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An analysis of approximate controllability for Hilfer fractional delay differential equations of Sobolev type without uniqueness

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Abstract. This study focused on the approximate controllability results for the Hilfer fractional delay evolution equations of the Sobolev type without uniqueness. Initially, the Lipschitz condition is derived from the hypothesis, which is represented by a measure of noncompactness, in particular, nonlinearity. We also examined the continuity of the solution map of the Sobolev type of Hilfer fractional delay evolution equation and the topological structure of the solution set. Furthermore, we prove the approximate controllability of the fractional evolution equation of the Sobolev type with delay. Finally, we provided an example to illustrate the theoretical results.

Keywords: approximate controllability, Hilfer fractional evolution equations, condensing map, reachable set, Sobolev type, measure of noncompactness.

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1 Introduction

In recent years, fractional calculus has improved mathematical modeling, with specific results demonstrating that it is an excellent technique for conveying the recurring features of various ideas. Because fractional systems have been proven to be important equipment for displaying a range of highly detailed marvels in a huge variety of disciplines in physics and engineering areas, this combination has recently got a huge amount of attention; one can refer to books [19, 24, 28]. Furthermore, Sobolev-type differential systems appear in the mathematical modelling of various physical experiences, such as fluid flow through fissured rocks, the propagation of small amplitude long waves, thermodynamics, shear in second-order fluids, and medicine, among others. For more details, readers can refer to the recent research articles [2, 6, 20]. Hilfer [12] suggested a generalized Riemann–Liouville fractional derivative, denoted as Hilfer fractional derivative, which incorporates the Riemann–Liouville fractional derivative and the Caputo fractional derivative. This operator was discovered through a theoretical simulation of dielectric relaxation in glass-forming materials. After that, several researchers investigated fractional differential equations involving Hilfer fractional derivatives. In [11], the authors established the existence of mild solution for evolution equations involving Hilfer fractional derivative. Recently, the authors of [16] analyzed the controllability results of Hilfer fractional neutral differential equations with infinite delay through the measures of noncompactness theory. In the analysis and design of control systems, the concept of exact and approximate controllability is an effective instrument.

The authors of [13] considered an initial-value problem comprising generalization and analyzed the existence as well as the uniqueness of its solution. They first presented an approximation sequence using a successive substitution approach, and then demonstrated that the sequence uniformly converges to the unique solution of the regularised ψ-Hilfer fractional differential equation. In [1], the authors studied the Lagrangian and derived the classical equations of motion using the Euler–Lagrange equations of integer order. Furthermore, the generalized Lagrangian is introduced by using noninteger, so-called fractional, derivative operators. Then the resulting fractional Euler–Lagrangian equations are generated and solved numerically.

A semigroup-theoretic development of theories for the analogs of deterministic evolution equations is both powerful and beneficial within a unified context. In the case of infinite dimensional systems, two basic concepts of exact controllability and approximate controllability emerged in the applications. Exact controllability enables to steer the system to an arbitrary final state, while approximate controllability means that the system can be steered to an arbitrarily small neighbourhood of the final state. Approximate controllability is essentially a weaker notion than exact controllability, and it gives the possibility of steering the system to states, which form the dense subspace in the state space. However, in the case of infinite dimensional systems, exact controllability appears rather exceptionally but in the case of finite dimensional systems, notions of exact and approximate controllability coincide. The controllability of nonlinear deterministic systems is well known in the literature. The idea of controllability has enormous influence in mathematical control theory and engineering because it is closely related to pole assignment, structural decomposition, observer design, etc. In [21], the author established sufficient conditions for the approximate controllability of certain classes of abstract evolution equations with nonlocal initial conditions.

Following are our article's significant contributions:

1. We establish a set of sufficient conditions for the approximate controllability results for Hilfer fractional delay differential equations of Sobolev type without uniqueness under the assumption that the corresponding linear system is approximately controllable.
2. In [23], the authors commented on an error present in the recent and extensive literature on the exact controllability of abstract control differential problems. But, in our paper, we establish only sufficient conditions for the approximate controllability results of fractional differential systems to avoid these kinds of errors.
3. It is assumed that C_0-semigroup $S(t)$ is compact, and consequently, the associated linear control system is not exactly controllable but only approximately controllable.
4. We show that our result has no analog for the concept of complete controllability. To the best of our knowledge, the approximate controllability for Hilfer fractional delay differential system of Sobolev type has not been studied in this connection by using Gronwall's inequality and Lipschitz's nonlinearity condition.
5. In the end, we give an example of a system that is not completely controllable but is approximately controllable.
6. We combine the ideas of the Hilfer fractional derivative with the mathematical formulation of the Caputo derivative and Riemann–Liouville derivative.
7. "An analysis of approximate controllability for Hilfer fractional delay differential systems of Sobolev type without uniqueness in Banach space" is a problem that, as far as we know, has not been looked at before. It has not been combined with any of the operators looked at in this article, so that will be one of the new things we look at in this study.

Moreover, in [17], the author provided a detailed study on the approximate controllability for semilinear functional differential equations without uniqueness by using the fixed point theory for multivalued maps with nonconvex values. The author proved that the nonlinear problem is approximately controllable, provided that the corresponding linear problem is approximately controllable. Additionally, the author obtained some results on the continuity of the solution map and the topological structure of the solution set for the considered problem. Further, the authors of [18] investigated the control systems governed by abstract Volterra equations without uniqueness in a Banach space. Recently, in [26], the author proved the approximate controllability of a class of second-order functional evolution differential equations without uniqueness by using the fixed point theory for the multivalued maps with nonconvex values. Motivated by the above consideration, we are generalizing them to approximate controllability for Hilfer fractional delay differential equations of Sobolev type without uniqueness by employing the fixed point theory for multivalued maps with nonconvex values.

To this purpose,

(i) Section 2 presents the definitions of the Riemann–Liouville derivative, Caputo derivative, Hilfer fractional derivative, measures of noncompactness, semigroups, and control systems.

(ii) We discuss the topological structure in Section 3. This approach appears to be valid in the sense that many applications of classical topological dynamics to the study of differential equation solutions can now be carried out without the need for the uniqueness assumption.

(iii) Further, we establish the outcomes of approximate controllability with delay by utilizing ANR-space, and AR space in Section 4.

(iv) Finally, in Section 5, we provide an application to demonstrate our main arguments, and some inferences are established in the end.

This method seems to work in the sense that many applications of classical topological dynamics results to the study of differential equation solutions can now be carried out without the need for the uniqueness assumption. So, we choose the Sobolev type of Hilfer fractional delay differential systems, which has the following form:

$$\begin{aligned} &D_{0^+}^{\nu,\mu}\big[\mathcal{K}x(t)\big] = Ax(t) + G\big(t, x(t), x_t\big) + Bu(t), \quad t \in J' := (0, b], \\ &I_{0^+}^{(1-\nu)(1-\mu)} x(t) = p(t), \quad t \in [-\hbar, 0], \end{aligned} \tag{1}$$

where $D_{0^+}^{\nu,\mu}$ is the Hilfer fractional derivative and whose order $\mu \in (1/2, 1)$ and type $\nu \in [0, 1]$, $x(t)$ takes values in a Hilbert space $\mathscr{X}$. Let $J = [0, b]$ and A be an infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geqslant 0}$. $u(t) \in U$, and the bounded linear operator $B : L^2(J, U) \to L^2(J, \mathscr{X})$, where U is also a Hilbert space. The history x_t is characterized as $x_t(\theta) := x(t + \theta)$ for $\theta \in [-\hbar, 0]$.

2 Preliminaries

In this section, the essential basic preliminaries, definitions, notations, and lemmas of fractional calculus and multivalued maps, which are needed to establish the main results, are presented.

Throughout this paper, by $C(J, \mathscr{X})$ and $C(J', \mathscr{X})$ we denote the spaces of all continuous functions from J to $\mathscr{X}$ and J' to $\mathscr{X}$, respectively. Assume that $\gamma = \nu + \mu - \nu\mu$, then $(1 - \gamma) = (1 - \nu)(1 - \mu)$. Now, characterize $C_{1-\gamma}(J, \mathscr{X}) = \{x \in C(J', \mathscr{X})$: $t^{1-\gamma}x(t) \in C(J, \mathscr{X})\}$, $\|\cdot\|_{C_{1-\gamma}}$ represented by $\|x\|_{C_{1-\gamma}} = \sup\{t^{1-\gamma}\|x(t)\|, t \in J'\}$. Obviously, $C_{1-\gamma}(J, \mathscr{X})$ is a Banach space.

