Nonlinear Analysis: Modelling and Control, Vol. 28, No. 3, 538–553
https://doi.org/10.15388/namc.2023.28.31866

Analysis and soliton solutions of biofilm model by new extended direct algebraic method

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Received: April 2, 2022 / **Revised:** February 18, 2023 / **Published online:** March 29, 2023

Abstract. In this paper, the examination of soliton solutions of the biofilm model with the help of a new extended direct algebraic method is expressed. Besides the exact solutions, the existence of these solutions is also discussed with the help of the Schauder fixed point theorem. The nonlinear dynamical biofilm model which we consider in this paper is basically the bistable Allen–Cahn equation with quartic potential. For more physical understanding, the 3D plots of the solutions are presented. The different types of hyperbolic, trigonometric, and rational soliton solutions are gained. In the biofilm model, different parameters belong to certain spaces and give the exact solutions by applying the technique of new extended direct algebraic method. Exact solutions of the biofilm model are highlighted along with their restriction.

Keywords: biofilms model, existence of result, Schauder fixed point theorem, new extended direct algebraic method, soliton solutions.

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1 Introduction

In the 18th century, Euler, d'Alembert, Lagrange, and Laplace discussed partial differential equations (PDEs) in the field of continuum mechanics and analysis of models in physical science. The prime contribution of the PDEs in the natural world includes the vibration of solids, the phenomenon of electromagnetic radiation, structural properties of molecules including the interaction between photons and electrons, fluid dynamics, propagation of sound, heat flow and diffusion, etc. In short, the key defining properties of PDEs are in geometry and analysis. Infinite-dimensional group representation, homogenous spaces, mathematical physics, and quantum field theory are the areas of coeval research in mathematics in which PDEs plays a crucial role. PDEs act as a bridge between applied mathematics and physical science. Also, PDEs are the basis of the advancement of mathematical ideas in pure mathematics; see [23]. Abridge of pivotal functioning of PDEs in other areas of mathematics include differential geometry, calculus of variation, Riemannian surfaces, algebric geometry, general relativity, quantum mechanics, electrostatics, electrodynamics, etc.

Physical sciences act as a skeleton for many dominant ideas in mathematics. For example, the foundation of calculus lies in the motion of bodies. Mathematical equations describe the theory behind the physical phenomenon. In short, they act as a fabric for physical phenomena, i.e., the electrodynamical phenomena have been explained by Maxwell's equations. The description of the mechanical system has been provided by Newton's equation, Schrödinger's equations have their application in the field of quantum mechanics, and so on. The primary role of PDEs is also found in the field of mathematical modelings like financial modeling, biological modeling, and so on. Mathematical modeling summarizes the physical observations of the physical system. For example, Newton's equations are the model for mechanical systems. In short, the simplified version of physical systems in mathematical terms is expressed by mathematical modeling; see [17].

PDEs are further divided into linear and nonlinear branches. Our work is basically on nonlinear PDEs. The traits of nonlinear PDEs are connected with the nonlinear laws of nature. They are used in stochastic game theory, gravitational behavior, Fluid dynamics, Poincaré and Calabi conjectures, non-Newtonian fluid, and so on. The roots of the nonlinear PDEs such as Monge–Ampère, minimal surfaces, Yamabe equations originate from differential geometry. There are different techniques for finding solutions to nonlinear PDEs. We apply the technique of new extended direct algebraic method (NEDAM) to study the behavior of microbes forming biofilm on the surfaces; see [14, 18]. Jornet consider biofilm formation via the stochastic bistable Allen–Cahn partial differential equation [14], Tijani construct an unconditionally positive nonstandard finite difference scheme for a mathematical model of biofilm formation using the Allen–Cahn equation [29], and Tijani apply the finite difference scheme to model biofilm growth [30], but Iqbal et al. construct the different types of soliton solutions of stochastic Allen–Chan equation by extended fan-sub technique [13]. In this modern era of research, finding soliton solutions is an important field to describe the physical behavior of the nonlinear PDEs. There are many different techniques to find the soliton solutions such as G'/G expansion method

