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New results of global Mittag-Leffler synchronization on Caputo fuzzy delayed inertial neural networks*

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Abstract. This article is devoted to discussing the problem of global Mittag-Leffler synchronization (GMLS) for the Caputo-type fractional-order fuzzy delayed inertial neural networks (FOFINNs). First of all, both inertial and fuzzy terms are taken into account in the system. For the sake of reducing the influence caused by the inertia term, the order reduction is achieved by the measure of variable substitution. The introduction of fuzzy terms can evade fuzziness or uncertainty as much as possible. Subsequently, a nonlinear delayed controller is designed to achieve GMLS. Utilizing the inequality techniques, Lyapunov's direct method for functions and Razumikhin theorem, the criteria for interpreting the GMLS of FOFINNs are established. Particularly, two inferences are presented in two special cases. Additionally, the availability of the acquired results are further confirmed by simulations.

Keywords: Caputo derivative, global Mittag-Leffler synchronization, fuzzy inertial neural networks, variable substitution.

1 Introduction

Fractional integral, as an extension of traditional-order integral, can be traced back to the 17th century [19]. In contrast to the usual infinitesimal calculus, fractional calculus

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has been proved to offer a more powerful auxiliary instrument to describe the properties of memory and genetics in assorted categories of materials and movement processes.

In recent years, neural networks (NNs) have been proverbially studied and effectively used in various fields [22, 25] including disease diagnosis, control engineering, signal processing, pattern recognition, etc. Compared with integral order NNs, the dynamical behavior analysis of fractional-order neural networks (FONNs) is more specific in simulating biological NNs, which has fascinated plenty of interest. It plays an indispensable role in many research fields. In 1986, Babcock and Westervelt promulgated the essence of inertia, thereupon then generating inertial neural networks (INNs) [2]. The difference is that it contains the second derivative of the state variables. The inertia term was originally used in inductive circuits. Therefore, inertia term may lead to unstable, spontaneous concussion, chaos, and other complex behaviors. For instance, it is a powerful implement for chaos and bifurcation control in the dynamic behaviors of recurrent NNs with inertia terms. As a result of its widespread application prospects and diverse biological backgrounds, INNs have been intensively researched, and they obtained many meaningful results. Cui et al. [6] derived global asymptotic stability criteria for INNs by the linear matrix inequalities. The exponential synchronization for INNs was considered through the new hybrid control scheme in [11]. Wei and Cao [23] obtained sufficient conditions of exponential synchronization and quasi synchronization for the inertia memristor NNs by selecting a feedback controller. The standards of synchronization for INNs were established by integral inequality method [32].

In 1965, Zadeh proposed fuzzy logic. It is generally known that approximation and fuzziness appear in actual modeling. However, fuzzy logic takes these factors into account, and it is one of the most common and important tools for modeling real-world problems. Just because of this, it is extremely practical to add fuzzy terms to INNs, fuzzy inertial neural network (FINNs) is then constructed. It is highly pivotal to probe into some synchronization of FINNs. Zhen and Zhang [26] studied the maximum method to solve the interrogation of synchronization for FINNs. As can be noticed, fractional-order inertial neural networks (FOINNs) better described the dynamic properties of NNs in comparison with integer-order INNs, which inspired many scholars concerning this problem to acquire a great number of accomplishments. Using the properties of fractional integral and derivative, the Mittag-Leffler (ML) stability of FOINNs with delays were explored in [12]. In the Riemann–Liouville's sense, Gu et al. [8] proposed the model of FOINNs with delays. [5] further submitted the basis for judging GMLS of FOINNs under the sense of Caputo, which was expressed in the form of algebraic inequality. In [11,17,20,21,23,28,33,34], various dynamic behaviors of NNs were discussed, including but not limited to, fixed-time synchronization, ML synchronization, lag synchronization, finite-time synchronization, exponential synchronization, quasisynchronization, asymptotic stability, mean-square exponential synchronization, and H_∞ synchronization. Some types of delays include proportional delays, leakage delays, transmission delays, discrete delays, and hybrid delays [6, 16, 23, 29, 30].

Stimulated by the above discussions, the intention of the text is to research the GMLS of FOFINNs by designing a feedback controller. By choosing the suitable variable replacement, the original inertial system is converted into the β-order conventional differential system, which can reach the purpose of order reduction. Utilizing the proper Lyapunov functions and applying the inequality techniques, some adequate criteria are established for GMLS, which are characterized by the algebraic inequalities. The following innovations are listed:

(i) The critical models focus on FOFINNs, and fuzzy logic can better handle the phenomena of uncertainty, approximation, and ambiguity.
(ii) The inertial system is converted into two conventional systems by variable transformations, which can overcome the difficulties due to inertial terms. By designing delayed-feedback controller to establish GMLS criteria for FOFINNs.
(iii) The criteria are concise and easier to demonstrate, and the results are simulated with the MATLAB toolbox. The accuracy of the results is confirmed in an intuitive way, and the validity of the criteria is illustrated by utilizing the data and results.

2 Preliminaries and model description

Subsequently, the basic concepts and lemmas are given, and the model of FOFINNs is explained in detail.

Definition 1. (See [13].) For the function $u \in C^n([0, +\infty], \mathbb{R})$, the fractional derivative of Caputo with order β is

$$ {}_{t_0}^{c}D_t^{\beta} u(t) = \frac{1}{\Gamma(n-\beta)} \int_{t_0}^{t} \frac{u^{(n)}(s)}{(t-s)^{\beta-n+1}} \, \mathrm{d}s, $$

where $t \geqslant t_0$, $n-1 \leqslant \beta < n$.

Definition 2. (See [19].) The Mittag-Leffler function of a parameter is represented as

$$ E_{\beta}(\varsigma) = \sum_{k=0}^{\infty} \frac{\varsigma^k}{\Gamma(k\beta+1)}, $$

where $0 < \beta < 1$, ς is a complex set.

Lemma 1. (*See* [4].) *For* $0 < \beta < 1$, $\lambda \in \mathbb{R}$, *suppose that* $Q(t) \in C[0, +\infty)$. *If*

$$ {}_{0}^{c}D_t^{\beta} Q(t) \leqslant -\lambda Q(t), $$

then

$$ Q(t) \leqslant Q(0) E_{\beta}\left(-\lambda t^{\beta}\right). $$

Lemma 2. (*See* [10].) *If* $\phi(t)$ *is continuous and differentiable,* $0 < \beta < 1$, *then*

$$ {}_{t_0}^{c}D_t^{\beta}\left(\phi^2(t)\right) \leqslant 2\phi(t)\, {}_{t_0}^{c}D_t^{\beta}\phi(t). $$

Lemma 3. (*See* [15].) *If* k_ζ, $\tilde{k}_\zeta$ *are the two states of the system, let* $\breve{h}_{\delta\zeta}, \breve{s}_{\delta\zeta} \in \mathbb{R}$. *Then*

$$\left| \bigwedge_{\zeta=1}^{n} \breve{s}_{\delta\zeta}\phi_\zeta(k_\zeta) - \bigwedge_{\zeta=1}^{n} \breve{s}_{\delta\zeta}\phi_\zeta(\tilde{k}_\zeta) \right| \leqslant \sum_{\zeta=1}^{n} |\breve{s}_{\delta\zeta}| \left|\phi_\zeta(k_\zeta) - \phi_\zeta(\tilde{k}_\zeta)\right|,$$

$$\left| \bigvee_{\zeta=1}^{n} \breve{h}_{\delta\zeta}\phi_\zeta(k_\zeta) - \bigwedge_{\zeta=1}^{n} \breve{h}_{\delta\zeta}\phi_\zeta(\tilde{k}_\zeta) \right| \leqslant \sum_{\zeta=1}^{n} |\breve{h}_{\delta\zeta}| \left|\phi_\zeta(k_\zeta) - \phi_\zeta(\tilde{k}_\zeta)\right|.$$

