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Fixed point theorems of new generalized C-conditions for (ψ, γ)-mappings in modular metric spaces and its applications

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Abstract. This paper introduces new generalizations of the C-condition for (ψ, γ)-mappings in modular metric spaces. We extend the fixed point results for such mappings yielding the generalized C-condition in metric spaces to modular ones. We proved the existence and uniqueness of solutions in modular metric spaces for these kinds of mappings. We give an example to emphasize that our results work in the difference between modular metric spaces and usual ones. Moreover, we consider some initial and boundary value problems to support the results obtained here. We examine the existence and uniqueness of the solutions for the problems in modular metric spaces.

Keywords: fixed point theorem, modular metric space, C-condition, (ψ, γ)-mappings, Cauchy problem.

1 Introduction

The fixed point theory has maintained as an attractive topic since Banach introduced and proved the well-known Banach fixed point theorem in 1922. In addition, to providing the existence and uniqueness of solutions for distinct types of problems, the result and its applications have been taken place in a lot of branches of applied science such as physics, chemistry, engineering, image processing, economics, etc. Many different versions of this principle have been extended in distinct spaces.

On the other hand, extending metric spaces to modular spaces (MS) has become a significant issue since Nakano [22], Musielak, and Orlicz [20] introduced the concept of MS and worked on some theories about them. For other results on MS, see [17, 18].

The idea of modular metric spaces (MMS) has been worked on by Chistyakov [8, 9]. He introduced basic definitions and properties of modular metric (MM). In [10], he presented significant fixed point results for the contractions in MMS. After Chistyakov's

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works, many authors have studied fixed point theory in MMS. Mongkolkeha and Kumam [19] presented significant fixed point results for the contractive mappings. Aksoy et al. [5] stated and proved fixed point theorems for Meir–Keeler-type contractions. Ege and Alaca [12] obtained some fixed point results and gave an application to homotopy. Alaca et al. [6] introduced fixed point theorems in modular ultrametric spaces. Abdou [3] attained fixed points of Kannan mappings. Recently, Karapınar et al. [16] considered interpolative Meir–Keeler maps and proved fixed point theorem for them. The papers mentioned above demonstrate the significance and use of MMS in the literature. For some other related papers in MMS, see [1, 2, 21, 26].

Now some information on C-condition are given. For the first time, Suzuki defined the C-condition as follows.

Definition 1. (See [24].) Consider a self-map F on a given metric space (Ω, d). F yields the C-condition if

$$\frac{1}{2}d(\tau_1, F\tau_1) \leqslant d(\tau_1, \tau_2) \quad\Longrightarrow\quad d(F\tau_1, F\tau_2) \leqslant d(\tau_1, \tau_2) \quad \forall \tau_1, \tau_2 \in \Omega.$$

Then he proved the following theorem for mappings satisfying the C-condition in a compact metric space.

Theorem 1. (*See* [25].) *Suppose that F is a self-mapping on a compact metric space (Ω, d). If*

$$\frac{1}{2}d(\tau_1, F\tau_1) \leqslant d(\tau_1, \tau_2) \quad\Longrightarrow\quad d(F\tau_1, F\tau_2) \leqslant d(\tau_1, \tau_2) \quad \forall \tau_1, \tau_2 \in \Omega$$

holds, then F is of a fixed point.

After, Popescu extended Suzuki's theorem and gave the below result in his work.

Theorem 2. (*See* [23].) *Assume that $F : \Omega \to \Omega$ is a self-mapping where (Ω, d) is complete. Let*

$$\frac{1}{2}d(\tau_1, F\tau_1) \leqslant d(\tau_1, \tau_2)$$

implies

$$\begin{aligned} d(F\tau_1, F\tau_2) \leqslant{}& Ad(\tau_1, \tau_2) + B\big[d(\tau_1, F\tau_1) + d(\tau_2, F\tau_2)\big] \\ &+ C\big[d(\tau_1, F\tau_2) + d(\tau_2, F\tau_1)\big], \end{aligned}$$

where $A + 2B + 2C = 1$, $A \geqslant 0$, and $B, C > 0$. Then F is of exactly one fixed point.

Definition 2. (See [13].) Let ψ be defined and continuous on $[0, \infty)$, and let ψ yields the followings:

(ψ1) ψ is nondecreasing,
(ψ2) $\psi(t) = 0 \Leftrightarrow t = 0$.

Then it is named an altering distance function. The set of such functions is denoted by Ψ.

Lemma 1. (*See* [27].) *Let* ψ *be in* Ψ*, and a continuous function* γ *be defined on* $[0,\infty)$. *Then* $\gamma(0)=0$ *whenever* $\psi(\xi)>\gamma(\xi)$ *for all* $\xi>0$.

Recently, Gupta et al. [14] have defined the C-condition for (ψ,γ)-mappings denoted by C_γ^ψ-condition and obtained fixed point results for those maps.

Definition 3. (See [14].) Consider a metric space (Ω,d) and take a self-map F on Ω. If

$$\frac{1}{2}d(\tau_1,F\tau_1)\leqslant d(\tau_1,\tau_2) \quad\Longrightarrow\quad \psi\bigl(d(F\tau_1,F\tau_2)\bigr)\leqslant\gamma\bigl(d(\tau_1,\tau_2)\bigr) \quad \forall\tau_1,\tau_2\in\Omega,$$

where γ is defined and continuous on $[0,\infty)$ and $\psi\in\Psi$, then F is said to satisfy the C_γ^ψ-condition.

In addition to this, Gupta et al. [15] introduced a generalized C-condition for the mappings ψ and γ. They also obtained some fixed point results of a map yielding this condition.

Definition 4. (See [15].) Take the map $F:\Omega\to\Omega$ on a metric space (Ω,d). If

$$\frac{1}{2}d(\tau_1,F\tau_1)\leqslant d(\tau_1,\tau_2)$$

implies

$$\begin{aligned}\psi\bigl(d(F\tau_1,F\tau_2)\bigr)\leqslant\gamma\biggl(\max\biggl\{&d(\tau_1,\tau_2),\frac{1}{2}\bigl(d(\tau_1,F\tau_1)+d(\tau_2,F\tau_2)\bigr),\\&\frac{1}{2}\bigl(d(\tau_1,F\tau_2)+d(\tau_2,F\tau_1)\bigr)\biggr\}\biggr)\quad\forall\tau_1,\tau_2\in\Omega,\end{aligned}$$

then F is said to yield the generalized C-condition. Here ψ is in Ψ, and γ is defined and continuous on $[0,\infty)$.

In the light of the above, any fixed point results about generalized C-condition in MMS have not been obtained. We, therefore, focus on (ψ,γ)-mappings yielding the generalized C-condition in MMS. This paper aims to introduce some fixed point theorems and results on the given MMS for (ψ,γ)-mappings. Moreover, we prove the existence and uniqueness of solutions for these mappings. An example is considered in the difference of MMS from metric ones. This example illustrates the importance of this paper. For applications, some Cauchy problems with initial and boundary conditions of 1st-order differential equations are examined. Fixed point results for these problems are given in the considered MMS. As a result, the existence and uniqueness of solutions for the considered problem are attained.

2 Fundamentals of modular metric spaces

This part gives some definitions and features of MMS.

For $\eta>0$ and $\Omega\neq$, define a function m as $m:(0,\infty)\times\Omega\times\Omega\to[0,\infty]$. We can write the function m as $m(\eta,\tau_1,\tau_2)=m_\eta(\tau_1,\tau_2)$ for all $\eta>0$ and $\tau_1,\tau_2\in\Omega$ so that $m_\eta:\Omega\times\Omega\to[0,\infty]$. For more detail, see [11].

Definition 5. (See [8,9].) Suppose that $m_\eta : \Omega \times \Omega \to [0,\infty]$ yields the followings:

(m1) $\tau_0 = \tau_1 \ \Leftrightarrow \ m_\eta(\tau_0, \tau_1) = 0$,
(m2) $m_\eta(\tau_0, \tau_1) = m_\eta(\tau_1, \tau_0)$,
(m3) $m_{\eta+\mu}(\tau_0, \tau_1) \leqslant m_\eta(\tau_0, \tau_2) + m_\mu(\tau_2, \tau_1)$

for all $\tau_0, \tau_1, \tau_2 \in \Omega$ and $\eta, \mu > 0$. Then m_η is called a metric modular (MM) on Ω.

A MM is called pseudomodular if m satisfies $m_\eta(\tau, \tau) = 0$ for all $\eta > 0$ instead of (m1).