[4, 31], first integral method [5], Kudryashov method [3, 8], generalized logistic equation method [22, 34], Riccati mapping method [2, 33], ϕ^6-model expansion method [27, 35], He's variational method [20], generalized exponential rational function method [12], Hirota bilinear method [11], modified exponential rational functional method [1], a new auxiliary equation [28], Riccati–Bernoulli sub-ODE method [7, 16, 19], etc. But in this study, we apply the new modified extended direct algebraic method and the existence of the solutions on the bistable Allen–Cahn equation with quartic potential.

This manuscript basically deals with the nonlinear PDE in the form of biofilm model. We used the technique of new extended direct algebraic method to find the more general and exact soliton solutions of biofilm model. The dynamical nature of the new extended direct algebraic method illuminates the soliton solutions as well as the hyperbolic and trigonometric function solutions; see [10, 25, 32]. The plague layer that is usually formed on teeth is one simple examples of a biofilm. In short, the collection of one or different types of microorganisms that adhere on different surfaces form a biofilm. These microbes consist of bacteria, fungi, algae, etc.; see [14–24]. So, these results are new and affected in the living reproducing microbes that penchant to live in colonies form a biofilm.

The paper is organized in the following manner. The next section expresses the statement of the problem, then Section 3 present the existence theory, summary and application of the method are highlighted in Sections 4 and 5. Restriction of solutions in certain cases is encoded in Section 6, and Sections 6 and 7 describe the graphical interpretation and conclusion.

2 Statement of the problem

The nonlinear dynamical biofilm model that we consider in this paper is basically the bistable Allen–Cahn equation (with quartic potential); [21]

$$c_t = D\Delta c + rc(1-c)(c-\theta), \quad t > 0, \tag{1}$$

where $c(t, x, y)$ is the density of microbes on the rectangular surface $[e, g] \times [n, m]$ (i.e., $(x, y) \in [e, g] \times [n, m]$, and times $t \geqslant 0$). $D > 0$ is the diffusion coefficient, $r > 0$ is the growth rate, the parameter $\theta \in [0, 1]$ encodes the survival rate of microbes [14]. The goal of the article is subdivided into two main parts. The one is to show the classical behavior of the solutions of the above differential equation, and the second is to formulate the interval in which solutions exist by existence theory.

3 Classical existence of at least one solution of biofilm model

In this section, we discussed the existence of solution of Eq. (1) and their set of continuity, and especially, it is an important consequence if we can give some information on the bound of the set of existence of solution without knowing the solution itself. The integral representation of Eq. (1) is

$$c = c_0(x,y) + \int_0^t \big[D\Delta c(x,y,\tau) + r\big(c^2(x,y,\tau) - \theta c(x,y,\tau) \\ - c^3(x,y,\tau) + \theta c^2(x,y,\tau)\big)\big]\,\mathrm{d}\tau. \tag{2}$$

The formation of Eq. (2) leads us to rewrite it in the form of fixed operator

$$\begin{aligned}\mathcal{C} &= \mathcal{C}\big(c(x,y,t)\big) \\ &= c_0(x,y) + \int_0^t \big[D\Delta c(x,y,\tau) + \big(c^2(x,y,\tau) - \theta c(x,y,\tau) \\ &\quad - c^3(x,y,\tau) + \theta c^2(x,y,\tau)\big)\big]\,\mathrm{d}\tau.\end{aligned} \tag{3}$$

The operator equation is a fixed point representation of the problem for $c(x,y,t)$. Hence, any fixed function $c^*(x,y,t)$ of $\mathcal{C}(x,y,t)$ will turn out to be a solution of not only Eq. (3) but also of Eqs. (1), (2). Before we proceed for the construction of fixed points of Eq. (3), we must look for the topological spaces. Here we choose the space of continuous functions $\boldsymbol{C}$ equipped with supremum norm, i.e., $m^* \in \boldsymbol{C}[0,\rho] \Rightarrow \|m^*\|_{\boldsymbol{C}} = \max_{[0,\rho]} |m^*|$. Assume that functions c, D, r, θ are continuous and, consequently, locally bounded. Hence,

$$s,\ \boldsymbol{p}, R > 0.$$

For the existence of solution, we shall apply the Schauder fixed point theorem, which is a generalization of Brouwer fixed point theorem for infinite dimensional space; see [9,15]. There are two important steps by Schauder fixed point theorem, which are the implication of the statement of theorem.