2.1 Model description

The Caputo-type FOFINNs with time delay is investigated as follows:

$$\begin{aligned}
&{}_0^cD_t^{2\beta}\tilde{k}_\delta(t) \\
&\quad= -\rho_\delta\, {}_0^cD_t^{\beta}\tilde{k}_\delta(t) - \sigma_\delta\tilde{k}_\delta(t) + \sum_{\zeta=1}^{n} c_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t)\big) \\
&\qquad+ \sum_{\zeta=1}^{n} b_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big) + \bigvee_{\zeta=1}^{n} \breve{h}_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big) + \bigwedge_{\zeta=1}^{n} \breve{s}_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big) \\
&\qquad+ \bigvee_{\zeta=1}^{n} T_{\delta\zeta}\varpi_\zeta + \bigwedge_{\zeta=1}^{n} H_{\delta\zeta}\varpi_\zeta + I_\delta(t),
\end{aligned} \tag{1}$$

where $0 < \beta < 1$, $\tilde{k}_\delta(t)$ stands for the state variable of the δth neuron, $\rho_\delta, \sigma_\delta > 0$, τ indicates the time delay, $\breve{s}_{\delta\zeta}$, $H_{\delta\zeta}$, $\breve{h}_{\delta\zeta}$, and $T_{\delta\zeta}$ represent the elements of the fuzzy feedback MIN template and MAX template, $\bigwedge$ and $\bigvee$ stand for the fuzzy AND and fuzzy OR, $I_\delta(t)$ is the inputs.

By proposing the transformation of variable

$$\tilde{r}_\delta(t) = {}_0^cD_t^{\beta}\tilde{k}_\delta(t) + \tilde{k}_\delta(t),$$

afterwards, system (1) can be reformulated as

$$\begin{aligned}
&{}_0^cD_t^{\beta}\tilde{r}_\delta(t) \\
&\quad= -(\rho_\delta - 1)\tilde{r}_\delta(t) - (\sigma_\delta - \rho_\delta + 1)\tilde{k}_\delta(t) + \sum_{\zeta=1}^{n} c_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t)\big) \\
&\qquad+ \sum_{\zeta=1}^{n} b_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big) + \bigvee_{\zeta=1}^{n} \breve{h}_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big) + \bigwedge_{\zeta=1}^{n} \breve{s}_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big) \\
&\qquad+ \bigvee_{\zeta=1}^{n} T_{\delta\zeta}\varpi_\zeta + \bigwedge_{\zeta=1}^{n} H_{\delta\zeta}\varpi_\zeta + I_\delta(t), \\
&{}_0^cD_t^{\beta}\tilde{k}_\delta(t) = \tilde{r}_\delta(t) - \tilde{k}_\delta(t).
\end{aligned} \tag{2}$$

Similarly, we apply the variable transformation

$$r_\delta(t) = {}_0^cD_t^{\beta}k_\delta(t) + k_\delta(t).$$

The response system of (2) is presented as

$$\begin{aligned}
&{}_0^cD_t^\beta r_\delta(t)\\
&\quad= -(\rho_\delta - 1)r_\delta(t) - (\sigma_\delta - \rho_\delta + 1)k_\delta(t) + \sum_{\zeta=1}^n c_{\delta\zeta}\phi_\zeta\bigl(k_\zeta(t)\bigr)\\
&\qquad+ \sum_{\zeta=1}^n b_{\delta\zeta}\phi_\zeta\bigl(k_\zeta(t-\tau)\bigr) + \bigvee_{\zeta=1}^n \breve{h}_{\delta\zeta}\phi_\zeta\bigl(k_\zeta(t-\tau)\bigr) + \bigwedge_{\zeta=1}^n \breve{s}_{\delta\zeta}\phi_\zeta\bigl(k_\zeta\bigl(t-\tau\bigr)\bigr)\\
&\qquad+ \bigvee_{\zeta=1}^n T_{\delta\zeta}\varpi_\zeta + \bigwedge_{\zeta=1}^n H_{\delta\zeta}\varpi_\zeta + I_\delta(t) + v_\delta(t),\\
&{}_0^cD_t^\beta k_\delta(t) = r_\delta(t) - k_\delta(t) + u_\delta(t).
\end{aligned} \tag{3}$$

Let $e_\delta(t) = k_\delta(t) - \tilde{k}_\delta(t),\ z_\delta(t) = r_\delta(t) - \tilde{r}_\delta(t)$, then we can get error systems

$$\begin{aligned}
&{}_0^cD_t^\beta z_\delta(t)\\
&\quad= -(\rho_\delta - 1)z_\delta(t) - (\sigma_\delta - \rho_\delta + 1)e_\delta(t) + \sum_{\zeta=1}^n c_{\delta\zeta}\bigl(\phi_\zeta\bigl(k_\zeta(t)\bigr) - \phi_\zeta\bigl(\tilde{k}_\zeta(t)\bigr)\bigr)\\
&\qquad+ \sum_{\zeta=1}^n b_{\delta\zeta}\bigl(\phi_\zeta\bigl(k_\zeta(t-\tau)\bigr) - \phi_\zeta\bigl(\tilde{k}_\zeta(t-\tau)\bigr)\bigr) + \bigvee_{\zeta=1}^n \breve{h}_{\delta\zeta}\phi_\zeta\bigl(k_\zeta(t-\tau)\bigr)\\
&\qquad- \bigvee_{\zeta=1}^n \breve{h}_{\delta\zeta}\phi_\zeta\bigl(\tilde{k}_\zeta(t-\tau)\bigr) + \bigwedge_{\zeta=1}^n \breve{s}_{\delta\zeta}\phi_\zeta\bigl(k_\zeta(t-\tau)\bigr) - \bigwedge_{\zeta=1}^n \breve{s}_{\delta\zeta}\phi_\zeta\bigl(\tilde{k}_\zeta(t-\tau)\bigr)\\
&\qquad+ v_\delta(t),\\
&{}_0^cD_t^\beta e_\delta(t) = z_\delta(t) - e_\delta(t) + u_\delta(t).
\end{aligned} \tag{4}$$

Assumption 1. There exist positive real numbers L_ζ that satisfy*:*

$$\bigl|\phi_\zeta(k) - \phi_\zeta(\tilde{k})\bigr| \leqslant L_\zeta|k - \tilde{k}|, \quad \zeta = 1, 2, \dots, n.$$

Definition 3. (See [24].) Systems (2) and (3) achieve GMLS if $\|\cdot\|_\varrho$ is ϱ-norm, $\|e(t)\|_\varrho = (\sum_{\delta=1}^n |e_\delta^\varrho(t)|)^{1/\varrho}$, $\|z(t)\|_\varrho = (\sum_{\delta=1}^n |z_\delta^\varrho(t)|)^{1/\varrho}$, $\psi(e(0)) \geqslant 0$, $\varphi(z(0)) \geqslant 0$, $\psi(0) = 0$, $\varphi(0) = 0$, and $\epsilon > 0$ such that

$$\bigl\|e(t)\bigr\|_\varrho + \bigl\|z(t)\bigr\|_\varrho \leqslant \bigl\{\bigl[\psi\bigl(e(0)\bigr) + \varphi\bigl(z(0)\bigr)\bigr]E_\beta\bigl(-\lambda t^\beta\bigr)\bigr\}^{1/\epsilon}.$$

3 Main results

The sufficient criteria of the GMLS for systems (2) and (3) are derived by using the appropriate controllers and inequality techniques. Hence, the controllers of system (4) are designed as

$$u_\delta(t) = -\eta_\delta e_\delta(t), \qquad v_\delta(t) = -\gamma_\delta z_\delta(t) - \xi_\delta z_\delta(t-\tau), \tag{5}$$

where $\eta_\delta > 0$, $\gamma_\delta > 0$, $\xi_\delta > 0$.

