Nonlinear Analysis: Modelling and Control, Vol. 28, No. 3, 597–612
https://doi.org/10.15388/namc.2023.28.32123

Vilnius University Press

Existence of multiple positive solutions for a third-order boundary value problem with nonlocal conditions

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Received: February 3, 2023 / **Revised:** April 19, 2023 / **Published online:** April 26, 2023

Abstract. We study the existence of multiple positive solutions for a nonlinear third-order differential equation subject to various nonlocal boundary conditions. The boundary conditions that we study contain Stieltjes integral and include the special cases of m-point conditions and integral conditions. The main tool in the proof of our result is Krasnosel'skii's fixed point theorem. To illustrate the applicability of the obtained results, we consider examples.

Keywords: third-order nonlinear boundary value problems, nonlocal boundary conditions, existence of positive solutions, Green's function, Krasnosel'skii's fixed point theorem.

1 Introduction

We study the existence of multiple positive solutions for the nonlinear third-order differential equation

$$x''' + f(t, x) = 0, \quad t \in (0, 1), \tag{1}$$

subject to the nonlocal boundary conditions

$$x(0) = 0, \quad x'(0) = 0, \qquad x(1) = \lambda[x], \tag{2}$$

where $\lambda[x] = \int_0^1 x(t)\,\mathrm{d}\Lambda(t)$ is a linear functional on $C[0,1]$ given by Stieltjes integral with Λ a suitable function of bounded variation. Boundary conditions (2) include as special cases multipoint conditions and integral conditions. We do not suppose that $\lambda[x] \geqslant 0$ for all $x \geqslant 0$, but we allow $\mathrm{d}\Lambda$ to be a signed measure. This makes it possible for us to use coefficients in multipoint conditions of both signs and to use functions in integral conditions that may change the sign. Of course, we need the requirement $\lambda[x] \geqslant 0$ for any positive solution x.

Throughout the paper, we assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. By a positive solution of (1), (2) we understand $C^3[0,1]$ function, which is positive on $0 < t < 1$ and satisfies differential equation (1) for $0 < t < 1$ and boundary conditions (2). Since $f(t,x)$ is not defined for $x < 0$, every solution of (1), (2) is nonnegative. Using additional conditions on $\mathrm{d}\Lambda$, we will show that every nonnegative nontrivial solution of (1), (2) is positive.

In fact, our main result states that for each given positive integer n, we can specify f so that problem (1), (2) has at least n positive solutions. The approach we will use to get this result is one that is commonly used. First, we rewrite problem (1), (2) as an equivalent integral equation by constructing the corresponding Green's function. Then we define an operator in a suitable cone of nonnegative continuous functions, and hence the problem turns into finding fixed points of the operator. Finally, we prove the existence of multiple fixed points in the cone using Krasnosel'skii's cone compression and expansion theorem of norm type [2, 5], which we state here for convenience.

Theorem 1. *Let E be a Banach space and $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that*

(A) $\|Tx\| \leqslant \|x\|$ *for all* $x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \geqslant \|x\|$ *for all* $x \in K \cap \partial\Omega_2$, *or*
(B) $\|Tx\| \geqslant \|x\|$ *for all* $x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \leqslant \|x\|$ *for all* $x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The study of the existence of solutions to boundary value problems is often associated with rewriting the problem as an equivalent integral equation by the construction of the corresponding Green's functions. Thus, the concept of Green's functions plays an important role in the theory of boundary value problems. A survey of results on the Green's functions for stationary problems with nonlocal boundary conditions is presented in [8]. Green's functions for third-order boundary value problems with different additional conditions were studied in [7].

There are several reasons for studying the existence of positive solutions for problem (1), (2). First, boundary value problems with nonlocal boundary conditions have attracted increasing attention in recent decades due to their important role in various branches of applied mathematics. Nonlocal boundary conditions are often the result of measurements at multiple locations that can be combined to get more accurate models. Let us mention some recent achievements in the field of nonlocal problems. In [1], the authors investigate the Sturm–Liouville problem with one classical and another nonlocal two-point boundary condition. In [9], the authors obtain asymptotic formulas for eigenvalues and eigenfunctions of the one-dimensional Sturm–Liouville equation with one classical-type Dirichlet boundary condition and integral-type nonlocal boundary condition. In [11], the author proves the existence of multiple positive solutions of nonlinear second-order nonlocal boundary value problems with a nonlinear term having derivative dependence. In [6], the authors solve the two-dimensional nonlinear elliptic equation with the boundary integral condition depending on two parameters. The next reason is that namely positive solutions to boundary value problems frequently occur in applications. In [3], the authors investigate the existence of positive solutions for a cantilever equation subject to nonlocal and nonlinear boundary conditions. In [4], the authors study the existence of nonnegative solutions for a system of impulsive differential equations subject to nonlinear, nonlocal boundary conditions. In [10], the author considers the existence of positive solutions to a higher-order boundary value problem with nonlocal conditions. In [12], the authors establish the existence of multiple positive solutions of nonlinear second-order equations subject to various nonlocal boundary conditions. In [13], the authors give a new unified method of establishing the existence of multiple positive solutions for a large number of nonlinear differential equations of arbitrary order with any allowed number of nonlocal boundary conditions. In [14], the authors study the existence of positive solutions for second-order equations subject to various nonlocal boundary conditions. We would like to highlight the papers [10, 12, 13] that motivated the present investigation and from which many ideas were taken. For instance, we use a similar cone of nonnegative functions in the proof of our main result, but we use somewhat different conditions on the nonlinearity in the equation.