Let instead of (m3), m yields the following:

$$m_{\eta+\mu}(\tau_0, \tau_1) \leqslant \frac{\eta}{\eta+\mu} m_\eta(\tau_0, \tau_2) + \frac{\mu}{\eta+\mu} m_\mu(\tau_2, \tau_1) \quad \forall \eta, \mu > 0.$$

Then it is named convex. Furthermore, any convex MM yields

$$m_\eta(\tau_0, \tau_1) \leqslant \frac{\mu}{\eta} m_\mu(\tau_0, \tau_1) \leqslant m_\mu(\tau_0, \tau_1) \quad \forall \eta, \mu > 0,$$

for all $\tau_0, \tau_1 \in \Omega$ and $0 < \mu \leqslant \eta$ [8]. In general, a MM yields

$$m_{\eta_2}(\tau_0, \tau_1) \leqslant m_{\eta_1}(\tau_0, \tau_1), \quad 0 < \eta_1 \leqslant \eta_2,$$

for all $\tau_0, \tau_1 \in \Omega$.

Definition 6. (See [8,9].) m is said to be a strict on Ω, provided that for $\tau_1, \tau_2 \in \Omega$ with $\tau_1 \neq \tau_2$, $m_\eta(\tau_1, \tau_2) > 0$ for all $\eta > 0$, or equivalently, if $m_\eta(\tau_1, \tau_2) = 0$ for some $\eta > 0$, then $\tau_1 = \tau_2$.

Definition 7. (See [8].) Consider a MM m on Ω and $\tau_0 \in \Omega$. The followings are MMS around τ_0:

$$\begin{aligned} \Omega_m &= \Omega_m(\tau_0) = \bigl\{\tau \in \Omega\colon\ m_\eta(\tau, \tau_0) \to 0 \ \text{ as } \eta \to \infty\bigr\}, \\ \Omega_m^* &= \Omega_m^*(\tau_0) = \bigl\{\tau \in \Omega\colon\ \exists \eta = \eta(\tau) > 0 \ \text{ s.t. } m_\eta(\tau, \tau_0) < \infty\bigr\}. \end{aligned} \tag{1}$$

Definition 8. (See [8, 10].) Consider the MMS Ω_m and Ω_m^* above. The following statements hold for both spaces:

- The sequence $\{h_n\}$ in Ω_m is m-convergent to a point $h \in \Omega$, named as the modular limit of $\{h_n\}$, if and only if $m_\eta(h_n, h) \to 0$ as $n \to \infty$ for some $\eta > 0$.
- $\{h_n\}$ in Ω_m is m-Cauchy if $m_\eta(h_n, h_m) \to 0$ as $n, m \to \infty$ for some $\eta > 0$.
- Consider S to be a nonempty subset of Ω_m. Provided that every m-Cauchy sequence in S is m-convergent in S, then S is m-complete.
- $S \subseteq \Omega_m$ and $\theta : S \to \mathbb{R}^+$ is a function on S. θ is called lower semicontinuous on S if
$$\lim_{n\to\infty} m_\eta(h_n, h) = 0 \quad \Longrightarrow \quad \theta(h) \leqslant \liminf_{n\to\infty} \bigl(\theta(h_n)\bigr).$$

Lemma 2. (*See* [10].) *For any pseudomodular m on Ω, the MMS given in* (1) *are closed w.r.t. m-convergent. Moreover, the limit of any strict modular is unique.*

Note that if $\lim_{n\to\infty} m_\eta(h_n, h) = 0$ for some $\eta > 0$, then $\lim_{n\to\infty} m_\nu(h_n, h) = 0$ for all $\nu < \eta$.

3 Main results

In this part, we define two new generalized C-conditions for (ψ,γ)-mappings in MMS. Then some theorems about the fixed points for the given mappings are proved in the considered MMS.

Definition 9. Any self-mapping F on Ω_m^* is named as a generalized C-condition for (ψ,γ)-mappings, provided that F holds the following:

$$\frac{1}{2}m_\eta(\tau_0,F\tau_0)\leqslant m_\eta(\tau_0,\tau_1)\quad\Longrightarrow\quad \psi\bigl(m_\eta(F\tau_0,F\tau_1)\bigr)\leqslant\gamma\bigl(M(\tau_0,\tau_1)\bigr)\tag{2}$$

for all $\tau_0,\tau_1\in\Omega_m^*$. Here γ is defined and continuous on $[0,\infty)$, and ψ is in Ψ.

We construct two fixed point theorems of new generalized C-conditions for different $M(\tau_0,\tau_1)$ given below:

$$\begin{aligned}M_1(\tau_0,\tau_1)=\max\biggl\{&m_\eta(\tau_0,\tau_1),\ \frac{m_\eta(\tau_0,F\tau_0)(1+m_\eta(\tau_1,F\tau_1))}{1+m_\eta(\tau_0,\tau_1)},\\&\frac{m_\eta(\tau_1,F\tau_0)(1+m_\eta(\tau_1,F\tau_0))}{1+m_\eta(\tau_0,\tau_1)}\biggr\}\end{aligned}\tag{3}$$

and

$$\begin{aligned}M_2(\tau_0,\tau_1)=\max\biggl\{&m_\eta(\tau_0,\tau_1),\ \frac{m_\eta(\tau_0,F\tau_0)(1+m_\eta(\tau_1,F\tau_1))}{1+m_\eta(\tau_0,\tau_1)},\\&\frac{m_\eta(\tau_1,F\tau_1)m_\eta(\tau_0,F\tau_0)}{1+m_\eta(F\tau_0,F\tau_1)},\\&\frac{m_\eta(\tau_1,F\tau_1)m_\eta(\tau_1,F\tau_0)}{1+m_\eta(\tau_1,F\tau_0)+m_\eta(\tau_0,F\tau_1)}\biggr\}.\end{aligned}$$

Now we shall give our main theorems for mappings satisfying condition (2) for $M(\tau_0,\tau_1)=M_1(\tau_0,\tau_1)$ and $M(\tau_0,\tau_1)=M_2(\tau_0,\tau_1)$.

Theorem 3. *Assume that m_η is strict and convex MM on Ω and Ω_m^* is m-complete. Let $F:\Omega_m^*\to\Omega_m^*$ be satisfying condition* (2) *for $M(\tau_0,\tau_1)=M_1(\tau_0,\tau_1)$ with $\psi(\tau)>\gamma(\tau)$ for all $\tau>0$, where $\psi\in\Psi$, and $\gamma:[0,\infty)\to[0,\infty)$ is continuous. Suppose that for all $\eta>0$, $m_\eta(h,Fh)<\infty$ holds for a given $h\in\Omega_w^*$. Then F has at least one fixed point. In addition, if $m_\eta(h,g)<\infty$ for all $\eta>0$ and $h,g\in\Omega_m^*$, then F is of a unique fixed point, which belongs to Ω_m^*.*

Proof. Take $h_0\in\Omega_m^*$ satisfying $m_\eta(h_0,Fh_0)<\infty$. Then consider a sequence $\{h_n\}\in\Omega_m^*$ as $h_n=F^nh_0$ for all $n\in\mathbb{N}$.

Assume that $h_n=h_{n+1}$ holds for any $n\in\mathbb{N}$. In this case, h_n is the fixed point. Hence, assume that $h_n\neq h_{n+1}$ holds for all $n\in\mathbb{N}$.

Putting $\tau_0=h_n$ and $\tau_1=h_{n+1}$ in (2) yields

$$\frac{1}{2}m_\eta(h_n,Fh_n)=\frac{1}{2}m_\eta(h_n,h_{n+1})\leqslant m_\eta(h_n,h_{n+1}).$$

This implies

$$\begin{aligned}
&\psi\big(m_\eta(Fh_n, Fh_{n+1})\big)\\
&\quad= \psi\big(m_\eta(h_{n+1}, h_{n+2})\big)\\
&\quad\leqslant \gamma\left(\max\left\{m_\eta(h_n, h_{n+1}), \frac{m_\eta(h_n, Fh_n)(1+m_\eta(h_{n+1}, Fh_{n+1}))}{1+m_\eta(h_n, h_{n+1})},\right.\right.\\
&\qquad\qquad \left.\left.\frac{m_\eta(h_{n+1}, Fh_n)(1+m_\eta(h_{n+1}, Fh_{n+1}))}{1+m_\eta(h_n, h_{n+1})}\right\}\right)\\
&\quad\leqslant \gamma\left(\max\left\{m_\eta(h_n, h_{n+1}), \frac{m_\eta(h_n, h_{n+1})(1+m_\eta(h_{n+1}, h_{n+2}))}{1+m_\eta(h_n, h_{n+1})},\right.\right.\\
&\qquad\qquad \left.\left.\frac{m_\eta(h_{n+1}, h_{n+1})(1+m_\eta(h_{n+1}, h_{n+2}))}{1+m_\eta(h_n, h_{n+1})}\right\}\right).
\end{aligned} \tag{4}$$

Then we have

$$\begin{aligned}
&\psi\big(m_\eta(h_{n+1}, h_{n+2})\big)\\
&\quad\leqslant \gamma\left(\max\left\{m_\eta(h_n, h_{n+1}), \frac{m_\eta(h_n, h_{n+1})(1+m_\eta(h_{n+1}, h_{n+2}))}{1+m_\eta(h_n, h_{n+1})}\right\}\right).
\end{aligned} \tag{5}$$