Theorem 1. *When $\theta > 1$ and $\mu_1 - \mu_2\theta > 0$, systems* (2) *and* (3) *can realize GMLS, where*

$$\mu_1 = \min_{1\leqslant\delta\leqslant n}\Bigg\{1+2\eta_\delta-|\rho_\delta-\sigma_\delta-1|-\sum_{\zeta=1}^{n}|c_{\zeta\delta}|L_\delta,\\ 2\rho_\delta-3+2\gamma_\delta-\xi_\delta-|\rho_\delta-\sigma_\delta-1|-\sum_{\zeta=1}^{n}\big(|c_{\delta\zeta}|+|b_{\delta\zeta}|+|\breve{h}_{\delta\zeta}|+|\breve{s}_{\delta\zeta}|\big)L_\zeta\Bigg\},$$

$$\mu_2 = \max_{1\leqslant\delta\leqslant n}\Bigg\{\sum_{\zeta=1}^{n}\big(|b_{\zeta\delta}|+|\breve{h}_{\zeta\delta}|+|\breve{s}_{\zeta\delta}|\big)L_\delta,\ \xi_\delta\Bigg\}.$$

Proof. Let us construct $V(\cdot)$ as

$$V(t) = \sum_{\delta=1}^{n} e_\delta^2(t) + \sum_{\delta=1}^{n} z_\delta^2(t),$$

then

$${}_0^cD_t^\beta V(t) = {}_0^cD_t^\beta\Bigg(\sum_{\delta=1}^{n}\big(e_\delta^2(t)+z_\delta^2(t)\big)\Bigg).$$

On the basis of Lemma 2, we get

$${}_0^cD_t^\beta V(t) \leqslant 2\sum_{\delta=1}^{n} e_\delta(t)\,{}_0^cD_t^\beta e_\delta(t) + 2\sum_{\delta=1}^{n} z_\delta(t)\,{}_0^cD_t^\beta z_\delta(t). \tag{6}$$

Substituting equation (4) into the right end of equation (6), one obtains

$$\begin{aligned}
&{}_0^cD_t^\beta V(t)\\
&\quad\leqslant 2\sum_{\delta=1}^{n} e_\delta(t)\big(z_\delta(t)-e_\delta(t)-\eta_\delta e_\delta(t)\big)\\
&\qquad+2\sum_{\delta=1}^{n} z_\delta(t)\Bigg(-(\rho_\delta-1)z_\delta(t)-(\sigma_\delta-\rho_\delta+1)e_\delta(t)\\
&\qquad+\sum_{\zeta=1}^{n} c_{\delta\zeta}\big(\phi_\zeta\big(k_\zeta(t)\big)-\phi_\zeta\big(\tilde{k}_\zeta(t)\big)\big)+\sum_{\zeta=1}^{n} b_{\delta\zeta}\big(\phi_\zeta\big(k_\zeta(t-\tau)\big)-\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big)\big)\\
&\qquad+\bigvee_{\zeta=1}^{n}\breve{h}_{\delta\zeta}\phi_\zeta\big(k_\zeta(t-\tau)\big)-\bigvee_{\zeta=1}^{n}\breve{h}_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big)+\bigwedge_{\zeta=1}^{n}\breve{s}_{\delta\zeta}\phi_\zeta\big(k_\zeta(t-\tau)\big)\\
&\qquad-\bigwedge_{\zeta=1}^{n}\breve{s}_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big)-\gamma_\delta z_\delta(t)-\xi_\delta z_\delta(t-\tau)\Bigg).
\end{aligned} \tag{7}$$

Based on Assumption 1 and Lemma 3, we have

$$\Bigg|\bigwedge_{\zeta=1}^{n}\breve{s}_{\delta\zeta}\phi_\zeta\big(k_\zeta(t-\tau)\big)-\bigwedge_{\zeta=1}^{n}\breve{s}_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big)\Bigg| \leqslant \sum_{\zeta=1}^{n}|\breve{s}_{\delta\zeta}|L_\zeta\big|e_\zeta(t-\tau)\big|, \tag{8}$$

similarly,

$$\left|\bigvee_{\zeta=1}^{n}\breve{h}_{\delta\zeta}\phi_\zeta\big(k_\zeta(t-\tau)\big)-\bigvee_{\zeta=1}^{n}\breve{h}_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big)\right|\leqslant\sum_{\zeta=1}^{n}|\breve{h}_{\delta\zeta}|L_\zeta\big|e_\zeta(t-\tau)\big|, \tag{9}$$

$$\big|\phi_\zeta\big(k_\zeta(t)\big)-\phi_\zeta(\tilde{k}_\zeta(t)\big|\leqslant L_\zeta\big|e_\zeta(t)\big|, \tag{10}$$

$$\big|\phi_\zeta\big(k_\zeta(t-\tau)\big)-\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big)\big|\leqslant L_\zeta\big|e_\zeta(t-\tau)\big|. \tag{11}$$