The paper contains three sections besides the Introduction. In Section 2, we rewrite the main problem as an equivalent integral equation by constructing the corresponding Green's function. Also, we give some inequalities for the Green's function here. In Section 3, we prove our main theorem on the existence of multiple positive solutions for the problem. In Section 4, we consider examples to illustrate the applicability of our main result.

2 Equivalent integral equation

Proposition 1. *Assume that $\lambda[t^2] \neq 1$. A function $x = x(t)$ is a solution of boundary value problem* (1), (2) *if and only if x is a solution of the integral equation*

$$x(t) = \int_0^1 H(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1. \tag{3}$$

Here $H(t,s)$ denotes Green's function for the equation $x''' = 0$ with boundary conditions (2) *and is explicitly given by*

$$H(t,s) = G(t,s) + \frac{t^2}{1-\lambda[t^2]}\,\Gamma(s),$$

where

$$G(t,s) = \frac{1}{2}\begin{cases} s(1-t)((1-s)t + (t-s)), & 0 \leqslant s \leqslant t \leqslant 1, \\ t^2(1-s)^2, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

and $\Gamma(s) = \int_0^1 G(t,s)\,\mathrm{d}\Lambda(t)$. By a solution of (3) *we understand $C[0,1]$ function that satisfies integral equation* (3) *for $0 \leqslant t \leqslant 1$.*

Proof. Suppose that $x(t)$ is a solution to problem (1), (2), then $x'''(t) + f(t, x(t)) = 0$ or $x'''(t) + h(t) = 0$, where $h(t) \equiv f(t, x(t))$. Integrating the equation $x'''(t) + h(t) = 0$ thrice, we get

$$x(t) = x(0) + tx'(0) + \frac{1}{2}t^2x''(0) - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,\mathrm{d}s.$$

Since $x(0) = 0$ and $x'(0) = 0$, we have $x(t) = t^2x''(0)/2 - \int_0^t (t-s)^2 h(s)\,\mathrm{d}s/2$. Let us find $x''(0)$, using the condition $x(1) = \lambda[x]$. So $x''(0) = \int_0^1 (1-s)^2 h(s)\,\mathrm{d}s + 2\,\lambda[x]$. Thus we get

$$\begin{aligned} x(t) &= \frac{1}{2}\int_0^1 t^2(1-s)^2 h(s)\,\mathrm{d}s + t^2\lambda[x] - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,\mathrm{d}s \\ &= t^2\lambda[x] + \frac{1}{2}\int_0^t s(1-t)\big((1-s)t + (t-s)\big)h(s)\,\mathrm{d}s + \frac{1}{2}\int_t^1 t^2(1-s)^2 h(s)\,\mathrm{d}s \\ &= t^2\lambda[x] + \int_0^1 G(t,s)h(s)\,\mathrm{d}s = t^2\int_0^1 x(t)\,\mathrm{d}\Lambda(t) + \int_0^1 G(t,s)h(s)\,\mathrm{d}s. \end{aligned}$$

Since

$$\begin{aligned} \int_0^1 x(t)\,\mathrm{d}\Lambda(t) &= \int_0^1 t^2\left(\int_0^1 x(s)\,\mathrm{d}\Lambda(s)\right)\mathrm{d}\Lambda(t) + \int_0^1\left(\int_0^1 G(t,s)h(s)\,\mathrm{d}s\right)\mathrm{d}\Lambda(t) \\ &= \int_0^1 t^2\,\mathrm{d}\Lambda(t)\int_0^1 x(s)\,\mathrm{d}\Lambda(s) + \int_0^1\left(\int_0^1 G(t,s)\,\mathrm{d}\Lambda(t)\right)h(s)\,\mathrm{d}s, \end{aligned}$$

we get

$$\int_0^1 x(t)\,\mathrm{d}\Lambda(t) = \frac{\int_0^1(\int_0^1 G(t,s)\,\mathrm{d}\Lambda(t))h(s)\,\mathrm{d}s}{1 - \int_0^1 t^2\,\mathrm{d}\Lambda(t)},$$

and therefore,

$$x(t) = \int_0^1 G(t,s)h(s)\,\mathrm{d}s + \frac{t^2}{1-\lambda[t^2]}\int_0^1 \Gamma(s)h(s)\,\mathrm{d}s = \int_0^1 H(t,s)f\big(s,x(s)\big)\,\mathrm{d}s,$$

or $x(t)$ is a solution of (3).

Now let $x(t)$ be a solution of integral equation (3). To show that $x(t)$ is a solution to problem (1), (2), one can differentiate thrice equation (3) and verify the continuity. □

Remark 1. In the proof of Proposition 1, Green's function H was constructed directly. In [7], the reader can find another way of constructing Green's functions for such types of problems.

Now we prove some inequalities for the functions G and H.

