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An immunity-structured SEIRS epidemic model with variable infectivity and spatial heterogeneity

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Abstract. A mathematical model is proposed for the spread of an epidemic disease of age-dependent infectivity through an asexual population with spatial heterogeneity, assuming that some individuals recover from the disease with temporary immunity, another part recover with permanent immunity, and the last part recover with no immunity. The demographic changes such as births and deaths due to natural causes and the chronological age of individuals are not taken into account. The model is based on a system of partial integro-differential equations including a differential equation to describe the evolution of individuals who have recovered with temporary immunity. The existence and uniqueness of the globally defined solution is proved, and its long-time behaviour is studied.

Keywords: epidemic models, coupled parabolic systems, infectivity, immunity, reaction-diffusion systems.

1 Introduction

The objective of this work is to analyse a mathematical model, which describes the spread of an epidemic disease through an asexual population taking into account the spatial dispersal of individuals, infection age (time since infection), and immunity duration after recovery from the disease. Many studies [8, 9, 11], [10, 14–16, 18, 20, 22] have been devoted to study the spread of epidemics in an asexual spatially homogeneous population taking into account the infection-age-dependent infectivity. Epidemic models with spatial diffusion but without age structure are treated in [1–3, 21].

Epidemic models with spatial and infection-age-dependent heterogeneity are studied in [4,5]. According to [4], individuals recover from the disease with permanent immunity, while in work [5], it is assumed that people recover with no immunity. In [4, 5], the population is divided into three classes: susceptible (who are not infected but capable of becoming infected), exposed (who are infected, but during the latent period (time elapsed from an infection moment to infectiousness), are not yet infectious), and infectious (who can transmit the disease to susceptibles through contacts with them). Both of these works assume that the diffusion coefficients of all classes are constants and that the infectivity of the disease is independent of the infection age. In [4], the infectivity is a positive function of the location variable and is independent of the infection age, while in [5], it is a positive constant. The mortality rate is a constant in [5] and it is a function of the location variable and infection age in [4]. In works [4, 5], it is also supposed that the disease spreads over a short enough period of time to disregard demographic changes such as births, natural death, and chronological age of individuals. In works [4,5], the infective class is structured with infection age $\tau \in [0, T]$, $T < \infty$.

In many epidemic diseases, natural immunity after recovery is temporary, and recovered individuals lose their immunity and return to the class of susceptibles after an average protected period. For example, according to [12], natural immunity to HCoV-OC43 and HCoV-HKU1 infection appears to wane within one year, while SARS-CoV-1 infection can induce longer-lasting immunity. In this work, we consider a SEIRS model for the spread of an epidemic disease through an asexual population. As in [4, 5], we disregard births, natural death, and chronological age of individuals and, contrary to works [4, 5], take into account the dependence of the infectivity on the infection age and assume that individuals spread over the Ω habitat bounded with the surface Σ with diffusion coefficients depending on the location variable. We also take into account temporary immunity of recovered individuals. We divide the population into four classes: susceptible, infected, temporary immune (composed of individuals recovered with temporary immunity), and removed (who recovered with permanent immunity). We divide the class of infected individuals into two subclasses: (i) exposed (who are infected, but during the latency period, are not yet infectious) and (ii) infectious (who can transmit the disease to the other individuals through contacts with them). We assume that individuals, who have recovered without any immunity and those whose temporary immunity has ended, immediately return to the class of susceptibles.

Our aim is to study the existence, uniqueness, and long-time behaviour of the classical solution to this model for two classes of smoothness of the model data.

The plan of this work is the following. In Section 2, we describe the model. In Section 3, we prove the existence and uniqueness theorem. Section 4 contains the long-time behaviour of the solution for the model data class given in Section 2. In Section 5, we consider the model with improved smoothness of the model data, prove the existence and uniqueness theorem, and find the long-time behaviour of its classical solution. Some remarks in Section 6 conclude the paper.

2 The model

Let $S = S(x, t)$ denote the density of susceptible individuals at time t at the position $x \in \Omega \subset \mathbb{R}^n$, $n \geqslant 2$, and let $I = I(x, t, \tau)$ and $R = R(x, t, \tau_1)$ be densities of the infected (exposed with $\tau \in [0, \tau_*]$ and infectious with $\tau \in (\tau_*, T]$) and recovered individuals with the disease and immunity age τ and τ_1, respectively, at the position $x \in \Omega$ at time t. Here $\tau \in [0, T]$ is time passed since the infection moment, and T is the disease infectivity period, τ_* is the latent period, $\tau_1 \in [0, T_1]$ is time past since the recovery moment with temporary immunity, and T_1 is the period of the temporary immunity. Assume that $R_1 = R_1(x,t)$ is the density of the individuals recovered with permanent immunity at time t at the position $x \in \Omega$. Let $u = u(x,t)$ be the infection rate at time t at the position $x \in \Omega$ (rate at which susceptibles catch the disease, which infectivity is $k = k(x,\tau)$, $x \in \Omega$), and let $\gamma I(\cdot,\cdot,T)$ denote the rate at which individuals recovered without any immunity enter the susceptible class. Similarly, assume that $\gamma_1 I(\cdot,\cdot,T)$ is the rate at which recovered individuals enter the class of individuals recovered with temporary immunity. Let $(1-\gamma-\gamma_1)I(\cdot,\cdot,T)$ denote the rate at which the recovered individuals enter the class of individuals recovered with permanent immunity. We also assume that all individuals after the expiration of the protected period T_1 immediately return to the susceptible class and that $\nu = \nu(\tau)$ means disease mortality. Our model consists of the following coupled systems:

$$\begin{aligned}
&\partial_t S - \operatorname{div}(\kappa_s \nabla S) = \gamma I(\cdot,\cdot,T) + R(\cdot,\cdot,T_1) - uS \quad \text{in } \Omega \times (0,\infty),\\
&\partial_{\mathbf{n}} S = 0 \quad \text{on } \Sigma \times (0,\infty),\\
&S(\cdot,0) = S_0 \quad \text{in } \overline{\Omega},
\end{aligned} \tag{1}$$

$$\begin{aligned}
&\partial_t I + \partial_\tau I - \operatorname{div}(\kappa_i \nabla I) = -\nu I \quad \text{in } \Omega \times (0,\infty) \times (0,T],\\
&\partial_{\mathbf{n}} I = 0 \quad \text{on } \Sigma \times (0,\infty) \times (0,T],\\
&I(\cdot,0,\cdot) = I_0 \quad \text{in } \overline{\Omega} \times [0,T],\\
&I(\cdot,\cdot,0) = uS \quad \text{in } \overline{\Omega} \times [0,\infty),
\end{aligned} \tag{2}$$

$$\begin{aligned}
&\partial_t R + \partial_{\tau_1} R - \operatorname{div}(\kappa_1 \nabla R) = 0 \quad \text{in } \Omega \times (0,\infty) \times (0,T_1],\\
&\partial_{\mathbf{n}} R = 0 \quad \text{on } \Sigma \times (0,\infty) \times (0,T_1],\\
&R(\cdot,0,\cdot) = R_0 \quad \text{in } \overline{\Omega} \times [0,T_1],\\
&R(\cdot,\cdot,0) = \gamma_1 I(\cdot,\cdot,T) \quad \text{in } \overline{\Omega} \times [0,\infty),
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\partial_t R_1 - \operatorname{div} \kappa_1 \nabla R_1 = (1-\gamma-\gamma_1) I(\cdot,\cdot,T) \quad \text{in } \Omega \times (0,\infty),\\
&\partial_{\mathbf{n}} R_1 = 0 \quad \text{on } \Sigma \times (0,\infty),\\
&R_1(\cdot,0) = R_{10} \quad \text{in } \overline{\Omega},
\end{aligned} \tag{4}$$

$$u = \int_{\tau_*}^{T} k(\cdot,\tau) I(\cdot,\cdot,\tau)\,\mathrm{d}\tau \quad \text{in } \overline{\Omega} \times [0,\infty), \tag{5}$$

where $\Sigma = \partial\Omega$, T, T_1, γ, γ_1 are positive constants such that $T < T_1$, $\gamma + \gamma_1 < 1$, ∂_t, ∂_τ, and ∂_{τ_1} stand for partial derivatives, $\partial_{\mathbf{n}}$ with $\mathbf{n} = \mathbf{n}(\mathbf{x})$, $x \in \Sigma$, denotes the outward normal derivative, ∇ and div are the gradient and divergence operators, $\kappa_s = \kappa_s(x)$, $\kappa_i = \kappa_i(x)$, and $\kappa_1 = \kappa_1(x)$ denote the diffusivity of the susceptible, infected (exposed and infectious), and recovered individuals with temporary and permanent immunity, respectively, $S_0 = S_0(x)$, $I_0 = I_0(x,\tau)$, $R_0 = R_0(x,\tau_1)$, $R_{10} = R_{10}(x)$ are the initial functions. Condition $\partial_{\mathbf{n}} f|_{\Sigma} = 0$, where $f = S, I, R$, and R_1 ensure that the population remains confined to Ω for all time.

The main novelty of our model is the introduction of a class of individuals of density $R(x, t, \tau_1)$ whose age of temporary immunity at moment t at point x is $\tau_1 \in [0, T_1]$, where T_1 is a finite period of temporary immunity, and $R(x, t, \tau_1)$ satisfies system (3).

Knowing the model data κ_s, κ_i, κ_1, k, γ, γ_1, T, T_1, τ_*, and initial functions S_0, I_0, R_0, equations (1)–(3) and (5) can be applied, for example, to model the spread of COVID-19 or influenza infection in a human population.

It is trivial to observe that Eqs. (1)–(3), (5) decouple from system (4). Since individuals of the removed class do not affect the development of the disease, and the determination of I and the initial function R_{10} completely determine R_1, we shall not consider system (4) further.

We add to system (1)–(3) and (5) the following compatibility conditions:

$$I_0(\cdot, 0) = S_0 \int_0^T k(\cdot, \tau) I_0(\cdot, \tau)\, \mathrm{d}\tau, \quad R_0(\cdot, 0) = \gamma_1 I_0(\cdot, T) \quad \text{in } \overline{\Omega}.$$

Set

$$I(x, t, \tau) = F(x, t, \tau) \begin{cases} \exp\{-\int_{\tau-t}^{\tau} \nu(s)\, \mathrm{d}s\}, & x \in \overline{\Omega},\ 0 \leqslant t \leqslant \tau \leqslant T, \\ \exp\{-\int_0^{\tau} \nu(s)\, \mathrm{d}s\}, & x \in \overline{\Omega},\ t \geqslant \tau \in [0, T], \end{cases} \tag{6}$$

and insert function (6) into (2) to get

$$\begin{aligned} &\partial_t F + \partial_\tau F = \operatorname{div}(\kappa_i \nabla F) \quad \text{in } \Omega \times (0, \infty) \times (0, T], \\ &\partial_{\mathbf{n}} F = 0 \quad \text{on } \Sigma \times (0, \infty) \times (0, T], \\ &F(\cdot, 0, \cdot) = I_0 \quad \text{in } \overline{\Omega} \times [0, T], \\ &F(\cdot, \cdot, 0) = uS \quad \text{in } \overline{\Omega} \times [0, \infty). \end{aligned} \tag{7}$$

Inserting function (6) into Eq. (5), we have

$$u(\cdot, t) = \begin{cases} \int_{\tau^*}^{T} k(\cdot, \tau) \exp\{-\int_{\tau-t}^{\tau} \nu(s)\, \mathrm{d}s\} F(\cdot, t, \tau)\, \mathrm{d}\tau, & t \in [0, \tau^*], \\ \int_{\tau^*}^{t} k(\cdot, \tau) \exp\{-\int_0^{\tau} \nu(s)\, \mathrm{d}s\} F(\cdot, t, \tau)\, \mathrm{d}\tau & \\ \quad + \int_t^T k(\cdot, \tau) \exp\{-\int_{\tau-t}^{\tau} \nu(s)\, \mathrm{d}s\} F(\cdot, t, \tau)\, \mathrm{d}\tau, & t \in (\tau^*, T], \\ \int_{\tau^*}^{T} k(\cdot, \tau) \exp\{-\int_0^{\tau} \nu(s)\, \mathrm{d}s\} F(\cdot, t, \tau)\, \mathrm{d}\tau, & t \geqslant T, \end{cases} \tag{8}$$

in $\overline{\Omega}$. Knowing F, we can find I by Eq. (6) and then construct u by Eq. (8).