Definition 1. The fractional integral of order $\mu > 0$ for a function $G : [a, +\infty) \to \mathbb{R}$ is defined as

$$I_{a^+}^{\mu} G(t) = \frac{1}{\Gamma(\mu)} \int_a^t \frac{G(s)}{(t - s)^{1-\mu}}\, \mathrm{d}s, \quad t > a,\ \mu > 0,$$

provided the right side is point-wise defined on $[a, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2. The Riemann–Liouville derivative of order $\mu > 0$ for a function $G : [a, +\infty) \to \mathbb{R}$ is defined as

$$ {}^{L}D^{\mu}_{a^+} G(t) = \frac{1}{\Gamma(m-\mu)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_a^t \frac{G(s)}{(t-s)^{\mu+1-m}} \,\mathrm{d}s, \quad t > a,\ m-1 < \mu < m. $$

Definition 3. The Caputo derivative of order $\mu > 0$ for a function $G : [a, +\infty) \to \mathbb{R}$ is defined as

$$ {}^{C}D^{\mu}_{a^+}\ G(t) = \frac{1}{\Gamma(m-\mu)} \int_a^t \frac{G^m(s)}{(t-s)^{\mu+1-m}} \,\mathrm{d}s, \quad t > a,\ m-1 < \mu < m. $$

Definition 4. The Hilfer fractional derivative of order $0 < \mu < 1$ and $0 \leqslant \nu \leqslant 1$ with the lower limit b is defined as

$$ D^{\nu,\mu}_{a^+} G(t) = \bigl(I^{\nu(1-\mu)}_{a^+} D\bigl(I^{(1-\nu)(1-\mu)}_{a^+} G\bigr)\bigr)(t), \quad \text{where } D = \frac{\mathrm{d}}{\mathrm{d}t}. $$

Remark 1. (See [12].) The Hilfer fractional derivative corresponds to the classical Riemann–Liouville fractional derivative and the classical Caputo fractional derivative:

$$ D^{\nu,\mu}_{a+} G(t) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\mu}_{0+} G(t) = {}^{L}D^{\mu}_{0+} G(t), & \nu = 0,\ 0 < \mu < 1,\ a = 0; \\ I^{1-\mu}_{0+} \frac{\mathrm{d}}{\mathrm{d}t} G(t) = {}^{C}D^{\mu}_{0+} G(t), & \nu = 1,\ 0 < \mu < 1,\ a = 0. \end{cases} $$

We introduce the following assumptions on the operators $A : D(A) \subset \mathscr{X} \to \mathscr{X}$ and $\mathcal{K} : D(\mathcal{K}) \subset \mathscr{X} \to \mathscr{X}$:

(i) $D(\mathcal{K}) \subset D(A)$ and $\mathcal{K}$ is bijective.
(ii) A and $\mathcal{K}$ are closed linear operators.
(iii) $\mathcal{K}^{-1} : \mathscr{X} \to D(A)$ is continuous.

Furthermore, from (i) and (ii) we see that $\mathcal{K}^{-1}$ is closed. From (iii) and closed graph theorem we have the boundedness of the linear operator $A\mathcal{K}^{-1} : \mathscr{X} \to \mathscr{X}$. Let $\|\mathcal{K}^{-1}\| = \widetilde{\mathcal{K}}_1$ and $\|\mathcal{K}\| = \widetilde{\mathcal{K}}_2$.

We introduce the Wright function $\mathcal{M}_\mu$, which is defined by

$$ \mathcal{M}_\mu(\varrho) = \sum_{m=1}^{\infty} \frac{(-\varrho)^{m-1}}{(m-1)\Gamma(1-m\mu)}, \quad 0 < \mu < 1,\ \varrho \in \mathbb{C}, $$

and satisfies

$$ \int_0^\infty \varrho^v \mathcal{M}_\mu(\varrho) d\varrho = \frac{\Gamma(1+v)}{\Gamma(1+\mu v)} \quad \text{for } \varrho \geqslant 0. $$

Lemma 1. (*See* [12,30].) *The operators $T_{\nu,\mu}(t)$ and $Q_\mu(t)$ have the following properties.*

(i) *$Q_\mu(t)$ is continuous in the uniform operator topology for $t \geqslant 0$, and $\{Q_\mu(t)\colon t \geqslant 0\}$ is uniformly bounded, i.e., there exists $M > 1$ such that the following holds: $\sup_{t\in[0,\infty)} |S(t)| < M$.*

(ii) *For any fixed $t > 0$, $T_{\nu,\mu}(t)$ and $Q_\mu(t)$ are linear and bounded operators, and*

$$\left\|Q_\mu(t)x\right\| \leqslant \frac{Mt^{\mu-1}\|x\|}{\Gamma(\mu)}, \qquad \left\|T_{\nu,\mu}(t)x\right\| \leqslant \frac{Mt^{\gamma-1}\|x\|}{\Gamma(\nu(1-\mu)+\mu)}.$$

(iii) *$\{T_{\nu,\mu}(t)\colon t > 0\}$ and $\{Q_\mu(t)\colon t > 0\}$ are strongly continuous.*

Let $\mathcal{Z}$ be a Banach space. Denote by $\mathscr{B}(\mathcal{Z})$ the collection of nonempty bounded subsets of $\mathcal{Z}$. We now see the well-known definition of measures of noncompactness.

Definition 5. (See [3].) Let $(\mathcal{A}, \leqslant)$ be a partially ordered set. A function $\alpha : \mathscr{B}(\mathcal{Z}) \to \mathcal{A}$ is called a measure of noncompactness in $\mathcal{Z}$. If $\alpha(\overline{\text{co}}\,\Psi) = \alpha(\Psi)$ for $\Psi \in \mathscr{P}(\mathcal{Z})$, where $\overline{\text{co}}\,\Psi$ is closure of the convex hull of Ψ.

The measure of noncompactness (MNC) function α is said to be

(i) Monotone if $\Psi_1, \Psi_2 \in \mathscr{P}(\mathcal{Z})$, $\Psi_1 \subset \Psi_2$ imply $\alpha(\Psi_1) \leqslant \alpha(\Psi_2)$;

(ii) Nonsingular if $\alpha(\{c\} \cup \Psi) = \alpha(\Psi)$ for any $c \in \mathcal{Z}$ and $\Psi \in \mathscr{P}(\mathcal{Z})$;

(iii) Invariant with respect to union with compact sets if $\alpha(\mathscr{L} \cup \Psi) = \alpha(\Psi)$ for every relatively compact set $\mathscr{L} \subset \mathcal{Z}$ and $\Psi \in \mathscr{P}(\mathcal{Z})$;

(iv) If $\mathcal{A}$ is a cone in a normed space, we say that α is algebraically semiadditive. Then $\alpha(\Psi_1 + \Psi_2) \leqslant \alpha(\Psi_1) + \alpha(\Psi_2)$ for any $\Psi_1, \Psi_2 \in \mathscr{P}(\mathcal{Z})$;

(v) Regular if $\alpha(\Psi) = 0$ is equivalent to the relative compactness of Ψ.

The Hausdorff measure of noncompactness $\mathscr{R}(\cdot)$ is a significant illustration of measure of noncompactness and is defined as follows:

$$\mathscr{R}(\Psi) = \inf\{\epsilon\colon \Psi \text{ has a finite } \epsilon\text{-net}\}.$$

Moreover, we can apply the measure of noncompactness. For any bounded set $\mathcal{U} \in C(J, \mathcal{Z})$, the modulus of fiber noncompactness of $\mathcal{U}$ is defined by

$$\omega_K(\mathcal{U}) = \sup_{t\in J} \mathrm{e}^{-Kt} \mathscr{R}\big(\mathcal{U}(t)\big), \tag{2}$$

where K denotes a positive constant. The modulus of equicontinuity of $\mathcal{U}$ is

$$\wp_C(\mathcal{U}) = \lim_{\phi\to 0} \sup_{y\in\mathcal{U}} \max_{|t_1-t_2|<\phi} \left\|y(t_1) - y(t_2)\right\|_{\mathcal{Z}}. \tag{3}$$

From [15] these MNCs satisfy all properties stated in Definition 5 except regularity. Now, consider the function $\mathscr{R}^* : \mathscr{B}(C(J, \mathcal{Z})) \to \mathbb{R}^2_+$,

$$\mathscr{R}^*(\Psi) = \max_{\mathcal{U}\in\mathscr{R}(\Psi)} \big(\omega_K(\mathcal{U}), \wp_C(\mathcal{U})\big),$$

where the measure of noncompactness ω_K and $\wp_C$ are given in (2) and (3), respectively, $\Delta(\Psi)$ is the collection of all countable subsets of Ψ, the maximum is taken in the sense of the partial order in the cone $\mathbb{R}^2_+$. Using the considerations in [15], $\mathscr{R}^*$ is well defined, that is, the maximum is achieved in $\Delta(\Psi)$, and $\mathscr{R}^*$ is an measure of noncompactness in the space $C(J, \mathcal{Z})$, which satisfies all properties in Definition 5 (see [15]).

Definition 6. (See [15].) A continuous map $\mathcal{G} : \mathcal{Y} \subseteq \mathcal{Z} \to \mathcal{Z}$ is said to be condensing with respect to an MNC α (α-condensing) if for any bounded set $\Psi \subset \mathcal{Y}$, the relation $\alpha(\Psi) \leqslant \alpha(\mathcal{G}(\Psi))$ implies the relative compactness of Ψ.

Let α be a monotone nonsingular measure of noncompactness in $\mathcal{Z}$. The application of the topological degree theory for condensing maps (see, e.g., [15]) yields the following fixed point principle.

Remark 2. Let $(\mathcal{Z}, \leqslant, \|\cdot\|)$ be a partially ordered complete normed linear space such that the order relation $\leqslant$ and the norm $\|\cdot\|$ are compatible. Suppose that $\mathcal{G} : \mathcal{Z} \to \mathcal{Z}$ is a partially continuous, nondecreasing, partially bounded, and partially condensing mapping. If $\mathcal{Z}$ is regular and there exists an element $x_0 \in \mathcal{Z}$ such that $x_0 \leqslant \mathcal{G}x_0$ or $x_0 \geqslant \mathcal{G}x_0$, then $\mathcal{G}$ has a fixed point x^*, and the sequence $\mathcal{G}^m x_0$ of successive iterations converges monotonically to x^*.