Substituting (8)–(11) into (7) gives

$$\begin{aligned}
{}_0^cD_t^\beta V(t)\leqslant{}& 2\sum_{\delta=1}^{n}e_\delta(t)\big(z_\delta(t)-e_\delta(t)-\eta_\delta e_\delta(t)\big)\\
&+2\sum_{\delta=1}^{n}z_\delta(t)\big(-(\rho_\delta-1)z_\delta(t)-(\sigma_\delta-\rho_\delta+1)e_\delta(t)\big)\\
&+2\sum_{\delta=1}^{n}z_\delta(t)\Bigg(\sum_{\zeta=1}^{n}c_{\delta\zeta}L_\zeta\big|e_\zeta(t)\big|+\sum_{\zeta=1}^{n}b_{\delta\zeta}L_\zeta\big|e_\zeta(t-\tau)\big|\\
&+\sum_{\zeta=1}^{n}|\breve{h}_{\delta\zeta}|L_\zeta\big|e_\zeta(t-\tau)\big|+\sum_{\zeta=1}^{n}|\breve{s}_{\delta\zeta}|L_\zeta\big|e_\zeta(t-\tau)\big|-\gamma_\delta z_\delta(t)-\xi_\delta z_\delta(t-\tau)\Bigg)\\
={}&2\sum_{\delta=1}^{n}e_\delta(t)z_\delta(t)-2\sum_{\delta=1}^{n}(1+\eta_\delta)e_\delta^2(t)-2\sum_{\delta=1}^{n}(\rho_\delta-1)z_\delta^2(t)\\
&-2\sum_{\delta=1}^{n}\gamma_\delta z_\delta^2(t)+2\sum_{\delta=1}^{n}(\rho_\delta-\sigma_\delta-1)e_\delta(t)z_\delta(t)\\
&+2\sum_{\delta=1}^{n}\sum_{\zeta=1}^{n}|c_{\delta\zeta}|L_\delta z_\delta(t)\big|e_\zeta(t)\big|-2\sum_{\delta=1}^{n}\xi_\delta z_\delta(t-\tau)z_\delta(t)\\
&+2\sum_{\delta=1}^{n}\sum_{\zeta=1}^{n}\big(|b_{\delta\zeta}|+|\breve{h}_{\delta\zeta}|+|\breve{s}_{\delta\zeta}|\big)L_\zeta\big|e_\zeta(t-\tau)\big|z_\delta(t)\\
\leqslant{}&2\sum_{\delta=1}^{n}\frac{e_\delta^2(t)+z_\delta^2(t)}{2}-2\sum_{\delta=1}^{n}(1+\eta_\delta)e_\delta^2(t)-2\sum_{\delta=1}^{n}(\rho_\delta-1)z_\delta^2(t)\\
&-2\sum_{\delta=1}^{n}\gamma_\delta z_\delta^2(t)+2\sum_{\zeta=1}^{n}|\rho_\delta-\sigma_\delta-1|\frac{e_\delta^2(t)+z_\delta^2(t)}{2}\\
&+2\sum_{\delta=1}^{n}\sum_{\zeta=1}^{n}|c_{\delta\zeta}|L_\zeta\frac{e_\zeta^2(t)+z_\delta^2(t)}{2}+2\sum_{\delta=1}^{n}\xi_\delta\frac{z_\delta^2(t)+z_\delta^2(t-\tau)}{2}\\
&+2\sum_{\delta=1}^{n}\sum_{\zeta=1}^{n}\big(|b_{\delta\zeta}|+|\breve{h}_{\delta\zeta}|+|\breve{s}_{\delta\zeta}|\big)L_\zeta\frac{e_\zeta^2(t-\tau)+z_\delta^2(t)}{2}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -\sum_{\delta=1}^{n}\left(1+2\eta_\delta-|\rho_\delta-\sigma_\delta-1|-\sum_{\zeta=1}^{n}|c_{\zeta\delta}|L_\delta\right)e_\delta^2(t)\\
&\quad-\sum_{\delta=1}^{n}\left(2\rho_\delta-3+2\gamma_\delta-\xi_\delta-|\rho_\delta-\sigma_\delta-1|\right.\\
&\quad\left.-\sum_{\zeta=1}^{n}\left(|c_{\delta\zeta}|+|b_{\delta\zeta}|+|\breve{h}_{\delta\zeta}|+|\breve{s}_{\delta\zeta}|\right)L_\zeta\right)z_\delta^2(t)\\
&\quad+\sum_{\delta=1}^{n}\left(\sum_{\zeta=1}^{n}\left(|b_{\zeta\delta}|+|\breve{h}_{\zeta\delta}|+|\breve{s}_{\zeta\delta}|\right)L_\delta\right)e_\delta^2(t-\tau)+\sum_{\delta=1}^{n}\xi_\delta z_\delta^2(t-\tau)\\
&\leqslant -\mu_1 V(t)+\mu_2 V(t-\tau).
\end{aligned}
$$

Employing the Razumikhin theorem in [3], one has

$$ {}_0^cD_t^\beta V(t)\leqslant -(\mu_1-\mu_2\theta)V(t), $$

then from Lemma 1

$$ V(t)\leqslant V(0)E_\beta\big(-(\mu_1-\mu_2\theta)t^\beta\big). $$

Therefore,

$$
\begin{aligned}
&\|e(t)\|_2+\|z(t)\|_2\\
&\quad=\left(\sum_{\delta=1}^{n}e_\delta^2(t)\right)^{1/2}+\left(\sum_{\delta=1}^{n}z_\delta^2(t)\right)^{1/2}\leqslant\left[2\left(\sum_{\delta=1}^{n}e_\delta^2(t)+\sum_{\delta=1}^{n}z_\delta^2(t)\right)\right]^{1/2}\\
&\quad=\big(2V(t)\big)^{1/2}\leqslant\big(2v(0)E_\beta\big(-(\mu_1-\mu_2\theta)t^\beta\big)\big)^{1/2}\\
&\quad=\big\{2\big[\|e(0)\|_2^2+\|z(0)\|_2^2\big]E_\beta\big(-(\mu_1-\mu_2\theta)t^\beta\big)\big\}^{1/2}.
\end{aligned}
\tag{12}
$$

Based on Definition 3, systems (2) and (3) actualize GMLS. □

If the model does not contain fuzzy logic terms, where $\breve{h}_{\delta\zeta}=\breve{s}_{\delta\zeta}=T_{\delta\zeta}=H_{\delta\zeta}=0$, then systems (2) and (3) are degenerated into FONNs

$$
\begin{aligned}
{}_0^cD_t^\beta\tilde{r}_\delta(t)&=-(\rho_\delta-1)\tilde{r}_\delta(t)-(\sigma_\delta-\rho_\delta+1)\tilde{k}_\delta(t)\\
&\quad+\sum_{\zeta=1}^{n}c_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t)\big)+\sum_{\zeta=1}^{n}b_{\delta\zeta}\phi_\zeta\big(\tilde{k}_\zeta(t-\tau)\big)+I_\delta(t),\\
{}_0^cD_t^\beta\tilde{k}_\delta(t)&=\tilde{r}_\delta(t)-\tilde{k}_\delta(t),
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
{}_0^cD_t^\beta r_\delta(t)&=-(\rho_\delta-1)r_\delta(t)-(\sigma_\delta-\rho_\delta+1)k_\delta(t)\\
&\quad+\sum_{\zeta=1}^{n}c_{\delta\zeta}\phi_\zeta\big(k_\zeta(t)\big)+\sum_{\zeta=1}^{n}b_{\delta\zeta}\phi_\zeta\big(k_\zeta(t-\tau)\big)+I_\delta(t)+v_\delta(t),\\
{}_0^cD_t^\beta k_\delta(t)&=r_\delta(t)-k_\delta(t)+u_\delta(t).
\end{aligned}
\tag{14}
$$

Corollary 1. *If there exists $\theta > 1$ and $\mu_3 - \mu_4\theta > 0$, then systems* (13) *and* (14) *attain GMLS under controller* (5)*, where*

$$\mu_3 = \min_{1\leqslant\delta\leqslant n}\Biggl\{1 + 2\eta_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{n}|c_{\zeta\delta}|L_\delta,\\ 2\rho_\delta - 3 + 2\gamma_\delta - \xi_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{n}\bigl(|c_{\delta\zeta}| + |b_{\delta\zeta}|\bigr)L_\zeta\Biggr\},$$

$$\mu_4 = \max_{1\leqslant\delta\leqslant n}\Biggl\{\sum_{\zeta=1}^{n}|b_{\zeta\delta}|L_\delta,\, \xi_\delta\Biggr\}.$$