Proposition 2. *For all* $(t,s) \in [0,1] \times [0,1]$, *we have*

$$0 \leqslant G(t,s) \leqslant \frac{s(1-s)^2}{2(2-s)} = \Phi(s).$$

If $(t,s) \in (0,1) \times (0,1)$, *then*

$$G(t,s) > 0.$$

Proof. Inequalities $G(t,s) \geqslant 0$ and $G(t,s) > 0$ are obvious.

Let us find the maximum of $G(t,s)$ for each s with respect to t.

For $0 \leqslant s \leqslant t \leqslant 1$, the maximum occurs at $t = 1/(2-s)$ and is equal to $s(1-s)^2/(2(2-s))$.

If $0 \leqslant t \leqslant s \leqslant 1$, the maximum occurs at $t = s$ and is equal to $s^2(1-s)^2/2$.

Since for all $s \in [0,1]$,

$$\frac{s^2(1-s)^2}{2} = \frac{s(2s-s^2)(1-s)^2}{2(2-s)} \leqslant \max_{0 \leqslant s \leqslant 1}\left(2s-s^2\right)\frac{s(1-s)^2}{2(2-s)} = \frac{s(1-s)^2}{2(2-s)},$$

we get the proof. □

Proposition 3. *For all* $(t,s) \in [1/2, 13/14] \times [0,1]$, *we have*

$$G(t,s) \geqslant \frac{1}{4}\frac{s(1-s)^2}{2(2-s)} = \frac{1}{4}\Phi(s).$$

Proof. For $M_1 = \{(t,s)\colon\ 1/2 \leqslant t \leqslant 13/14,\ 0 \leqslant s \leqslant 1,\ s \leqslant t\}$, we have

$$\min_{M_1} \frac{G(t,s)}{\Phi(s)} = \frac{13}{49}.$$

If $M_2 = \{(t,s)\colon\ 1/2 \leqslant t \leqslant 13/14,\ 0 \leqslant s \leqslant 1,\ t \leqslant s\}$, then

$$\min_{M_2} \frac{G(t,s)}{\Phi(s)} = \frac{1}{4}.$$

Therefore,

$$\frac{G(t,s)}{\Phi(s)} \geqslant \frac{1}{4} \quad \text{for } \frac{1}{2} \leqslant t \leqslant \frac{13}{14},\ 0 \leqslant s \leqslant 1.$$ □

Proposition 4. *If* $\lambda[t^2] < 1$ *and* $\Gamma(s) \geqslant 0$ *for* $s \in [0,1]$, *then for all* $(t,s) \in [0,1] \times [0,1]$, *we have*

$$0 \leqslant H(t,s) \leqslant \Phi(s) + \frac{\Gamma(s)}{1-\lambda[t^2]} = \Phi_1(s). \tag{4}$$

If $(t,s) \in (0,1) \times (0,1)$, *then* $H(t,s) > 0$.

Proof. Inequalities $H(t,s) \geqslant 0$ and $H(t,s) > 0$ are obvious.

Let $(t,s) \in [0,1] \times [0,1]$ and consider

$$H(t,s) = G(t,s) + \frac{t^2}{1-\lambda[t^2]}\,\Gamma(s) \leqslant \Phi(s) + \frac{\Gamma(s)}{1-\lambda[t^2]}. \qquad \square$$

Proposition 5. *If* $\lambda[t^2] < 1$ *and* $\Gamma(s) \geqslant 0$ *for* $s \in [0,1]$*, then for all* $(t,s) \in [1/2, 13/14] \times [0,1]$*, we have*

$$H(t,s) \geqslant \frac{1}{4}\Phi_1(s). \tag{5}$$

Proof. Let $(t,s) \in [1/2, 13/14] \times [0,1]$ and consider

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{t^2}{1-\lambda[t^2]}\Gamma(s) \geqslant \frac{1}{4}\Phi(s) + \frac{\min_{1/2 \leqslant t \leqslant 13/14} t^2}{1-\lambda[t^2]}\,\Gamma(s) \\ &= \frac{1}{4}\Phi(s) + \frac{1}{4}\frac{\Gamma(s)}{1-\lambda[t^2]} = \frac{1}{4}\Phi_1(s). \qquad \square \end{aligned}$$

Proposition 6. *If* $\lambda[t^2] < 1$ *and* $\Gamma(s) \geqslant 0$ *for* $s \in [0,1]$*, then every nonnegative nontrivial solution* $x(t)$ *of* (1), (2) *is positive.*

Proof. Suppose that there exists $t_0 \in (0,1)$ such that $x(t_0) = 0$. Since boundary value problem (1), (2) is equivalent to integral equation (3), we get

$$x(t_0) = \int_0^1 H(t_0,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s = 0.$$

Since $H(t_0,s)f(s,x(s)) \geqslant 0$ for all $s \in [0,1]$, then

$$H(t_0,s) f\bigl(s, x(s)\bigr) = 0 \quad \text{for all } s \in [0,1].$$

Since $H(t_0,s) > 0$ for all $s \in (0,1)$, we get that $x'''(s) = -f(s,x(s)) = 0$ for all $s \in (0,1)$. Hence, $x(s)$ is a polynomial of degree at most two. Since $x(s)$ satisfies boundary conditions (2), it follows that $x(s) = 0$ for all $s \in [0,1]$. We get the contradiction. $\square$