Theorem 1. (*See* [15].) *Let $\mathcal{E}$ be a bounded convex closed subset of $\mathcal{Z}$, and let $\mathcal{G} : \mathcal{E} \to \mathcal{E}$ is an α-condensing map. Then the fixed point set of $\mathcal{G}$,* $\operatorname{Fix}(\mathcal{G}) := \{x = \mathcal{G}(x)\}$, *is nonempty compact set.*

Now, we consider the nonlinearity $G : J \times \mathscr{Z} \times C_{1-\gamma}([-\hbar, 0], \mathscr{Z}) \to \mathscr{Z}$ in (1). Define

$$\|\psi\|_\hbar = \|\psi\|_{C_{1-\gamma}([-\hbar,0],\mathscr{Z})} := \sup_{t\in[-\hbar,0]} \bigl\{t^{1-\gamma}\bigl\|\psi(t)\bigr\|\bigr\},$$

where $\|\cdot\| = \|\cdot\|_{\mathscr{Z}}$.

Some of following hypotheses are:

(H1) The semigroup $S(\cdot)$ generated by A is compact, i.e., $S(t)$ is a compact operator for each $t > 0$.

(H2) $G(t, \cdot, \cdot)$ is continuous for each $t \in J$, and $G(\cdot, \eta, \psi)$ is measurable for each $\eta \in \mathscr{Z}$, $\psi \in C_{1-\gamma}([-\hbar, 0], \mathscr{Z})$.

(H3) There exist $e_1, e_2, e_3 \in L^1(J)$ such that

$$\bigl\|G(t, \eta, \psi)\bigr\| \leqslant e_1(t)\|\eta\| + e_2(t)\|\psi\|_\hbar + e_3(t)$$

for any $(\eta, \psi) \in \mathscr{Z} \times C_{1-\gamma}([-\hbar, 0], \mathscr{Z})$.

(H4) There exist $\xi_1, \xi_2 : J \times J \to \mathbb{R}$ such that $\xi_1(t, \cdot), \xi_2(t, \cdot) \in L^1(0, t)$ for all $t > 0$ and

$$\mathscr{R}\bigl(Q_\mu(t-s)G(s, \Psi, \Psi')\bigr) \leqslant \xi_1(t, s)\mathscr{R}(\Psi) + \xi_2(t, s) \sup_{-\hbar\leqslant\theta\leqslant 0} \mathscr{R}\bigl(\Psi'(\theta)\bigr)$$

for all bounded subsets $\Psi \subset \mathscr{Z}$, $\Psi' \subset C_{1-\gamma}([-\hbar, 0], \mathscr{Z})$ and for almost everywhere $t, s \in J$.

Remark 3. Note that (H4) can be deduced from (H3) when $\mathscr{X} = \mathbb{R}^m$, that is, the locally bounded property implies that the set $Q_\mu(t-s)G(s,\Psi,\Psi')$ is bounded in $\mathbb{R}^m$. Then for every $t,s \in J$, $Q_\mu(t-s)G(s,\Psi,\Psi')$ is precompact. Especially, if $Q_\mu(t)$ is compact for $t>0$, then (H4) is testified obviously with $\xi_1 = \xi_2 = 0$.

Definition 7. (See [29].) A function $x \in C([-\hbar,b];\mathscr{X})$ is a mild solution of (1) corresponding to control u if

$$x(t) = \begin{cases} \varphi(t) \quad \text{for } t \in [-\hbar,0], \\ \mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0) + \int_0^t \mathcal{K}^{-1}Q_\mu(t-s)Bu(s)\,\mathrm{d}s \\ \quad + \int_0^t \mathcal{K}^{-1}Q_\mu(t-s)G(s,x(s),x_s)\,\mathrm{d}s \quad \text{for } t \in J', \end{cases} \tag{4}$$

where

$$T_{\nu,\mu}(t) = I_{0+}^{\nu(1-\mu)}Q_\mu(t), \quad Q_\mu(t) = t^{\mu-1}V_\mu(t), \quad V_\mu(t) = \int_0^\infty \mu\theta\mathcal{M}_\mu(\theta)S(t^\mu\theta)d\theta.$$

For the sake of convenience, we write (4) as

$$x(t) = \begin{cases} \varphi(t) \quad \text{for } t \in [-\hbar,0], \\ \mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0) + \int_0^t \mathcal{K}^{-1}(t-s)^{\mu-1}V_\mu(t-s)Bu(s)\,\mathrm{d}s \\ \quad + \int_0^t \mathcal{K}^{-1}(t-s)^{\mu-1}V_\mu(t-s)G(s,x(s),x_s)\,\mathrm{d}s \quad \text{for } t \in J'. \end{cases}$$

Assume that

$$\mathcal{S}_p = \{z \in C_{1-\gamma}(J,\mathscr{X})\colon\ z(0) = p(0)\}.$$

For all $z \in \mathcal{S}_p$, we set

$$z[p](t) = \begin{cases} p(t), & t \in [-\hbar,0], \\ z(t), & t \in J'. \end{cases}$$

For any $u \in L^2(J,U)$, denote by $\mathcal{G}_u$ the operator acting on $\mathcal{S}_p$ such that

$$\mathcal{G}_u(x)(t) = \mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0) + \int_0^t \mathcal{K}^{-1}Q_\mu(t-s)Bu(s)\,\mathrm{d}s + \int_0^t \mathcal{K}^{-1}Q_\mu(t-s)G(s,x(s),x[p]_s)\,\mathrm{d}s. \tag{5}$$

Define a mapping $\mathscr{F} : L^1(J,\mathscr{X}) \to C_{1-\gamma}(J,\mathscr{X})$ as

$$\mathscr{F}(g)(t) = \int_0^t \mathcal{K}^{-1}Q_\mu(t-s)g(s)\,\mathrm{d}s. \tag{6}$$

Moreover, putting

$$\mathscr{V}_G(x)(t) = G\big(t, x(t), x[p]_t\big), \tag{7}$$

we have

$$\mathcal{G}_u(x) = T_{\nu,\mu}(\cdot)p(0) + \mathscr{F}\big(Bu + \mathscr{V}_G(x)\big).$$

It is evident that $x \in \mathcal{S}_p$ is a fixed point of $\mathcal{G}_u$ if and only if $x[p]$ is a mild solution of (1).

Lemma 2. *Suppose* (H2) *and* (H3) *hold. Then $\mathcal{G}_u(\{z_m\})$ is relatively compact for every $\{z_m\} \subset \mathcal{S}_p$ satisfying $\omega_K(\{z_m\}) = 0$. We note that $\wp_C(\mathcal{G}_u(\{z_m\})) = 0$.*

Proof. We state as a fact that $\{g_m\} \subset L^1(J, \mathscr{Z})$ is a semicompact sequence, that is, there exists $r \in L^1(J)$ such that $\|g_m(t)\| \leqslant r(t)$ for all m and for almost everywhere $t \in J$; $\mathscr{R}(\{g_m(t)\}) = 0$ for almost everywhere $t \in J$. Then $\mathscr{F}(g_m)$ is relatively compact in $C_{1-\gamma}(J, \mathscr{Z})$ (see [15]).

Now we consider $\{z_m\} \subset \mathcal{S}_p$ to be bounded sequence such that $\omega_K(\{z_m\}) = 0$. From (H3) we have that $g_m(t) = G(t, z_m(t), z_m[p]_t)$ satisfies the estimate

$$\begin{aligned}\big\|g_m(t)\big\| &\leqslant e_1(t)\big\|z_m(t)\big\| + e_2(t)\Big(\sup_{s\in[0,t]}\big\|z_m(s)\big\| + \|p\|_\hbar\Big) + e_3(t)\\ &\leqslant r(t) := M_1\big[e_1(t) + e_2(t)\big] + e_2(t)\|p\|_\hbar + e_3(t),\end{aligned}$$

where M_1 is an upper bound for $\{z_m\}$ in $C_{1-\gamma}(J, \mathscr{Z})$. As $\omega_K(\{z_m\}) = 0$, one has $\mathscr{R}(\{z_m(t)\}) = 0$ for all $t \in J$; that is, $\{z_m(t)\}$ is relatively compact for any $t \in J$. Then it is obvious that $\{z_m[p]_s\}$ is a relatively compact set in $C_{1-\gamma}([-\hbar, 0]; \mathscr{Z})$. Since $G(t, \cdot, \cdot)$ is continuous, we get that $G(t, z_m(t), z_m[p]_t)$ is relatively compact for a.e. $t \in J$. Thus $\{g_m\}$ is a semicompact sequence, and then

$$\mathcal{G}_u\big(\{z_m\}\big) = \mathscr{F}\big(\{g_m\}\big) + \mathscr{F}(Bu) + T_{\nu,\mu}(t)p(0)$$

is relatively compact in $C_{1-\gamma}(J, \mathscr{Z})$. For instance, $\mathcal{G}_u(\{z_m\})$ is an equicontinuous set, or likewise, $\wp_C(\mathcal{G}_u(\{z_m\})) = 0$. This completes the proof of this lemma. □

Note. (See [15].) Now choosing K the definition of ω_K in (2) such that

$$\Lambda := 2\sup_{t\in J}\int_0^t \mathcal{K}^{-1}\mathrm{e}^{-K(t-s)}\big[\xi_1(t,s) + \xi_2(t,s)\big]\,\mathrm{d}s < 1. \tag{8}$$

We will prove that $\mathcal{G}_u$ is $\mathscr{R}^*$-condensing. To prove this, we must have the following proposition.