Now we assume that $m_\eta(h_{n+1}, h_{n+2}) > m_\eta(h_n, h_{n+1}) > 0$. One can see that

$$\frac{1+m_\eta(h_{n+1}, h_{n+2})}{1+m_\eta(h_n, h_{n+1})} < \frac{m_\eta(h_{n+1}, h_{n+2})}{m_\eta(h_n, h_{n+1})}. \tag{6}$$

Putting (6) into (4) gives

$$\begin{aligned}
&\psi\big(m_\eta(h_{n+1}, h_{n+2})\big)\\
&\quad\leqslant \gamma\left(\max\left\{m_\eta(h_n, h_{n+1}), \frac{m_\eta(h_n, h_{n+1})(m_\eta(h_{n+1}, h_{n+2}))}{m_\eta(h_n, h_{n+1})}\right\}\right),\\
&\quad\leqslant \gamma\Big(\max\Big\{m_\eta(h_n, h_{n+1}), \big(m_\eta(h_{n+1}, h_{n+2})\big)\Big\}\Big).
\end{aligned}$$

Since $m_\eta(h_{n+1}, h_{n+2}) > m_\eta(h_n, h_{n+1})$, we have

$$\psi\big(m_\eta(h_{n+1}, h_{n+2})\big) \leqslant \gamma\big(m_\eta(h_{n+1}, h_{n+2})\big),$$

which is a contradiction. Thus, $m_\eta(h_{n+1}, h_{n+2}) \leqslant m_\eta(h_n, h_{n+1})$.

Since the sequence $m_\eta(h_n, h_{n+1}) \geqslant 0$ is nonincreasing, it has a limit

$$\lim_{n\to\infty} m_\eta(h_n, h_{n+1}) = l \geqslant 0.$$

Taking the limit of both sides in (5) yields

$$\psi(l) \leqslant \gamma\left(\max\left\{l, \frac{l}{1+l}(1+l)\right\}\right) = \gamma(l),$$

which gives a contradiction. Hence, $l = 0$, that is, $\lim_{n\to\infty} m_\lambda(h_n, h_{n+1}) = 0$.

Here we shall prove that $\{h_n\}$ is m-Cauchy. Suppose, on the contrary, $\{h_n\}$ is not m-Cauchy. This means that for $\epsilon > 0$, we can find two subsequences $\{h_{n_k}\}$ and $\{h_{m_k}\}$ of $\{h_n\}$ satisfying $n_k > m_k > k$ such that n_k is the smallest index for which

$$m_\eta(h_{m_k}, h_{n_k}) \geqslant \epsilon, \qquad m_\eta(h_{m_k}, h_{n_{k-1}}) < \epsilon$$

for all $\eta > 0$. Thereby,

$$\begin{aligned}\epsilon &\leqslant m_{4\eta}(h_{m_k}, h_{n_k}) \\ &\leqslant m_{2\eta}(h_{m_k}, h_{m_{k+1}}) + m_\eta(h_{m_{k+1}}, h_{n_{k+1}}) + m_\eta(h_{n_{k+1}}, h_{n_k}).\end{aligned} \tag{7}$$

Taking the limit of both sides in (7) gives $\lim_{k\to\infty} m_\eta(h_{m_{k+1}}, h_{n_{k+1}}) \geqslant \epsilon$. We can write

$$m_\eta(h_{m_k}, h_{n_k}) \leqslant w_{\eta/2}(h_{m_k}, h_{n_{k-1}}) + w_{\eta/2}(h_{n_{k-1}}, h_{n_k}).$$

Then we obtain $\lim_{k\to\infty} m_\eta(h_{m_k}, h_{n_k}) \leqslant \epsilon$. Similarly,

$$m_\eta(h_{m_k}, h_{n_{k+1}}) \leqslant w_{\eta/2}(h_{m_k}, h_{n_{k-1}}) + w_{\eta/4}(h_{n_{k-1}}, h_{n_k}) + w_{\eta/4}(h_{n_k}, h_{n_{k+1}}).$$

As $k \to \infty$, we get $\lim_{k\to\infty} m_\eta(h_{m_k}, h_{n_{k+1}}) \leqslant \epsilon$.

If we put $x_1 = h_{n_k}$ and $x_2 = h_{m_k}$ in (2), then

$$\frac{1}{2} m_\eta(h_{n_k}, Fh_{n_k}) = \frac{1}{2} m_\eta(h_{n_k}, h_{n_{k+1}}) \leqslant m_\eta(h_{n_k}, h_{m_k})$$

is satisfied. This implies that

$$\begin{aligned}&\psi\bigl(m_\eta(Fh_{n_k}, Fh_{m_k})\bigr) \\ &\quad= \psi\bigl(m_\eta(h_{n_{k+1}}, Fh_{m_{k+1}})\bigr), \\ &\quad\leqslant \gamma\biggl(\max\biggl\{m_\eta(h_{n_k}, h_{m_k}), \frac{m_\eta(h_{n_k}, Fh_{n_k})(1 + m_\eta(h_{m_k}, Fh_{m_k}))}{1 + m_\eta(h_{n_k}, h_{m_k})}, \\ &\qquad\qquad \frac{m_\eta(h_{m_k}, Fh_{n_k})(1 + m_\eta(h_{m_k}, Fh_{n_k}))}{1 + m_\eta(h_{n_k}, h_{m_k})}\biggr\}\biggr).\end{aligned}$$

As $k \to \infty$, we obtain

$$\psi(\epsilon) \leqslant \lim_{k\to\infty} \psi\bigl(m_\eta(h_{n_{k+1}}, h_{m_{k+1}})\bigr) \leqslant \gamma\biggl(\max\biggl\{\epsilon, \frac{\epsilon}{1+\epsilon}(1+\epsilon)\biggr\}\biggr) \leqslant \gamma(\epsilon),$$

a contradiction. $\{h_n\}$ is, therefore, a m-Cauchy sequence on a m-complete Ω_m^*. Then the sequence $\{h_n\}$ is m-convergent to some $h \in \Omega_m^*$, that is, $\lim_{n\to\infty} m_\eta(h_n, h) = 0$. This limit is unique since m is a strict MM.

Now it will be proved that h yields $Fh = h$. Indeed, putting $\tau_0 = h_n$ and $\tau_1 = h$ in (2) gives

$$\frac{1}{2} m_\eta(h_n, Fh_n) = \frac{1}{2} m_\eta(h_n, h_{n+1}) \leqslant m_\eta(h_{n_k}, h).$$

This implies that

$$\begin{aligned}\psi\big(m_\eta(Fh_n,Fh)\big) &= \psi\big(m_\eta(h_{n+1},Fh)\big),\\ &\leqslant \gamma\bigg(\max\bigg\{m_\eta(h_n,h),\, \frac{m_\eta(h_n,Fh_n)(1+m_\eta(h,Fh))}{1+m_\eta(h_n,h)},\\ &\qquad \frac{m_\eta(h,Fh_n)(1+m_\eta(h,Fh_n))}{1+m_\eta(h_n,h)}\bigg\}\bigg),\\ &\leqslant \gamma\bigg(\max\bigg\{m_\eta(h_n,h),\, \frac{m_\eta(h_n,h_{n+1})(1+m_\eta(h,Fh))}{1+m_\eta(h_n,h)},\\ &\qquad \frac{m_\eta(h,h_{n+1})(1+m_\eta(h,h_{n+1}))}{1+m_\eta(h_n,h)}\bigg\}\bigg).\end{aligned}$$

As $n \to \infty$, we get

$$\psi\big(m_\eta(h,Fh)\big) \leqslant \gamma(0) < \psi(0).$$

Consequently, we obtain $m_\eta(h,Fh)=0$. Since m is a strict MM, we have $Fh=h$.

Now we will demonstrate that F is of exactly one fixed point. To prove this, take two fixed points of F as $h,g \in \Omega_m^*$ such that $h \neq g$. For $h,g \in \Omega_m^*$, $Fh=h$ and $Fg=g$ hold.

Putting $\tau_0=h$ and $\tau_1=g$ in (2) yields

$$0=\frac{1}{2}m_\eta(h,Fh)=\frac{1}{2}m_\eta(h,h) \leqslant m_\eta(h,g).$$

It implies that

$$\begin{aligned}\psi\big(m_\eta(Fh,Fg)\big) &= \psi\big(m_\eta(h,g)\big),\\ &\leqslant \gamma\bigg(\max\bigg\{m_\eta(h,g),\, \frac{m_\eta(h,Fh)(1+m_\eta(g,Fg))}{1+m_\eta(h,g)},\\ &\qquad \frac{m_\eta(g,Fh)(1+m_\eta(g,Fh))}{1+m_\eta(h,g)}\bigg\}\bigg),\\ &\leqslant \gamma\bigg(\max\bigg\{m_\eta(h,g),\, \frac{m_\eta(h,h)(1+m_\eta(g,g))}{1+m_\eta(h,g)},\\ &\qquad \frac{m_\eta(g,h)(1+m_\eta(g,h)}{1+m_\eta(h,g)}\bigg\}\bigg),\\ &= \gamma\big(m_\eta(h,g)\big).\end{aligned}$$

Since, for all $\tau>0$, $\psi(\tau)>\gamma(\tau)$, then $m_\eta(h,g)=0$. Hence, we have $h=g$ because m is a strict MM. This completes the proof. $\square$

If we take $M(\tau_1,\tau_2)=M_2(\tau_1,\tau_2)$ in (2), the following theorem can be proved in a similar way with Theorem 3.