3 Existence of multiple positive solutions

We work in the Banach space $C[0,1]$ endowed with the norm

$$\|x\| = \max_{0 \leqslant t \leqslant 1} \bigl|x(t)\bigr|, \quad x \in C[0,1].$$

Let us define a cone K in $C[0,1]$ by

$$K = \left\{ x \in C[0,1]\colon\ x(t) \geqslant 0,\ \min_{1/2 \leqslant t \leqslant 13/14} x(t) \geqslant \frac{1}{4}\|x\|,\ \lambda[x] \geqslant 0 \right\}$$

and an integral operator $T : K \to C[0,1]$ by

$$(Tx)(t) = \int_0^1 H(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1.$$

Boundary value problem (1), (2) has a solution x if and only if x is a fixed point of the operator T. Note that $T : K \to C[0,1]$ is a completely continuous operator.

Proposition 7. *If* $0 \leqslant \lambda[t^2] < 1$ *and* $\Gamma(s) \geqslant 0$ *for* $s \in [0,1]$, *then* $T(K) \subset K$.

Proof. From inequality (4) it follows that for $x \in K$, $(Tx)(t) \geqslant 0$ on $[0,1]$. For $x \in K$, we have from (4) that

$$(Tx)(t) = \int_0^1 H(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s \leqslant \int_0^1 \Phi_1(s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s,$$

and therefore,

$$\|Tx\| \leqslant \int_0^1 \Phi_1(s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s. \tag{6}$$

If $x \in K$, we have by (5) and (6)

$$\begin{aligned}\min_{1/2 \leqslant t \leqslant 13/14} (Tx)(t) &= \min_{1/2 \leqslant t \leqslant 13/14} \int_0^1 H(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s \\ &\geqslant \frac{1}{4} \int_0^1 \Phi_1(s)\, f\bigl(s, x(s)\bigr)\,\mathrm{d}s \geqslant \frac{1}{4}\|Tx\|.\end{aligned}$$

Now we need to prove that if $x \in K$, then $\lambda[Tx] \geqslant 0$. Consider

$$\int_0^1 H(t,s)\,\mathrm{d}\Lambda(t) = \Gamma(s) + \frac{\Gamma(s)}{1 - \lambda[t^2]} \int_0^1 t^2\,\mathrm{d}\Lambda(t) = \Gamma(s) + \frac{\lambda[t^2]}{1 - \lambda[t^2]}\Gamma(s) \geqslant 0.$$

If $x \in K$, we get

$$\begin{aligned}\lambda[Tx] &= \int_0^1 \left(\int_0^1 H(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s \right) \mathrm{d}\Lambda(t) = \int_0^1 \left(\int_0^1 H(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}\Lambda(t) \right) \mathrm{d}s \\ &= \int_0^1 \left(\int_0^1 H(t,s)\,\mathrm{d}\Lambda(t) \right) f\bigl(s, x(s)\bigr)\,\mathrm{d}s \geqslant 0.\end{aligned}$$

□

Further, we assume that $0 \leqslant \lambda[t^2] < 1$ and $\Gamma(s) \geqslant 0$ for $s \in [0,1]$. Let us denote

$$I_1 = \left(\max_{0\leqslant t\leqslant 1}\int_0^1 H(t,s)\,\mathrm{d}s\right)^{-1}, \qquad I_2 = \left(\max_{0\leqslant t\leqslant 1}\int_{1/2}^{13/14} H(t,s)\,\mathrm{d}s\right)^{-1}.$$

Obviously, $I_1 < I_2$.

The following two propositions will be used in the proof of our main result.

Proposition 8. *Suppose that there exists $r > 0$ such that $f(t,x) \leqslant I_1 r$ for $(t,x) \in [0,1]\times[0,r]$. If $x \in K$ with $\|x\| = r$, then $\|Tx\| \leqslant r$.*

Proof. If $x \in K$ with $\|x\| = r$, then for $t \in [0,1]$, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 H(t,s) f\bigl(s,x(s)\bigr)\,\mathrm{d}s \leqslant I_1 r\int_0^1 H(t,s)\,\mathrm{d}s \\ &\leqslant I_1 r \max_{0\leqslant t\leqslant 1}\int_0^1 H(t,s)\,\mathrm{d}s = r,\end{aligned}$$

or $\|Tx\| \leqslant r$. □

Proposition 9. *Suppose that there exists $r > 0$ such that $f(t,x) \geqslant I_2 r$ for $(t,x) \in [0,1]\times[r/4,r]$. If $x \in K$ with $\|x\| = r$, then $\|Tx\| \geqslant r$.*