Proposition 1. (*See* [27].) *If $\{w_m\} \subset L^1(J, \mathscr{Z})$ such that $\|w_m(t)\| \leqslant \nu(t)$ for a.e. $t \in J$ and for some $\nu \in L^1(J)$, then $\mathscr{R}(\{\int_0^t w_m(s)\,\mathrm{d}s\}) \leqslant 2\int_0^t \mathscr{R}(\{w_m(s)\})\,\mathrm{d}s$ for $t \in J$.*

Lemma 3. *Suppose* (H2)–(H4) *are satisfied, then $\mathcal{G}_u$ is $\mathscr{R}^*$-condensing.*

Proof. From (H2)–(H3) we conclude that $\mathcal{G}_u$ is a continuous mapping. Assume that $\varPsi \subset \mathcal{S}_p$ is bounded set such that

$$\mathscr{R}^*(\varPsi) \leqslant \mathscr{R}^*\big(\mathcal{G}_u(\varPsi)\big). \tag{9}$$

We will prove that $\varPsi$ is relatively compact in $C_{1-\gamma}(J, \mathscr{X})$. By the definition of $\mathscr{R}^*$ there exists $\{z_m\} \subset \varPsi$ such that

$$\begin{aligned} \mathscr{R}^*\big(\mathcal{G}_u(\varPsi)\big) &= \big(\omega_K\big(\mathcal{G}_u\big(\{z_m\}\big)\big), \wp_C\big(\mathcal{G}_u\big(\{z_m\}\big)\big)\big) \\ &\geqslant \big(\omega_K\big(\{z_m\}\big), \wp_C\big(\{z_m\}\big)\big). \end{aligned} \tag{10}$$

Now, we first give an estimate for $\omega_K(\mathcal{G}_u(\{z_m\}))$. From (H4) and Proposition 1 one can get

$$\begin{aligned} &\mathscr{R}\big(\mathcal{G}_u\big(\{z_m\}\big)(t)\big) \\ &\quad \leqslant \mathscr{R}\Bigg(\Bigg\{\int_0^t \mathcal{K}^{-1}Q_\mu(t-s)G\big(s, z_m(s), z_m[p]_s\big)\,\mathrm{d}s\Bigg\}\Bigg) \\ &\quad \leqslant 2\int_0^t \mathscr{R}\big(\big\{\mathcal{K}^{-1}Q_\mu(t-s)G\big(s, z_m(s), z_m[p]_s\big)\big\}\big)\,\mathrm{d}s \\ &\quad \leqslant 2\int_0^t \mathcal{K}^{-1}\Big[\xi_1(t,s)\mathscr{R}\big(\big\{z_m(s)\big\}\big) + \xi_2(t,s)\sup_{\tau\in[-\hbar,0]}\mathscr{R}\big(\big\{z_m[p](s+\tau)\big\}\big)\Big]\,\mathrm{d}s \\ &\quad \leqslant 2\int_0^t \mathcal{K}^{-1}\Big[\xi_1(t,s)\mathscr{R}\big(\big\{z_m(s)\big\}\big) + \xi_2(t,s)\sup_{\varsigma\in[0,s]}\mathscr{R}\big(\big\{z_m(\varsigma)\big\}\big)\Big]\,\mathrm{d}s \\ &\quad \leqslant 2\int_0^t \mathcal{K}^{-1}\big[\xi_1(t,s) + \xi_2(t,s)\big]\sup_{\varsigma\in[0,s]}\mathscr{R}\big(\big\{z_m(\varsigma)\big\}\big)\,\mathrm{d}s. \end{aligned}$$

Next,

$$\begin{aligned} &\mathrm{e}^{-Kt}\mathscr{R}\big(\mathcal{G}_u\big(\{z_m\}(t)\big)\big) \\ &\quad \leqslant 2\int_0^t \mathrm{e}^{-K(t-s)}\mathcal{K}^{-1}\big[\xi_1(t,s) + \xi_2(t,s)\big]\sup_{\varsigma\in[0,s]}\mathrm{e}^{-K\varsigma}\mathscr{R}\big(\big\{z_m(\varsigma)\big\}\big)\,\mathrm{d}s \\ &\quad \leqslant 2\omega_K\big(\{z_m\}\big)\int_0^t \mathcal{K}^{-1}\mathrm{e}^{-K(t-s)}\big[\xi_1(t,s) + \xi_2(t,s)\big]\,\mathrm{d}s. \end{aligned}$$

The last inequality implies

$$\omega_K\big(\mathcal{G}_u\big(\{z_m\}\big)\big) \leqslant \varLambda\omega_K\big(\{z_m\}\big).$$

From (10) we conclude that $\omega_K(\{z_m\}) \leqslant \Lambda\omega_K(\{z_m\})$. Then $\omega_K(\{z_m\}) = 0$ due to the fact that $\Lambda < 1$ as chosen in (8). This turns out that $\omega_K(\mathcal{G}_u(\{g_m\})) = 0$. In view of Lemma 2, one can get $\wp_C(\mathcal{G}_u(\{z_m\})) = 0$. Again, from (10) one can get $\mathscr{R}^*(\mathcal{G}_u(\Psi)) = 0$. Hence $\mathscr{R}^*(\Psi) = 0$ due to (9). The proof is complete. □

Theorem 2. *Assume that* (H2)–(H4) *are satisfied. Then the solution set of* (1) *is nonempty and compact. Further, any solution of* (1) *satisfies the following estimate*:

$$\sup_{\varsigma\in(0,t]} \|x(\varsigma)\|_{\mathscr{X}} \leqslant \left(\mathbb{C}^* + v_2\|Bu\|_{L^1(J,\mathscr{X})}\right) \times \exp\left\{\frac{M\widetilde{\mathcal{K}}_1 b^{1-\gamma}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[e_1(s)+e_2(s)\right]\mathrm{d}s\right\} \tag{11}$$

for $t \in J'$, *where*

$$v_1 = \frac{M\widetilde{\mathcal{K}}_1\widetilde{\mathcal{K}}_2}{\Gamma(\nu(1-\mu)+\mu)}, \qquad v_2 = \frac{M\widetilde{\mathcal{K}}_1 b^{\mu+1-\gamma}}{\Gamma(\mu+1)},$$
$$\mathbb{C}^* = v_1\|p(0)\| + v_2\left(\|e_3\|_{L^1(J)} + \|p\|_\hbar\|e_2\|_{L^1(J)}\right).$$

Proof. The solution operator $\mathcal{G}_u$ is $\mathscr{R}^*$-condensing due to Lemma 3. Let $\kappa \in C_{1-\gamma}(J,\mathscr{X})$ be the solution of integral equation

$$\kappa(t) = v_1\|p(0)\| + \widetilde{\mathcal{K}}_1\frac{M}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[e_1(s)+e_2(s)\right]\kappa(s)\,\mathrm{d}s + v_2\left(\|e_3\|_{L^1(J)} + \|p\|_\hbar\|e_2\|_{L^1(J)} + \|Bu\|_{L^1(J,\mathscr{X})}\right),$$

and let

$$\mathcal{E} = \left\{z \in \mathcal{S}_p\colon \sup_{s\in[0,t]}\left\{s^{1-\gamma}\|z(s)\|\right\} \leqslant \kappa(t),\ t \in J'\right\}.$$

Then it is easy to check that $\mathcal{E}$ is a bounded, closed, and convex set. Further, if $z \in \mathcal{E}$, then