Theorem 4. *Suppose that the modular m_η is strict and convex MM on Ω and Ω_m^* is m-complete. Assume that $F : \Omega_m^* \to \Omega_m^*$ is holding condition* (2) *for $M(\tau_0, \tau_1) = M_2(\tau_0, \tau_1)$ with $\psi(\tau) > \gamma(\tau)$ for all $\tau > 0$, where $\psi \in \Psi$, and $\gamma : [0, \infty) \to [0, \infty)$ is continuous. If $m_\eta(h, Fh) < \infty$ yields for $h \in \Omega_m^*$ and all $\eta > 0$, then F is of a fixed point, which belongs to Ω_m^*. Moreover, providing that $m_\eta(h, g) < \infty$ for all $h, g \in \Omega_m^*$ and $\eta > 0$, then F is of exactly one fixed point.*

Now we give an example in MMS, which is not a metric space. The purpose here is to illustrate that our results work in the difference of MMS from metric spaces.

Example 1. Let us take $\Omega = [-1, 1]$ and the mapping $F\tau = -\tau$. We consider the modular metric space $\Omega_m^* = \{\tau \in \Omega\colon \exists \eta = \eta(\tau) > 0 \text{ s.t. } m_\eta(\tau, \tau_0) < \infty\}$. In Ω_m^*, we take the strict and convex modular metric $m_\eta(\tau_0, \tau_1) = \mathrm{e}^{-\eta}|\tau_0 - \tau_1| + |\tau_0| + |\tau_1|$ that is not metric. Notice that F maps Ω_m^* into itself, i.e., $F : \Omega_m^* \to \Omega_m^*$.

First of all, we show that F satisfies condition (2), where $M(\tau_0, \tau_1) = M_1(\tau_0, \tau_1)$ with $\psi(\tau) > \gamma(\tau)$ for all $\tau > 0$:

$$\begin{aligned}
\frac{1}{2} m_\eta(\tau_0, F\tau_0) = \frac{1}{2} m_\eta(\tau_0, -\tau_0) &= \frac{1}{2}\bigl(\mathrm{e}^{-\eta}|\tau_0 + \tau_0| + |\tau_0| + |-\tau_0|\bigr),\\
&= \mathrm{e}^{-\eta}|\tau_0| + |\tau_0| \leqslant \mathrm{e}^{-\eta}|\tau_0 - \tau_1 + \tau_1| + |\tau_0|,\\
&\leqslant \mathrm{e}^{-\eta}|\tau_0 - \tau_1| + |\tau_0| + |\tau_1| = m_\eta(\tau_0, \tau_1).
\end{aligned}$$

Now we compute $m_\eta(F\tau_0, F\tau_1)$ and $M_1(\tau_0, \tau_1)$ separately. For $m_\eta(F\tau_0, F\tau_1)$, we can write

$$\begin{aligned}
m_\eta(F\tau_0, F\tau_1) &= m_\eta(-\tau_0, -\tau_1)\\
&= \mathrm{e}^{-\eta}|\tau_1 - \tau_0| + |-\tau_0| + |-\tau_1|\\
&= \mathrm{e}^{-\eta}|\tau_0 - \tau_1| + |\tau_0| + |\tau_1|.
\end{aligned}$$

Since $\mathrm{e}^{-\eta} \leqslant 1$, we can write

$$\begin{aligned}
m_\eta(F\tau_0, F\tau_1) &\leqslant |\tau_0 - \tau_1| + |\tau_0| + |\tau_1|\\
&\leqslant |\tau_0| + |\tau_1| + |\tau_0| + |\tau_1|\\
&\leqslant 4.
\end{aligned}$$

Since $\tau_0, \tau_1 \in [-1, 1]$, we get $m_\eta(F\tau_0, F\tau_1) \leqslant 4$. For $M_1(\tau_0, \tau_1)$ given in (3), we can write

$$\begin{aligned}
M_1(\tau_0, \tau_1) = \max\biggl\{&\mathrm{e}^{-\eta}|\tau_0 - \tau_1| + |\tau_0| + |\tau_1|,\\
&\frac{(\mathrm{e}^{-\eta}|\tau_0 + \tau_0| + |\tau_0| + |\tau_0|)(1 + \mathrm{e}^{-\eta}|\tau_1 + \tau_1| + |\tau_1| + |\tau_1|)}{1 + \mathrm{e}^{-\eta}|\tau_0 - \tau_1| + |\tau_0| + |\tau_1|},\\
&\frac{(\mathrm{e}^{-\eta}|\tau_1 + \tau_0| + |\tau_1| + |\tau_0|)(1 + \mathrm{e}^{-\eta}|\tau_1 + \tau_0| + |\tau_1| + |\tau_0|)}{1 + \mathrm{e}^{-\eta}|\tau_0 - \tau_1| + |\tau_0| + |\tau_1|}\biggr\}.
\end{aligned}$$

Since $\mathrm{e}^{-\eta} \leqslant 1$ and $1 + \mathrm{e}^{-\eta}|\tau_0 - \tau_1| + |\tau_0| + |\tau_1| > 1$, we have

$$M_1(\tau_0, \tau_1) \leqslant \max\bigl\{2\bigl(|\tau_0| + |\tau_1|\bigr), 4|\tau_0|\bigl(1 + 4|\tau_1|\bigr), 2\bigl(|\tau_0| + |\tau_1|\bigr)\bigl(1 + 2\bigl(|\tau_0| + |\tau_1|\bigr)\bigr)\bigr\}.$$

Since $\tau_0, \tau_1 \in [-1, 1]$, we obtain $M_1(\tau_0, \tau_1) \leqslant 20$.

Now we take the functions $\psi(\tau) = \tau$ and $\gamma(\tau) = \tau/2$ such that $\psi(\tau) > \gamma(\tau)$ for all $\tau > 0$. It is clear that $\psi \in \Psi$ and $\gamma : [0, \infty) \to [0, \infty)$ is continuous. Therefore, we obtain $\psi(4) = 4 \leqslant \gamma(20) = 10$. This means that $m_\eta(\tau_0, F\tau_0)/2 \leqslant m_\eta(\tau_0, \tau_1)$ implies that $\psi(m_\eta(F\tau_0, F\tau_1)) \leqslant \gamma(M_1(\tau_0, \tau_1))$. F hence satisfies condition (2).

If there exists an element $h_0 \in \Omega_m^*$ satisfying $m_\eta(h_0, Fh_0) < \infty$, then F has et least one fixed point. Indeed,

$$m_\eta(h_0, Fh_0) = m_\eta(h_0, -h_0) = 2\bigl(\mathrm{e}^{-\eta}|h_0| + |h_0|\bigr) \leqslant 4|h_0| \leqslant 4 < \infty$$

is satisfied. So F has a fixed point.

For the uniqueness of the fixed point, we show that for all $h, g \in \Omega_m^*$, $m_\eta(h, g) < \infty$ holds. Indeed,

$$m_\eta(h, g) = \mathrm{e}^{-\eta}|h - g| + |h| + |g| \leqslant 2\bigl(|h| + |g|\bigr) \leqslant 4 < \infty$$

holds. Consequently, all hypothesis of Theorem 3 are yielded. Hence, F is of exactly one fixed point, which is 0, i.e., $F0 = 0$.

4 Applications

This part covers some problems for 1st-order differential equations with initial and boundary conditions. The existence and uniqueness of solutions for these equations are investigated. For this aim, we apply the results on fixed points given in Theorems 3 and 4.

(i) Consider the problem below:

$$\begin{aligned} &z'(t) = \zeta\bigl(t, z(t)\bigr), \quad t \in [0, T], \\ &z(0) = -z(T), \end{aligned} \tag{8}$$

where $\zeta : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a kind of Carathéodory function.

We assume that the following hypotheses yield for both problems:

(H1) For all $z \in \mathbb{R}$, the function $\zeta(t, z(t))$ is Lebesgue-measurable on $[0, T]$, and for any point $z* \in \mathbb{R}$, $\int_0^T \varphi(|\zeta(s, z*)|/\eta)\, \mathrm{d}s < \infty$ holds.