Proof. If $x \in K$ with $\|x\| = r$, then we have $\min_{1/2\leqslant s\leqslant 13/14} x(s) \geqslant \|x\|/4 = r/4$ and $x(s) \in [r/4, r]$ for every $s \in [1/2, 13/14]$. Hence,

$$\begin{aligned}\|Tx\| &= \max_{0\leqslant t\leqslant 1}\int_0^1 H(t,s) f\bigl(s,x(s)\bigr)\,\mathrm{d}s \geqslant \max_{0\leqslant t\leqslant 1}\int_{1/2}^{13/14} H(t,s) f\bigl(s,x(s)\bigr)\,\mathrm{d}s \geqslant \\ &\geqslant I_2 r \max_{0\leqslant t\leqslant 1}\int_{1/2}^{13/14} H(t,s)\,\mathrm{d}s = r.\end{aligned}$$

□

Theorem 2. *Suppose that there exist $2m$ constants $0 < r_1 < r_2 < \cdots < r_{2m-1} < r_{2m}$, and let $\alpha(n) = 2n - (1-(-1)^n)/2$, $\beta(n) = 2n - (1+(-1)^n)/2$, where $1 \leqslant n \leqslant m$. If*

$$f(t,x) \leqslant I_1 r_{\alpha(n)} \quad \textit{for } (t,x) \in [0,1]\times\bigl[0, r_{\alpha(n)}\bigr]$$

and

$$f(t,x) \geqslant I_2 r_{\beta(n)} \quad \textit{for } (t,x) \in [0,1]\times\left[\frac{r_{\beta(n)}}{4}, r_{\beta(n)}\right],$$

then boundary value problem (1), (2) *has at least m positive solutions $x_n(t)$ such that $r_{2n-1} \leqslant \|x_n\| \leqslant r_{2n}$.*

Proof. If $\Omega_k = \{x \in C[0,1]\colon \|x\| < r_k\}$, $1 \leqslant k \leqslant 2m$, then by Propositions 8, 9 we have

$$\|Tx\| \leqslant \|x\| \quad \text{for } x \in K \cap \partial\Omega_{\alpha(n)}$$

and

$$\|Tx\| \geqslant \|x\| \quad \text{for } x \in K \cap \partial\Omega_{\beta(n)}.$$

If follows from Theorem 1 that T has a fixed point in each of the sets $K \cap (\overline{\Omega}_{2n} \setminus \Omega_{2n-1})$. Thus, boundary value problem (1), (2) has at least m positive solutions. □

4 Examples

Boundary conditions (2) cover many cases and include m-point and integral conditions.

For instance, if $\Lambda(t) = \int_0^t g(s)\,\mathrm{d}s$, where $g : [0,1] \to \mathbb{R}$ is continuous, we have $\lambda[x] = \int_0^1 x(t)\,\mathrm{d}(\int_0^t g(s)\,\mathrm{d}s) = \int_0^1 x(t)g(t)\,\mathrm{d}t$, and thus, we get integral boundary condition $x(1) = \int_0^1 x(t)g(t)\,\mathrm{d}t$. We do not suppose that $g(t) \geqslant 0$, but, of course, conditions $0 \leqslant \lambda[t^2] = \int_0^1 t^2 g(t)\,\mathrm{d}t < 1$ and $\Gamma(s) = \int_0^1 G(t,s)g(t)\,\mathrm{d}t \geqslant 0$ are necessary.

If $\Lambda(t) = \sum_{i=1}^{m-2} \alpha_i \chi(t-\tau_i)$, where $\tau_i < t$, $\alpha_i \in \mathbb{R}$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{m-2} < 1$, and

$$\chi(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leqslant 0, \end{cases}$$

we have

$$\lambda[x] = \int_0^1 x(t)\,\mathrm{d}\left(\sum_{i=1}^{m-2} \alpha_i \chi(t-\tau_i)\right) = \sum_{i=1}^{m-2} \alpha_i \int_0^1 x(t)\,\mathrm{d}\chi(t-\tau_i) = \sum_{i=1}^{m-2} \alpha_i x(\tau_i),$$

and we get m-point boundary conditions $x(0) = x'(0) = 0$, $x(1) = \sum_{i=1}^{m-2} \alpha_i, x(\tau_i)$. We do not suppose that all α_i are nonnegative. We allow coefficients α_i to be of both signs, but conditions $0 \leqslant \lambda[t^2] < 1$ and $\Gamma(s) \geqslant 0$ are necessary.

4.1 Problems with integral boundary conditions

Example 1. Consider the problem

$$\begin{gathered} x''' + (t^2+1), x^3 = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \qquad x(1) = \frac{5}{2}\int_0^1 x(t)\,\mathrm{d}t. \end{gathered} \tag{7}$$

We have $f(t,x) = (t^2+1)x^3$ and $g(t) \equiv 5/2$. Let us find

$$\lambda[t^2] = \int_0^1 t^2 g(t)\,\mathrm{d}t = \frac{5}{2}\int_0^1 t^2\,\mathrm{d}t = \frac{5}{6} \in [0,1),$$

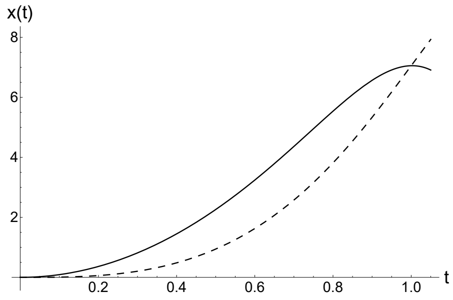

Figure 1. The solution of (7).