Let a constant $\beta \in (0, 1)$ and assume that the surface $\Sigma \in C^2$ and given functions S_0, I_0, R_0, κ_s, κ_i, κ_1, ν, k satisfy the following conditions of smoothness (called (H_1) hypotheses):

(i) $S_0 \in C(\overline{\Omega})$, $S_0(x) \geqslant 0$ in $\overline{\Omega}$ and is continuously differentiable in a neighbourhood of surface Σ excluding the Σ itself.

(ii) $I_0 \in C^{0,1}(\overline{\Omega} \times [0,T])$ and $R_0 \in C^{0,1}(\overline{\Omega} \times [0,T_1])$ are nonnegative and continuously differentiable in x in a neighbourhood of surface Σ excluding the surface Σ itself.

(iii) $\kappa_s, \kappa_i, \kappa_1 \in C^{1+\beta}(\overline{\Omega})$ and are positive in $\overline{\Omega}$.

(iv) $\nu \in C([0,T])$ and is nonnegative, $k \in C^{1,1}(\overline{\Omega} \times [\tau^*,T]) \cap C(\overline{\Omega} \times [\tau^*,T])$ and is positive.

Definition 1. Collection (S,I,R,u) is called a classical solution of problem (1)–(3), (5) if $S \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty)])$, $\partial_{\mathbf{n}} S$ is continuous on $\Sigma \times (0,\infty)$, $I \in C^{2,1,1}(\Omega \times (((0,\infty) \times (0,T]) \setminus \{t=\tau\})) \cap C(\overline{\Omega} \times [0,\infty) \times [0,T])$, $\partial_{\mathbf{n}} I$ is continuous on $\Sigma \times (0,\infty) \times (0,T]$, $R \in C^{2,1,1}(\overline{\Omega} \times (((0,\infty) \times (0,T_1]) \setminus \{t=\tau_1\})) \cap C(\overline{\Omega} \times [0,\infty) \times [0,T_1])$, $\partial_{\mathbf{n}} R$ is continuous on $\Sigma \times (0,\infty) \times (0,T_1]$, $u \in C^{1,1}(\Omega \times (0,\infty) \cap C(\overline{\Omega} \times [0,\infty))$ and if this collection satisfies equations (1)–(3), (5) and their initial and boundary conditions.

We also use the definition of the classical solution to system (1), (3), (6), (7), (8).

Definition 2. Collection (S,F,R,u) is called a classical solution of problem (1), (3), (7), (8) if $S \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty)])$, $\partial_{\mathbf{n}} S$ is continuous on $\Sigma \times (0,\infty)$, $F \in C^{2,1,1}(\Omega \times (((0,\infty) \times (0,T]) \setminus \{t=\tau\})) \cap C(\overline{\Omega} \times [0,\infty) \times [0,T])$, $\partial_{\mathbf{n}} F$ is continuous on $\Sigma \times (0,\infty) \times (0,T]$, $R \in C^{2,1,1}(\overline{\Omega} \times (((0,\infty) \times (0,T_1]) \setminus \{t=\tau_1\})) \cap C(\overline{\Omega} \times [0,\infty) \times [0,T_1])$, $\partial_{\mathbf{n}} R$ is continuous on $\Sigma \times (0,\infty) \times (0,T_1]$, $u \in C^{1,1}(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$ and if this collection satisfies equations (1), (3), (7), and (8) and their initial and boundary conditions.

3 Existence, uniqueness, and estimates of the classical solution to system (1)–(3), (5)

Consider the linear parabolic system

$$\begin{aligned} &\partial_t f - \operatorname{div}(\varphi \nabla f) + cf = q \quad \text{in } \Omega \times (0,t^*],\ t^* < \infty, \\ &\partial_{\mathbf{n}} f = \psi \quad \text{on } \Sigma \times (0,t^*], \\ &f|_{t=0} = f_0 \quad \text{in } \overline{\Omega}, \end{aligned} \tag{9}$$

where $f = f(x,t)$, $q = q(x,t)$, $\psi = \psi(x,t)$, $c = c(x,t)$, $\varphi = \varphi(x)$, and $f_0 = f_0(x)$ are given functions. We apply to this system a well-known result on the existence and uniqueness of the classical solution of linear parabolic equations. Let $\Gamma = \Gamma(\xi,x;t,t')$ be the fundamental solution of the differential equation $\partial_t f - \operatorname{div}(\varphi \nabla f) + cf = 0$, and let $\mathbf{n}(\xi)$ be a unit outward normal vector to surface Σ at point ξ.

Theorem 1. (*See* [6, *Chap.* V, *Sect.* 3, *Thm.* 2 *and Cor.* 2]). *Let* $\Sigma \in C^{1+\beta}$, $\varphi \in C^{1+\beta}(\overline{\Omega})$ *and is positive in* $\overline{\Omega}$, $c \in C^{\beta,0}(\overline{\Omega} \times [0,t^*])$, $\psi \in C^{0,0}(\Sigma \times [0,t^*])$, $q \in C^{\beta,0}(\overline{\Omega} \times [0,t^*])$, $0 < t^* < \infty$, $f_0 \in C(\Omega_*)$, $\Omega_* \supset \overline{\Omega}$, *and satisfies the condition*

$$\left| \int_{\Omega_*} \frac{\partial \Gamma(\xi,x;t,0)}{\partial \mathbf{n}(\xi)} f_0(x)\,\mathrm{d}x \right| \leqslant C t^{-\epsilon}, \quad \epsilon \in \left(\frac{1}{2}, 1\right),\ C = \text{const}. \tag{10}$$

Then system (9) *has a unique solution* $f \in C^{2,1}(\Omega \times (0, t^*]) \cap C(\overline{\Omega} \times [0, t^*])$*, which is continuous in* x*, uniformly in* $\overline{\Omega} \times [0, t^*]$.

It is shown [6, Chap. V, Sect. 3, Cor. 2] that condition (10) is fulfilled if function f_0 is continuously differentiable in a neighbourhood of the surface $\Sigma \in C^{1+\beta}$. If a surface Σ belongs to the class C^2, then (see [19, Vol. IV, Part II, Chap. II, Sect. 101]) for sufficiently small $\delta > 0$, it is possible to construct surfaces $\Sigma_\delta^\pm$, parallel to Σ, for any point $\xi \in \Sigma$ assigning a point $\bar{\xi} = \xi \pm \delta\mathbf{n}(\xi) \in \Sigma_\delta^\pm$, where $\mathbf{n}(\xi) = \mathbf{n}(\bar{\xi})$. Let $\Omega_\delta = \{x \in \mathbb{R}^n\colon \operatorname{dist}\{x, \Sigma\} < \delta\} \subset \Omega_*$, $\delta > 0$, be a neighbourhood of Σ, and let $\Omega_\delta^+ = \Omega_\delta \setminus \overline{\Omega}$, $\Omega_\delta^- = \Omega_\delta \cap \Omega$ with the surfaces $\partial\Omega_\delta^+ = \Sigma \cup \Sigma_\delta^+$, $\partial\Omega_\delta^- = \Sigma \cup \Sigma_\delta^-$, respectively. The following statement shows that, by increasing the smoothness of the surface Σ, the condition of the continuous differentiability of f_0 in Ω_δ can be weakened.

Lemma 1. *Let* $\Sigma \in C^2$*, and let a nonnegative function* $f_0(y)$ *be continuous in* Ω_* *and continuously differentiable in* $\Omega_\delta \setminus \Sigma$*. Suppose that there exist the normal derivatives of function* $f_0(y)$*,*

$$\lim_{\substack{y=\xi\pm s\mathbf{n}(\xi)\\ s\to+0}} \partial_{\mathbf{n}(\xi)} f_0(y),$$

that are continuous on Σ*. Then*

$$\int\limits_{\Omega_\delta} \frac{\partial\Gamma(\xi, y, t, 0)}{\partial\mathbf{n}(\xi)} f_0(y)\,\mathrm{d}y \leqslant \frac{C}{t^\varepsilon}, \quad C = \text{const},\ \varepsilon \in \left(0, \frac{1}{2}\right),\ \xi \in \Sigma,$$

and this integral is a continuous function on the surface Σ.

Proof. The fundamental solution Γ can be represented by the formula $\Gamma(\xi, y, t, t') = \Gamma_0(\xi - y,\, y, t, t') + \Gamma'(\xi, y, t, t')$, where

$$\Gamma_0(\xi - y,\, y, t, t') = \bigl(4\pi\varphi(y)(t - t')\bigr)^{-n/2} \exp\biggl\{-\frac{|\xi - y|^2}{4\varphi(y)(t - t')}\biggr\}$$

is the principal term of the fundamental solution Γ. Therefore, it is enough to prove the continuity of the integral

$$\int\limits_{\Omega_\delta} \sum_{i=1}^n \frac{\partial\Gamma_0(\xi - y,\, y, t, 0)}{\partial\xi_i} n_i(\xi) f_0(y)\,\mathrm{d}y$$
$$= \int\limits_{\Omega_\delta} \sum_{i=1}^n \frac{\partial\Gamma_0(\xi - y,\, y, t, 0)}{\partial\xi_i} \bigl(n_i(\xi) - n_i(\bar{y}) + n_i(\bar{y})\bigr) f_0(y)\,\mathrm{d}y.$$

It is easy to prove that

$$\biggl|\int\limits_{\Omega_\delta} \sum_{i=1}^n \frac{\partial\Gamma_0(\xi - y,\, y, t, 0)}{\partial\xi_i} \bigl(n_i(\xi) - n_i(\bar{y})\bigr) f_0(y)\,\mathrm{d}y\biggr| \leqslant \frac{c}{t^\varepsilon}, \quad \varepsilon \in \left(0, \frac{1}{2}\right),$$

$$\int\limits_{\Omega_\delta} \sum_{i=1}^{n} \frac{\partial \Gamma_0(\xi - y,\, y, t, 0)}{\partial \xi_i} n_i(\bar{y}) f_0(y)\,\mathrm{d}y$$
$$= \int\limits_{\Omega_\delta} H(\xi - y,\, y, t, 0) f_0(y)\,\mathrm{d}y - \int\limits_{\Omega_\delta} \sum_{i=1}^{n} \frac{\partial \Gamma_0(\xi - y,\, y, t, 0)}{\partial y_i} n_i(\bar{y}) f_0(y)\,\mathrm{d}y,$$

where $\bar{y}$ is a point on the surface Σ, which realizes the distance of the point y to the surface Σ,

$$H(\xi - y,\, y, t, 0) = \left(\frac{|\xi - y|^2}{4t} - \frac{n}{2\varphi(y)}\right) \Gamma_0(\xi - y,\, y, t, 0)\big(\varphi_y(y), n(\bar{y})\big).$$