If the model does not contain time delay, then systems (2) and (3) are transformed into the following FONNs:

$$\begin{aligned}{}_0^cD_t^\beta\tilde r_\delta(t) &= -(\rho_\delta - 1)\tilde r_\delta(t) - (\sigma_\delta - \rho_\delta + 1)\tilde k_\delta(t)\\ &\quad + \sum_{\zeta=1}^{n}c_{\delta\zeta}\phi_\zeta\bigl(\tilde k_\zeta(t)\bigr) + \bigvee_{\zeta=1}^{n}\breve h_{\delta\zeta}\phi_\zeta\bigl(\tilde k_\zeta(t)\bigr) + \bigwedge_{\zeta=1}^{n}\breve s_{\delta\zeta}\phi_\zeta\bigl(\tilde k_\zeta(t)\bigr)\\ &\quad + \bigvee_{\zeta=1}^{n}T_{\delta\zeta}\varpi_\zeta + \bigwedge_{\zeta=1}^{n}H_{\delta\zeta}\varpi_\zeta + I_\delta(t),\\ {}_0^cD_t^\beta\tilde k_\delta(t) &= \tilde r_\delta(t) - \tilde k_\delta(t),\end{aligned} \tag{15}$$

and

$$\begin{aligned}{}_0^cD_t^\beta r_\delta(t) &= -(\rho_\delta - 1)r_\delta(t) - (\sigma_\delta - \rho_\delta + 1)k_\delta(t)\\ &\quad + \sum_{\zeta=1}^{n}c_{\delta\zeta}\phi_\zeta\bigl(k_\zeta(t)\bigr) + \bigvee_{\zeta=1}^{n}\breve h_{\delta\zeta}\phi_\zeta\bigl(k_\zeta(t)\bigr) + \bigwedge_{\zeta=1}^{n}\breve s_{\delta\zeta}\phi_\zeta\bigl(k_\zeta(t)\bigr)\\ &\quad + \bigvee_{\zeta=1}^{n}T_{\delta\zeta}\varpi_\zeta + \bigwedge_{\zeta=1}^{n}H_{\delta\zeta}\varpi_\zeta + I_\delta(t) + v_\delta(t),\\ {}_0^cD_t^\beta k_\delta(t) &= r_\delta(t) - k_\delta(t) + u_\delta(t).\end{aligned} \tag{16}$$

The controller design of system (16) is described below:

$$u_\delta(t) = -\eta_\delta e_\delta(t), \qquad v_\delta(t) = -\gamma_\delta z_\delta(t),$$

where $\eta_\delta > 0$, $\gamma_\delta > 0$.

Corollary 2. *Systems* (15) *and* (16) *achieve GMLS if*

$$\lambda = \min_{1\leqslant\delta\leqslant n}\Biggl\{1 + 2\eta_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{n}\bigl(|c_{\zeta\delta}| + |\breve h_{\zeta\delta}| + |\breve s_{\zeta\delta}|\bigr)L_\delta,\\ 2\rho_\delta - 3 + 2\gamma_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{n}\bigl(|c_{\delta\zeta}| + |\breve h_{\delta\zeta}| + |\breve s_{\delta\zeta}|\bigr)L_\zeta\Biggr\}\\ > 0.$$

Proof. $V(\cdot)$ is designed as mentioned below:

$$V(t) = \sum_{\delta=1}^{n} e_\delta^2(t) + \sum_{\delta=1}^{n} z_\delta^2(t).$$

By calculations we have

$$\begin{aligned}
{}_0^cD_t^\beta V(t) &= {}_0^cD_t^\beta \left(\sum_{\delta=1}^{n} e_\delta^2(t) + \sum_{\delta=1}^{n} z_\delta^2(t) \right) \\
&\leqslant 2\sum_{\delta=1}^{n} e_\delta(t) {}_0^cD_t^\beta e_\delta(t) + 2\sum_{\delta=1}^{n} z_\delta(t) {}_0^cD_t^\beta z_\delta(t) \\
&\leqslant 2\sum_{\delta=1}^{n} e_\delta(t)\bigl(z_\delta(t) - e_\delta(t) - \eta_\delta e_\delta(t)\bigr) \\
&\quad + 2\sum_{\delta=1}^{n} z_\delta(t)\bigl(-(\rho_\delta - 1)z_\delta(t) - (\sigma_\delta - \rho_\delta + 1)e_\delta(t)\bigr) \\
&\quad + 2\sum_{\delta=1}^{n} z_\delta(t)\left(\sum_{\zeta=1}^{n} |c_{\delta\zeta}| L_\zeta \bigl|e_\zeta(t)\bigr| + \sum_{\zeta=1}^{n} |\breve{h}_{\delta\zeta}| L_\zeta \bigl|e_\zeta(t)\bigr| \right. \\
&\quad \left. + \sum_{\zeta=1}^{n} |\breve{s}_{\delta\zeta}| L_\zeta \bigl|e_\zeta(t)\bigr| - \gamma_\delta z_\delta(t) \right) \\
&= 2\sum_{\delta=1}^{n} e_\delta(t) z_\delta(t) - 2\sum_{\delta=1}^{n} (1+\eta_\delta) e_\delta^2(t) + 2\sum_{\delta=1}^{n} (1-\rho_\delta) z_\delta^2(t) \\
&\quad - 2\sum_{\delta=1}^{n} \gamma_\delta z_\delta^2(t) + 2\sum_{\delta=1}^{n} (\rho_\delta - \sigma_\delta - 1) e_\delta(t) z_\delta(t) \\
&\quad + 2\sum_{\delta=1}^{n}\sum_{\zeta=1}^{n} \bigl(|c_{\delta\zeta}| + |\breve{h}_{\delta\zeta}| + |\breve{s}_{\delta\zeta}|\bigr) L_\zeta \bigl|e_\zeta(t)\bigr| z_\delta(t) \\
&\leqslant 2\sum_{\delta=1}^{n} \frac{e_\delta^2(t) + z_\delta^2(t)}{2} - 2\sum_{\delta=1}^{n} (1+\eta_\delta) e_\delta^2(t) + 2\sum_{\delta=1}^{n} (1-\rho_\delta) z_\delta^2(t) \\
&\quad - 2\sum_{\delta=1}^{n} \gamma_\delta z_\delta^2(t) + 2\sum_{\delta=1}^{n} |\rho_\delta - \sigma_\delta - 1| \frac{e_\delta^2(t) + z_\delta^2(t)}{2} \\
&\quad + 2\sum_{\delta=1}^{n}\sum_{\zeta=1}^{n} \bigl(|c_{\delta\zeta}| + |\breve{h}_{\delta\zeta}| + |\breve{s}_{\delta\zeta}|\bigr) L_\zeta \frac{e_\zeta^2(t) + z_\delta^2(t)}{2} \\
&\leqslant -\sum_{\delta=1}^{n} \left(1 + 2\eta_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{n} \bigl(|c_{\zeta\delta}| + |\breve{h}_{\zeta\delta}| + |\breve{s}_{\zeta\delta}|\bigr) L_\delta \right) e_\delta^2(t)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{\delta=1}^{n}\left(2\rho_\delta-3+2\gamma_\delta-|\rho_\delta-\sigma_\delta-1|-\sum_{\zeta=1}^{n}\big(|c_{\delta\zeta}|+|\breve{h}_{\delta\zeta}|+|\breve{s}_{\delta\zeta}|\big)L_\zeta\right)z_\delta^2(t)\\
&\leqslant -\lambda\sum_{\delta=1}^{n}\big(e_\delta^2(t)+z_\delta^2(t)\big)=-\lambda V(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda=\min_{1\leqslant\delta\leqslant n}\Bigg\{&1+2\eta_\delta-|\rho_\delta-\sigma_\delta-1|-\sum_{\zeta=1}^{n}\big(|c_{\zeta\delta}|+|\breve{h}_{\zeta\delta}|+|\breve{s}_{\zeta\delta}|\big)L_\delta,\\
&2\rho_\delta-3+2\gamma_\delta-|\rho_\delta-\sigma_\delta-1|-\sum_{\zeta=1}^{n}\big(|c_{\delta\zeta}|+|\breve{h}_{\delta\zeta}|+|\breve{s}_{\delta\zeta}|\big)L_\zeta\Bigg\}.
\end{aligned}
$$

Based on Lemma 1, one obtains

$$
V(t)\leqslant V(0)E_\beta\big(-\lambda t^\beta\big), \tag{17}
$$

thus, similar to (12), according to (17), we have

$$
\begin{aligned}
\|e(t)\|_2+\|z(t)\|_2 &\leqslant \left[2\left(\sum_{\delta=1}^{n}e_\delta^2(t)+\sum_{\delta=1}^{n}z_\delta^2(t)\right)\right]^{1/2}\leqslant\big(2V(0)E_\beta\big(-\lambda t^\beta\big)\big)^{1/2}\\
&=\big\{2\big[\|e(0)\|_2^2+\|z(0)\|_2^2\big]E_\beta\big(-\lambda t^\beta\big)\big\}^{1/2}.
\end{aligned} \tag{18}
$$

According to Definition 3, (18) reveals that systems (15) and (16) achieve GMLS. □

Remark 1. The integer-order INNs have been discussed, and many research results have been obtained [6, 9, 11, 23, 26, 32]. In contrast to the dynamic properties of integer-order INNs, FOINNs have more infinite memory characteristics and can better describe the dynamic behavior of neurons, which can be preferably applied to realistic fields. The model in this paper has fuzzy logic, which is an effective tool to deal with some factors that may appear in the model, for instance, approximation, uncertainty, and fuzziness.