Theorem. *Suppose B is a closed convex and bounded subset of Banach space $\boldsymbol{C}$, a function $\mathcal{C}$ continuous and maps the set B to itself, and $\mathcal{C}(B)$ is relatively compact. Then $\mathcal{C}$ has at least one fixed point in B.*

In view of the above result, the following two conditions must be verified:

- $\mathcal{C}(x,y,t) : B \to B$,
- $\mathcal{C}(B)$ is relatively compact.

The set B is closed, convex, and bounded subset of $\boldsymbol{C}[0,\rho]$, and it is defined by

$$B_R(\Theta) = \big\{c,\ c \in \boldsymbol{C}[0,\rho]\big\},$$

where capital theta Θ symbol is for zero element of function space $\boldsymbol{C}$, and also, R is the radius of the ball and define the complete length of continuity of the solution.

To verify the first condition $\mathcal{C} : B \to B$, we shall use $B = B_R(\Theta)$. For this purpose, we apply the norm of Eq. (3) on both sides, and further simplification leads to

$$\|\mathcal{C}\| \leqslant s + \rho\big[D\boldsymbol{p} + r\big(R^2 + \theta R + R^3 + \theta R^2\big)\big],$$

where $\rho = (t - 0)$, length of continuity. For mapping the ball into itself, we have the following condition:

$$\rho = \frac{R - s}{D\boldsymbol{p} + r(R^2 + \theta R + R^3 + \theta R^2)}.$$

This will hold true if $R > S$. The minimum value of $[D\boldsymbol{p} + r(R^2 + \theta R + R^3 + \theta R^2)]$ will leads to the largest length of continuity of solution, for that we get the condition

$$\theta^2 \geqslant \theta - 1.$$

Now this step is essential for the accomplishment of the existence of solution by Schauder's well-known result for the existence in infinite dimensional setting. For this, we again consider Eq. (3)

$$\mathcal{C} = c_0(x, y) + \int_0^t \bigl[D\Delta c(x, y, \tau) + r\bigl(c^2(x, y, \tau) - \theta c(x, y, \tau) - c^3(x, y, \tau) + \theta c^2(x, y, \tau)\bigr)\bigr]\,\mathrm{d}\tau.$$

The equicontinuity of the solution is checked at two points t and t^*. Therefore, we have the following equation:

$$\begin{aligned}
&\mathcal{C}_i(x, y, t) - \mathcal{C}_i(x, y, t^*) \\
&\quad = c_0(x, y) + \int_0^t \bigl[D\Delta c_i(x, y, \tau) + r\bigl(c_i^2(x, y, \tau) - \theta c_i(x, y, \tau) \\
&\qquad - c_i^3(x, y, \tau) + \theta c_i^2(x, y, \tau)\bigr)\bigr]\,\mathrm{d}\tau \\
&\qquad - c_0(x, y) - \int_0^{t^*} \bigl[D\Delta c_i(x, y, \tau) + r\bigl(c_i^2(x, y, \tau) - \theta c_i(x, y, \tau) \\
&\qquad - c_i^3(x, y, \tau) + \theta c_i^2(x, y, \tau)\bigr)\bigr]\,\mathrm{d}\tau.
\end{aligned}$$

After applying the norm and further simplification, we get

$$\begin{aligned}
\bigl\|\mathcal{C}_i(x, y, t) - \mathcal{C}_i(x, y, t^*)\bigr\| &\leqslant \bigl[Dp|t - t^*| + r\bigl(R^2|t - t^*| + \theta R|t - t^*| \\
&\quad + R^3|t - t^*| + \theta R^2|t - t^*|\bigr)\bigr] \\
\bigl\|\mathcal{C}_i(x, y, t) - \mathcal{C}_i(x, y, t^*)\bigr\| &\leqslant \bigl[\bigl(D\boldsymbol{p} + r\bigl(R^2 + \theta R + R^3 + \theta R^2\bigr)\bigr)|t - t^*|\bigr].
\end{aligned}$$

Remark 1. The condition of continuity holds.