It is easy to verify that the integral with the kernel H is a bounded and continuous function on the surface Σ. Then

$$\int\limits_{\Omega_\delta} \sum_{i=1}^{n} \frac{\partial \Gamma_0(\xi - y,\, y, t, 0)}{\partial y_i} n_i(\bar{y}) f_0(y)\,\mathrm{d}y$$
$$= \int\limits_{\Omega_\delta^+} \sum_{i=1}^{n} \frac{\partial \Gamma_0(\xi - y, y, t, 0)}{\partial y_i} n_i(\bar{y}) f_0(y)\,\mathrm{d}y + \int\limits_{\Omega_\delta^-} \sum_{i=1}^{n} \frac{\partial \Gamma_0(\xi - y, y, t, 0)}{\partial y_i} n_i(\bar{y}) f_0(y)\,\mathrm{d}y$$
$$= -\int\limits_{\Sigma_\delta^-} \Gamma_0(\xi - \eta, \eta, t, 0) f_0(\eta)\,\mathrm{d}\Sigma_\eta^- + \int\limits_{\Sigma_\delta^+} Z_0(\xi - \eta, \eta, t, 0) f_0(\eta)\,d\Sigma_\eta^+$$
$$- \int\limits_{\Omega_\delta} \sum_{i=1}^{n} \Gamma_0(\xi - y,\, y, t, 0)\big(n_i(\bar{y}) f_0(y)\big)_{y_i}\,\mathrm{d}y.$$

By virtue of the assumption of smoothness of the surface Σ, function $\sum_{i=1}^{n}(n_i(\bar{y}))_{y_i}$ is continuous. Moreover, conditions of lemma show that $\sum_{i=1}^{n} n_i(\bar{y})(f_0(y))_{y_i} = \partial_{\mathbf{n}(\bar{y})} f_0(y)$ converges to a continuous function on the Σ if y approaches the Σ from within Ω or from without Ω. Therefore, the last integral is also a continuous function. The proof is complete. □

In order to obtain a global existence result, we apply the method of steps and Lemma 1. The following proposition gives the existence and uniqueness of the solution to system (1)–(3), (5):

Theorem 2. *Let assumptions* (i)–(iv) *hold. Then system* (1)–(3)*,* (5) *has a unique globally defined nonnegative solution* (S, I, R, u) *such that*

(i) $S \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$,
(ii) $I \in C^{2,1,1}(\Omega \times (((0,\infty) \times (0,T]) \setminus \{t = \tau\})) \cap C(\overline{\Omega} \times [0,\infty) \times [0,T])$,
(iii) $R \in C^{2,1,1}(\Omega \times (((0,\infty) \times (0,T_1]) \setminus \{t = \tau_1\})) \cap C(\overline{\Omega} \times [0,\infty) \times [0,T_1])$,
(iv) $u(x,t) \in C^{1,1}(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$.

Proof. In order to prove the existence of the solution, we first consider system $(7)_{1,2,3}$ for $0 \leqslant t \leqslant \tau$ on the characteristic lines $\tau = t + \alpha$, $\alpha = \text{const} \in [0, T]$, of the operator

$\partial_t + \partial_\tau$ and system $(3)_{1,2,3}$ for $0 \leqslant t \leqslant \tau_1$ on the characteristic lines $\tau_1 = t + \alpha_1$, $\alpha_1 = \text{const} \in [0, T_1]$, of the operator $\partial_t + \partial_{\tau_1}$. Denoting $A(x,t;\alpha) := F(x,t,\, t+\alpha)$ and $A_1(x,t;\alpha_1) := R(x,t,\, t+\alpha_1)$, we have the following equations:

$$\begin{aligned} &\partial_t A(\cdot,\cdot;\alpha) - \operatorname{div}\bigl(\kappa_i \nabla A(\cdot,\cdot;\alpha)\bigr) = 0 \\ &\qquad \text{in } \Omega \times (0, T-\alpha],\ 0 \leqslant \alpha < T, \\ &\partial_{\mathbf{n}} A(\cdot,\cdot;\alpha) = 0 \quad \text{on } \Sigma \times (0, T-\alpha],\ 0 \leqslant \alpha < T, \\ &A(\cdot,0;\alpha) = I_0(x,\alpha) \quad \text{in } \overline{\Omega},\ 0 \leqslant \alpha \leqslant T, \end{aligned} \tag{11}$$

$$\begin{aligned} &\partial_t A_1(\cdot,\cdot;\alpha_1) - \operatorname{div}\bigl(\kappa_1 \nabla A_1(\cdot,\cdot;\alpha_1)\bigr) = 0, \\ &\qquad \text{in } \Omega \times (0, T_1-\alpha_1],\ 0 \leqslant \alpha_1 < T_1, \\ &\partial_{\mathbf{n}} A_1(\cdot,\cdot;\alpha_1) = 0 \quad \text{on } \Sigma \times (0, T_1-\alpha_1],\ 0 \leqslant \alpha_1 < T_1, \\ &A_1(\cdot,0;\alpha_1) = R_0(\cdot,\alpha_1) \quad \text{in } \overline{\Omega},\ 0 \leqslant \alpha_1 \leqslant T_1. \end{aligned} \tag{12}$$

Assumptions (ii) and (iii) of hypotheses (H_1) and Theorem 1 ensure the existence and uniqueness of the solution $A(\cdot,\cdot;\alpha) \in C^{2,1}(\Omega \times (0,\, T-\alpha])$ with $\alpha \in [0,T)$ and $A(\cdot,\cdot;\alpha) \in C(\overline{\Omega} \times [0,\, T-\alpha])$ with $0 \leqslant \alpha \leqslant T$ to system (11). Similarly, Assumptions (ii) and (iii) of hypotheses (H_1) and Theorem 1 guarantee the existence and uniqueness of the solution $A_1(\cdot,\cdot;\alpha_1) \in C^{2,1}(\Omega \times (0,\, T_1-\alpha_1])$ with $\alpha_1 \in [0,T_1)$ and $A_1(\cdot,\cdot;\alpha_1) \in (\overline{\Omega} \times [0, T_1-\alpha_1])$ with $0 \leqslant \alpha_1 \leqslant T_1$ to system (12). Moreover, these solutions can be represented using the potential theory (see [6, Chap. V, Sect. 3]) and are Hölder continuous in x, uniformly in $(\overline{\Omega} \times [0,\, T-\alpha])$ and $(\overline{\Omega} \times [0,\, T_1-\alpha_1])$, respectively (see [17, Chap. 2, Thm. 1.2]). The nonnegativity follows from the positivity lemma (see [17, Chap. 2, Sect. 2.2]).

In order to prove that A and A_1 are continuously differentiable in α and α_1, respectively, we consider system (11) with A and I_0 replaced by $\partial_\alpha A$ and $\partial_\alpha I_0$, respectively, and system (12) with A_1 and R_0 replaced by $\partial_{\alpha_1} A_1$ and $\partial_{\alpha_1} R_0$, respectively. Again, by virtue of assumptions (ii) and (iii) of hypotheses (H_1) and Theorem 1, each of these two new systems has a unique solution $\partial_\alpha A(\cdot,\cdot;\alpha) \in C^{2,1}(\Omega \times (0,\, T-\alpha])$ with $\alpha \in [0,T)$ and $\partial_{\alpha_1} A_1(\cdot,\cdot;\alpha_1) \in C^{2,1}(\Omega \times (0,\, T_1-\alpha_1])$ with $\alpha_1 \in [0,T_1)$. Direct computation shows that function $F = F(x,t,\tau) = A(x,t;\tau-t)$ satisfies equations $(7)_{1,2,3}$ for $\tau - t \geqslant 0$, and that function I defined by $(6)_1$ is a solution to system $(2)_{1,2,3}$ and lies in $C^{2,1,1}(\Omega \times (0,\tau] \times (0,T]) \cap C(\overline{\Omega} \times [0,\tau] \times [0,T])$. Similarly, function $R(\cdot,\cdot,\cdot) = A_1(x,\tau_1;\tau_1-t) \in C^{2,1,1}(\Omega \times (0,\tau_1] \times (0,T_1]) \cap C(\overline{\Omega} \times [0,\tau_1] \times [0,T_1])$ satisfies system $(3)_{1,2,3}$ for $\tau_1 - t \geqslant 0$.

Second, denoting $B(x,\tau;\alpha) := F(x,\, \tau+\alpha,\, \tau)$ and $B_1(x,\tau_1;\alpha_1) := R(x,\, \tau_1+\alpha_1,\, \tau_1)$, we rewrite systems $(7)_{1,2,4}$ and $(3)_{1,2,4}$ on the characteristic lines $t = \alpha + \tau$, $\alpha \geqslant 0$, and $t = \alpha_1 + \tau_1$, $\alpha_1 \geqslant 0$, of the operators $\partial_t + \partial_\tau$ and $\partial_t + \partial_{\tau_1}$, respectively:

$$\begin{aligned} &\partial_\tau B(\cdot,\cdot;\alpha) - \operatorname{div}\bigl(\kappa_i \nabla B(\cdot,\cdot;\alpha)\bigr) = 0 \quad \text{in } \Omega \times (0,T],\ \alpha \geqslant 0, \\ &\partial_{\mathbf{n}} B(\cdot,\cdot;\alpha) = 0 \quad \text{on } \Sigma \times (0,T],\ \alpha \geqslant 0, \\ &B(\cdot,0;\alpha) = S(\cdot,\alpha)u(\cdot,\alpha) \quad \text{in } \overline{\Omega},\ \alpha \geqslant 0, \end{aligned} \tag{13}$$

$$\begin{aligned}
&\partial_\tau B_1(\cdot,\cdot;\alpha_1) - \operatorname{div}\bigl(\kappa_i \nabla B_1(\cdot,\cdot;\alpha_1)\bigr) = 0 \quad \text{in } \Omega\times(0,T_1],\ \alpha_1 \geqslant 0,\\
&\partial_{\mathbf{n}} B_1(\cdot,\cdot;\alpha_1) = 0 \quad \text{on } \Sigma\times(0,T_1],\ \alpha_1 \geqslant 0,\\
&B_1(\cdot,0;\alpha_1) = \gamma_1 I(\cdot,\alpha_1,T) \quad \text{in } \overline{\Omega},\ \alpha_1 \geqslant 0.
\end{aligned} \tag{14}$$

By virtue of Eq. $(6)_1$, the function $I(\cdot,t,\tau) = F(\cdot,t,\tau)\exp\{-\int_{\tau-t}^{\tau}\nu(s)\,\mathrm{d}s\}$, where $F(\cdot,t,\tau) := A(\cdot,t;\tau-t)$ is known for $(t,\tau)\in[0,\tau_*]\times[\tau_*,T]$. Changing variables, function u determined by Eq. (8) can be reduced to

$$u(\cdot,t) = \int\limits_{\tau^*-t}^{T-t} k(\cdot,y+t)\exp\Biggl\{-\int\limits_{y}^{y+t}\nu(s)\,\mathrm{d}s\Biggr\}A(\cdot,t;y)\,\mathrm{d}y, \quad 0\leqslant t\leqslant \tau^*,$$

which shows that u is known for $t\in[0,\tau_*]$ and that $u\in C^{1,1}(\Omega\times(0,\tau^*]\cap C(\overline{\Omega}\times[0,\tau_*])$. Moreover, u is Hölder continuous in x, uniformly in $\overline{\Omega}\times[0,\tau_*]$. Because $I(x,t,T) = A(x,t;T-t)\exp\{-\int_{T-t}^{T}\nu(s)\,\mathrm{d}s\}$ and $R(x,t,T_1) = A_1(x,\tau;T_1-t)$ are Hölder continuous in x, uniformly in $\overline{\Omega}\times[0,T]$ and $\overline{\Omega}\times[0,T_1]$, respectively, assumptions (i) and (iii) of hypotheses (H_1) and Theorem 1 show that system (1) has a unique solution $S\in C^{2,1}(\Omega\times(0,\tau_*])\cap C(\overline{\Omega}\times[0,\tau_*])$, which by virtue of the positivity lemma is nonnegative. Moreover, S is Hölder continuous in x, uniformly in $\overline{\Omega}\times[0,\tau^*]$, and can be represented by the formula (see [6, Chap. V, Sect. 3])

$$\begin{aligned}
S(x,t) &= \int\limits_0^t\int\limits_\Sigma \Gamma(x,t,\xi,s)\mu(\xi,s)\,\mathrm{d}\Sigma_\xi\,\mathrm{d}s\\
&\quad + \int\limits_{\Omega_0}\Gamma(x,t,y,0)S_0(y)\,\mathrm{d}y + \int\limits_0^t\int\limits_\Omega \Gamma(x,t,y,s)\overline{f}(y,s)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned} \tag{15}$$