Remark 2. By utilizing variable transformation of reduced order, the FOFINNs model is represented as a β-order fuzzy Caputo equation. Adopting Lyapunov theory and algebraic inequality technique, the delayed feedback controller is selected to reach GMLS of the derive-response systems. The proposed model is less conservative and more general.

Remark 3. In contrast to the sampled-date control [7], predefined-time control [9], linear feedback control [34], event-triggered impulsive control [28], intermittent pinning control [11], pinning control [27], hybrid control [30], and sliding mode control [21], in this paper, the nonlinear delayed feedback controller is designed to effectively implement synchronization.

Remark 4. Different from the maximum-value approach [26], matrix inequalities [5,31], and matrix measure approach [14], in this paper, the new algebraic criteria for GMLS is established by using inequality technique, which makes the results easier to verify and more convenient to calculate in practical applications.

Remark 5. Compared with the models [8,12,31], the models of FOINNs are based on the sense of Riemann–Liouville derivative. However, this paper studies the GMLS problem under the derivative of Caputo. Different from literature [1, 5, 18], fuzzy terms are added on the basis of the above model in this paper. Therefore, the considered model is more general and less conservative.

4 Numerical simulations

Some simulations are provided in this part to testify the availability and feasibility of the criterion.

Example 1. Consider two-dimensional FOFINNs with time delay. We select the parameters of system (1): $\beta = 0.8$, $\rho_1 = 2$, $\rho_2 = 2$, $\sigma_1 = 0.25$, $\sigma_2 = 0.25$, $c_{11} = 1$, $c_{12} = -1.5$, $c_{21} = 1.5$, $c_{22} = 0.2$, $b_{11} = -0.5$, $b_{12} = -1$, $b_{21} = 2$, $b_{22} = -1$, $\breve{h}_{11} = -0.1$, $\breve{h}_{12} = 1$, $\breve{h}_{21} = -0.5$, $\breve{h}_{22} = 1$, $\breve{s}_{11} = 2$, $\breve{s}_{12} = 1.5$, $\breve{s}_{21} = -0.15$, $\breve{s}_{22} = -0.5$, $\phi_\zeta(\cdot) = \tanh(\cdot)$, $\tau = 1$. The Lipchitz constants $L_1 = L_2 = 0.5$.

We select the parameters of controller (5): $\gamma_1 = 6$, $\gamma_2 = 6$, $\xi_1 = 3$, $\xi_2 = 4$, $\eta_1 = 4$, $\eta_2 = 6$. Then we give two groups of initial values: $\tilde{k}_1(0) = 1$, $\tilde{k}_2(0) = -1.5$, $\tilde{r}_1(0) = -3.4$, $\tilde{r}_2(0) = 0.2$, $k_1(0) = -1.5$, $k_2(0) = 1.1$, $r_1(0) = -2.2$, $r_2(0) = -1.3$.

By computations, when $\delta = 1$, we get $\sum_{\zeta=1}^{2} |c_{\zeta\delta}| L_\delta = 1.25$, $\sum_{\zeta=1}^{2} |c_{\delta\zeta}| = 2.5$, $\sum_{\zeta=1}^{2} |b_{\delta\zeta}| = 1.5$, $\sum_{\zeta=1}^{2} |b_{\zeta\delta}| = 2.5$, $\sum_{\zeta=1}^{2} |\breve{h}_{\delta\zeta}| = 1.1$, $\sum_{\zeta=1}^{2} |\breve{h}_{\zeta\delta}| = 0.6$, $\sum_{\zeta=1}^{2} |\breve{s}_{\delta\zeta}| = 3.5$, $\sum_{\zeta=1}^{2} |\breve{s}_{\zeta\delta}| = 2.15$.

Moreover, when $\delta = 2$, $\sum_{\zeta=1}^{2} |c_{\zeta\delta}| L_\delta = 0.85$, $\sum_{\zeta=1}^{2} |c_{\delta\zeta}| = 1.7$, $\sum_{\zeta=1}^{2} |b_{\delta\zeta}| = 3$, $\sum_{\zeta=1}^{2} |b_{\zeta\delta}| = 2$, $\sum_{\zeta=1}^{2} |\breve{h}_{\delta\zeta}| = 1.5$, $\sum_{\zeta=1}^{2} |\breve{h}_{\zeta\delta}| = 2$, $\sum_{\zeta=1}^{2} |\breve{s}_{\delta\zeta}| = 0.65$, $\sum_{\zeta=1}^{2} |\breve{s}_{\zeta\delta}| = 2$. Therefore, we can get the following result.

We have

$$\begin{aligned}
\mu_1 = \min_{1\leqslant\delta\leqslant 2} \Biggl\{ & 1 + 2\eta_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{2} |c_{\zeta\delta}| L_\delta, \\
& 2\rho_\delta - 3 + 2\gamma_\delta - \xi_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{2} \bigl(|c_{\delta\zeta}| + |b_{\delta\zeta}| + |\breve{h}_{\delta\zeta}| + |\breve{s}_{\delta\zeta}|\bigr) L_\zeta \Biggr\} \\
= 4.825, &
\end{aligned}$$

$$\mu_2 = \max_{1\leqslant\delta\leqslant 2} \Biggl\{ \sum_{\zeta=1}^{2} \bigl(|b_{\zeta\delta}| + |\breve{h}_{\zeta\delta}| + |\breve{s}_{\zeta\delta}|\bigr) L_\delta,\ \xi_\delta \Biggr\} = 4.$$

Let $\theta = 1.01 > 1$, then we have $\mu_1 - \mu_2\theta = 0.785 > 0$. Hence, it has been demonstrated that all the conditions specified in Theorem 1 are confirmed and fulfilled.

Through the above analysis, systems (2) and (3) can achieve GMLS. The trajectories of state variable are portrayed in Figs. 1 and 2. As can be seen from the trajectories curves in the figures, the trajectories of the state variables are finally coincident. The error curves of systems (2) and (3) are exhibited in Fig. 3.