Clearly, $\|\mathcal{C}_i(x, y, t) - \mathcal{C}_i(x, y, t^*)\| \to 0$ as $t \to t^*$, and even for a special pair (x^*, y^*, t^*), $\|\mathcal{C}_i(x, y, t) - \mathcal{C}_i(x^*, y^*, t^*)\| \to 0$ as $(x, y, t) \to (x^*, y^*, t^*)$. So, the family $\mathcal{C}_i(x, y, t)$ is turn out to be equicontinuous, and Arzelà–Ascoli theorem is applicable. By Arzelà–Ascoli theorem there exist a subsequence $\mathcal{C}_{i_j}(x, y, t)$ of $\mathcal{C}_i$ such that $\mathcal{C}_{i_j}$ is uniformly convergent, so, the operator $\mathcal{C}(x, y, t)$ turns out to be a relatively compact operator, and Schauder theorem is applicable; see [9, 15].

4 Summary of new extended direct algebraic method

Consider the following PDEs:

$$T(\beta, \beta_x, \beta_y, \beta_t, \beta_{xx}, \beta_{xy}, \dots) = 0, \tag{4}$$

where $\beta = \beta(x, y, t)$, also, the subscript denotes the partial derivative w.r.t. x, y, t using the transformation

$$\beta(x, y, t) = \eta(\xi), \quad \xi = x + y - \omega t.$$

After applying the transformation, Eq. (4) is converted into ordinary differential equation (ODE)

$$G = (\eta, \eta', \eta'', \dots) = 0, \tag{5}$$

where prime denote derivative w.r.t. ξ. The solution of Eq. (5) has the form [6, 25]

$$\eta(\xi) = \sum_{u=0}^{j} a_u Q^u(\xi), \quad a_u \neq 0, \tag{6}$$

where a_u $(0 \leqslant u \leqslant j)$ are the constant to be determined, j is the positive integer, and its value is determined by using the balancing principle in Eq. (5). $Q(\xi)$ satisfies the ODE in the form

$$Q'(\xi) = \ln A\big(\alpha + \delta Q(\xi) + \gamma Q^2(\xi)\big), \quad A \neq 0, 1. \tag{7}$$

Here α, δ, γ are constants, and the general solutions of Eq. (7) are given as follows.

Case 1. When $\mu = \delta^2 - 4\alpha\gamma < 0$ and $\gamma \neq 0$,

$$Q_1(\xi) = -\frac{\delta}{2\gamma} + \frac{\sqrt{-\mu}}{2\gamma} \tan_A\left(\frac{\sqrt{-\mu}}{2}\xi\right), \tag{8}$$

$$Q_2(\xi) = -\frac{\delta}{2\gamma} - \frac{\sqrt{-\mu}}{2\gamma} \cot_A\left(\frac{\sqrt{-\mu}}{2}\xi\right), \tag{9}$$

$$Q_3(\xi) = -\frac{\delta}{2\gamma} + \frac{\sqrt{-\mu}}{2\gamma}\big(\tan_A\left(\sqrt{-\mu}\xi\right) \pm \sqrt{pq}\sec_A(\sqrt{-\mu}\xi)\big), \tag{10}$$

$$Q_4(\xi) = -\frac{\delta}{2\gamma} + \frac{\sqrt{-\mu}}{2\gamma}\big(-\cot_A\left(\sqrt{-\mu}\xi\right) \pm \sqrt{pq}\csc_A(\sqrt{-\mu}\xi)\big), \tag{11}$$

$$Q_5(\xi) = -\frac{\delta}{2\gamma} + \frac{\sqrt{-\mu}}{4\gamma}\left(\tan_A\left(\frac{\sqrt{-\mu}}{4}\xi\right) - \cot_A\left(\frac{\sqrt{-\mu}}{4}\xi\right)\right). \tag{12}$$