$$\Gamma(s)=\int_0^1 G(t,s)g(t)\,\mathrm{d}t=\frac{5}{2}\int_0^1 G(t,s)\,\mathrm{d}t=\frac{5}{12}(1-s)^2 s\geqslant 0,\quad s\in[0,1],$$

$$I_1=\frac{24}{5}=4.8,\qquad I_2=\frac{19208}{1245}\approx 15.428.$$

If we choose $r_1=1$, $r_2=32$, we get

$$f(t,x)\leqslant I_1 r_1\quad\text{for } (t,x)\in[0,1]\times[0,r_1],$$
$$f(t,x)\geqslant I_2 r_2\quad\text{for } (t,x)\in[0,1]\times[r_2/4,r_2].$$

Thus, by Theorem 2, boundary value problem (7) has at least one positive solution $x(t)$ such that $1\leqslant\|x\|\leqslant 32$. This solution (solid line), together with its antiderivative multiplied by $5/2$ (dashed line), is depicted in Fig. 1. This figure was obtained by using the program Wolfram Mathematica 11.1. The initial conditions for this solution are $x(0)=x'(0)=0$, $x''(0)\approx 18.11$.

Example 2. Consider the problem

$$\begin{gathered}x'''+f(t,x)=0,\quad t\in(0,1),\\ x(0)=x'(0)=0,\qquad x(1)=\int_0^1 t\,x(t)\,\mathrm{d}t,\end{gathered}\tag{8}$$

with

$$f(t,x)=\begin{cases}33x^2, & 0\leqslant x\leqslant 1,\\ 717(x-1)^2+33, & 1\leqslant x\leqslant 2,\\ \frac{80}{2\sqrt{6}-1}(\sqrt{x-1}-1)+750, & 2\leqslant x\leqslant 25,\\ y(x),\quad y(25)=830, & 25\leqslant x,\end{cases}$$

where $y:[25,\infty)\to[0,\infty)$ is a continuous function. We have $g(t)=t$. Let us find

$$\lambda[t^2]=\int_0^1 t^2 g(t)\,\mathrm{d}t=\int_0^1 t^3\,\mathrm{d}t=\frac{1}{4}\in[0,1),$$

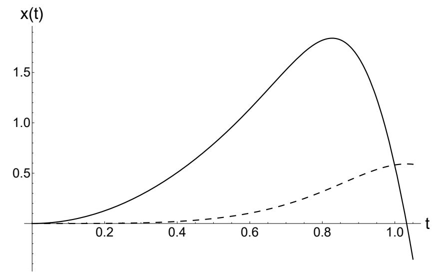

Figure 2. The solution $x_1(t)$.

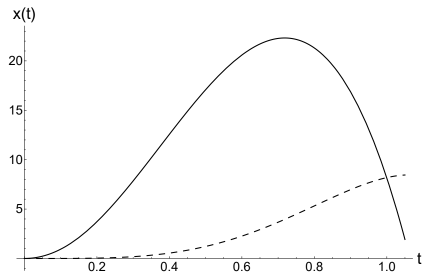

Figure 3. The solution $x_2(t)$.

$$\Gamma(s) = \int_0^1 G(t,s)\, g(t)\,\mathrm{d}t = \int_0^1 t\,G(t,s)\,\mathrm{d}t = \frac{1}{24}s\bigl(2-3s+s^3\bigr) \geqslant 0, \quad s \in [0,1],$$

$$I_1 \approx 33.371, \qquad I_2 \approx 88.548.$$

If we choose $r_1 = 1$, $r_2 = 8$, $r_3 = 8.4$, $r_4 = 25$, we get

$$\begin{aligned} f(t,x) &\leqslant I_1 r_1 \quad \text{for } (t,x) \in [0,1] \times [0, r_1], \\ f(t,x) &\geqslant I_2 r_2 \quad \text{for } (t,x) \in [0,1] \times \left[\frac{r_2}{4}, r_2\right], \\ f(t,x) &\geqslant I_2 r_3 \quad \text{for } (t,x) \in [0,1] \times \left[\frac{r_3}{4}, r_3\right], \\ f(t,x) &\leqslant I_1 r_4 \quad \text{for } (t,x) \in [0,1] \times [0, r_4]. \end{aligned}$$

Therefore, by Theorem 2, boundary value problem (8) has at least two positive solutions $x_1(t)$ and $x_2(t)$ such that

$$1 \leqslant \|x_1\| \leqslant 8, \qquad 8.4 \leqslant \|x_2\| \leqslant 25.$$

Solutions $x_1(t)$ and $x_2(t)$ (solid lines), together with $\int_0^t s\,x_1(s)\,\mathrm{d}s$ and $\int_0^t s\,x_2(s)\,\mathrm{d}s$ (dashed lines), are depicted in Figs. 2 and 3, respectively. The initial conditions for these solutions are $x_1(0) = 0$, $x_1'(0) = 0$, $x_1''(0) \approx 6.32$ and $x_2(0) = 0$, $x_2'(0) = 0$, $x_2''(0) \approx 188$.