Here $t\in[0,\tau^*]$, Γ is a fundamental solution of the equation $\partial_t S - \operatorname{div}(\kappa\nabla S) + uS = 0$, $\overline{f}(y,s) = \gamma I(y,s,T) + R(y,s,T_1)$, $\Omega_0\supset\overline{\Omega}$, function S_0 is extended on $\Omega_0\setminus\overline{\Omega}$ preserving the same smoothness, nonnegativity, and notation, μ is a continuous and bounded solution of the integral equation (see [6, Chap. V, Sec. 3])

$$\mu(\xi,t) = \int\limits_0^t\int\limits_\Sigma Q_1(\xi,t,\eta,s)\mu(\eta,s)\,\mathrm{d}\Sigma_\eta\,\mathrm{d}s + \psi(\xi,t),$$

$$\psi(\xi,t) = \int\limits_{\Omega_0} Q_1(\xi,t,y,0)S_0(y)\,\mathrm{d}y + \int\limits_0^t\int\limits_\Omega Q_1(\xi,t,y,s)\overline{f}(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

$$Q_1(\xi,t,\eta,s) = -2\frac{\partial\Gamma(\xi,t,\eta,s)}{\partial\mathbf{n}(\xi)}, \quad \xi\in\Sigma,\ t\in\times[0,\tau^*].$$

We note that function S expressed by formula (15) is defined in the whole space $\mathbb{R}^n$ and is continuously differentiable in $\Omega_0 \setminus \Sigma$. Moreover, it has continuous normal derivatives $\partial_{\mathbf{n}(\bar{y})} S(\bar{y},t) = \lim_{y\to\bar{y}\in\Sigma} \partial_{\mathbf{n}(\bar{y})} S(y,t)$ regardless of whether the variable y approaches Σ along the normal $\mathbf{n}(\bar{y})$ from inside or outside domain Ω. The similar equality for normal derivatives of function $A(\cdot,\cdot;\alpha)$, $\alpha\in[0,T]$, is true.

Next, we consider system (13). For every $\tau\in[0,T]$, we can extend function k on $\Omega_\delta \setminus \overline{\Omega}$ preserving the same smoothness, nonnegativity, and definition (see [13, Chap. IV, Sect. 4]). Since function S expressed by Eq. (15) is defined in Ω_0 and function uS for any fixed $t\in[0,T]$ satisfies conditions of Lemma 1, an application of Theorem 1 shows that linear system (13) has a unique solution $B\in C^{2,1}(\Omega\times(0,T])\cap C(\overline{\Omega}\times[0,T])$, which obviously is nonnegative, Hölder continuous in x, uniformly in $\overline{\Omega}\times[0,T]$, and can be represented by the formula (see [6, Chap. V, Sec. 3])

$$B(x,\tau;\alpha) = \int_0^\tau \int_\Sigma \Gamma(x,\tau,\xi,s)\varphi(\xi,s;\alpha)\,\mathrm{d}\Sigma_\xi\,\mathrm{d}s + \int_{\Omega_0} \Gamma(x,\tau,y,0)\overline{B}_0(y,\alpha)\,\mathrm{d}y,$$

where $\alpha\in[0,\tau^*]$, function $\overline{B}_0(y,\alpha)=u(y,\alpha)S(y,\alpha)$ is continuous in Ω_0 and continuously differentiable in $\overline{\Omega}_0\setminus\Sigma$ for $\alpha\in(0,\tau^*]$, Γ is a fundamental solution of Eq. (13), $\Omega_0\supset\overline{\Omega}$, and φ is a continuous and bounded solution of the integral equation

$$\varphi(\xi,\tau;\alpha) = \int_0^\tau \int_\Sigma Q_1(\xi,\tau,\eta,s)\varphi(\eta,s;\alpha)\,\mathrm{d}\Sigma_\eta\,\mathrm{d}s + \phi(\xi,\tau;\alpha),$$

where $\xi\in\Sigma$, $\tau\in\times[0,T]$,

$$\phi(\xi,\tau;\alpha) = \int_{\Omega_0} Q_1(\xi,\tau,y,0)\overline{B}_0(y,\alpha)\,\mathrm{d}y,\, Q_1(\xi,\tau,\eta,s) = -2\frac{\partial\Gamma(\xi,\tau,\eta,s)}{\partial\mathbf{n}(\xi)}.$$

By arguments used to find the smoothness of $\partial_\alpha A(x,t;\alpha)$, it is easy to see that $\partial_\alpha B(\cdot,\cdot;\alpha)\in C^{2,1}(\Omega\times(0,T])\cap C(\overline{\Omega}\times[0,T])$ with $\alpha\in(0,\tau_*]$. Direct computation shows that function $F=F(x,t,\tau)=B(x,\tau;t-\tau)$ satisfies Eqs. $(7)_{1,2,4}$ and that function I, determined by $(6)_2$ with $F(x,t,\tau)=B(x,\tau;t-\tau)$, is known in $\overline{\Omega}\times[\tau,\,\tau+\tau_*]\times[0,T]$, satisfies Eq. $(2)_{1,2,4}$, and lies in $C^{2,1,1}(\Omega\times(\tau,\,\tau+\tau_*]\times(0,T])\cap C(\overline{\Omega}\times[\tau,\tau+\tau_*]\times[0,T])$. In particular, we have found function $I(\cdot,T)\in C^{2,1}(\Omega\times(T,\,T+\tau_*])\cap C(\overline{\Omega}\times[T,\,T+\tau_*])$. Then by virtue of Eq. (8), function u is known in $(\overline{\Omega}\times[\tau^*,2\tau^*]$ and lies in $C^{1,1}(\Omega\times(\tau_*,2\tau_*])\cap C(\overline{\Omega}\times[\tau_*,2\tau_*])$. Moreover, function u is continuously differentiable in t at $t=\tau^*$ since from Eq. (8) it follows that

$$\lim_{t\to\tau*-0}\partial_t u(\cdot,t) = \lim_{t\to\tau*-0}\int_{\tau^*}^T \partial_t\left(k(\cdot,\tau)\exp\left\{-\int_{\tau-t}^\tau \nu(s)\,\mathrm{d}s\right\}A(\cdot,t;\tau-t)\right)\mathrm{d}\tau$$

and

$$\begin{aligned}\lim_{t\to\tau*+0}\partial_t u(\cdot,t) &= \lim_{t\to\tau*+0}\partial_t\Biggl(\int\limits_{\tau^*}^{t} k(\cdot,\tau)\exp\Biggl\{-\int\limits_0^{\tau}\nu(s)\,\mathrm{d}s\Biggr\}B(\cdot,\tau;t-\tau)\,\mathrm{d}\tau \\ &\quad + \int\limits_t^T k(\cdot,\tau)\exp\Biggl\{-\int\limits_{\tau-t}^{\tau}\nu(s)\,\mathrm{d}s\Biggr\}A(\cdot,t,\tau-t)\,\mathrm{d}\tau\Biggr) \\ &= \lim_{t\to\tau*+0}\int\limits_{\tau_*}^{T}\partial_t\Biggl(k(\cdot,\tau)\exp\Biggl\{-\int\limits_{\tau-t}^{\tau}\nu(s)\,\mathrm{d}s\Biggr\}A(\cdot,t;\tau-t)\Biggr)\,\mathrm{d}\tau.\end{aligned}$$

Arguing similarly as above, we prove the existence and uniqueness of a nonnegative solution $B_1(\cdot,\cdot;\alpha_1)\in C^{2,1}(\Omega\times(0,T_1])\cap C(\overline{\Omega}\times[0,T_1])$ with $\alpha_1\in[0,\tau_*]$ to system (14). By argument above, we can also prove that

$$\partial_\alpha B_1(\cdot,\cdot;\alpha_1)\in C^{2,1}\bigl(\Omega\times(0,T_1]\bigr)\cap C\bigl(\overline{\Omega}\times[0,T_1]\bigr)$$

for $\alpha_1\in(0,\tau_*]$. Direct computation shows that function $R=R(x,t,\tau_1)=B_1(x,\tau_1;t-\tau_1)$ satisfies $(3)_{1,2,4}$. It is easy to see that R lies in $C^{2,1,1}(\Omega\times(\tau_1,\,\tau_1+\tau_*]\times(0,T_1])\cap C(\overline{\Omega}\times[\tau_1,\,\tau_1+\tau_*]\times[0,T_1])$. Hence, $R(\cdot,T_1)\in C^{2,1}(\Omega\times(T_1,\,T_1+\tau_*])\cap C(\overline{\Omega}\times[T_1,\,T_1+\tau_*])$ is known.

Since $u\in C^{1,1}(\Omega\times(0,2\tau_*])\cap C(\overline{\Omega}\times[0,2\tau_*])$ is known, we can find a unique nonnegative function $S\in C^{2,1}(\Omega\times(0,2\tau_*])\cap C(\overline{\Omega}\times[0,2\tau_*])$ from Eqs. (15) and (1). Then from Eqs. (13) we find a unique nonnegative function $B(\cdot,\cdot;\alpha)\in C^{2,1}(\Omega\times(0,T])\cap C(\overline{\Omega}\times[0,T])$ with $\alpha\in[\tau_*,2\tau_*]$, and by Eqs. $(6)_2$ and $(2)_{1,2,4}$ we determine $I\in C^{2,1,1}(\Omega\times(\tau+\tau_*,\,\tau+2\tau_*]\times(0,T])\cap C(\overline{\Omega}\times[\tau+\tau_*,\,\tau+2\tau_*]\times[0,T])$. This allows us to construct function $u\in C^{1,1}(\Omega\times[2\tau_*,3\tau_*])\cap C(\overline{\Omega}\times[2\tau_*,3\tau_*])$.

Continuing this process, we find the solution (S,I,R,u) for $x\in\overline{\Omega}$, $\tau\in[0,T]$, and any $t>0$. Thus we have proved the existence and uniqueness of the solution (S,B,B_1) to Eqs. (1), (13), (14) and the existence of the solution (S,I,R,u) to Eqs. (1)–(3), (5).