Case 2. When $\mu = \delta^2 - 4\alpha\gamma > 0$ and $\gamma \neq 0$,

$$Q_6(\xi) = -\frac{\delta}{2\gamma} - \frac{\sqrt{\mu}}{2\gamma}\tanh_A\left(\frac{\sqrt{\mu}}{2}\xi\right), \tag{13}$$

$$Q_7(\xi) = -\frac{\delta}{2\gamma} - \frac{\sqrt{\mu}}{2\gamma}\coth_A\left(\frac{\sqrt{\mu}}{2}\xi\right), \tag{14}$$

$$Q_8(\xi) = -\frac{\delta}{2\gamma} + \frac{\sqrt{\mu}}{2\gamma}\left(-\tanh_A(\sqrt{\mu}\xi) \pm \iota\sqrt{pq}\,\mathrm{sech}_A(\sqrt{\mu}\xi)\right), \tag{15}$$

$$Q_9(\xi) = -\frac{\delta}{2\gamma} + \frac{\sqrt{\mu}}{2\gamma}\left(-\coth_A(\sqrt{\mu}\xi) \pm \sqrt{pq}\,\mathrm{csch}_A(\sqrt{\mu}\xi)\right), \tag{16}$$

$$Q_{10}(\xi) = -\frac{\delta}{2\gamma} - \frac{\sqrt{\mu}}{4\gamma}\left(\tanh_A\left(\frac{\sqrt{\mu}}{4}\xi\right) + \coth_A\left(\frac{\sqrt{\mu}}{4}\xi\right)\right). \tag{17}$$

Case 3. When $\alpha\gamma > 0$ and $\delta = 0$,

$$Q_{11}(\xi) = \sqrt{\frac{\alpha}{\gamma}}\tan_A(\sqrt{\alpha\gamma}\xi), \tag{18}$$

$$Q_{12}(\xi) = -\sqrt{\frac{\alpha}{\gamma}}\cot_A(\sqrt{\alpha\gamma}\xi), \tag{19}$$

$$Q_{13}(\xi) = \sqrt{\frac{\alpha}{\gamma}}\left(\tan_A(2\sqrt{\alpha\gamma}\xi) \pm \sqrt{pq}\,\sec_A(2\sqrt{\alpha\gamma}\xi)\right), \tag{20}$$

$$Q_{14}(\xi) = \sqrt{\frac{\alpha}{\gamma}}\left(-\cot_A(2\sqrt{\alpha\gamma}\xi) \pm \sqrt{pq}\,\csc_A(2\sqrt{\alpha\gamma}\xi)\right), \tag{21}$$

$$Q_{15}(\xi) = \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\left(\tan_A\left(\frac{\sqrt{\alpha\gamma}}{2}\xi\right) - \cot_A\left(\frac{\sqrt{\alpha\gamma}}{2}\xi\right)\right). \tag{22}$$

Case 4. When $\alpha\gamma < 0$, $\delta = 0$, and $\gamma = \alpha$,

$$Q_{16}(\xi) = -\sqrt{-\frac{\alpha}{\gamma}}\tanh_A(\sqrt{-\alpha\gamma}\xi), \tag{23}$$

$$Q_{17}(\xi) = -\sqrt{-\frac{\alpha}{\gamma}}\coth_A(\sqrt{-\alpha\gamma}\xi), \tag{24}$$

$$Q_{18}(\xi) = \sqrt{-\frac{\alpha}{\gamma}}\left(-\tanh_A(2\sqrt{-\alpha\gamma}\xi) \pm \iota\sqrt{pq}\,\mathrm{sech}_A(2\sqrt{-\alpha\gamma}\xi)\right), \tag{25}$$