Now let us discuss the difference between conditions on nonlinearity f in Theorem 2 and in [10, Thm. 2.2]. According to [10], the problem has at least two positive solutions if certain conditions are fulfilled, two of which are the following inequalities:

$$0 \leqslant \lim_{x\to 0+} \frac{f(x)}{x} < \mu_1, \qquad 0 \leqslant \lim_{x\to\infty} \frac{f(x)}{x} < \mu_1,$$

where μ_1 is called the principal characteristic value of operator T or the principal eigenvalue of the corresponding boundary value problem. We see that the first inequality is fulfilled, but the second one is not fulfilled because $\lim_{x\to\infty} f(x)/x$ can be every nonnegative number in our example. Also, the author would like to mention that our conditions allow us to get an estimate of the norm for positive solutions to the problem.

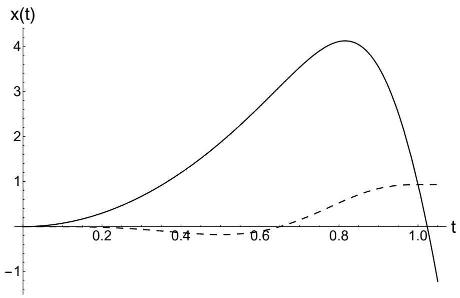

Figure 4. The solution of (9).

Example 3. Consider the problem

$$x''' + x^5 = 0, \quad t \in (0,1),$$
$$x(0) = x'(0) = 0, \qquad x(1) = -\int_0^1 x(t)\sin 2\pi t \, \mathrm{d}t. \tag{9}$$

We have $f(t,x) = x^5$ and $g(t) = -\sin 2\pi t$. Note that g changes its sign on $[0,1]$. Let us find

$$\lambda[t^2] = \int_0^1 t^2 g(t)\,\mathrm{d}t = -\int_0^1 t^2 \sin 2\pi t\,\mathrm{d}t = \frac{1}{2\pi} \in [0,1),$$
$$\Gamma(s) = \int_0^1 G(t,s)\,g(t)\,\mathrm{d}t = -\int_0^1 G(t,s)\sin 2\pi t\,\mathrm{d}t = \frac{\sin^2 \pi s}{4\pi^3} \geqslant 0, \quad s \in [0,1],$$
$$I_1 \approx 37.197, \qquad I_2 \approx 95.68.$$

If we choose $r_1 = 2$, $r_2 = 20$, we get

$$f(t,x) \leqslant I_1 r_1 \quad \text{for } (t,x) \in [0,1] \times [0, r_1],$$
$$f(t,x) \geqslant I_2 r_2 \quad \text{for } (t,x) \in [0,1] \times \left[\frac{r_2}{4}, r_2\right].$$

Thus, by Theorem 2, boundary value problem (9) has at least one positive solution $x(t)$ such that $2 \leqslant \|x\| \leqslant 20$. This solution (solid line), together with $-\int_0^t x(s)\sin 2\pi s\,\mathrm{d}s$ (dashed line), is depicted in Fig. 4. The initial conditions for this solution are $x(0) = x'(0) = 0$, $x''(0) \approx 14.95$.

4.2 Three-point problems

If $\lambda[x] = \alpha x(\tau)$, where $\alpha \in \mathbb{R}$ and $\tau \in (0,1)$, then we get three-point boundary conditions $x(0) = x'(0) = 0$, $x(1) = \alpha x(\tau)$. Let us determine the restrictions to be

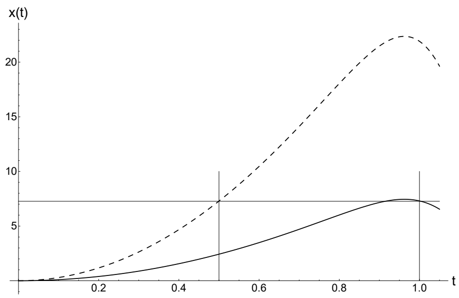

Figure 5. The solution $x(t)$ of (10).

placed on α so that the conditions $0 \leqslant \lambda[t^2] < 1$ and $\Gamma(s) \geqslant 0$, $s \in [0,1]$, are satisfied. Since $\lambda[t^2] = \alpha\tau^2$, we get $0 \leqslant \alpha\tau^2 < 1$. Since $\Gamma(s) = \int_0^1 G(t,s)\,\mathrm{d}\Lambda(t) = \alpha\,G(\tau,s)$, then for $\Gamma(s) \geqslant 0$, we need

$$\alpha s(1-\tau)\big((1-s)\tau + (\tau - s)\big) \geqslant 0 \quad \text{for } 0 \leqslant s \leqslant \tau < 1,$$
$$\alpha\tau^2(1-s)^2 \geqslant 0 \quad \text{for } 0 < \tau \leqslant s \leqslant 1,$$

or $\alpha > 0$. The total requirement is therefore

$$0 \leqslant \alpha\tau^2 < 1.$$

Example 4. Consider the problem

$$\begin{aligned} &x''' + \mathrm{e}^x - 1 = 0, \quad t \in (0,1), \\ &x(0) = x'(0) = 0, \qquad x(1) = 3\,x\left(\frac{1}{2}\right). \end{aligned} \tag{10}$$