The proof of the uniqueness is standard, and we skip it. □

4 Long-time behaviour of the solution (S,I,R,u) to system (1)–(3), (5)

In this section, we show that the total number of infected individuals and the total number of individuals recovered with temporary immunity for any diffusion coefficients eventually tend to zero and that the spatial averages of the infected individuals and of those who recover with temporary immunity asymptotically converge to zero, provided that all diffusion coefficients are equal. Denote

$$\phi(x,t)=\int\limits_0^T F(x,t,\tau)\,\mathrm{d}\tau,\qquad \phi_0(x)=\int\limits_0^T I_0(x,\tau)\,\mathrm{d}\tau,$$

$$\psi(x,t) = \int_0^{T_1} R(x,t,\tau_1)\,\mathrm{d}\tau_1, \qquad \psi_0(x) = \int_0^{T_1} R_0(x,\tau_1)\,\mathrm{d}\tau_1$$

and integrate Eqs. $(7)_1$ and $(3)_1$ over $(0,T)$ and $(0,T_1)$, respectively, to have

$$\begin{aligned} &\partial_t\phi = -F(\cdot,\cdot,T) + uS + \operatorname{div}(\kappa\nabla\phi) \quad \text{in } \Omega\times(0,\infty),\\ &\partial_{\mathbf{n}}\phi = 0 \quad \text{on } \Sigma\times(0,\infty),\\ &\phi(\cdot,0) = \phi_0 \quad \text{in } \Omega, \end{aligned} \tag{16}$$

$$\begin{aligned} &\partial_t\psi = -R(\cdot,\cdot,T_1) + \gamma_1 r(T)F(\cdot,\cdot,T) + \operatorname{div}(\kappa\nabla\psi),\\ &\partial_{\mathbf{n}}\psi = 0 \quad \text{on } \Sigma\times(0,\infty),\\ &\psi(\cdot,0) = \psi_0 \quad \text{in } \Omega. \end{aligned} \tag{17}$$

Set

$$P(x,t) = S(x,t) + \phi(x,t) + \psi(x,t), \qquad P_0(x) = S_0(x) + \phi_0(x) + \psi_0,$$

$$P_0^* = \max_{\overline{\Omega}} P_0(x), \qquad I_0^* = \max_{\overline{\Omega}\times[0,T]} I_0, \qquad k^* = \max_{\overline{\Omega}\times[0,T]} k, \qquad R_0^* = \max_{\overline{\Omega}\times[0,T_1]} R_0,$$

$$\omega_i := \max\bigl(I_0^*, k^*(P_0^*)^2\bigr), \qquad \omega_1 := \max(R_0^*, \gamma_1\omega_i), \qquad \rho(t,T) = 1 - (\gamma+\gamma_1)\zeta(t,T),$$

$$\zeta(t,\tau) = \begin{cases} \exp\{-\int_{\tau-t}^{\tau}\nu(s)\,\mathrm{d}s\} & \text{if } 0\leqslant t\leqslant\tau,\\ r(\tau) = \exp\{-\int_0^{\tau}\nu(s)\,\mathrm{d}s\} & \text{if } t\geqslant\tau, \end{cases}$$

$$Z(t,\tau) = \int_\Omega I(x,t,\tau)\,\mathrm{d}x, \qquad Q(t,\tau_1) = \int_\Omega R(x,t,\tau_1)\,\mathrm{d}x,$$

$$Z_0(\tau) = \int_\Omega I_0(x,\tau)\,\mathrm{d}x, \qquad Q_0(\tau_1) = \int_\Omega R_0(x,\tau_1)\,\mathrm{d}x.$$

Lemma 2. *Let (S,I,R,u) be a solution to system* (1)–(3), (5) *guaranteed by Theorem* 2. *Then the integrals $\int_\Omega S(x,t)\,\mathrm{d}x$, $\int_\Omega \phi(x,t)\,\mathrm{d}x$, and $\int_\Omega \psi(x,t)\,\mathrm{d}x$ with $t\geqslant 0$ do not exceed $\int_\Omega P_0(x)\,\mathrm{d}x$, and the integrals $\int_0^t Z(s,T)\,\mathrm{d}s$, $\int_0^t Z(s,0)\,\mathrm{d}s$, $\int_0^t Q(s,T_1)\,\mathrm{d}s$, $\int_0^t\int_\Omega u(x,s)S(x,s)\,\mathrm{d}x\,\mathrm{d}s$, $\int_0^t\int_{\tau^*}^T Z(s,\tau)\,\mathrm{d}\tau\,\mathrm{d}s$, $\int_0^t\int_\Omega u(x,s)\,\mathrm{d}x\,\mathrm{d}s$ are uniformly bounded for all $t\geqslant 0$.*

Moreover:

(i) *these six temporal integrals converge as $t\to\infty$,*
(ii) *there exist nonnegative limits of $\int_\Omega S(x,t)\,\mathrm{d}x$, $\int_\Omega \phi(x,t)\,\mathrm{d}x$, and $\int_\Omega \psi(x,t)\,\mathrm{d}x$ as $t\to\infty$,*
(iii) $\lim_{t\to\infty}\int_0^T Z(t,\tau)\,\mathrm{d}\tau = \lim_{t\to\infty}\int_0^T\int_\Omega F(x,t,\tau)\,\mathrm{d}x\,\mathrm{d}\tau = \lim_{t\to\infty}\int_\Omega u(x,t)\,\mathrm{d}x = \lim_{t\to\infty}\int_0^{T_1} Q(t,\tau_1)\,\mathrm{d}\tau_1 = 0.$

If $\kappa_s = \kappa_i = \kappa_1 =: \kappa$, then $P\leqslant P_0^*$ and $u\leqslant k^*P_0^*$ in $\overline{\Omega}\times[0,\infty)$, $I\leqslant F\leqslant\omega_i$ in $\overline{\Omega}\times[0,\infty)\times[0,T]$, and $R\leqslant\omega_1$ in $\overline{\Omega}\times[0,\infty)\times[0,T_1]$.

Proof. Assuming equal diffusion coefficients, we add Eqs. (16) and (17) to Eqs. (1) to get

$$\begin{aligned} &\partial_t P - \operatorname{div}(\kappa \nabla P) = -\rho(\cdot, T) F(\cdot, \cdot, T) \qquad \text{in } \Omega \times (0, \infty), \\ &\partial_{\mathbf{n}} P = 0 \qquad \text{on } \Sigma \times (0, \infty), \\ &P(\cdot, 0) = P_0 \qquad \text{in } \overline{\Omega}. \end{aligned} \tag{18}$$

An application of the positivity lemma to this system yields $P \leqslant P_0^*$ in $\overline{\Omega} \times [0, \infty)$, which shows that

$$u(x,t) \leqslant k^* \int_0^T I(x,t,\tau)\,\mathrm{d}\tau \leqslant k^* P_0^*, \quad x \in \overline{\Omega},\ t \geqslant 0,$$

and the positive lemma applied to Eqs. (11)–(14) yields $I \leqslant \omega_i$, $R \leqslant \omega_1$.

In the case of any diffusion coefficients, we integrate Eqs. $(16)_1$ and $(17)_1$ over Ω, add to Eq. $(1)_1$ integrated over Ω, and integrate their sum over $(0, t)$ to have

$$\begin{aligned} \int_\Omega P(x,t)\,\mathrm{d}x &= \int_\Omega P_0(x)\,\mathrm{d}x - \int_0^t \rho(s,T) \int_\Omega F(x,s,T)\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \int_\Omega P_0(x)\,\mathrm{d}x - (1-\gamma-\gamma_1) \int_0^t \int_\Omega F(x,s,T)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Since the left-hand side of the above equality is nonnegative and the temporal integrals in the equality and inequality above do not decrease in t, they are bounded for all $t \geqslant 0$ and converge as $t \to \infty$. This yields the existence of the nonnegative $\lim_{t\to\infty} \int_\Omega P(x,t)\,\mathrm{d}x$.

If we integrate system Eq. $(17)_1$ over $\Omega \times (0, t)$, we obtain

$$\int_\Omega \psi(x,t)\,\mathrm{d}x = \int_\Omega \psi_0(x)\,\mathrm{d}x + \gamma_1 \int_0^t \int_\Omega \zeta(s,T) F(x,s,T)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t Q(s,T_1)\,\mathrm{d}s.$$

Because the first temporal integral on the right-hand side of this equality converges as $t \to \infty$ and the left-hand side is nonnegative, the second temporal integral on the right-hand side is bounded for all $t \geqslant 0$ and is not decreasing in t. Therefore, it converges, and hence, the integral $\int_\Omega \psi(x,t)\,\mathrm{d}x$ converges to a nonnegative limit as $t \to \infty$.

Integration of Eq. $(1)_1$ over $\Omega \times (0, t)$ yields

$$\begin{aligned} \int_\Omega S(x,t)\,\mathrm{d}x = \int_\Omega S_0(x)\,\mathrm{d}x &+ \int_0^t \biggl(\int_\Omega \gamma\zeta(s,T) F(x,s,T)\,\mathrm{d}x + Q(s,T_1) \biggr)\,\mathrm{d}s \\ &- \int_0^t \int_\Omega u(x,s) S(x,s)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Since the first temporal integral on the right-hand side of this equality converges as $t \to \infty$ and the left-hand side is nonnegative, the second temporal integral on the right-hand side is bounded for all $t \geqslant 0$. Because it does not decrease, it converges as $t \to \infty$. This and the equality above yield the existence of the nonnegative $\lim_{t\to\infty} \int_\Omega S(x,t)\,\mathrm{d}x$.

By integrating Eqs. (2) and (3) over Ω we derive the systems

$$\begin{aligned} &\partial_t Z + \partial_\tau Z = -\nu Z \qquad \text{in } (0,\infty)\times(0,T],\\ &Z(0,\cdot) = Z_0,\ \tau\in[0,T],\\ &Z(\cdot,0) = \int_\Omega u(x,\cdot)S(x,\cdot)\,\mathrm{d}x \quad \text{in } [0,\infty), \end{aligned}$$

$$\begin{aligned} &\partial_t Q + \partial_{\tau_1} Q = 0 \quad \text{in } (0,\infty)\times(0,T_1],\\ &Q(0,\cdot) = Q_0 \quad \text{in } [0,T_1],\\ &Q(\cdot,0) = \gamma_1 Z(\cdot,T) \quad \text{in } [0,\infty) \end{aligned}$$

and solve them to have

$$Z(t,\tau) = \begin{cases} Z_0(\tau-t)\exp\{-\int_{\tau-t}^{\tau}\nu(s)\,\mathrm{d}s\}, & t\leqslant\tau\leqslant T,\\ Z(t-\tau,0)r(\tau), & 0\leqslant\tau\leqslant t, \end{cases} \tag{19}$$

$$Q(t,\tau_1) = \begin{cases} Q_0(\tau_1-t), & t\leqslant\tau_1,\ \tau_1\in[0,T_1],\\ \gamma_1 Z(t-\tau_1,0), & t\geqslant\tau_1,\ \tau_1\in[0,T_1]. \end{cases} \tag{20}$$

Similarly, from system (7) we get

$$\int_\Omega F(x,t,\tau)\,\mathrm{d}x = \begin{cases} \int_\Omega F(x,0,\tau-t)\,\mathrm{d}x = Z_0(\tau-t), & t\leqslant\tau,\\ \int_\Omega F(x,t-\tau,0)\,\mathrm{d}x = Z(t-\tau,0), & \tau\leqslant t. \end{cases}$$