$$\begin{aligned}
&\left\|\mathcal{G}_u(z)(t)\right\| \\
&= \sup_{t\in J'} t^{1-\gamma}\left\|\mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0) + \int_0^t \mathcal{K}^{-1}Q_\mu(t-s)\left[Bu(s) + G\left(s, z(s), z[p]_s\right)\right]\mathrm{d}s\right\| \\
&\leqslant \left\|\mathcal{K}^{-1}\frac{M}{\Gamma(\nu(1-\mu)+\mu)}\mathcal{K}p(0)\right\| + b^{1-\gamma}\int_0^t \left\|\mathcal{K}^{-1}Q_\mu(t-s)Bu(s)\right\|\mathrm{d}s \\
&\quad + b^{1-\gamma}\int_0^t \left\|\mathcal{K}^{-1}Q_\mu(t-s)G\left(s, z(s), z[p]_s\right)\right\|\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant v_1\big\|p(0)\big\| + v_2\|Bu\|_{L^1(J,\mathscr{X})} \\
&\quad + \widetilde{\mathcal{K}}_1 \frac{Mb^{1-\gamma}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\big[e_1(s)\big\|y(s)\big\| + e_2(s)\big\|y[p]_s\big\|_{\hbar} + e_3(s)\big]\,\mathrm{d}s \\
&\leqslant v_1\big\|p(0)\big\| \\
&\quad + \widetilde{\mathcal{K}}_1 \frac{Mb^{1-\gamma}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\Big[e_1(s)\big\|z(s)\big\| + e_2(s)\Big(\sup_{\varsigma\in[0,s]}\big\|z(\varsigma)\big\| + \|p\|_{\hbar}\Big)\Big]\,\mathrm{d}s \\
&\quad + v_2\big(\|e_3\|_{L^1(J)} + \|Bu\|_{L^1(J,\mathscr{X})}\big) \\
&\leqslant v_1\big\|p(0)\big\| + \widetilde{\mathcal{K}}_1 \frac{Mb^{1-\gamma}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\big[e_1(s)+e_2(s)\big] \sup_{\varsigma\in[0,s]}\big\|z(\varsigma)\big\|\,\mathrm{d}s \\
&\quad + v_2\big(\|e_3\|_{L^1(J)} + \|p\|_{\hbar}\|e_2\|_{L^1(J)} + \|Bu\|_{L^1(J,\mathscr{X})}\big) \\
&\quad \leqslant v_1\big\|p(0)\big\| + \widetilde{\mathcal{K}}_1 \frac{M}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\big[e_1(s)+e_2(s)\big]\kappa(s)\,\mathrm{d}s \\
&\quad + v_2\big(\|e_3\|_{L^1(J)} + \|p\|_{\hbar}\|e_2\|_{L^1(J)} + \|Bu\|_{L^1(J,\mathscr{X})}\big) \\
&= \kappa(t).
\end{aligned}
$$

Due to the fact that κ is increasing, we get $\|\mathcal{G}_u(z)(\rho)\| \leqslant \kappa(\rho) \leqslant \kappa(t)$ for all $0 \leqslant \rho \leqslant t$. Accordingly, $\mathcal{G}_u(z) \in \mathcal{E}$, i.e., $\mathcal{G}_u(\mathcal{E}) \subset \mathcal{E}$. Hence, we get the conclusion of existence result by the application of Theorem 1. If we consider x as a solution of (1), then by the same estimate as for $\mathcal{G}_u$ one can get

$$
\begin{aligned}
\big\|x(t)\big\| &\leqslant v_1\big\|p(0)\big\| + \widetilde{\mathcal{K}}_1 \frac{Mb^{1-\gamma}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\big[e_1(s)+e_2(s)\big] \sup_{\varsigma\in[0,s]}\big\|x(\varsigma)\big\|\,\mathrm{d}s \\
&\quad + v_2\big(\|e_3\|_{L^1(J)} + \|p\|_{\hbar}\|e_2\|_{L^1(J)} + \|Bu\|_{L^1(J,\mathscr{X})}\big) \\
&\leqslant \big(\mathbb{C}^* + v_2\|Bu\|_{L^1(J,\mathscr{X})}\big) \\
&\quad + \widetilde{\mathcal{K}}_1 \frac{Mb^{1-\gamma}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\big[e_1(s)+e_2(s)\big] \sup_{\varsigma\in[0,s]}\big\|x(\varsigma)\big\|\,\mathrm{d}s. \qquad (12)
\end{aligned}
$$

R.H.S. of (12) is increasing with respect to t, and we obtain

$$
\begin{aligned}
\sup_{\varsigma\in J'}\|x(\varsigma)\| &\leqslant \big(\mathbb{C}^* + v_2\|Bu\|_{L^1(J,\mathscr{X})}\big) \\
&\quad + \widetilde{\mathcal{K}}_1 \frac{Mb^{1-\gamma}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\big[e_1(s)+e_2(s)\big] \sup_{\varsigma\in[0,s]}\big\|x(\varsigma)\big\|\,\mathrm{d}s.
\end{aligned}
$$

Therefore, one can obtain estimate (11) by referring the Gronwall's inequality. The proof is complete. □

3 Topological outcome

Let $\mathscr{X}$ and $\mathscr{Y}$ be metric spaces. A multi-valued map $\mathcal{F}:\mathscr{X}\to\mathscr{P}(\mathscr{Y})$ is said to be:

(i) upper semicontinuous (u.s.c) if the set $\mathcal{F}_+^{-1}(U)=\{z\in\mathscr{X}\colon \mathcal{F}(z)\subset U\}$ is open for any open set $U\subset\mathscr{Y}$;

(ii) closed if its graph $\varGamma_{\mathcal{F}}=\{(z,y)\colon y\in\mathcal{F}(z)\}$ is a closed subset of $\mathscr{X}\times\mathscr{Y}$.

The multivalued map $\mathcal{F}$ is called closed quasicompact if its restriction to any compact set is compact. For sufficient condition of upper semicontinuity, we need the following statement.

Lemma 4. (*See* [7].) *Let $\mathscr{X}$ and $\mathscr{Y}$ be metric spaces, and let $\mathcal{F}:\mathscr{X}\to\mathscr{P}(\mathscr{Y})$ be a quasicompact multimap with compact values. Then $\mathcal{F}$ is upper semicontinuous.*

Consider the solution multimap

$$\mathcal{W}:L^2(J,U)\to\mathscr{P}\big(C_{1-\gamma}(J,\mathscr{Z})\big),\quad \mathcal{W}(u)=\big\{x\colon x=\mathcal{G}_u(x)\big\}.\tag{13}$$

Proposition 2. *From* (H1) *the restriction of operator $\mathscr{F}$, given by* (5)*, on $L^2(J,\mathscr{Z})$ is compact, i.e., if $\varPsi\subset L^2(J,\mathscr{Z})$ is a bounded set, then $\mathscr{F}(\varPsi)$ is relatively compact in $C_{1-\gamma}(J,\mathscr{Z})$.*

Assumption (H3) can be extended in the following way to obtain further properties of the solution multimap $\mathcal{W}$:

(H5) The nonlinearity G satisfies (H3) with $e_1,e_2,e_3\in L^2(J)$.

Lemma 5. *From assumptions* (H1)*,* (H2) *and* (H5) *the solution multivalued map $\mathcal{W}$, discussed in* (13)*, is completely continuous, that is, it is upper semicontinuous and assigns each bounded set into a relatively compact set.*

Proof. We split the proof into two steps.

Step 1. Let $\mathscr{E}$ be a bounded set in $L^2(J,U)$. We show that $\mathcal{W}(\mathscr{E})$ is relatively compact in $C_{1-\gamma}(J,\mathscr{Z})$. Suppose $\{x_m\}\subset\mathcal{W}(\mathscr{E})$. Then there exists $\{u_m\}\subset\mathscr{E}$ such that

$$\begin{aligned}x_m(t)&=\mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0)\\&\quad+\int_0^t\mathcal{K}^{-1}Q_\mu(t-s)\big[Bu_m(s)+G\big(s,x_m(s),x_m[p]_s\big)\big]\,\mathrm{d}s.\end{aligned}$$

Above inequality can also be written as

$$x_m(t)=\mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0)+\mathscr{F}(g_m+Bu_m)(t),\tag{14}$$

where $\mathscr{F}$ is the operator defined in (6), $g_m(t) = G(t, x_m(t), x_m[p]_t)$. We observe that $\{Bu_m\}$ is a bounded set in $L^2(J, \mathscr{Z})$ since B is a bounded linear operator. This implies by Proposition 2 that $\{\mathscr{F}(Bu_m)\}$ is relatively compact in $C_{1-\gamma}(J, \mathscr{Z})$. On the other side, in view of some regular estimates, we can obtain that $\{x_m\}$ is a bounded in $C_{1-\gamma}(J, \mathscr{Z})$. Therefore, assumption (H5) means that $\{g_m\}$ is also bounded in $L^2(J, \mathscr{Z})$, and one ensure that $\{\mathscr{F}(g_m + Bu_m)\}$ is compact. In view of (14), we conclude that $\{x_m\}$ is compact as well.

Step 2. We prove that $\mathcal{W}$ is u.s.c. According to Lemma 13, $\mathcal{W}$ has a closed graph. Suppose $u_m \to u$ in $L^2(J, U)$ and x_m in $\mathcal{W}(u_m)$, $x_m \to x$ in $C_{1-\gamma}(J, \mathscr{Z})$. We claim that $x \in \mathcal{W}(u)$. We obtain

$$\begin{aligned} x_m(t) &= \mathcal{K}^{-1} T_{\nu,\mu}(t) \mathcal{K} p(0) \\ &\quad + \int_0^t \mathcal{K}^{-1} Q_\mu(t-s) \big[Bu_m(s) + G\big(s, x_m(s), x_m[p]_s\big) \big] \, \mathrm{d}s. \end{aligned} \tag{15}$$

Since $G(t, \cdot, \cdot)$ is continuous, we have that $g_m(s) = G(s, x_m(s), x_m[p]_s)$ converges to $g(s) = G(s, x(s), x[p]_s)$ for a.e. $s \in J$. Due to the fact that $\{g_m\}$ is integrably bounded, the Lebesgue dominated convergence theorem implies $g_m - g \to 0$ in $L^1(J, \mathscr{Z})$. Moreover, since B is bounded, we notify that $Bu_m - Bu \to 0$ in $L^1(J, \mathscr{Z})$. Therefore, from (15) we get

$$\begin{aligned} x(t) &= \mathcal{K}^{-1} T_{\nu,\mu}(t) \mathcal{K} p(0) \\ &\quad + \int_0^t \mathcal{K}^{-1} Q_\mu(t-s) \big[Bu(s) + G\big(s, x(s), x[p]_s\big) \big] \, \mathrm{d}s, \quad t \geqslant 0. \end{aligned}$$

The proof is now completed. □

Definition 8. (See [15, 17].) A subset $\mathcal{B}$ of a metric space $\mathscr{X}$ is said to be contractible in $\mathscr{X}$ if the inclusion map $i_{\mathcal{B}} : \mathcal{B} \to \mathscr{X}$ is null-homotopic, i.e., there exist $z_0 \in \mathscr{X}$ and a continuous map $h : \mathcal{B} \times [0, 1] \to \mathscr{X}$ such that $h(z, 0) = y$ and $h(z, 1) = z_0$ for any $z \in \mathcal{B}$.