(H2) There exists $L > 0$ such that $|\zeta(t, z_1) - \zeta(t, z_2)| \leqslant L|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{R}$ and $t \in [0, T]$.

Problem (8) is equivalent to

$$z(t) = \int_0^T G(t, s)\bigl[\zeta\bigl(s, z(s)\bigr) + z(s)\bigr]\, \mathrm{d}t,$$

where the Green function, $G(t,s)$, is defined by

$$G(t,s) = \begin{cases} \frac{e^{L+s-t}}{e^T+1}, & 0 \leqslant s < t \leqslant T, \\ \frac{-e^{s-t}}{e^T+1}, & 0 \leqslant t < s \leqslant T. \end{cases} \tag{9}$$

Now consider continuous, convex, nondecreasing, and unbounded function defined by $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(x) = 0 \Leftrightarrow x = 0$. Let us denote a set of real-valued functions defined on $[0,T]$ by $\Omega' := \{z \mid z : [0,L] \to \mathbb{R}\}$. For all $z_1, z_2 \in \Omega'$ and $\eta > 0$, we consider the following MM $m : (0,\infty) \times \Omega' \times \Omega' \to [0,\infty]$:

$$\begin{aligned} &m_\eta(z_1,z_2) \\ &\quad = \sup_\pi \sum_{j=1}^{i} \varphi\left(\frac{|[z_1(t_j) + z_2(t_{j-1})] - [z_2(t_j) + z_1(t_{j-1})]|}{\eta(t_j - t_{j-1})}\right)(t_j - t_{j-1}), \end{aligned} \tag{10}$$

where $\pi = \{t_j\}_{j=0}^{i}$ are partitions of $[0,T]$. Note that $m_\eta(z_1,z_2)$ is a pseudomodular and, moreover, convex on Ω' [8, 10].

We take Ω'_m as the space of bounded generalized φ-variations mappings given by

$$\Omega'_m = \Omega'_m(z_0) = \bigl\{z \in \Omega' \colon \exists \eta = \eta(z) > 0 \text{ s.t. } m_\eta(z,z_0) < \infty\bigr\},$$

which is a convex pseudomodular metric space, and denoted by $GV_\varphi([0,L])$ [7]. Here

$$z \in \Omega'_m \iff m_\eta(z,z_0) = \sup_\pi \sum_{j=1}^{i} \varphi\left(\frac{|z(t_j) - z(t_{j-1})|}{\eta(t_j - t_{j-1})}\right)(t_j - t_{j-1}) < \infty$$

for $\eta = \eta(z) > 0$.

Define a set for real valued functions with anti-periodic condition on $[0,T]$ as

$$\Omega_1 := \bigl\{z \bigm| z : [0,T] \to \mathbb{R},\ z(0) = -z(T)\bigr\} \subset \Omega'.$$

Note that the MM m_η in (10) is strict and convex on Ω_1.

For problem (8), we take the following MMS:

$$\Omega^*_m := \Omega'_m \cap \Omega_1 = \bigl\{z \in \Omega'_m \colon z(0) = -z(L)\bigr\}. \tag{11}$$

Now we give some lemmas to be utilized in proving the existence and uniqueness of the solution for problem (8).

Lemma 3. (*See* [10, 11].) *Ω^*_m given in* (11) *is m-complete.*

Lemma 4. (*See* [4].) *If φ yields $\varphi(\xi)/\xi \to \infty$ as $\xi \to \infty$, then $m_1(z,0)$ is named as the φ-variation of z. If, in addition, $m_1(z,0) < \infty$, then z has a bounded φ-variation on the interval* $[0,L]$. *Here we may write $m_\eta(z_1,z_2)$ as*

$$m_\eta(z_1,z_2) = m_\eta(z_1 - z_2,\, 0) = m_1\left(\frac{z_1 - z_2}{\eta}, 0\right), \quad \eta > 0.$$

For the function ω on Ω_m^*, we have the following:

$$\begin{aligned}\omega \in \Omega_m^* \quad &\Longleftrightarrow \quad m_\eta(\omega, 0) = m_1\left(\frac{\omega}{\eta}, 0\right) < \infty, \quad \eta > 0 \\ &\Longleftrightarrow \quad \omega \in AC[0,T], \quad \omega' \in L^1[0,T] \\ &\qquad \text{and} \quad m_\eta(\omega, 0) = \int_0^T \varphi\left(\frac{|\omega'(t)|}{\eta}\right) \mathrm{d}t < \infty,\end{aligned}$$

where $AC(I)$ and $L^1(I)$ denote the spaces of absolutely continuous and Lebesgue-integrable functions on I, respectively.

Define the function

$$Fz(t) = \int_0^T G(t,s)\big[\zeta(s,z) + z\big]\,\mathrm{d}s,$$

where $t \in I = [0,T]$, $z \in \Omega_m^*$, and $G(t,s)$ is given in (9). Let z yield $Fz = z$. Then $z \in GV_\varphi(I)$ is a solution of problem (8). Our aim is to get all hypotheses in Theorem 3 satisfied, which gives that (8) is of a unique solution on Ω_w^*.

Lemma 5. (*See* [4].) *If the function* $\zeta(t, z(t))$ *yields conditions* (H1) *and* (H2)*, then* F *maps* Ω_m^* *into itself, i.e.,* $F : \Omega_m^* \to \Omega_m^*$. *Moreover,* F *satisfies* $(Fz)' = \zeta(t,z) + z - Fz$.

Theorem 5. *Suppose that* $z_0, z_1 \in GV_\varphi([0,T])$ *are two functions with* $z_0 \leqslant z_1$ *and* $\zeta(t, z_0(t)) \leqslant z_1'(t)$ *for all* $t \in I = [0,T]$. *If* $\zeta(t, z_0)$ *yields conditions* (H1) *and* (H2)*, then problem* (8) *has a unique solution,* $z_0 \in GV_\varphi(I)$.

Proof. Let $z_0(t) \leqslant z_1(t)$ and $\zeta(t, z_0(t)) \leqslant z_1'(t)$. Then we can write $z_0(t) + \zeta(t, z_0(t)) \leqslant z_1(t) + z_1'(t)$. Moreover, we have

$$Fz_0(t) = \int_0^T G(t,s)\big[f\big(s, z_0(s)\big) + z_0(s)\big]\,\mathrm{d}x \leqslant z_1(t).$$

First of all, we will show $m_\eta(z_0, Fz_0)/2 \leqslant m_\eta(z_0, z_1)$.

$$m_\eta(z_0, Fz_0) = m_\eta(z_0 - Fz_0,\, 0) = \int_0^T \varphi\left(\frac{|(z_0(t) - Fz_0(t))'|}{\eta}\right) \mathrm{d}t.$$

Since $(Fz_0)'(t) = \zeta(t, z_0) + z_0 - Fz_0$ and $z_0' = \zeta(t, z_0)$, $|(z_0(t) - Fz_0(t))'| = |z_0'(t) - (Fz_0)'(t)| = |z_0(t) - Fz_0(t)|$, we have

$$m_\eta(z_0, Fz_0) = \int_0^T \varphi\left(\frac{|z_0(t) - Fz_0(t)|}{\eta}\right) \mathrm{d}t.$$

By $Fz_0(t) \leqslant z_1(t)$ we find $|z_0(t) - Fz_0(t)| \leqslant |z_0(t) - z_1(t)|$. For $|z_0(t) - z_1(t)|$, we can write

$$\begin{aligned}
\big|z_0(t) - z_1(t)\big| &= \left|\int_0^T G(t,s)\big(\big[\zeta\big(s, z_0(s)\big) + z_0(s) - \zeta\big(s, z_1(s)\big) - z_1(s)\big]\big)\,\mathrm{d}s\right|,\\
&\leqslant \int_0^T \big|G(t,s)\big|\big|\big(\big[\zeta\big(s, z_0(s)\big) + z_0(s) - \zeta\big(s, z_1(s)\big) - z_1(s)\big]\big)\big|\,\mathrm{d}s,\\
&\leqslant \big|\big(\big[\zeta\big(s, z_0(s)\big) + z_0(s) - \zeta\big(s, z_1(s)\big) - z_1(s)\big]\big)\big| \int_0^T \big|G(t,s)\big|\,\mathrm{d}s,
\end{aligned}$$

where $\int_0^T |G(t,s)|\,\mathrm{d}s = (\mathrm{e}^T - 1/(\mathrm{e}^T + 1) \leqslant 1$.