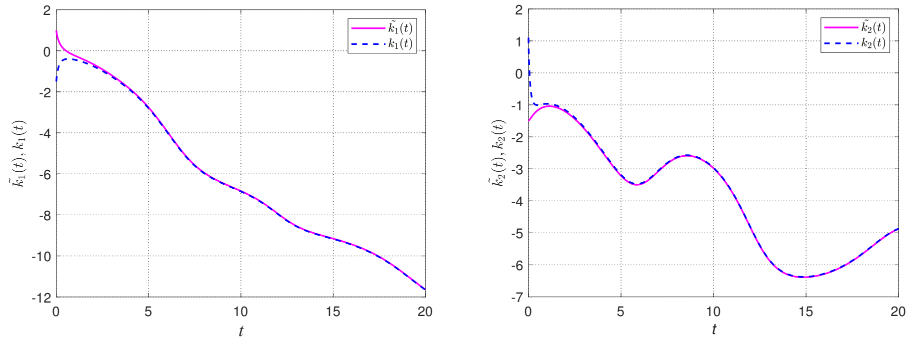

Figure 1. Trajectories of systems (2) and (3).

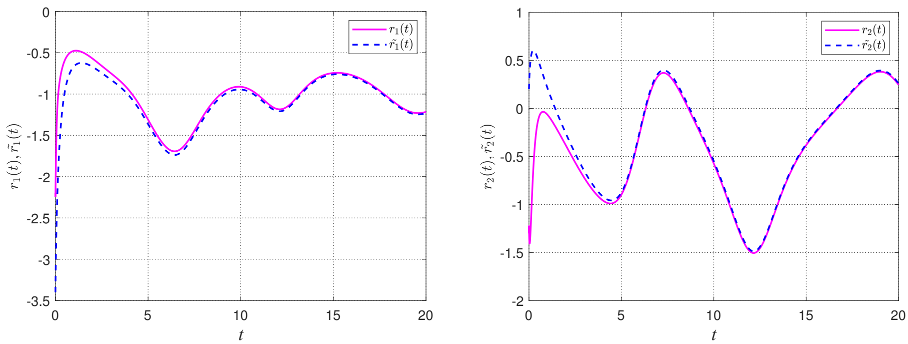

Figure 2. Trajectories of systems (2) and (3).

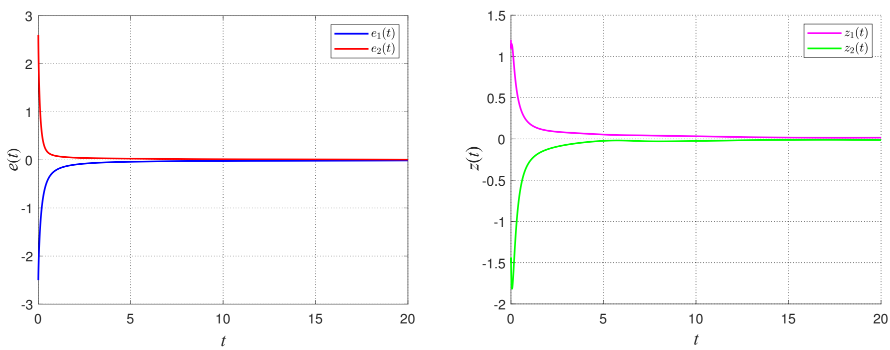

Figure 3. Synchronization error of systems (2) and (3).

Example 2. Consider three-dimensional FOFINNs with time delay. About the relevant parameters of the systems (2) and (3), we select: $\beta = 0.9$, $\rho_1 = 1.5$, $\rho_2 = 1$, $\rho_3 = 1$, $\sigma_1 = 1.5$, $\sigma_2 = 0.5$, $\sigma_3 = 1.5$, $c_{11} = 0.7$, $c_{12} = -0.4$, $c_{21} = 1$, $c_{22} = -0.5$, $c_{13} = -0.25$, $c_{23} = -1$, $c_{31} = -0.25$, $c_{32} = 0.01$, $c_{33} = -0.3$, $b_{11} = 0.1$, $b_{12} = -0.5$, $b_{21} = -0.8$, $b_{22} = -1$, $b_{13} = -0.15$, $b_{23} = -0.1$, $b_{31} = -0.7$, $b_{32} = 2$, $b_{33} = -0.8$, $\breve{h}_{11} = -1$, $\breve{h}_{12} = 0.5$, $\breve{h}_{21} = -0.5$, $\breve{h}_{22} = -0.1$, $\breve{h}_{13} = 1.8$, $\breve{h}_{23} = -1$, $\breve{h}_{31} = 0.3$, $\breve{h}_{32} = -0.15$, $\breve{h}_{33} = 0.3$, $\breve{s}_{11} = -0.4$, $\breve{s}_{12} = -0.5$, $\breve{s}_{21} = -0.1$, $\breve{s}_{22} = 1$, $\breve{s}_{13} = -0.1$, $\breve{s}_{23} = -0.5$, $\breve{s}_{31} = 2$, $\breve{s}_{32} = 0.9$, $\breve{s}_{33} = -0.9$, $\tau = 1$. The Lipchitz constants $L_1 = L_2 = L_3 = 0.5$, $\phi_\zeta(\cdot) = \tanh(\cdot)$.

Under controller (5), we select: $\gamma_1 = 5.5$, $\gamma_2 = 6$, $\gamma_3 = 6.2$, $\xi_1 = 2.2$, $\xi_2 = 2$, $\xi_3 = 1.5$, $\eta_1 = 4$, $\eta_2 = 3.5$, $\eta_3 = 5$.

Then we select the initial values: $\tilde{k}_1(0) = 2.5$, $\tilde{k}_2(0) = 4.5$, $\tilde{k}_3(0) = -0.5$, $\tilde{r}_1(0) = -2.4$, $\tilde{r}_2(0) = 3.2$, $\tilde{r}_3(0) = -2.5$, $k_1(0) = 0.5$, $k_2(0) = -1.1$, $k_3(0) = -4.9$, $r_1(0) = 2.2$, $r_2(0) = 4.6$, $r_3(0) = 1.9$.

Through simple calculation, we have: when $\delta = 1$, $\sum_{\zeta=1}^{2} |c_{\zeta\delta}| L_\delta = 0.975$, $\sum_{\zeta=1}^{2} |c_{\delta\zeta}| = 1.35$, $\sum_{\zeta=1}^{2} |b_{\delta\zeta}| = 0.75$, $\sum_{\zeta=1}^{2} |b_{\zeta\delta}| = 1.6$, $\sum_{\zeta=1}^{2} |\breve{h}_{\delta\zeta}| = 3.3$, $\sum_{\zeta=1}^{2} |\breve{h}_{\zeta\delta}| = 1.8$, $\sum_{\zeta=1}^{2} |\breve{s}_{\delta\zeta}| = 1$, $\sum_{\zeta=1}^{2} |\breve{s}_{\zeta\delta}| = 2.5$.

Furthermore, we get: when $\delta = 2$, $\sum_{\zeta=1}^{2} |c_{\zeta\delta}| L_\delta = 0.455$, $\sum_{\zeta=1}^{2} |c_{\delta\zeta}| = 2.5$, $\sum_{\zeta=1}^{2} |b_{\delta\zeta}| = 1.9$, $\sum_{\zeta=1}^{2} |b_{\zeta\delta}| = 3.5$, $\sum_{\zeta=1}^{2} |\breve{h}_{\delta\zeta}| = 1.6$, $\sum_{\zeta=1}^{2} |\breve{h}_{\zeta\delta}| = 0.75$, $\sum_{\zeta=1}^{2} |\breve{s}_{\delta\zeta}| = 1.6$, $\sum_{\zeta=1}^{2} |\breve{s}_{\zeta\delta}| = 2.4$.