Definition 9. (See [15, 17].) A subset $\mathcal{B}$ of a metric space $\mathscr{X}$ is called R_δ-set if $\mathcal{B}$ can be represented as the intersection of decreasing sequence of compact contractible sets.

A multivalued $\mathcal{F} : \mathscr{Z} \to \mathscr{P}(\mathscr{X})$ is said to be an R_δ-map only if $\mathcal{F}$ is u.s.c. and for every $x \in \mathscr{Z}$, $\mathcal{F}(x)$ is an R_δ-set in $\mathscr{X}$.

Lemma 6. (*See* [15, 17].) *Suppose $\mathscr{Z}$ is a metric space, E is a Banach space, and $f : \mathscr{Z} \to E$ is a proper map. That is, $f^{-1}(L)$ is compact for each compact set $L \subset E$, f is continuous. Then there exists a sequence $\{f_m\}$ of mappings from $\mathscr{Z}$ into E such that*

(i) *f_m is proper, and $\{f_m\}$ uniformly converges to f on $\mathscr{Z}$.*
(ii) *For a given point $z_0 \in E$ and for all z in a neighborhood $\mathscr{N}(z_0)$ of z_0 in E, there exists exactly one solution x_m of the equation $f_m(x) = z$.*

It is necessary to have the following theorem to prove this lemma.

Lemma 7 [Lasota–Yorke approximation theorem]. (*See* [5, 17].) *Let E be a normed space and $g : \mathscr{X} \to E$ a continuous map. Then for each $\varepsilon > 0$, there is a locally Lipschitz map $g_\varepsilon : \mathscr{X} \to E$ such that $\|g_\varepsilon(x) - g(x)\|_E < \varepsilon$ for each $x \in \mathscr{X}$.*

Theorem 3. *Suppose the assumptions of Lemma 5 are satisfied. Then for each $u \in L^2(J, U)$, $\mathcal{W}(u)$ is an R_δ-set.*

Proof. As the nonlinearity $G(t, \cdot, \cdot)$ in our problem is continuous, by Lemma 7 one can take a sequence $\{G_m\}$ such that $G_m(t, \cdot, \cdot)$ are locally Lipschitz functions and

$$\big\|G_m(t, \eta, \psi) - G(t, \eta, \psi)\big\| \leqslant \varepsilon_m$$

for any $t \in J$ and $\eta \in \mathscr{X}$, $\psi \in C_{1-\gamma}([-\hbar, 0], \mathscr{X})$. In the above inequality, $\varepsilon_m \to 0$ as $m \to \infty$. Without loss of generality, we can assume that

$$\big\|G_m(t, \eta, \psi)\big\| \leqslant e_1(t)\|\eta\| + e_2(t)\|\psi\|_\hbar + e_3(t) + 1 \quad \text{for all } m.$$

Consider the equation

$$x(t) = \mathcal{K}^{-1} z^*(t)\mathcal{K} + \int_0^t \mathcal{K}^{-1} Q_\mu(t-s)\big[Bu(s) + G\big(s, x(s), x[p]_s\big)\big]\,\mathrm{d}s. \tag{16}$$

From previous section we get existence result for (16). In addition, since $G_m(t, \cdot, \cdot)$ is a locally Lipschitz property, the solution of (16) is unique.

Let

$$\mathscr{H}(x) = (I - \mathcal{G}_u)(x), \qquad \mathscr{H}_m(x)(t) = (I - \mathcal{G}_{um})(x)$$

$$\mathcal{G}_{um}(x) = \mathcal{K}^{-1} T_{\nu,\mu}(t)\mathcal{K}p(0) + \int_0^t \mathcal{K}^{-1} Q_\mu(t-s)\big[Bu(s) + G\big(s, x(s), x[p]_s\big)\big]\,\mathrm{d}s.$$

Then one claims that the maps $\mathscr{H}$ and $\mathscr{H}_m$ are proper. Absolutely, we will verify this assertion, e.g., for $\mathscr{H}$. We take $\mathscr{H}^{-1}(L)$ compact for any compact set $L \subset C_{1-\gamma}(J, \mathscr{X})$. Assume that $(I - \mathcal{G}_u)(\mathcal{U}) = L$ and $\{x_m\} \subset \mathcal{U}$ is any sequence. Then there exists a sequence $\{z_m\} \subset L$ such that $x_m - \mathcal{G}_u(x_m) = z_m$. That is,

$$x_k(t) = \mathcal{K}^{-1} T_{\nu,\mu}(t)\mathcal{K}p(0) + z_m(t) + \int_0^t \mathcal{K}^{-1} Q_\mu(t-s)\big[Bu(s) + g_m(s)\big]\,\mathrm{d}s,$$

where $g_m(s) = G(s, x_m(s), x_m[p]_s)$, $s \in J$.

Referring (H5) and the fact that $\{z_m\}$ is bounded in $C_{1-\gamma}(J, \mathscr{X})$, we observe that $\{x_m\}$ is bounded in $C_{1-\gamma}(J, \mathscr{X})$. Thus, $\{g_m\}$ is continuous and bounded in $L^2(I, \mathscr{X})$. From Proposition 2 we conclude that $\{\mathscr{F}(g_m)\}$ is compact. Hence $\{x_m\}$ is relatively compact, and $\mathcal{U}$ is a compact set. Moreover, $\{\mathscr{H}_m\}$ converges to $\mathscr{H}$ uniformly in $C_{1-\gamma}(J, \mathscr{X})$, and $\mathscr{H}_m(x) = z$ has a unique solution $z \in \mathcal{S}_p$ from (16). Then we conclude that $\mathcal{W}(u) = \mathscr{H}^{-1}(0)$ is an R_δ-set. The proof is complete. □

4 Approximate controllability results

As discussed in Section 2, the Hilfer fractional system

$$\begin{aligned} &D_{0+}^{\nu,\mu}\big[\mathcal{K}x(t)\big] = Ax(t) + G\big(t, x(t), x_t\big) + q(t), \quad t \in J' := (0, b], \\ &I_{0+}^{(1-\nu)(1-\mu)}x(t) = p(t), \quad t \in [-\hbar, 0], \end{aligned} \tag{17}$$

has at least one mild solution $x = x(\cdot, q)$ for all $q \in L^2(J, \mathscr{X})$. Also, in view of the results reviewed in Section 3, we see that the solution map

$$\mathcal{W}(q) = \big\{x(\cdot, q)\colon \text{ the solution of system (17)}\big\}$$

is an R_δ-map. Furthermore, by (H2) and (H3) the map $\mathscr{V}_G$ defined by (7) is continuous. Therefore, $\mathscr{V}_G$ is also an R_δ-map.

Define a linear operator $\Upsilon : L^2(J, \mathscr{X}) \to \mathscr{X}$ by

$$\Upsilon(w) = \int_0^b \mathcal{K}^{-1} Q_\mu(b - s) w(s)\, \mathrm{d}s.$$

Let $\mathcal{O} = \{w \in L^2(J, \mathscr{X})\colon \Upsilon w = 0\}$, we have that $\mathcal{O}$ is a closed subspace of $L^2(J, \mathscr{X})$. Suppose $\mathcal{O}^\perp$ is the orthogonal space of $\mathcal{O} \in L^2(J, \mathscr{X})$ and Q is the projection from $L^2(J, \mathscr{X})$ into $\mathcal{O}^\perp$. Let $R[B]$ be the range of B. We need the following assumption:

(H6) For any $r_1 \in L^2(J, \mathscr{X})$, there exists $r_2 \in R[B]$ such that $\Upsilon(r_1) = \Upsilon(r_2)$.

By assumption (H6) we have that $\{x + \mathcal{O}\} \cap R[B] \neq \emptyset$ for any $x \in \mathcal{O}^\perp$. Hence, by the proof of [22, Lemma 1] the following mapping P from $\mathcal{O}^\perp$ to $R[B]$

$$\begin{aligned} \mathsf{P}x = \big\{x^*\colon\ & x^* \in \{x + \mathcal{O}\} \cap R[B], \text{ and} \\ & \|x^*\|_{L^2(J,\mathscr{X})} = \min\big\{\|z\|_{L^2(J,\mathscr{X})}\colon z \in \{x + \mathcal{O}\} \cap R[B]\big\}\big\} \end{aligned}$$

is well defined. Furthermore, P is linear and bounded.