Thereby, we can write

$$\begin{aligned}
|z_0 - z_1| &\leqslant \big|\big(\big[\zeta\big(s, z_0(s)\big) + z_0(s) - \zeta\big(s, z_1(s)\big) - z_1(s)\big]\big)\big|,\\
&\leqslant \big|\zeta\big(s, z_0(s)\big) - \zeta\big(s, z_1(s)\big)\big| + \big|z_0(s) - z_1(s)\big|,\\
&\leqslant L|z_0 - z_1| + |z_0 - z_1| = (L+1)|z_0 - z_1|.
\end{aligned}$$

Furthermore, using $|z_0 - z_1| \leqslant \int_0^T |(z_0(s) - z_1(s))'|\,\mathrm{d}s$ gives

$$|z_0 - z_1| \leqslant (L+1)|z_0 - z_1| \leqslant (L+1)\int_0^T \big|\big(z_0(s) - z_1(s)\big)'\big|\,\mathrm{d}s.$$

Since $|z_0(t) - Fz_0(t)| \leqslant |z_0(t) - z_1(t)|$, using monotonicity of φ provides us with

$$\varphi\left(\frac{|(z_0(t) - Fz_0(t))'|}{\eta}\right) \leqslant \varphi\left((L+1)\int_0^T \frac{|(z_0(s) - z_1(s))'|}{\eta}\right)\mathrm{d}s.$$

Let $T(L+1) = a/2 < 1/2$. By Jensen's inequality we acquire

$$\varphi\left((L+1)\int_0^T \frac{|(z_0(s) - z_1(s))'|}{\eta}\right)\mathrm{d}s \leqslant T(L+1)\int_0^T \varphi\left(\frac{|(z_0(s) - z_1(s))'|}{\eta}\right)\mathrm{d}s,$$

and, therefore,

$$m_\eta(z_0, Fz_0) \leqslant T(L+1)m_\eta(z_0 - z_1,\, 0) = T(L+1)m_\eta(z_0, z_1).$$

Then the following holds:

$$\frac{1}{2}m_\eta(z_0, Fz_0) \leqslant m_\eta(z_0, Fz_0) \leqslant m_\eta(z_0, z_1). \tag{12}$$

Now we will show that (12) implies $m_\eta(Fz_0, Fz_1) \leqslant \gamma(M_1(z_0, z_1))$.

$$
\begin{aligned}
&m_\eta(Fz_0, Fz_1)\\
&\quad = m_\eta(Fz_0 - Fz_1, 0) = \int_0^T \varphi\biggl(\frac{|(Fz_0 - Fz_1)'|}{\eta}\biggr)\,\mathrm{d}t,\\
&\quad = \int_0^T \varphi\biggl(\frac{|(\zeta(t, z_0(t)) + z_0(t) - Fz_0(t)) - (\zeta(t, z_1(t)) + z_1(t) - Fz_1(t))|}{\eta}\biggr)\,\mathrm{d}t.
\end{aligned}
$$

Since

$$
\begin{aligned}
|Fz_0 - Fz_1| &= \biggl|\int_0^T G(t,s)\bigl(\bigl[\zeta\bigl(s, z_0(s)\bigr) + z_0(s) - \zeta\bigl(s, z_1(s)\bigr) - z_1(s)\bigr]\bigr)\,\mathrm{d}s\biggr|,\\
&\leqslant \int_0^T \bigl|G(t,s)\bigr|\bigl|\bigl(\bigl[\zeta\bigl(s, z_0(s)\bigr) + z_0(s) - \zeta\bigl(s, z_1(s)\bigr) - z_1(s)\bigr]\bigr)\bigr|\,\mathrm{d}s,\\
&\leqslant \bigl|\bigl(\bigl[\zeta\bigl(s, z_0(s)\bigr) + z_0(s) - \zeta\bigl(s, z_1(s)\bigr) - z_1(s)\bigr]\bigr)\bigr| \int_0^T \bigl|G(t,s)\bigr|\,\mathrm{d}s,\\
&\leqslant \bigl|\zeta\bigl(s, z_0(s)\bigr) - \zeta\bigl(s, z_1(s)\bigr)\bigr| + \bigl|z_0(s) - z_1(s)\bigr|,\\
&\leqslant L|z_0 - z_1| + |z_0 - z_1| = (L+1)|z_0 - z_1|,
\end{aligned}
$$

then we can write

$$
\begin{aligned}
&\bigl|\bigl(\zeta\bigl(t,\ z_0(t) + z_0(t) - Fz_0(t)\bigr)\bigr) - \bigl(\zeta\bigl(t, z_1(t) + z_1(t) - Fz_1(t)\bigr)\bigr)\bigr|,\\
&\quad \leqslant \bigl|\zeta\bigl(t, z_0(t)\bigr) - \zeta\bigl(t, z_1(t)\bigr)\bigr| + \bigl|z_0(t) - z_1(t)\bigr| + \bigl|Fz_0(t) - Fz_1(t)\bigr|,\\
&\quad \leqslant L|z_0 - z_1| + |z_0 - z_1| + (L+1)|z_0 - z_1| = 2(L+1)|z_0 - z_1|,\\
&\quad \leqslant 2(L+1)\int_0^T \bigl|\bigl(z_0(s) - z_1(s)\bigr)'\bigr|\,\mathrm{d}s.
\end{aligned}
$$

Using monotonicity, the convexity of φ and Jensen's inequality give

$$
\begin{aligned}
&\int_0^T \varphi\biggl(\frac{|\zeta(t,\ z_0(t) + z_0(t) - Fz_0(t)) - \zeta(t,\ z_1(t) + z_1(t) - Fz_1(t))|}{\eta}\biggr)\,\mathrm{d}t,\\
&\quad \leqslant \int_0^T \varphi\Biggl(2(L+1)\int_0^T \frac{|(z_0(t) - z_1(t))'|}{\eta}\Biggr)\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}
$$

$$\leqslant 2(L+1)\int_0^T\int_0^T \varphi\Big(\frac{|(z_0(t)-z_1(t))'|}{\eta}\Big)\,\mathrm{d}s\,\mathrm{d}t,$$
$$= 2(L+1)Tm_\eta(z_0-z_1,\,0) = 2T(L+1)m_\eta(z_0,z_1).$$

Hence, we get

$$m_\eta(Fz_0,Fz_1) \leqslant am_\eta(z_0,z_1),$$

where $2T(L+1)=a<1$.

Let $\psi(\xi)=\xi$ and $\gamma(\xi)=a\xi$. Then ψ is in Ψ, and γ is continuous on $[0,\infty)$, yielding $\psi(\xi)>\gamma(\xi)$ for all $\xi>0$. Hence, F yields condition (2) for $M(z_1,z_2)=M_1(z_1,z_2)$.

Now we want to prove that the problem has a unique solution. Since $m_\eta(z_0,Fz_0)\leqslant m_\eta(z_0,z_1)$ from (12), it is adequate to demonstrate that $m_\eta(z_0,z_1)<\infty$ for all $z_0,z_1\in\Omega_m^*$:

$$m_\eta(z_0,z_1) = m_\eta(z_0-z_1,\,0) = \int_0^T \varphi\bigg(\frac{|(z_0(t)-z_1(t))'|}{\eta}\bigg)\,\mathrm{d}t,$$
$$= \int_0^T \varphi\bigg(\frac{|\zeta(t,z_0(t))-\zeta(t,z_1(t))|}{\eta}\bigg)\,\mathrm{d}t.$$

For a point $h_0\in\mathbb{R}$,

$$\begin{aligned}\big|\zeta(t,z_0(t))-\zeta\big(t,z_1(t)\big)\big| &= \big|\zeta\big(t,z_0(t)\big)-\zeta(t,h_0)-\zeta\big(t,z_1(t)\big)+\zeta(t,h_0)\big|,\\ &\leqslant \big|\zeta\big(t,z_0(t)\big)-\zeta(t,h_0)\big|+\big|\zeta\big(t,z_1(t)\big)-\zeta(t,h_0)\big|,\\ &\leqslant L\big|z_0(t)-h_0\big|+L\big|z_1(t)-h_0\big|,\\ &\leqslant L\big|z_0(t)\big|+L\big|z_1(t)\big|+2L|h_0|.\end{aligned}$$

Since $z_0,z_1\in\Omega_m^*$, there exist $\eta_1>0$ and $\eta_2>0$ such that $\int_0^T\varphi(|z_0'(t)|/\eta_1)\,\mathrm{d}t<\infty$ and $\int_0^T\varphi(|z_1'(t)|/\eta_2)\,\mathrm{d}t<\infty$. Taking $\eta_0=LT\eta_1+LT\eta_2+1$ such that $LT\eta_1/\eta_0+LT\eta_2/\eta_0+1/\eta_0=1$ and using the convexity of φ give

$$\begin{aligned}m_\eta(z_0,z_1) &= \int_0^T \varphi\bigg(\frac{|\zeta(t,z_0(t))-\zeta(t,z_1(t))|}{\eta_0}\bigg)\,\mathrm{d}t,\\ &\leqslant \frac{LT\eta_1}{\eta_0}\int_0^T\varphi\bigg(\frac{|z_0'(t)|}{\eta_1}\bigg)\,\mathrm{d}t+\frac{LT\eta_2}{\eta_0}\int_0^T\varphi\bigg(\frac{|z_1'(t)|}{\eta_2}\bigg)\,\mathrm{d}t+\frac{T}{\eta_0}\varphi\big(L|h_0|\big),\\ &<\infty.\end{aligned}$$

All hypotheses of Theorem 3 are satisfied, and hence, the solution of (8) is unique. $\square$

(ii) Consider the following problem:

$$\begin{aligned} &z'(t) = \zeta\bigl(t, z(t)\bigr), \quad t \in [0,1], \\ &z(0) = 0, \end{aligned} \tag{13}$$

where $\zeta : [0,1] \times \mathbb{R} \to \mathbb{R}$ is satisfying (H1) and (H2).