$$Q_{19}(\xi) = \sqrt{-\frac{\alpha}{\gamma}}\left(-\coth_A(2\sqrt{-\alpha\gamma}\xi) \pm \sqrt{pq}\,\mathrm{csch}_A(2\sqrt{-\alpha\gamma}\xi)\right), \tag{26}$$

$$Q_{20}(\xi) = -\frac{1}{2}\sqrt{-\frac{\alpha}{\gamma}}\left(\tanh_A\left(\frac{\sqrt{-\alpha\gamma}}{2}\xi\right) + \coth_A\left(\frac{\sqrt{-\alpha\gamma}}{2}\xi\right)\right). \tag{27}$$

Case 5. When $\delta = 0$ and $\gamma = \alpha$,

$$Q_{21}(\xi) = \tan_A(\alpha\xi), \tag{28}$$

$$Q_{22}(\xi) = -\cot_A(\alpha\xi), \tag{29}$$

$$Q_{23}(\xi) = \tan_A(2\alpha\xi) \pm \sqrt{pq}\sec_A(2\alpha\xi), \tag{30}$$

$$Q_{24}(\xi) = -\cot_A(2\alpha\xi) \pm \sqrt{pq}\csc_A(2\alpha\xi), \tag{31}$$

$$Q_{25}(\xi) = \frac{1}{2}\left(\tan_A\left(\frac{\alpha}{2}\xi\right) - \cot_A\left(\frac{\alpha}{2}\xi\right)\right). \tag{32}$$

Case 6. When $\delta = 0$ and $\gamma = -\alpha$,

$$Q_{26}(\xi) = -\tanh_A(\alpha\xi), \tag{33}$$

$$Q_{27}(\xi) = -\coth_A(\alpha\xi), \tag{34}$$

$$Q_{28}(\xi) = -\tanh_A(2\alpha\xi) \pm \iota\sqrt{pq}\,\mathrm{sech}_A(2\alpha\xi), \tag{35}$$

$$Q_{29}(\xi) = -\coth_A(2\alpha\xi) \pm \sqrt{pq}\,\mathrm{csch}_A(2\alpha\xi), \tag{36}$$

$$Q_{30}(\xi) = -\frac{1}{2}\left(\tanh_A\left(\frac{\alpha}{2}\xi\right) + \coth_A\left(\frac{\alpha}{2}\xi\right)\right). \tag{37}$$

Case 7. When $\delta^2 = 4\alpha\gamma$,

$$Q_{31}(\xi) = \frac{-2\alpha(\ln(A)\delta\xi + 2)}{\ln(A)\delta^2\xi}. \tag{38}$$

Case 8. When $\delta^2 = \nu$, $\alpha = h\nu$ ($h \neq 0$), and $\gamma = 0$,

$$Q_{32}(\xi) = A^{\nu\xi} - h. \tag{39}$$

Case 9. When $\delta = \gamma = 0$,

$$Q_{33}(\xi) = \ln(A)\,\alpha\xi. \tag{40}$$

Case 10. When $\delta = \alpha = 0$,

$$Q_{34}(\xi) = -\frac{1}{\ln(A)\gamma\xi}. \tag{41}$$

Case 11. When $\alpha = 0$, $\delta \neq 0$,

$$Q_{35}(\xi) = -\frac{p^\delta}{\gamma(\cosh_A(\delta\xi) - \sinh_A(\delta\xi) + p)}, \tag{42}$$

$$Q_{36}(\xi) = -\frac{\delta(\sinh_A(\delta\xi) + \cosh_A(\delta\xi))}{\gamma(\sinh_A(\delta\xi) + \cosh_A(\delta\xi) + q)}. \tag{43}$$

Case 12. When $\delta = \nu$, $\gamma = h\nu$ ($h \neq 0$), and $\alpha = 0$,

$$Q_{37}(\xi) = \frac{pA^{\nu\xi}}{p - hqA^{\nu\xi}}. \tag{44}$$

Remark 2.