Remark 4. We take hypothesis (H6) as in [25]; that is, it requires $r_2 \in R[B]$, it is slightly stronger than that in [22] ($r_2 \in \overline{R[B]}$). In fact, this requirement is necessary for our arguments when the solution to control system is not unique. Moreover, in application, (H6) is much easier to verify than the one assumed in [22].

For given $u_0 \in L^2(J, U)$, we establish the operator $\mathscr{J} : \mathcal{O}^\perp \to \mathscr{P}(\mathcal{O}^\perp)$ determined in

$$\mathscr{J} w = \mathsf{Q}Bu_0 - \mathsf{Q}\mathscr{V}_G \mathcal{W}\mathsf{P}w. \tag{18}$$

We have to verify that $\mathscr{J}$ has a fixed point. For this purpose, we need the following notions and facts in the sequel.

Definition 10. Let $\mathscr{X}$ be a metric space.

(i) $\mathscr{X}$ is called an absolute retract (AR-space) if for any metric space $\mathscr{Y}$ and any closed $\mathcal{A} \subset \mathscr{Y}$, every continuous function $g : \mathcal{A} \to \mathscr{X}$ extends to a continuous function $\tilde{g} : \mathscr{Y} \to \mathscr{X}$.
(ii) $\mathscr{X}$ is called an absolute neighborhood retract (ANR-space) if for any metric space $\mathscr{Y}$, any closed $\mathcal{A} \subset \mathscr{Y}$, and continuous $g : \mathcal{A} \to \mathscr{Y}$, there exists a neighborhood of $\mathbb{U} \supset \mathcal{A}$ and a continuous extension $\tilde{g} : \mathcal{A} \to \mathscr{Y}$ of g.

Obviously, if $\mathscr{X}$ is an AR-space, then $\mathscr{Y}$ is an ANR-space.

Proposition 3. (*See* [22].) *Let $\mathscr{C}$ be a convex set in a locally convex linear space $\mathscr{X}$. Then $\mathscr{C}$ is an* AR-*space.*

In particular, the last proposition states that every Banach space and its convex subsets are AR-spaces. The following theorem is the main tool for this section. For related results on fixed point theory for ANR-spaces, one can verify the papers [8, 10, 17].

Theorem 4. (*See* [9].)*Let $\mathscr{X}$ be an* AR-*space. Assume that $\phi : \mathscr{X} \to \mathscr{P}(\mathscr{X})$ may be factorized as*

$$\phi = \phi_m \circ \phi_{m-1} \circ \dots \circ \phi_1.$$

In the above equation, $\phi_\jmath : \mathscr{X}_{\jmath-1} \to \mathscr{P}(\mathscr{X}_\jmath)$, $\jmath = 1, 2, \dots, N$, are R_δ-maps, and $\mathscr{X}_\jmath$, $\jmath = 1, \dots, N-1$, are ANR-*spaces, $\mathscr{X}_0 = \mathscr{X}_N = \mathscr{X}$ are* AR-*spaces. If there is a compact set $\mathcal{U}$ such that $\phi(\mathscr{X}) \subset \mathcal{U} \subset \mathscr{X}$, then ϕ has a fixed point.*

Theorem 5. *If hypotheses* (H1), (H2), (H5)*, and* (H6) *hold, then the operator $\mathscr{J}$ defined in* (18) *has a fixed point in $\mathcal{O}^\perp$, provided that*

$$\frac{M\widetilde{\mathcal{K}}_1 b^{(1-2\gamma+2\mu)/2}}{\sqrt{2\mu-1}\Gamma(\mu)}\|\mathsf{P}\|\|e_1+e_2\|_{L^2(J,\mathscr{X})} \times \exp\left\{\frac{M\widetilde{\mathcal{K}}_1 b^{\mu+1-\gamma}}{\Gamma(\mu+1)}\|e_1+e_2\|_{L^1(J,\mathscr{X})}\right\} < 1. \tag{19}$$

Proof. The operator $\mathscr{J}$ can be factorized as

$$\mathscr{J} = \mathcal{I} \circ \mathsf{Q} \circ \mathscr{V}_G \circ \mathcal{W} \circ \mathsf{P},$$

where $\mathcal{I}(w) = \mathsf{Q}Bu_0 - w$ is a single-valued and continuous mapping. It is easy to see that all component in above presentation is R_δ-map. Therefore, in order to use Theorem 4, it suffice to prove that there exists a convex subsets $\mathcal{U} \subset \mathcal{O}^\perp$ such that $\mathscr{J}(\mathcal{U}) \subset \mathcal{U}$ and $\mathscr{J}(\mathcal{U})$ is a compact set. We look for $R > 0$ such that $\|\mathscr{J}(w)\|_{L^2(J,\mathscr{X})} \leqslant R$, provided $\|w\|_{L^2(I,\mathscr{X})} \leqslant R$, and then take

$$\mathcal{U} = \overline{B}_R \cap \mathcal{O}^\perp \tag{20}$$

thanks to the fact that $\mathcal{O}^\perp$ is a convex subset of $L^2(J,\mathscr{X})$. For $x \in \mathcal{W}(\mathsf{P}w)$, it follows from Theorem 2 that

$$\sup_{\varsigma\in[0,t]}\|x(\varsigma)\| \leqslant \left(\mathbb{C}^* + v_2\|\mathsf{P}w\|_{L^1(J,\mathscr{X})}\right) \times \exp\left\{\frac{M\widetilde{\mathcal{K}}_1 b^{1-\gamma}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[e_1(s)+e_2(s)\right]\mathrm{d}s\right\}.$$

From (11) and hypothesis (H3) one can get

$$\begin{aligned}
&\left\|\mathscr{V}_G(x)(t)\right\| \\
&\quad= \left\|G(t,x(t),x_t)\right\| \leqslant e_1(t)\left\|x(t)\right\| + e_2(t)\left\|x_t\right\|_{\hbar} + e_3(t) \\
&\quad\leqslant \left[e_1(t)+e_2(t)\right]\sup_{\varsigma\in[0,t]}\left\|x(\varsigma)\right\| + e_2(t)\|\|p\|_{\hbar} + e_3(t) \\
&\quad\leqslant \left(\mathbb{C}^* + v_2\|\mathsf{P}w\|_{L^1(J,\mathscr{X})}\right)\exp\left\{\frac{M\widetilde{\mathcal{K}}_1 b^{1-\gamma}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[e_1(s)+e_2(s)\right]\mathrm{d}s\right\} \\
&\qquad\times\left[e_1(t)+e_2(t)\right] + e_2(t)\|p\|_{\hbar} + e_3(t).
\end{aligned}$$

Taking into account that $\|\mathsf{Q}\| \leqslant 1$ for any $q \in \mathscr{J}(w)$, one can get

$$\begin{aligned}
\left\|q(t)\right\| = \left\|Bu_0(t)\right\| &+ \left(\mathbb{C}^* + v_2\|\mathsf{P}w\|_{L^1(J,\mathscr{X})}\right) \\
&\times\exp\left\{\frac{M\widetilde{\mathcal{K}}_1 b^{1-\gamma}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[e_1(s)+e_2(s)\right]\mathrm{d}s\right\} \\
&\times\left[e_1(t)+e_2(t)\right] + e_2(t)\|p\|_{\hbar} + e_3(t).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|q\|_{L^2(J,\mathscr{X})} \leqslant \|Bu_0\|_{L^2(J,\mathscr{X})} &+ \left(\mathbb{C}^* + \frac{M\widetilde{\mathcal{K}}_1 b^{\frac{1-2\gamma+2\mu}{2}}}{\sqrt{2\mu-1}\Gamma(\mu)}\|\mathsf{P}\|\|w\|_{L^2(J,\mathscr{X})}\right) \\
&\times\|e_1+e_2\|_{L^2(J,\mathscr{X})}\exp\left\{\frac{M\widetilde{\mathcal{K}}_1 b^{\mu+1-\gamma}}{\Gamma(\mu+1)}\|e_1+e_2\|_{L^1(J,\mathscr{X})}\right\} \\
&+\|e_2\|_{L^2(J,\mathscr{X})}\|p\|_{\hbar} + \|e_3\|_{L^2(J,\mathscr{X})}.
\end{aligned} \tag{21}$$

Thanks to assumption (19), (21) ensures the existence of a number $R > 0$ such that $\|q\|_{L^2(J,\mathscr{X})} \leqslant R$, provided $\|w\|_{L^2(J,\mathscr{X})} \leqslant R$. That is, $\mathscr{J}(\mathcal{U}) \subset \mathcal{U}$ with the closed bounded subset $\mathcal{U}$ denoted in (20). By Lemma 5 the set $\mathcal{W}\circ\mathsf{P}(\mathcal{U})$ is compact, and then $L = \mathscr{J}(\mathcal{U})$ is a compact set. Thus we get the desired conclusion. □

Remark 5. If the nonlinearity G is uniformly bounded with respect to the second and third arguments; that is, $\|G(t,\eta,\psi)\| \leqslant e_3(t)$ for a.e. $t \in J$ and all $\eta \in \mathscr{X}$, $\psi \in C_{1-\gamma}([-h,0],\mathscr{X})$, then condition (19) can be relaxed since in this case, $e_1 = e_2 = 0$. Let $\mathcal{R}_b(G) = \{x(b,u)\colon u \in L^2(J,U)\}$, the set of all terminal state of solutions to system (1).