We can write the integral equation for the problem given in (13) as

$$z(t) = \int\limits_0^1 R(t,s)\zeta\bigl(s, z(s)\bigr)\,\mathrm{d}t,$$

where $R(t,s)$ is given as

$$R(t,s) = \begin{cases} 1, & 0 \leqslant s \leqslant t \leqslant 1, \\ 0, & 0 \leqslant t < s \leqslant 1. \end{cases} \tag{14}$$

For problem (13), consider the following set:

$$\Omega_2 := \bigl\{ z \bigm| z : [0,L] \to \mathbb{R},\ z(0) = 0 \bigr\} \subset \Omega'.$$

Lemma 6. *m_η in* (10) *is strict and convex on Ω_2.*

Proof. If $m_\eta(z_0, z_1) = 0 \Rightarrow z_0(t*) = z_1(t*)$ holds for all $z_0, z_1 \in \Omega_2$ and $t* \in [0,1]$, then the proof is done.

From (10) we can write for any $t_0, t_1 \in [0,T]$,

$$\varphi\biggl(\frac{|[z_0(t_0) + z_1(t_1)] - [z_0(t_1) + z_1(t_0)]|}{\eta(t_0 - t_1)}\biggr)(t_0 - t_1) \leqslant m_\eta(z_0, z_1).$$

Then we gain

$$\bigl|\bigl[z_0(t_0) + z_1(t_1)\bigr] - \bigl[z_0\bigl(t_1 + z_1(t_0)\bigr)\bigr]\bigr| \leqslant \varphi^{-1}\biggl(\frac{m_\eta(z_0, z_1)}{t_0 - t_1}\biggr)\eta(t_0 - t_1).$$

Since $\varphi^{-1}(0) = 0$, letting $m_\eta(z_0, z_1) = 0$ gives

$$\bigl|\bigl[z_0(t_0) + z_1(t_1)\bigr] - \bigl[z_0(t_1) + z_1(t_0)\bigr]\bigr| \leqslant \varphi^{-1}(0)\eta(t_0 - t_1) = 0,$$

which implies

$$\bigl[z_0(t_0) + z_1(t_1)\bigr] - \bigl[z_0\bigl(t_1 + z_1(t_0)\bigr)\bigr] = 0.$$

From here we acquire for any $t_0, t_1 \in [0,1]$,

$$z_0(t_0) - z_1(t_0) = z_0(t_1) - z_1(t_1).$$

Letting $t_1 = 0$ and $z_0(0) = z_1(0) = 0$ provides us with

$$z_0(t_0) - z_1(t_0) = z_0(0) - z_1(0) = 0.$$

Then we earn $z_0(t*) = z_1(t*)$ for all $t* \in [0,1]$. $\square$

Here we consider the following MMS Ω_m^{**}, where problem (13) will be studied:

$$\Omega_m^{**} := \Omega_m' \cap \Omega_2 = \{ z \in \Omega_m' \mid z(0) = 0 \}. \tag{15}$$

Lemma 7. *Ω_m^{**} given in* (15) *is m-complete.*

Proof. Take any m-Cauchy sequence $\{h_n\} \subset \Omega_m^{**}$. Then $m_\eta(h_n, h_m) \to 0$ whenever $n, m \to \infty$ for some $\eta = \eta(h_n) > 0$.

We shall show that $\{h_n\}$ is m-convergent in Ω_m^{**}. For $t \in (0,1)$ and $m, n \in \mathbb{N}$, one has

$$\begin{aligned} |h_n(t) - h_m(t)| &= |h_n(t) - h_n(0) - h_m(t) + h_m(0)|, \\ &= |h_n(t) + h_m(0) - (h_m(t) + h_n(0))|, \\ &\leqslant \varphi^{-1}\left(\frac{m_\eta(h_n, h_m)}{t-0}\right)\eta(t-0). \end{aligned} \tag{16}$$

Taking limit in (16) as $m, n \to \infty$ provides

$$\lim_{m,n\to\infty} |h_n(t) - h_m(t)| = 0,$$

and, consequently, $\{h_n\}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $h_n \to h \in \Omega_2$, where $h : [0,1] \to \mathbb{R}$ and $h(0) = 0$. In other words, for all $t \in [0,1]$,

$$\lim_{n\to\infty} |h_n(t) - h(t)| = 0.$$

Now we want to prove that $\lim_{n\to\infty} m_\eta(h_n(t), h(t)) = 0$. Since m_η is semicontinuous, $m_\eta(h_n, h) \leqslant \liminf_{m\to\infty} m_\eta(h_n, h_m)$ for all $n \in \mathbb{N}$. Since h_n is m-Cauchy, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $m_\eta(h_n, h_m) < \epsilon$ for all $n, m \geqslant n_0$. Hence, $\limsup_{m\to\infty} m_\eta(h_n, h_m) \leqslant \sup_{m \geqslant n_0} m_\eta(h_n, h_m) < \epsilon$ for all $n \geqslant n_0$. Then we earn

$$m_\eta(h_n, h) \leqslant \liminf_{m\to\infty} m_\eta(h_n, h_m) \leqslant \limsup_{m\to\infty} m_\eta(h_n, h_m) < \epsilon.$$

Since Ω_m^{**} is closed w.r.t. modular convergence, h_n is m-convergent to $h \in \Omega_m^{**}$. As a result, Ω_m^{**} is m-complete. □

Now consider the following function:

$$\mathcal{H}z(t) = \int_0^1 R(t,s)\zeta(s, z(s))\,\mathrm{d}s, \quad t \in [0,1], \tag{17}$$

where $z \in \Omega_m^{**}$, and $R(t,s)$ is given in (14). If we show that $\mathcal{H}z = z$, then z is a solution of (13). Our goal is to get all hypotheses in Theorem 4 satisfied, which gives the uniqueness of solution to (13) in Ω_m^{**}.

Lemma 8. *If the function $\zeta(\tau, z(\tau))$ yields conditions* (H1) *and* (H2)*, then $\mathcal{H}$ maps Ω_m^{**} into itself, i.e., $\mathcal{H} : \Omega_m^{**} \to \Omega_m^{**}$.*

Proof. We first take any $z \in \Omega_m^{**}$ and $z(0) = 0$. If we prove that $\mathcal{H}z \in \Omega_m^{**}$, then $\mathcal{H} : \Omega_m^{**} \to \Omega_m^{**}$. First of all, since $z(0) = 0$, then $\mathcal{H}z(0) = 0$, and so, $\mathcal{H}z \in \Omega_2$.

For the function $\zeta(\tau, z(\tau))$, we can write the following:

$$\begin{aligned}
\left|\zeta\bigl(\tau, z(\tau)\bigr)\right| &= \left|\zeta\bigl(\tau, z(\tau)\bigr) - f\zeta(\tau, z_*) + \zeta(\tau, z_*)\right|, \\
&\leqslant \left|\zeta\bigl(\tau, z(\tau)\bigr) - \zeta(\tau, z_*)\right| + \left|\zeta(\tau, z_*)\right|, \\
&\leqslant L\left|z(\tau) - z_*\right| + \left|\zeta(\tau, z_*)\right| \leqslant L\left|z(\tau)\right| + L|z_*| + \left|\zeta(\tau, z_*)\right|.
\end{aligned}$$

Since $z(0) = 0$, $z(\tau)$ can be written as $z(\tau) = \int_0^\tau z'(s)\,\mathrm{d}s$ for $\tau \in [0, 1]$. Then we may write

$$\left|\zeta\bigl(\tau, z(\tau)\bigr)\right| \leqslant L \int_0^\tau \left|z'(s)\right| \mathrm{d}s, + \left|\zeta(\tau, z_*)\right| + L|z_*|.$$

Since $z \in \Omega_m^{**}$, $\int_0^T \varphi(|z'(t)|/\eta_3)\,\mathrm{d}t < \infty$ for a given $\eta_3 > 0$. Moreover, (H1) states that $\int_0^T \varphi(|\zeta(\tau, z*)|/\eta_4)\,\mathrm{d}t < \infty$ for $\eta_4 > 0$.