Analogously, when $\delta = 3$, $\sum_{\zeta=1}^{2} |c_{\zeta\delta}| L_\delta = 0.775$, $\sum_{\zeta=1}^{2} |c_{\delta\zeta}| = 0.56$, $\sum_{\zeta=1}^{2} |b_{\delta\zeta}| = 3.5$, $\sum_{\zeta=1}^{2} |b_{\zeta\delta}| = 1.05$, $\sum_{\zeta=1}^{2} |\breve{h}_{\delta\zeta}| = 0.75$, $\sum_{\zeta=1}^{2} |\breve{h}_{\zeta\delta}| = 3.1$, $\sum_{\zeta=1}^{2} |\breve{s}_{\delta\zeta}| = 3.8$, $\sum_{\zeta=1}^{2} |\breve{s}_{\zeta\delta}| = 1.5$.

Then, by calculation we obtain

$$\begin{aligned}
\mu_1 = \min_{1 \leqslant \delta \leqslant 3} \Biggl\{ & 1 + 2\eta_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{3} |c_{\zeta\delta}| L_\delta, \\
& 2\rho_\delta - 3 + 2\gamma_\delta - \xi_\delta - |\rho_\delta - \sigma_\delta - 1| - \sum_{\zeta=1}^{3} \bigl(|c_{\delta\zeta}| + |b_{\delta\zeta}| + |\breve{h}_{\delta\zeta}| + |\breve{s}_{\delta\zeta}|\bigr) L_\zeta \Biggr\} \\
= 4.095 & \\
\mu_2 = \max_{1 \leqslant \delta \leqslant 3} \Biggl\{ & \sum_{\zeta=1}^{3} \bigl(|b_{\zeta\delta}| + |\breve{h}_{\zeta\delta}| + |\breve{s}_{\zeta\delta}|\bigr) L_\delta,\, \xi_\delta \Biggr\} = 3.325.
\end{aligned}$$

Let $\theta = 1.2 > 1$, we get that $\mu_1 - \mu_2\theta = 0.105 > 0$. Hence, the criteria of Theorem 1 are confirmed and fulfilled.

According to the above analysis, systems (2) and (3) can realize GMLS, and the simulation results are provided in figures. Figures 4–6 display the state curves of each state variable. The synchronization error trajectories of systems (2) and (3) are depicted in Fig. 7 under the feedback controller. In summary, the numerical simulations results of Example 2 further show the validity of Theorem 1.

Remark 6. To make the numerical examples more convincing, the cases of $n=2$, $\beta=0.8$ and $n=3$, $\beta=0.9$ are respectively considered. By utilizing the MATLAB toolbox, the obtained images further show that the calculated results are in good agreement with the theoretical results.

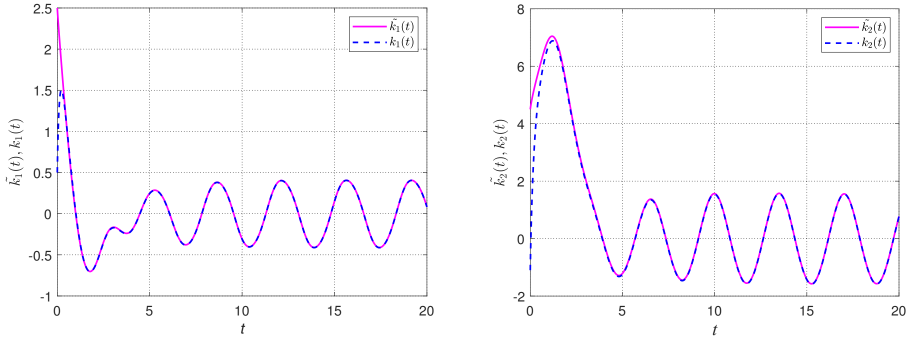

Figure 4. The state curves of systems (2) and (3).

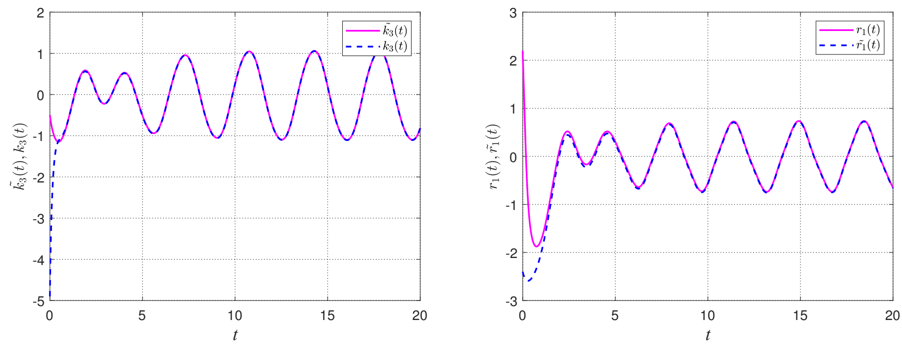

Figure 5. The state curves of systems (2) and (3).

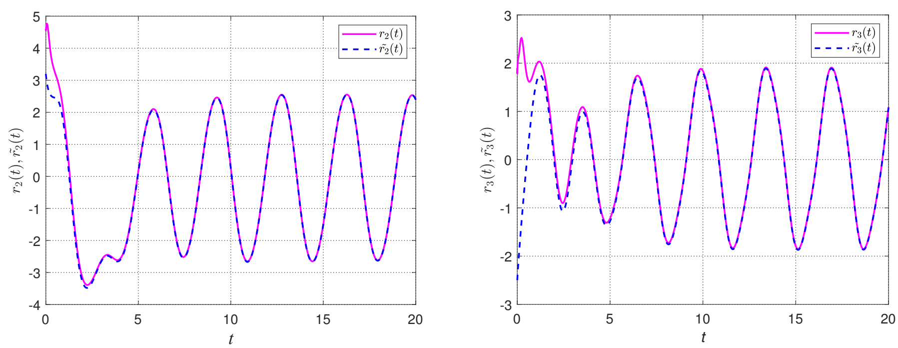

Figure 6. The state curves of systems (2) and (3).

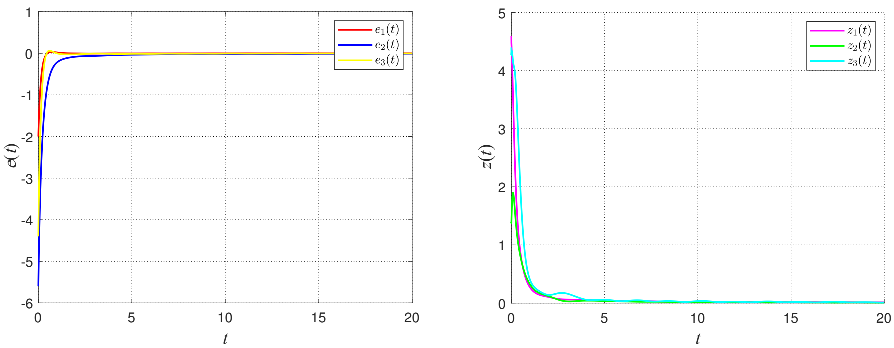

Figure 7. Synchronization error of systems (2) and (3).

5 Conclusions

This paper was undertaken to discuss the problem of synchronization for GMLS of FOFINNs with time delay. In light of the properties of the Caputo derivative, the inertial system is converted into a general fractional differential system through appropriate variable replacement. A simple and convenient controller is designed to implement GMLS. Applying the inequality techniques, Lyapunov's direct method for functions, and Razumikhin theorem, the criterion on Caputo FOFINNs has been established. The results are expressed as algebraic inequalities, which greatly reduce the computational complexity. According to two special cases of the model, corresponding inferences are obtained. Two numerical examples proved the availability of the theoretic results. The future research is to explore the GMLS of FOFINNs with generalized delays under the sliding mode control. In addition, we will furthermore investigate the problem of mean-square exponential synchronization and H_∞ synchronization.

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