The set $\mathcal{R}_b(G)$ is called the reachable set of the control system (1). When $G = 0$, the notation $\mathcal{R}_b(0)$ stands for the reachable set of the corresponding linear system.

Definition 11. (See [11].) The control system (1) is said to be exact controllable (or controllable) if $\mathcal{R}_b(G) = \mathscr{X}$. It is called approximately controllable if $\overline{\mathcal{R}_b(G)} = \mathscr{X}$.

It is shown in [22, Lemma 2] by assuming hypothesis (H6) that $\mathcal{R}_b(0) = \mathscr{X}$. Thus (H6) is a sufficient condition for the approximate controllability of the linear system associated with (1). One can find more details in [4, 14] for some other conditions. The following theorem is our main result in this section.

Theorem 6. *Under the hypotheses of Theorem* 5*, the control system* (1) *is approximately controllable if the corresponding linear system is.*

Proof. We prove that $\mathcal{R}_b(0) \subset \mathcal{R}_b(G)$. Consider $x_0 \in \mathcal{R}_b(0)$. Then there exists $u_0 \in L^2(J, U)$ such that $x_0 = \mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0) + \Upsilon B u_0$. Let w^* be a fixed point of $\mathscr{J}$, then we have

$$\mathsf{Q}Bu_0 = \mathsf{Q}\mathscr{V}_G\mathcal{W}\mathsf{P}w^* + w^*. \tag{22}$$

By the definition of P we conclude that $\mathsf{P}w^* \in \{w^* + \mathcal{O}\} \cap R[B]$, and then

$$\Upsilon\mathsf{P}w^* = \Upsilon w^*. \tag{23}$$

Moreover, Q is the projection from $L^2(J, \mathscr{X})$ into $\mathcal{O}^\perp$, then

$$\Upsilon\mathsf{Q}\breve{q} = \Upsilon\breve{q} \quad \text{for all } \breve{q} \in L^2(J, \mathscr{X}). \tag{24}$$

Combining (22)–(24) yields $\Upsilon Bu_0 = \Upsilon(g + \mathsf{P}w^*)$, where $g \in \mathscr{V}_G\mathcal{W}\mathsf{P}w^*$. Therefore,

$$\begin{aligned} x_0 &= \mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0) + \Upsilon B w_0 \\ &= \mathcal{K}^{-1}T_{\nu,\mu}(t)\mathcal{K}p(0) + \Upsilon(g + \mathsf{P}w^*) \\ &= x(b, \mathsf{P}w^*), \end{aligned}$$

where x is a solution of (17). Since $\mathsf{P}w^* \in R[B]$, there exists a function $u \in L^2(J, U)$ such that $\mathsf{P}w^* = Bu$. Then we have $x(\cdot, \mathsf{P}w^*) = x(\cdot, Bu) = y(\cdot, u)$, where y is a mild solution of system (1). This implies that $\mathcal{R}_b(0) \subset \mathcal{R}_b(G)$. This completes the proof. □

5 Example

As an application, we consider the following fractional system:

$$\begin{aligned} &D_{0+}^{\nu,2/3}\left[x(t,y) - \frac{\partial^2}{\partial y^2}x(t,y)\right] \\ &\quad = \frac{\partial^2}{\partial y^2}x(t,y) + G\big(t, x(t,y), x(t-\hbar, y)\big) + Bu(t,y), \\ &\quad y \in [0,\pi],\ t \in (0,b], \\ &x(t,0) = x(t,\pi) = 0, \quad t > 0, \\ &I_{0+}^{(1-\nu)/3}x(t,y) = p(t,y), \quad t \in [-\hbar, 0],\ y \in [0,\pi], \end{aligned} \tag{25}$$

where $D_{0+}^{\nu,2/3}$ represents the Hilfer fractional derivative, $\mu = 2/3$ is a order of the above system, and $\nu \in [0,1]$ is a type. $I_{0+}^{(1-\nu)/3}$ is the Riemann–Liouville integral of order $(1-\nu)/3$, $x, u \in C([0,b]; L^2(0,\pi))$ is the state function and the control function, respectively.

The characteristics in (25) is that the nonlinearity has neither Lipschitz property nor uniform boundedness compared to the existing results in literature.

Abstract form. Assume that $\mathscr{X} = U = L^2([0,\pi],\mathbb{R})$ and define the operators $A : D(A) \subset \mathscr{X} \to \mathscr{X}$ and $\mathcal{K} : D(\mathcal{K}) \subset \mathscr{X} \to \mathscr{X}$ respectively by $Ax = x''$ and $\mathcal{K}x = x - x''$ with domain

$$D(A) = D(\mathcal{K}) = \begin{cases} x \in \mathscr{X}, & x, x' \text{ are absolutely continuous,} \\ x'' \in \mathscr{X}, & x(0) = x(\pi) = 0. \end{cases}$$

Then A and $\mathcal{K}$ can be written, respectively, as

$$Ax = \sum_{m=1}^{\infty} m^2 \langle x, \vartheta_m \rangle \vartheta_m, \quad \vartheta \in D(A)$$

and

$$\mathcal{K}x = \sum_{m=1}^{\infty} \bigl(m^2 + 1\bigr) \langle x, \vartheta_m \rangle \vartheta_m, \quad \vartheta \in D(\mathcal{K}),$$

where $\vartheta_k(x) = \sqrt{2/\pi}\sin(mx)$, $m = 1, 2, \dots$, is the orthogonal set of eigenvectors of A. Additionally, for $x \in \mathscr{X}$, we have

$$\mathcal{K}^{-1}x = \sum_{m=1}^{\infty} \frac{1}{(1+m^2)} \langle x, \vartheta_m \rangle \vartheta_m,$$
$$A\mathcal{K}^{-1}x = \sum_{m=1}^{\infty} \frac{1}{(1+m^2)} \langle x, \vartheta_m \rangle \vartheta_m.$$

It is known that $A\mathcal{K}^{-1}$ is self-adjoint, and $A\mathcal{K}^{-1}$ the infinitesimal generator of an analytic semigroup $S(t)(t \geqslant 0)$ in $\mathscr{X}$ given by

$$S(t)x = \sum_{m=1}^{\infty} \mathrm{e}^{-m^2 t} \langle x, \vartheta_m \rangle \vartheta_m, \quad x \in \mathscr{X}.$$

In particular, $S(t)$ is a uniformly stable semigroup, and $\|S(t)\| \leqslant \mathrm{e}^{-t}$. We select B as in [22], that is, the intercept operator $B_{\gamma,b}$ is

$$B_{\gamma,b}w(t) = \begin{cases} 0, & 0 \leqslant t \leqslant \gamma, \\ w(t), & \gamma \leqslant t \leqslant b, \end{cases}$$

where $w \in L^2(0,b;L^2(0,\pi))$. It is known that $B = B_{\gamma,b}$ satisfies (H6). Then the linear system

$$\begin{aligned}
&D_{0+}^{\nu,2/3}\left[x(t,y) - \frac{\partial^2}{\partial y^2}x(t,y)\right] = \frac{\partial^2}{\partial y^2}x(t,y) + Bu(t,y),\\
&\quad y \in [0,\pi],\ t \in (0,b],\\
&x(t,0) = x(t,\pi) = 0, \quad t > 0,\\
&I_{0+}^{(1-\nu)/3}x(s,y) = p(s,y), \quad s \in [-\hbar,0],\ y \in [0,\pi],
\end{aligned}$$

is approximately controllable, that is, $\overline{\mathcal{R}_b(0)} = L^2(0,\pi)$. Concerning the nonlinearity G, we assume that $G : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. There exist functions $a_1, a_2, a_3 \in L^1(J)$ such that

$$\left\|G(t,\eta,\psi)\right\| \leqslant a_1(r)\|\psi\| + a_2(r)\|\psi'\|_\hbar + a_3(r),$$

for any $t \in [0,b]$, $\eta, \psi \in \mathbb{R}$.

Using the abstract results in previous sections, we conclude that the control system (25) is approximately controllable in $L^2(0,\pi)$, provided inequality (19) holds. Furthermore, the solution set depends upper semicontinuously on the control function u, and it is an $\mathcal{R}_\delta$ set.

6 Conclusion

In this paper, the existence of mild solution for Hilfer fractional delay differential equations of Sobolev type without uniqueness has been investigated using the fixed point theorem, and the topological structure of the solution map is discussed. The estimated outcomes for Hilfer fractional delay differential equations and approximate controllability results were calculated using the multivalued map, condensing, and the theorems and definitions related to the absolute neighborhood retract space and absolute retract space. An example is provided in the end to support the analytical findings.

1. Furthermore, future research combining Hilfer fractional derivatives with Volterra integrodifferential equations could provide useful insights into fractional calculus. While examining the performance of approximation controllability results, ideas from renormalization could be applied to fractional derivative prepossessing.
2. In the future, we will extend the current work to approximate controllability for Hilfer fractional delay impulsive differential equations of Sobolev type with nonlocal conditions via fixed point technique.
3. Some new work can also extend to the Hilfer fractional neutral differential evolution equation for the approximate controllability of a class of semilinear equations depending on the method of approximate technique with infinite delay.
4. A solution on the approximation controllability for stochastic delay evolution equations of Sobolev type with order $1 < r < 2$ with respect to uniqueness is taken up as follow-up work.

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