_1. In the other canonical region, every orbit distinct from the equilibrium points p_4 and p_1 has α-limit in p_4 and ω-limit in p_1. So the proof of the theorem is done. □

Proof of Theorem 5. By hypotheses the differential system (1) has no periodic orbits on the invariant sphere $\mathbb{S}^2$, and by Theorem 2 if $(a, b) \in R_1$, then the system has the equilibrium points p_i for $i = 1, \dots, 6$, being p_1 and p_2 two saddles, p_3 and p_4 two unstable equilibria, and p_5 and p_6 two stable equilibria. Then near the line L_1 but inside the region R_1, by continuity we have that the two stable separatrices of the saddle p_2 come one from the unstable equilibrium p_3 and the other from the unstable equilibrium p_4. Since the equilibrium points p_5 and p_6 bifurcate from the equilibrium point p_1, it follows that one of the two unstable separatrices of the saddle p_1 goes to the stable equilibrium p_5, and the other goes to the stable equilibrium p_6. On the line L_1 the two unstable separatrices of the saddle p_2 go to the stable equilibrium p_1. Again, by continuity one unstable separatrix of p_2 must go to the stable equilibrium p_5, and the other separatrix must go to the stable equilibrium p_6. Note that it is not possible that both unstable separatrices go either to p_5 or to p_6 because the local phase portraits at the points p_5 and p_6 are the same due to the symmetry S of the differential system (1). It only remains to know the α-limit of the two stable separatrices of the saddle p_1. Due to the previous results (see Fig. 2(c)), one comes from the unstable equilibrium p_3, and the other comes from the unstable equilibrium p_4. This completes the proof of the theorem. □

We note that we have computed numerically many phase portraits of the differential system (1) for different values of $(a, b) \in R_1$, and always we have obtained phase portraits topologically equivalent to the one described in Theorem 5.

Appendix: Some numerical computations

A polynomial differential system on the sphere $\mathbb{S}^2$

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z),$$

through the stereographic projection π^{-1}, becomes the following rational differential system:

$$\dot{u} = \frac{1 + u^2 + v^2}{2}(\bar{P} + u\bar{R}), \qquad \dot{v} = \frac{1 + u^2 + v^2}{2}(\bar{Q} + v\bar{R}) \tag{2}$$

on the plane $\mathbb{R}^2$, where

$$\bar{F} = F\left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2}\right).$$

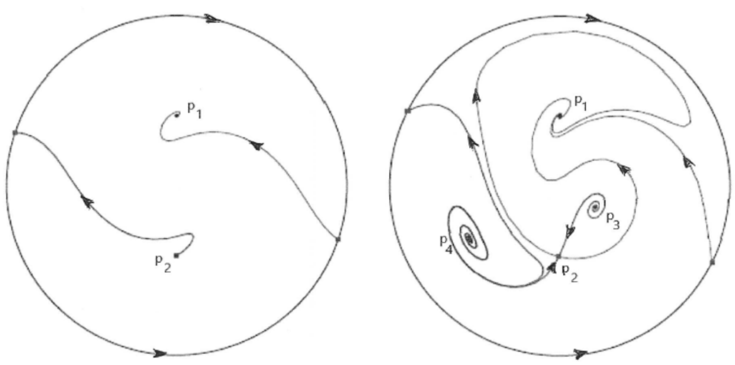
If t denotes the independent variable in the above differential system, then that system becomes polynomial introducing the new independent variable s through $\mathrm{d}s = (1 + u^2 + v^2)^{m-1}\mathrm{d}t$.

Now the differential system (1) written in the form (2) is

$$\begin{aligned} \dot{u} &= -a - 2au^2 - 36uv - 4bu^2v - au^4 + 4u^3v + 4uv^3 + av^4, \\ \dot{v} &= -2\big(1 + bu - 18u^2 + auv - bu^3 + buv^2 + u^4 + au^3v + auv^3 - v^4\big). \end{aligned} \tag{3}$$

We draw the phase portraits of the polynomial differential system (3) in the plane $\mathbb{R}^2$ in the Poincaré disc, i.e., roughly speaking, we identify the plane $\mathbb{R}^2$ with the interior of the unit disc and its boundary the circle $\mathbb{S}^1$ with the infinity of $\mathbb{R}^2$; for more details on the so called Poincaré compactification, see [13, Chap. 5]. Identifying the circle $\mathbb{S}^1$ of the infinity to a point, we have the phase portrait of the differential system (1) on the sphere $\mathbb{S}^2$.

In Fig. 3(a), we provide the phase portrait in the region $L_2 \cup R_3$, in Fig. 3(b), we provide the phase portrait in the region $L_1 \cup R_2$, and in Fig. 4, we provide a phase portrait in the region R_1.



(a) $(a, b) = (6, 2)$ (b) $(a, b) = (4, 8)$

Figure 3. The phase portrait of system (2).

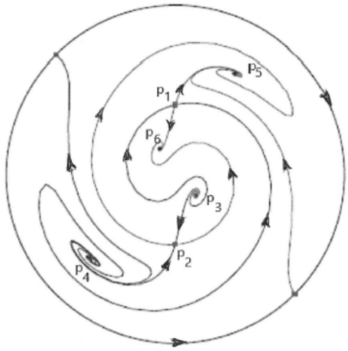


Figure 4. The phase portrait of system (2), $(a, b) = (2, 1)$.

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