By virtue of Eq. (19), we obtain for $t\geqslant T$,

$$\begin{aligned} \int_0^t Z(s,T)\,\mathrm{d}s &\leqslant \int_0^t\int_\Omega F(x,s,T)\,\mathrm{d}x\,\mathrm{d}s = C + \int_T^t\int_\Omega F(x,s,T)\,\mathrm{d}x\,\mathrm{d}s\\ &= C + \int_T^t\int_\Omega F(x,s-T,0)\,\mathrm{d}x\,\mathrm{d}s\\ &= C + \int_0^{t-T}\int_\Omega F(x,y,0)\,\mathrm{d}x\,\mathrm{d}y, \quad C = \int_0^T\int_\Omega F(x,s,T)\,\mathrm{d}x\,\mathrm{d}s, \end{aligned}$$

which shows that the integral $\int_0^t Z(y,0)\,\mathrm{d}y = \int_0^t \int_\Omega F(x,y,0)\,\mathrm{d}x\,\mathrm{d}y$ converges as $t \to \infty$. Then using Eq. (19), we get for $t \geqslant T$,

$$\begin{aligned}\int_\Omega u(x,t)\,\mathrm{d}x &\leqslant k^* \int_{\tau^*}^{T} Z(t,\tau)\,\mathrm{d}\tau \leqslant k^* \int_{\tau^*}^{T}\int_\Omega F(x,t,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &= k^* \int_{\tau^*}^{T}\int_\Omega F(x,t-\tau,0)\,\mathrm{d}x\,\mathrm{d}\tau = k^* \int_{t-T}^{t-\tau^*}\int_\Omega F(x,y,0)\,\mathrm{d}x\,\mathrm{d}y \\ &= k^* \int_0^{t-\tau^*}\int_\Omega F(x,y,0)\,\mathrm{d}x\,\mathrm{d}y - k^* \int_0^{t-T}\int_\Omega F(x,y,0)\,\mathrm{d}x\,\mathrm{d}y \to 0 \quad \text{as } t \to \infty.\end{aligned}$$

Similarly, we have for $t > T$,

$$\int_0^T Z(t,\tau)\,\mathrm{d}\tau \leqslant \int_0^T Z(t-\tau,0)\,\mathrm{d}\tau = \int_{t-T}^{t} Z(s,0)\,\mathrm{d}s \to 0 \quad \text{as } t \to \infty,$$

and for $t > T_1$,

$$\begin{aligned}\int_0^{T_1} Q(t,\tau_1)\,\mathrm{d}\tau_1 &= \int_0^{T_1} Q(t-\tau,0)\,\mathrm{d}\tau_1 = \int_{t-T_1}^{t} Q(y,0)\,\mathrm{d}y \\ &= \gamma_1 \int_{t-T_1}^{t} Z(y,T)\,\mathrm{d}y \leqslant \gamma_1 \int_{t-T_1}^{t}\int_\Omega F(x,y-T,0)\,\mathrm{d}x\,\mathrm{d}y \\ &= \gamma_1 \int_{t-T-T_1}^{t-T}\int_\Omega F(x,y,0)\,\mathrm{d}x\,\mathrm{d}y \to 0 \quad \text{as } t \to \infty.\end{aligned}$$

For $t \geqslant T$, we also have

$$\begin{aligned}\int_0^t \mathrm{d}s \int_0^T Z(s,\tau)\,\mathrm{d}\tau &= \int_0^T \mathrm{d}\tau \int_0^t Z(s,\tau)\,\mathrm{d}s = C + \int_0^T \mathrm{d}\tau \int_T^t Z(s,\tau)\,\mathrm{d}s \\ &= C + \int_0^T \mathrm{d}\tau \int_T^t Z(s-\tau,0)\,\mathrm{d}s = C + \int_0^T \mathrm{d}\tau \int_{T-\tau}^{t-\tau} Z(y,0)\,\mathrm{d}y \\ &\leqslant C + T\int_0^t Z(s,0)\,\mathrm{d}s < \infty, \quad C = \int_0^T \mathrm{d}s \int_0^T Z(s,\tau)\,\mathrm{d}\tau,\end{aligned}$$

which shows that $\int_0^t \mathrm{d}s \int_0^T Z(s,\tau)\,\mathrm{d}\tau$ converges as $t\to\infty$. Similarly,

$$\int_0^t \mathrm{d}s \int_\Omega u(x,s)\,\mathrm{d}x \leqslant k^* \int_0^t \mathrm{d}s \int_{\tau^*}^T Z(t,\tau)\,\mathrm{d}\tau < \infty.$$

The proof is complete. □

Lemma 3. *If all diffusion coefficients are equal and* (S,I,R,u) *is the solution to system* (1)–(3), (5) *guaranteed by Theorem* 2, *then* $\lim_{t\to\infty}\max_{\overline{\Omega}} P(\cdot,t) \leqslant P_0^*$.

Proof. Consider a sequence $P(\cdot,t_j) := S(\cdot,t_j) + \int_0^T F(\cdot,t_j,\tau)\,\mathrm{d}\tau + \int_0^{T_1} R(\cdot,t_j,\tau_1)\,\mathrm{d}\tau_1$ with $0=t_0<t_1<t_2<\cdots<t_j<\cdots$, $t_j\to\infty$ as $j\to\infty$ and rewrite Eqs. (18) as follows:

$$\begin{aligned}
&\partial_t P - \operatorname{div}\kappa\nabla P = -\rho(t,T)F(\cdot,\cdot,T) \quad \text{in } \Omega\times(t_j,\infty),\\
&\partial_{\mathbf{n}} P = 0 \quad \text{on } \Sigma\times(t_j,\infty),\\
&P(\cdot,t)|_{t=t_j} = P(\cdot,t_j) \quad \text{in } \overline{\Omega}.
\end{aligned}$$

The positivity lemma immediately yields $P\leqslant \max_{\overline{\Omega}} P(\cdot,t_j)$ in $\overline{\Omega}\times[t_j,\infty)$. Arguing as above, we also have

$$P \leqslant \max_{\overline{\Omega}} P(\cdot,t_{j+1}) \leqslant \max_{\overline{\Omega}} P(\cdot,t_j) \quad \text{in } \overline{\Omega}\times[t_{j+1},\infty).$$

Letting t_j run to infinity in this inequality, we observe that function $\max_{\overline{\Omega}} P(\cdot,t)$ does not increase in variable t. Moreover, it is bounded from above by P_0^* and from below by zero. Hence, it has a limit between zero and P_0^*. The proof is complete. □

Lemma 4. *Let all diffusion coefficients be equal, and let* (S,I,R,u) *be the solution to system* (1)–(3), (5) *guaranteed by Theorem* 2. *Then*

$$\lim_{t\to\infty} Z(t,\tau) = 0 \quad \textit{in } [0,T], \qquad \lim_{t\to\infty} Q(t,\tau_1)\,\mathrm{d}x = 0 \quad \textit{in } [0,T_1].$$

Proof. In the case of equal diffusion coefficients, Lemma 2 shows that

$$\begin{aligned}
Z(t,0) &= \int_\Omega S(x,t)u(x,t)\,\mathrm{d}x \leqslant P_0^* \int_\Omega \mathrm{d}x \int_0^T k(x,\tau)I(x,t,\tau)\,\mathrm{d}\tau\\
&\leqslant k^* P_0^* \int_0^T Z(t,\tau)\,\mathrm{d}\tau.
\end{aligned}$$

This and Lemma 2 yield

$$\lim_{t\to\infty} Z(t,0) = 0. \tag{21}$$

Then it follows from Eqs. (19) and (21) that

$$\lim_{t\to\infty} Z(t,\tau) = \lim_{t\to\infty} Z(t-\tau,0)r(\tau) = 0 \quad \text{for } \tau\in[0,T].$$

This and Eqs. (20) and (21) show that

$$\lim_{t\to\infty} Q(t,\tau_1) = \lim_{t\to\infty} Q(t-\tau_1,\,0) = \lim_{t\to\infty} \gamma_1 Z(t-\tau_1,\,T) = 0.$$

The proof is complete. □

Lemma 4 shows that, in the case of equal diffusion coefficients, the spatial averages $Z(t,\tau)/|\Omega|$ and $Q(t,\tau_1)/|\Omega|$ of the infected individuals and those who recover with temporary immunity, respectively, where $\tau \in [0,T]$, $\tau_1 \in [0,T_1]$, and $|\Omega|$ is the measure of the domain Ω, eventually converge to zero.

Remark 1. Let all diffusion coefficients be equal. Since $0 \leqslant r(T)\int_\Omega F(x,t,\tau)\,\mathrm{d}x \leqslant Z(t,\tau) \to 0$ as $t\to\infty$, then $\int_\Omega F(x,t,\tau)\,\mathrm{d}\tau \to 0$ as $t\to\infty$. This shows that, in the case of equal diffusion coefficients, the spatial average value of the infected individuals eventually extinguishes even if the mortality $\nu = 0$ in $[0,T]$.

Lemma 5. *Assume that (S,I,R,u) is the solution to system* (1)–(3), (5) *guaranteed by Theorem* 2*, and let the functions S, ϕ, and ψ be uniformly bounded. Then the limit $\lim_{t\to\infty}\int_\Omega S(x,t)\,\mathrm{d}x$ is positive.*

Proof. The proof of this lemma is based on the arguments used in the proofs of Lemmas 3.24 and 3.26 in [1], and for the sake of brevity of this article, we are forced to omit its details. □

Corollary 1. *Equations* (18) *show that, in the case of equal diffusion coefficients, functions S, ϕ, ψ are uniformly bounded by the constant $\max_{\overline{\Omega}} P(\cdot,T)$, and therefore, in this case, Lemma* 5 *is true.*

Define a spatial average of function S by the equality $\overline{S}(t) = \int_\Omega S(x,t)\,\mathrm{d}x/|\Omega|$, and let $S_\infty = \lim_{t\to\infty}\overline{S}(t)$.

Lemma 6. *Assume that (S,I,R,u) is the solution to system* (1)–(3), (5) *guaranteed by Theorem* 2*. Let functions S, I, R, and u be uniformly bounded. Then $S - S_\infty \to 0$ in $W_2^1(\Omega)$ as $t\to\infty$.*

Proof. Let positive constants $\overline{\omega}_s$, $\overline{\omega}_i$, $\overline{\omega}_1$, and ω^* be the upper bounds of functions S, I, R, and u, respectively. To prove this lemma, we may apply an argument used in [3]. We first prove that $\lim_{t\to\infty}\int_\Omega S^2(x,t)\,\mathrm{d}x$, $\lim_{t\to\infty}\int_0^t\int_\Omega S^2(x,t)\,\mathrm{d}x$, and $\lim_{t\to\infty}\int_0^t \mathrm{d}s \times \int_\Omega \kappa(x)|\nabla S(x,s)|^2\,\mathrm{d}x$ are finite. We multiply Eq. $(1)_1$ by $2S$, integrate over Ω, and then integrate by parts to have

$$\begin{aligned}\partial_t \int_\Omega S^2(x,t)\,\mathrm{d}x &= 2\int_\Omega \bigl(\gamma I(x,t,T) + R(x,t,T_1)\bigr) S(x,t)\,\mathrm{d}x \\ &\quad - 2\int_\Omega S^2(x,t)u(x,t)\,\mathrm{d}x - 2\int_\Omega \kappa\bigl|\nabla S(x,t)\bigr|^2\,\mathrm{d}x.\end{aligned}$$

Integration of this equation over $(0,t)$ yields

$$\int\limits_{\Omega} S^2(x,t)\,\mathrm{d}x = \int\limits_{\Omega} S_0^2(x)\,\mathrm{d}x + 2\int\limits_0^t \mathrm{d}s \int\limits_{\Omega} \bigl(\gamma I(x,s,T) + R(x,t,T_1)\bigr) S(x,s)\,\mathrm{d}x - 2\int\limits_0^t \mathrm{d}s \int\limits_{\Omega} S^2(x,t) u(x,s)\,\mathrm{d}s - 2\int\limits_0^t \mathrm{d}s \int\limits_{\Omega} \kappa(x)\bigl|\nabla S(x,s)\bigr|^2\,\mathrm{d}x.$$

Since S is bounded, the first temporal integral on the right-hand side of this equation by Lemma 2 converges. The sum of the second and third temporal integrals on the same side of this equation is bounded by the sum of the first two terms on the same side, and the third and fourth temporal integrals on the right-hand side do not decrease. Therefore, they have finite limits. Hence, there exists a finite nonnegative limit of the left-hand side as time tends to infinity, i.e., $\lim_{t\to\infty}\int_{\Omega} S^2(x,t)\,\mathrm{d}x < \infty$.