We have $f(t,x) = \mathrm{e}^x - 1$, $\alpha = 3$, $\tau = 1/2$, and $\alpha\tau^2 = 3/4 \in [0,1)$. Let us find

$$I_1 = 4, \qquad I_2 \approx 16.047.$$

If we choose $r_1 = 2$, $r_2 = 24$, we get

$$f(t,x) \leqslant I_1\,r_1 \quad \text{for } (t,x) \in [0,1] \times [0, r_1],$$
$$f(t,x) \geqslant I_2\,r_2 \quad \text{for } (t,x) \in [0,1] \times \left[\frac{r_2}{4}, r_2\right].$$

Thus, by Theorem 2, boundary value problem (10) has at least one positive solution $x(t)$ such that $2 \leqslant \|x\| \leqslant 24$. This solution (solid line), together with $3x(t)$ (dashed line), is depicted in Fig. 5. The initial conditions for this solution are $x(0) = x'(0) = 0$, $x''(0) \approx 19.5$.

4.3 Four-point problems

If $\lambda[x] = \alpha_1 x(\tau_1) + \alpha_2 x(\tau_2)$, where $\alpha_1 \alpha_2 \in \mathbb{R}$ and $0 < \tau_1 < \tau_2 < 1$, then we get four-point boundary conditions $x(0) = x'(0) = 0$, $x(1) = \alpha_1 x(\tau_1) + \alpha_2 x(\tau_2)$. Since $\lambda[t^2] = \alpha_1 \tau_1^2 + \alpha_2 \tau_2^2$, we get

$$0 \leqslant \alpha_1 \tau_1^2 + \alpha_2 \tau_2^2 < 1.$$

Since $\Gamma(s) = \alpha_1 G(\tau_1, s) + \alpha_2 G(\tau_2, s)$, we need

$$\begin{aligned}
&\alpha_1 s(1-\tau_1)\big((1-s)\tau_1 + (\tau_1 - s)\big) + \alpha_2 s(1-\tau_2)\big((1-s)\tau_2 + (\tau_2 - s)\big) \geqslant 0 \\
&\quad \text{for } 0 \leqslant s \leqslant \tau_1 < \tau_2, \\
&\alpha_1 \tau_1^2 (1-s)^2 + \alpha_2 s(1-\tau_2)\big((1-s)\tau_2 + (\tau_2 - s)\big) \geqslant 0 \quad \text{for } \tau_1 \leqslant s \leqslant \tau_2, \\
&\alpha_1 \tau_1^2 (1-s)^2 + \alpha_2 \tau_2^2 (1-s)^2 \geqslant 0 \quad \text{for } \tau_1 < \tau_2 \leqslant s \leqslant 1.
\end{aligned}$$

The total requirement is therefore

$$0 \leqslant \alpha_1 \tau_1^2 + \alpha_2 \tau_2^2 < 1 \tag{11}$$

and

$$\alpha_1 (1-\tau_1)^2 \tau_1 + \alpha_2 (1-\tau_2)\big((1-\tau_1)\tau_2 + (\tau_2 - \tau_1)\big) \geqslant 0. \tag{12}$$

Example 5. Consider the problem

$$\begin{aligned}
&x''' + x^5 \mathrm{e}^{3x} = 0, \quad t \in (0,1), \\
&x(0) = x'(0) = 0, \qquad x(1) = 2x\left(\frac{1}{3}\right) - \frac{1}{2} x\left(\frac{2}{3}\right).
\end{aligned} \tag{13}$$

We have $f(t,x) = x^5 \mathrm{e}^{3x}$, $\alpha_1 = 2$, $\alpha_2 = -1/2$, $\tau_1 = 1/3$, and $\tau_2 = 2/3$. Note that α_2 is negative. All constants satisfy inequalities (11), (12). Let us find

$$I_1 \approx 32.685, \qquad I_2 \approx 107.55.$$

If we choose $r_1 = 1$, $r_2 = 8$, we get

$$\begin{aligned}
&f(t,x) \leqslant I_1 r_1 \quad \text{for } (t,x) \in [0,1] \times [0, r_1], \\
&f(t,x) \geqslant I_2 r_2 \quad \text{for } (t,x) \in [0,1] \times \left[\frac{r_2}{4}, r_2\right].
\end{aligned}$$

Thus, by Theorem 2, boundary value problem (13) has at least one positive solution $x(t)$ such that $1 \leqslant \|x\| \leqslant 8$. This solution is depicted in Fig. 6. The initial conditions for this solution are $x(0) = x'(0) = 0$, $x''(0) \approx 5.0965$. We get $x(1) \approx 0.001$, $2x(1/3) \approx 0.566$, and $-x(2/3)/2 \approx -0.565$.

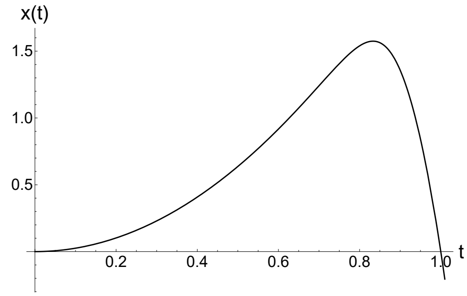

Figure 6. The solution of (13).

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