$$\sinh_A(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{2}, \qquad \cosh_A(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{2},$$
$$\tanh_A(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{pA^{\xi} + qA^{-\xi}}, \qquad \coth_A(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{pA^{\xi} - qA^{-\xi}},$$
$$\tan_A(\xi) = -\iota\frac{pA^{\iota\xi} - qA^{-\iota\xi}}{pA^{\iota\xi} + qA^{-\iota\xi}}, \qquad \cot_A(\xi) = \iota\frac{pA^{\iota\xi} + qA^{-\iota\xi}}{pA^{\iota\xi} - qA^{-\iota\xi}},$$

where $p, q > 0$ are arbitrary constants.

Now more than one exact solution of Eq. (4) can be obtained by combining Eq. (6) with Eqs. (8)–(44).

Remark 3. Due to the generic nature of the new extended direct algebraic method, many important method like G'/G-expansion method, the Kudryashov method, the modified Kurdryashov method, the tanh-function method, the extended tanh-function method, the direct algebraic method, and so on can be used [26].

5 Application of new extended direct algebraic method to biofilm model and physical solutions

Consider Eq. (1). Using the transformation

$$C(\xi) = c(t, x, y), \quad \xi = x + y - \omega t,$$

Eq. (1) becomes

$$-\omega C' = 2DC'' + rC^2 - rC\theta - rC^3 + r\theta C^2. \tag{45}$$

Balancing the highest-order derivative term to the highest-order nonlinear term in Eq. (45), we find $N = 1$. So, the solution of Eq. (45) has the form

$$C = b_0 + b_1 Q(\xi). \tag{46}$$

Now by using Eqs. (45), (46) we get the following set of algebraic equations:

$$-b_1^3 r = 0,$$
$$b_1^2 r - 3b_0 b_1^2 r + b_1^2\theta r = 0,$$
$$2b_0 b_1 r - b_1\theta r - 3b_0^2 b_1 r + 2b_0 b_1\theta r = 0,$$
$$\omega b_1 + rb_0^2 - rb_0\theta - rb_0^3 + rb_0^2\theta = 0,$$
$$-2Db_1 = 0,$$

$$b_0 = \frac{-2(1+\theta) \pm \sqrt{4(1+\theta)^2 - 4(3)(\theta)}}{2(-3)}, \quad b_1 = \frac{r(4\theta^3 + 3\theta^2 + 3\theta - 2)}{27\omega}. \tag{47}$$

Using Eq. (47), the abundant families of solutions for Eq. (1) are obtain as follows. If $d = b^2 - 12\theta > 0$, $b = 2(1+\theta)$, and $\theta \neq 0$, then