Because $\int_0^\infty \mathrm{d}s \int_{\Omega} \kappa(x)|\nabla S(x,s)|^2\,\mathrm{d}x$ converges, there exist two increasing sequences $\{t_k\}$ and $\{\bar t_k\}$, $t_{k+1} = t_k + h$, $h = \mathrm{const} > 0$, $\bar t_k \in (t_k, t_{k+1})$, $k = 1, 2, \dots,$ such that

$$\int\limits_{t_k}^{t_{k+1}} \int\limits_{\Omega} \kappa(x)\bigl|\nabla S(x,s)\bigr|^2\,\mathrm{d}x\,\mathrm{d}s = h\int\limits_{\Omega} \kappa(x)\bigl|\nabla S(x,\bar t_k)\bigr|^2\,\mathrm{d}x \to 0 \quad \text{as } k\to\infty.$$

Since $\kappa_* := \min_{\overline{\Omega}} \kappa > 0$, $\int_{\Omega} |\nabla S(x,\bar t_k)|^2\,\mathrm{d}x \to 0$ as $k\to\infty$.

Next, we multiply Eq. $(1)_1$ by $\partial_t S$, integrate over Ω, and use the upper bound for S to obtain

$$\int\limits_{\Omega} \bigl(\partial_t S(x,t)\bigr)^2\,\mathrm{d}x \leqslant \int\limits_{\Omega} \bigl(\gamma I(x,t,T) + R(x,t,T_1)\bigr)\partial_t S(x,t)\,\mathrm{d}x + \overline{\omega}_s \int\limits_{\Omega} u(x,t)\bigl|\partial_t S(x,t)\bigr)\bigr|\,\mathrm{d}x + \int\limits_{\Omega} \partial_t S(x,t)\,\mathrm{div}\,\kappa(x)\nabla S(x,t)\,\mathrm{d}x.$$

Young's inequality, integration by parts, and use of the boundary condition of system (1) show that

$$\int\limits_{\Omega} \bigl(\partial_t S(x,t)\bigr)^2\,\mathrm{d}x \leqslant \frac{1}{2\varepsilon}\int\limits_{\Omega} \bigl(\gamma I(x,t,T)^2 + R^2(x,t,T_1)\bigr)\,\mathrm{d}x + \eta\int\limits_{\Omega} \bigl(\partial_t S(x,t)\bigr)^2\,\mathrm{d}x + \overline{\omega}_s\biggl(\frac{1}{2\varepsilon}\int\limits_{\Omega} u(x,t)^2\,\mathrm{d}x + \frac{\varepsilon}{2}\int\limits_{\Omega} \bigl(\partial_t S(x,t)\bigr)^2\,\mathrm{d}x\biggr) - \frac{1}{2}\partial_t \int\limits_{\Omega} \kappa(x)\bigl|\nabla S(x,t)\bigr|^2\,\mathrm{d}x,$$

where $\eta = (1+\gamma)\varepsilon/2$. The upper bounds of $I(x,t,T)$, $R(x,t,T_1)$, and $u(x,t)$ yield

$$\mu \int_{\Omega} \big(\partial_t S(x,t)\big)^2 \, \mathrm{d}x \leqslant \frac{\max(\overline{\omega}_i, \overline{\omega}_1)}{2\varepsilon} \int_{\Omega} \big(\gamma I(x,t,T) + R(x,t,T_1)\big) \, \mathrm{d}x + \frac{\overline{\omega}_s \omega^*}{2\varepsilon} \int_{\Omega} u(x,t) \, \mathrm{d}x - \frac{1}{2}\partial_t \int_{\Omega} \kappa(x) \big|\nabla S(x,t)\big|^2 \, \mathrm{d}x, \tag{22}$$

where $\mu := 1 - (1+\gamma+\overline{\omega}_s)/2$. If we integrate this inequality over (t_1, t) with $t_1 > 0$, we obtain

$$\mu \int_{t_1}^{t} \bigg(\int_{\Omega} \big(\partial_s S(x,s)\big)^2 \, \mathrm{d}x \bigg) \, \mathrm{d}s \leqslant G(t) - \frac{1}{2} \int_{\Omega} \kappa(x) \big|\nabla S(x,t)\big|^2 \, \mathrm{d}x,$$

where

$$G(t) := \frac{\max(\overline{\omega}_i, \overline{\omega}_1)}{2\varepsilon} \int_{t_1}^{t} \mathrm{d}s \int_{\Omega} \big(\gamma I(x,s,T) + R(x,t,T_1)\big) \, \mathrm{d}x + \frac{\overline{\omega}_s \omega^*}{2\varepsilon} \int_{t_1}^{t} \mathrm{d}s \int_{\Omega} u(x,s) \, \mathrm{d}x + \frac{1}{2} \int_{\Omega} \kappa(x) \big|\nabla S(x,t_1)\big|^2 \, \mathrm{d}x.$$

The temporal integrals in $G(t)$ above converge as $t \to \infty$. Hence, $\lim_{t\to\infty} G(t)$ is finite. For $\varepsilon < 2(1+\gamma+\overline{\omega}_s)^{-1}$, the left-hand side of the inequality above is nonnegative, nondecreasing, and bounded from above by $\lim_{t\to\infty} G(t)$. Therefore, the integral on the left-hand side of the inequality above also converges to a finite limit, which is equal to or less then $\lim_{t\to\infty} G(t)/\mu$. Since $\kappa_* > 0$, then

$$\kappa_* \int_{\Omega} \big|\nabla S(x,t)\big|^2 \, \mathrm{d}x \leqslant \int_{\Omega} \kappa(x) \big|\nabla S(x,t)\big|^2 \, \mathrm{d}x \leqslant 2G(t) \leqslant 2 \lim_{t\to\infty} G(t)$$

for $t > t_1$, and, because S is bounded, $S(x,t) \in W_2^1(\Omega)$ for $t \geqslant 0$.

Integration of inequality (22) over (t_k, t), $t \in (t_k, t_{k+1}]$, yields

$$\begin{aligned}
&\mu \int_{t_k}^{t} \mathrm{d}s \int_{\Omega} \big(\partial_t S(x,s)\big)^2 \, \mathrm{d}x \\
&\quad \leqslant \frac{\max(\overline{\omega}_i, \overline{\omega}_1)}{2\varepsilon} \int_{t_k}^{t} \mathrm{d}s \int_{\Omega} \big(\gamma I(x,s,T) + R(x,t,T_1)\big) \, \mathrm{d}x + \frac{\overline{\omega}_s \omega^*}{2\varepsilon} \int_{t_k}^{t} \mathrm{d}s \int_{\Omega} u(x,t) \, \mathrm{d}x \\
&\qquad - \frac{1}{2} \int_{\Omega} \kappa(x) \big|\nabla S(x,t)\big|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\Omega} \kappa(x) \big|\nabla S(x,t_k)\big|^2 \, \mathrm{d}x.
\end{aligned}$$

All temporal integrals and the last term on the right-hand side of this inequality converge to zero as $t \to \infty$. Hence,

$$\lim_{t\to\infty} \int_\Omega \kappa(x)\bigl|\nabla S(x,t)\bigr|^2\,\mathrm{d}x = 0 \quad\text{and}\quad \lim_{t\to\infty} \int_\Omega \bigl|\nabla S(x,t)\bigr|^2\,\mathrm{d}x = 0$$

since $\kappa \geqslant \kappa_* > 0$. Then the Poincaré–Wirtinger inequality shows that

$$\int_\Omega \bigl(S(x,t) - \overline{S}(t)\bigr)^2\,\mathrm{d}x \leqslant K \int_\Omega \bigl|\nabla S(x,t)\bigr|^2\,\mathrm{d}x \to 0 \quad \text{as } t\to\infty, \tag{23}$$

where K is a constant independent of t, and $\overline{S}(t) = \int_\Omega S(x,t)\,\mathrm{d}x/|\Omega|$. Observe that by Lemma 5, $S_\infty > 0$. Thus $S - \overline{S}(t) \to 0$ in $W_2^1(\Omega)$ as $t\to\infty$. The proof is complete. □

In the case where $n = 1$, the Sobolev embedding theorem yields $\lim_{t\to\infty} S = S_\infty$ for all $x \in \Omega$.

Corollary 2. *In the case of equal diffusion coefficients,* $\overline{\omega}_s = P_0^*$, $\overline{\omega}_i = \omega_i$, $\overline{\omega}_1 = \omega_1$, $\omega^* = k^* P_0^*$. *Consequently, in this case, Lemma* 6 *is true.*

According to [4] and [5], densities of the infected individuals and susceptibles converge in $C(\overline{\Omega}\times[0,T])$ and $C(\overline{\Omega})$ to zero and a positive number, respectively, as $t\to\infty$. According to our model in the case of equal diffusion coefficients, density I for $\tau\in[0,T]$ and function R for $\tau_1 \in [0,T_1]$ converge to zero in $L_1(\Omega)$, while the density of susceptibles, S, converges in $L_1(\Omega)$ to a positive number. The claim that $\lim_{t\to\infty} S(x,t) > 0$, $\lim_{t\to\infty} I(x,t,\tau) = 0$ with $\tau\in[0,T]$, $\lim_{t\to\infty} R(x,t,\tau_1) = 0$ with $\tau_1\in[0,T_1]$, $\lim_{t\to\infty} u(x,t) = 0$ for all $x\in\overline{\Omega}$, and the dimension of the region Ω greater than one under the hypotheses (H_1) is an open problem.

In the next section, we improve the data smoothness of model (1)–(3) and (5) so that conditions of Theorem 5.3 from [13, Chap. IV, Sect. 5] would be satisfied, and, using Theorem 5.3 from [13, Chap. IV, Sect. 5], for any diffusion coefficients and any Ω dimension, prove the existence of a unique nonnegative globally defined solution (S,I,R,u) such that I and R converge to zero in $C(\overline{\Omega}\times[0,T])$ and $C(\overline{\Omega}\times[0,T_1])$, respectively, and S tends to a positive number in $C(\overline{\Omega})$ as $t\to\infty$.

5 System (1)–(3), (5) with improved smoothness of the model data

Assume that a constant $\beta\in(0,1)$, the surface Σ is of class $C^{2+\beta}$, and given functions S_0, I_0, R_0, κ_s, κ_i, ν, k satisfy the following smoothness conditions (called (H_2) hypotheses):

(i) $S_0 \in C^{2+\beta}(\overline{\Omega})$, $S_0 \geqslant 0$ in $\overline{\Omega}$, $\partial_n S_0 = 0$ on Σ,
(ii) $I_0 \in C^{2+\beta,1}(\overline{\Omega}\times[0,T])$, $I_0 \geqslant 0$ in $\overline{\Omega}\times[0,T]$, $\partial_n I_0 = 0$ on $\Sigma\times[0,T]$,
(iii) $R_0 \in C^{2+\beta,1}(\overline{\Omega}\times[0,T_1])$, $R_0 \geqslant 0$ in $\overline{\Omega}\times[0,T_1]$, $\partial_n R_0 = 0$ on $\Sigma\times[0,T_1]$,
(iv) $\kappa_s, \kappa_i \in C^{2+\beta}(\overline{\Omega})$ and are positive in $\overline{\Omega}$.
(v) $\nu \in C^{\beta/2}([0,T])$ and is positive in $[0,T]$,
(vi) $k \in C^{2+\beta,1}(\overline{\Omega}\times[\tau^*,T]) \cap C(\overline{\Omega}\times[\tau^*,T])$ and is positive in $\overline{\Omega}\times[\tau^*,T]$.