Now we take $\eta_5 = L\eta_3 + \eta_4 + 1$ so that $L\eta_3/\eta_5 + \eta_4/\eta_5 + 1/\eta_5 = 1$. Since $\varphi(\cdot)$ is convex and nondecreasing, then we have

$$\begin{aligned}
\varphi\biggl(\frac{|\zeta(\tau, z(\tau))|}{\eta_5}\biggr) &\leqslant \varphi\biggl(\frac{L\int_0^t |z'(s)|\,\mathrm{d}s + |\zeta(\tau, z_*)| + L|z_*|}{\eta_5}\biggr), \\
&= \varphi\biggl(\frac{L\eta_3}{\eta_5}\frac{|z'(s)|}{\eta_3} + \frac{\eta_4}{\eta_5}\frac{|\zeta(\tau, z_*)|}{\eta_4} + \frac{L}{\eta_5}|z_*|\biggr), \\
&\leqslant \frac{L\eta_3}{\eta_5}\varphi\biggl(\frac{\int_0^\tau |z'(s)|\,\mathrm{d}s}{\eta_3}\biggr) + \frac{\eta_4}{\eta_5}\varphi\biggl(\frac{|\zeta(\tau, z_*)|}{\eta_4}\biggr) + \frac{1}{\eta_5}\varphi\bigl(L|z_*|\bigr). \quad (18)
\end{aligned}$$

Integrating (18) from 0 to 1 gives

$$\begin{aligned}
\int_0^1 \varphi\biggl(\frac{|\zeta(s, z(s))|}{\eta_5}\biggr) \mathrm{d}s \leqslant \frac{L\eta_3}{\eta_5}\int_0^1 \varphi\biggl(\frac{|z'(s)|}{\eta_3}\biggr) \mathrm{d}s + \frac{\eta_4}{\eta_5}\int_0^1 \varphi\biggl(\frac{|\zeta(s, z_*)|}{\eta_4}\biggr) \mathrm{d}s \\
+ \frac{1}{\eta_5}\varphi\bigl(L|z_*|\bigr).
\end{aligned}$$

Hence, we get

$$\int_0^1 \varphi\biggl(\frac{|\zeta(s, z(s))|}{\eta_5}\biggr) \mathrm{d}s \leqslant A < \infty. \quad (19)$$

By using Jensen's inequality we can write

$$\varphi\biggl(\frac{1}{\eta_5}\int_0^1 \left|\zeta\bigl(s, z(s)\bigr)\right| \mathrm{d}s\biggr) \leqslant \int_0^1 \varphi\biggl(\frac{|\zeta(s, z(s))|}{\eta_5}\biggr) \mathrm{d}s. \quad (20)$$

Combining (19) and (20) gives

$$\varphi\left(\frac{1}{\eta_5}\int_0^1 \left|\zeta\left(s, z(s)\right)\right| \mathrm{d}s\right) \leqslant A. \tag{21}$$

From (21) we attain

$$\int_0^1 \left|\zeta\left(s, z(s)\right)\right| \mathrm{d}s \leqslant \eta_5 \varphi^{-1}(A) < \infty,$$

which provides us with $\zeta(\tau, z(\tau)) \in L^1[0,1]$. Since $(\mathcal{H}z)'(\tau) = \zeta(\tau, z(\tau))$, we obtain $(\mathcal{H}z)'(\tau) \in L^1[0,1]$.

Now it is enough to demonstrate that $\mathcal{H}z \in AC[0,1]$. Indeed, we have

$$m_\eta(\mathcal{H}z, 0) = \int_0^1 \varphi\left(\frac{|(\mathcal{H}z)'(s)|}{\eta}\right) \mathrm{d}s = \int_0^1 \varphi\left(\frac{|\zeta(s, z(s))|}{\eta}\right) \mathrm{d}s < \infty,$$

which yields $\mathcal{H}z \in AC[0,1]$. By Lemma 4 $\mathcal{H} \in \Omega_m^{**}$. Therefore, $\mathcal{H} : \Omega_m^{**} \to \Omega_m^{**}$. $\square$

Theorem 6. *Consider the map $\mathcal{H} : \Omega_m^{**} \to \Omega_m^{**}$ given by* (17)*. Suppose that the function $\zeta(t, z(t))$ yields* (H1) *and* (H2)*. Then problem* (13) *is of a unique solution.*

Proof. First of all, we want to show that $\mathcal{H}$ satisfies (2) with $M(z_1, z_2) = M_2(z_1, z_2)$.

For all $z_1, z_2 \in \Omega_m^{**}$, we have

$$\begin{aligned} m_\eta(z_1, \mathcal{H}z_1) &= m_\eta(z_1 - \mathcal{H}z_1,\, 0) \\ &= \int_0^1 \varphi\left(\frac{|(z_1(t) - \mathcal{H}z_1(t))'|}{\eta}\right) \mathrm{d}t = \int_0^1 \varphi\left(\frac{|z_1'(t) - (\mathcal{H}z_1)'(t)|}{\eta}\right) \mathrm{d}t, \\ &= \int_0^1 \varphi\left(\frac{|\zeta(t, z_1(t)) - \zeta(t, z_1(t))}{\eta}\right) \mathrm{d}t = 0. \end{aligned} \tag{22}$$

Then we gain

$$0 = \frac{1}{2} m_\eta(z_1, \mathcal{H}z_1) \leqslant m_\eta(z_1, z_2) \quad \forall z_1, z_2 \in \Omega_m^{**},$$

which implies that $\psi(m_\eta(\mathcal{H}z_1, \mathcal{H}z_2)) \leqslant \gamma(M_2(z_1, z_2))$.

Indeed,

$$\begin{aligned} & m_\eta(\mathcal{H}z_1, \mathcal{H}z_2) \\ &\quad = \int_0^1 \varphi\left(\frac{|(\mathcal{H}z_1(t) - \mathcal{H}u_z(t))'|}{\eta}\right) \mathrm{d}t = \int_0^1 \varphi\left(\frac{|\zeta(t, z_1(t)) - \zeta(t, z_2(t))|}{\eta}\right) \mathrm{d}t, \\ &\quad \leqslant \int_0^1 \varphi\left(\frac{L|z_1(t) - z_2(t)|}{\eta}\right) \mathrm{d}t \leqslant \int_0^1 \varphi\left(L \int_0^1 \frac{|(z_1(s) - z_2(s))'|}{\eta} \mathrm{d}s\right) \mathrm{d}t. \end{aligned} \tag{23}$$

Since $\varphi(\cdot)$ is convex with $\varphi(0) \leqslant 0$, for $a \leqslant 1$, it is known that $\varphi(ax) \leqslant a\varphi(x)$. Letting $L \leqslant 1$ in (23) and by Jensen's inequality, we acquire

$$m_\eta(\mathcal{H}z_1, \mathcal{H}z_2) \leqslant L \int_0^1 \varphi\left(\frac{|(z_1(s) - z_2(s))'|}{\eta}\right) \mathrm{d}s \leqslant L m_\eta(z_1, z_2).$$

Take $\psi(\tau) = \tau$, where $\psi \in \Psi$, and $\gamma(\tau) = L\tau$ with $L \leqslant 1$ so that γ is continuous on $[0, \infty)$. Then $\psi(t) > \gamma(t)$ holds for all $t > 0$. Hence, for $M(z_1, z_2) = M_2(z_1, z_2)$, condition (2) holds.

From (22) we have that $0 = m_\eta(z_1, \mathcal{H}z_1) < \infty$ holds for $\eta > 0$. This gives the existence of a solution for (13).

If for all $z_1, z_2 \in \Omega_m^{**}$, we have $m_\eta(z_1, z_2) < \infty$ accomplished, then the solution is unique.

For this purpose, let us take $z_1, z_2 \in \Omega_m^{**}$ with $z_1(0) = z_2(0) = 0$. Then $z_1 - z_2 \in \Omega_m^{**}$ as $(z_1 - z_2)(0) = z_1(0) - z_2(0) = 0$. Furthermore, since $z_1 - z_2 \in \Omega_m^{**}$ for any $\eta > 0$, we have

$$m_\eta(z_1 - z_2, 0) = \int_0^1 \varphi\left(\frac{|(z_1 - z_2)'(t)|}{\eta}\right) \mathrm{d}t < \infty.$$

Hence, we get

$$m_\eta(z_1, z_2) = m_\eta(z_1 - z_2,\, 0) = \int_0^1 \varphi\left(\frac{|(z_1(s) - z_2(s))'|}{\eta}\right) \mathrm{d}t < \infty.$$

Consequently, we conclude the proof. □

5 Conclusion

We define two generalized C-conditions for (ψ, γ)-mappings in MMS. For these conditions, we state the fixed point theorems and give their proofs. We also showed that the map given in Example 1 has a unique solution in the difference between MMS and metric ones. Furthermore, we consider two Cauchy problems on the MMS (11) and (15), respectively. By using the results obtained in Theorems 3 and 4, we provide the existence and also the uniqueness of the solutions for the considered problems. With these outcomes, we extend the results for the C-condition in metric spaces to MMS. Since the results found herein have not been investigated, these findings are new in the literature.

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