Theorem 3. *Under hypotheses* (H_2) (*assumptions* (i)–(vi)), *system* (1)–(3), (5) *has a nonnegative globally defined solution* (S, I, R, u) *such that*

(i) $S \in C^{2+\beta,1+\beta/2}(\overline{\Omega} \times [0,\infty))$,
(ii) $I \in C^{2+\beta,1,1}(\overline{\Omega} \times (([0,\infty) \times [0,T]) \setminus \{t = \tau\}))$,
(iii) $R \in C^{2+\beta,1,1}(\overline{\Omega} \times (([0,\infty) \times [0,T_1]) \setminus \{t = \tau_1\}))$,
(iv) $u \in C^{2+\beta,1}(\overline{\Omega} \times [0,\infty))$.

Proof. The proof of this theorem is based on the direct application of Theorem 5.3 from [13, Chap. IV, Sect. 5] to equations (11) for $t \in [0,\tau] \times [0,T]$, (12) for $t \in [0,\tau_1] \times [0,T_1]$, (13) for $t - \tau \in [j\tau^*, (j+1)\tau^*]$, and (14) for $t - \tau_1 \in [j\tau^*, (j+1)\tau^*]$, $j = 0, 1, 2, \dots,$ and therefore, we skip it. □

Further in this section, we consider the long time behaviour of the solution guaranteed by Theorem 3.

Lemma 7. *Under the hypotheses* (H_2), *I and* $R \to 0$ *uniformly in* $\overline{\Omega} \times [0,T]$ *and* $\overline{\Omega} \times [0,T_1]$, *respectively,* $u \to 0$ *and* $S \to S_\infty$ *uniformly in* $\overline{\Omega}$ *as* $t \to \infty$.

Proof. We first prove that S, I, R, and u are uniformly bounded. Set $\nu_* = \min_{[0,T]} \nu(\tau)$. Since $u(\cdot,\alpha)S(\cdot,\alpha) \in C^{2+\beta}(\overline{\Omega})$ and $\nu_* > 0$, the positivity lemma shows that the function $\tilde{I}$, determined by equations

$$
\begin{aligned}
&\partial_t \tilde{I} + \partial_\tau \tilde{I} - \operatorname{div}(\kappa_i \nabla \tilde{I}) = -\nu_* \tilde{I} \quad \text{in } \Omega \times (\tau + \tau_*, \infty) \times (0,T], \\
&\partial_{\mathbf{n}} \tilde{I} = 0 \quad \text{on } \Sigma \times (\tau + \tau_*, \infty) \times (0,T], \\
&\tilde{I}(\cdot,\cdot,0) = S(\cdot,\cdot)u(\cdot,\cdot) \quad \text{in } \overline{\Omega} \times [\tau_*, \infty)
\end{aligned}
\tag{24}
$$

written on the characteristic lines $\alpha = t - \tau$, is a majorant of function I for $t \geqslant \tau + \tau^*$, and it can be represented as

$$
\tilde{I}(x,t,\tau) = \int_\Omega G(x,x',\tau) S(x',\, t-\tau) u(x',\, t-\tau)\, \mathrm{d}x',
$$

where $G(x,x',\tau)$ is the Green function for system (24) written on the characteristic lines $t = \tau + \alpha$, $\alpha > 0$. It is well known (see e.g., [4, 5] and [7, Chap. VI, Sect. VI.2]) that for $x, x' \in \overline{\Omega}$ and $\tau \geqslant \tau' > 0$, function G is bounded, i.e., $|G(x,x',\tau)| \leqslant C(\tau') = \text{const}$. Hence,

$$
\begin{aligned}
\tilde{I}(x,t,T) &= \int_\Omega G(x,x',T) S(x',\, t-T) u(x',\, t-T)\, \mathrm{d}x' \\
&\leqslant C(\tau^*) \int_\Omega S(x',\, t-T) u(x',\, t-T)\, \mathrm{d}x'.
\end{aligned}
$$

The positivity lemma yields that $\tilde{S}$ determined in $[t^*, \infty)$ with $t^* \geqslant T + T_1$ by

$$
\tilde{S}(t) = \max_{\overline{\Omega}} S(x,t^*) + \int_{t^*}^{t} \Big(\gamma \max_{\overline{\Omega}} I(x,s,T) + \gamma_1 \max_{\overline{\Omega}} I(x,\, s-T_1,\, T) \Big)\, \mathrm{d}s
$$

is a majorant for function S. Moreover,

$$\tilde{S}(t) \leqslant \max_{\overline{\Omega}} S(x,t^*) + C(\tau^*)\gamma \int_{t^*}^{t} \int_{\Omega} S(x,\, s-T)u(x,\, s-T)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ C(\tau^*)\gamma_1 \int_{t^*}^{t} \int_{\Omega} S(x,\, s-T-T_1)u(x,\, s-T-T_1)\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant \max_{\overline{\Omega}} S(x,t^*) + C(\tau^*)(\gamma+\gamma_1) \int_{0}^{\infty} \int_{\Omega} S(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty$$

since by Lemma 2 the temporal integral in the above inequality converges and therefore function S is uniformly bounded. Then for $t \geqslant t^*$,

$$u(x,t) \leqslant k^* C(\tau_*) \int_{\tau_*}^{T} \int_{\Omega} S(x,\, t-\tau)u(x,\, t-\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leqslant k^* C(\tau_*) \int_{t-T}^{t-\tau_*} \int_{\Omega} S(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= k^* C(\tau_*) \left(\int_{0}^{t-\tau_*} \int_{\Omega} S(x,s)u(x,s)\,\mathrm{d}x\,\mathrm{d}s - \int_{0}^{t-T} \int_{\Omega} S(x,s)u(x,s)\,\mathrm{d}x\,\mathrm{d}s \right)$$
$$\to 0 \quad \text{as } t \to \infty$$

because by Lemma 2 both temporal integrals converge. Hence, function u is also uniformly bounded. Then for $t \geqslant t^*$,

$$I(x,t,\tau) \leqslant \max_{\overline{\Omega}} S(x,\, t-\tau)u(x,\, t-\tau) \to 0$$

and hence

$$R \leqslant \gamma_1 \max_{\Omega} I(x,\, t-T_1,\, T) \to 0 \quad \text{as } t \to \infty.$$

It is evident that the functions I and R are uniformly bounded, and therefore, Lemmas 5 and 6 can be used in the case under consideration. Denoting maximum of upper bounds of functions S, I, and R by ω and using Lemma 6, we get that the spatial average of S tends to a positive number S_∞ in $W_2^1(\Omega)$ as $t \to \infty$.

Second, since functions S, I, R, and u are uniformly bounded, [13, Chap. V, Sect. 7] yields that $|\nabla S|$ is also uniformly bounded. Then direct application of inequality (23) for $p > n \geqslant 2$ yields

$$\int_{\Omega} \big(S(x,t) - \overline{S}(t)\big)^p\,\mathrm{d}x \leqslant C_1 \int_{\Omega} \big(S(x,t) - \overline{S}(t)\big)^2\,\mathrm{d}x \leqslant C_1 K \int_{\Omega} |\nabla S|^2\,\mathrm{d}x,$$

$$\int_{\Omega} |\nabla S|^p \, \mathrm{d}x \leqslant C_2 \int_{\Omega} |\nabla S|^2 \, \mathrm{d}x,$$

and Eq. (23) shows that

$$\int_{\Omega} \big(S(x,t) - \overline{S}(t)\big)^p \, \mathrm{d}x + \int_{\Omega} |\nabla S|^p \, \mathrm{d}x \to 0 \quad \text{as } t \to \infty.$$

By the Sobolev embedding theorem, it follows that $\lim_{t\to\infty} S(x,t) = S_\infty$ for all $x \in \overline{\Omega}$. The proof is complete. □

It is easy to see that, in the case where $\min_{\overline{\Omega}} S_0 > 0$, the function $\hat{S}$ determined as

$$\hat{S}(t) = \min_{\overline{\Omega}} S_0 \exp\left\{ - \int_0^t \max_{\overline{\Omega}} u(x,s) \, \mathrm{d}s \right\}$$

is a minorant for S. Moreover,

$$\begin{aligned}
\hat{S}(t) &= C_1 \min_{\overline{\Omega}} S_0 \exp\left\{ - \int_{t^*}^t \max_{\overline{\Omega}} u(x,s) \, \mathrm{d}s \right\} \\
&\geqslant C_1 \min_{\overline{\Omega}} S_0 \exp\left\{ -k^* C(\tau_*) \int_{t^*}^t \int_{\tau_*}^T \int_{\Omega} S(x,\, s-\tau) u(x,\, s-\tau) \right\} \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}s \\
&\geqslant C_1 \min_{\overline{\Omega}} S_0 \exp\left\{ -k^* C(\tau_*)(T-\tau_*) \int_0^\infty \int_{\Omega} S(x,y) u(x,y) \, \mathrm{d}x \, \mathrm{d}y \right\} \\
&=: S_* > 0
\end{aligned}$$

since the temporal integral converges. Hence, $S_\infty \geqslant S_* > 0$.

Assume that $I_0 > 0$ in $\overline{\Omega} \times [0,T]$. Applying an argument similar to that used to prove Theorem 3.4 in [9, p. 137, Chap. 7, Sect. 3], we can prove that

$$S_\infty \leqslant \left(\int_{\tau_*}^T \min_{\overline{\Omega}} k(x,\tau) \, \mathrm{d}\tau \right)^{-1}.$$

Remark 2. Since $0 \leqslant r(T)F \leqslant I$ and $I \to 0$ uniformly in $\overline{\Omega} \times [0,T]$ as $t \to \infty$, then F also tends to zero uniformly in $\overline{\Omega} \times [0,T]$ as $t \to \infty$. This shows that under the (H_2) hypotheses the class of infected individuals eventually disappears even if the mortality rate ν is identically zero.

6 Concluding remarks

We conclude this work by summarizing main results. We proposed and analysed a mathematical model for the spread description of an epidemic disease of variable infectivity in an asexual infection-age-and-immunity-structured population with spatial dispersal. It is assumed that some individuals recover from the disease with temporary immunity, another part recover with permanent immunity, and the last part recover with no immunity. The demographic changes such as births and deaths due to natural causes and the chronological age of individuals are disregarded. The model is based on the system of partial integro-differential equations including a PDE for evolution description of individuals recovered with temporary immunity. The existence and uniqueness of the globally defined classical solution is proved. The long-time behaviour of its solution is studied for two classes (H_1 and H_2) of the model data smoothness.

In the case of model data of class H_1, we have proved that the total number of infected individuals and the total number of individuals recovered with temporary immunity for any diffusion coefficients eventually tend to zero and that for equal diffusion coefficients the spatial average of susceptible individuals tends to a positive number, while the spatial averages of the infected individuals and of those who recover with temporary immunity asymptotically converge to zero. The claim that $\lim_{t\to\infty} S(x,t) > 0$, $\lim_{t\to\infty} I(x,t,\tau) = 0$ with $\tau \in [0,T]$, $\lim_{t\to\infty} R(x,t,\tau_1) = 0$ with $\tau_1 \in [0,T_1]$, $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in \overline{\Omega}$, and the dimension of the region Ω greater than one under hypotheses (H_1) is an open problem.

In the case of the model data of class H_2, we have proved that for any diffusion coefficients, the density of susceptible individuals, S, eventually tends to a positive number uniformly in $\overline{\Omega}$, while densities I and R tend to zero uniformly in $\overline{\Omega} \times [0,T]$ and $\overline{\Omega} \times [0,T_1]$, respectively. The class of infected individuals disappears even if the mortality ν is identically zero.

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