$$\begin{aligned}
c_1(x,y,t) &= \ln\biggl[\frac{-2b^2+6\theta}{d} - \frac{3b}{\sqrt{d}}\tanh_A\biggl(\frac{\sqrt{d}}{2}(x+y-\omega t)\biggr)\biggr) \\
&\quad + \frac{6}{d}\biggl(\frac{-b}{2} - \frac{\sqrt{d}}{2}\tanh_A\biggl(\frac{\sqrt{d}}{2}(x+y-\omega t)\biggr)\biggr)^2\biggr], \\
c_2(x,y,t) &= \ln\biggl[\frac{-2b^2+6\theta}{d} - \frac{3b}{\sqrt{d}}\coth_A\biggl(\frac{\sqrt{d}}{2}(x+y-\omega t)\biggr) \\
&\quad + \frac{6}{d}\biggl(\frac{-b}{2} - \frac{\sqrt{d}}{2}\coth_A\biggl(\frac{\sqrt{d}}{2}(x+y-\omega t)\biggr)\biggr)^2\biggr], \\
c_3(x,y,t) &= \ln\biggl[\frac{-2b^2+6\theta}{d} - \frac{3b}{\sqrt{d}}\tanh_A\bigl(\sqrt{d}(x+y-\omega t)\bigr) \\
&\quad \pm \frac{3b\iota\sqrt{pq}}{\sqrt{d}}\operatorname{sech}_A\bigl(\sqrt{d}(x+y-\omega t)\bigr) \\
&\quad + \frac{6}{d}\biggl[-\frac{b}{2} - \frac{\sqrt{d}}{2}\tanh_A\bigl(\sqrt{d}(x+y-\omega t)\bigr) \\
&\quad \pm \iota\frac{\sqrt{pqd}}{2}\sec h_A\bigl(\sqrt{d}(x+y-\omega t)\bigr)\biggr]^2\biggr], \\
c_4(x,y,t) &= \ln\biggl[\frac{-2b^2+6\theta}{d} - \frac{3b}{\sqrt{d}}\cot_A\bigl(\sqrt{d}(x+y-\omega t)\bigr) \\
&\quad \pm \frac{3b\sqrt{pq}}{\sqrt{d}}\csc h_A\bigl(\sqrt{d}(x+y-\omega t)\bigr) \\
&\quad + \frac{6}{d}\biggl[-\frac{b}{2} - \frac{\sqrt{d}}{2}\coth_A\bigl(\sqrt{d}(x+y-\omega t)\bigr) \\
&\quad \pm \frac{\sqrt{pqd}}{2}\csc h_A\bigl(\sqrt{d}(x+y-\omega t)\bigr)\biggr]^2\biggr],
\end{aligned}$$

where $p, q > 0$ are the constants,

$$\begin{aligned}
c_5(x,y,t) &= \ln\biggl[\frac{-2b^2+6\theta}{d} - \frac{3b}{2\sqrt{d}}\tanh_A\biggl(\frac{\sqrt{d}}{4}(x+y-\omega t)\biggr) \\
&\quad - \frac{3b}{2\sqrt{d}}\coth_A\biggl(\frac{\sqrt{d}}{4}(x+y-\omega t)\biggr) \\
&\quad + \frac{6}{d}\biggl[-\frac{b}{2} - \frac{\sqrt{d}}{4}\tanh_A\biggl(\frac{\sqrt{d}}{4}(x+y-\omega t)\biggr) \\
&\quad - \frac{\sqrt{d}}{4}\coth_A\biggl(\frac{\sqrt{d}}{4}(x+y-\omega t)\biggr)\biggr]^2\biggr].
\end{aligned}$$

6 Physical restrictions of the biofilm model and behavior of the exact solutions

If $\theta \in (-\infty, +\infty)$, then most of the above described cases of the solutions will surely exist, and solution can easily be plotted. For biofilm model, $\theta \in [0, 1]$, so, in other than the following cases, the exact solutions are not of physical importance here. In the following surface plots the exact continuity regime can be seen for the biofilm model (1) for growth dynamics of microbes.

Graphs and their interpretation

In this section, we show some physical behaviors of soliton solutions that are extracted by the help of new modified extended direct algebraic method. Theses solution are newly and effected in the collection of one or different type of microorganism that adhere on different surfaces and form biofilm. These microbes consist of bacteria, fungi, algae. These plots show clearly singular, dark, and bright solitons. In these solitons surface plots the exact continuity regime can be seen for the biofilm model.

All the graphs are plotted taking different value of ω, p, q, θ. The wave-like formation of all the graphs basically encodes information about the structural behavior of microbes forming biofilm on the surface of rectangle possess variation in density. The peak point in all the figures highlights that at these places the biofilm thickness is more than in any other region of the surface, i.e., the maximum accumulation of microbes at these points make the wave-like formation in the biofilm. Graphically, the travelling wave solutions represent soliton solutions of our biofilm model. The formation of waves in Figs. 1, 2 clearly demonstrates that at some specific point the cluster of microbes on the rectangular surface changes the thickness of surface. Similarly, Figs. 3–7 physically demonstrated that at some region of rectangular surface the biofilm shows less thickness compared to other region.

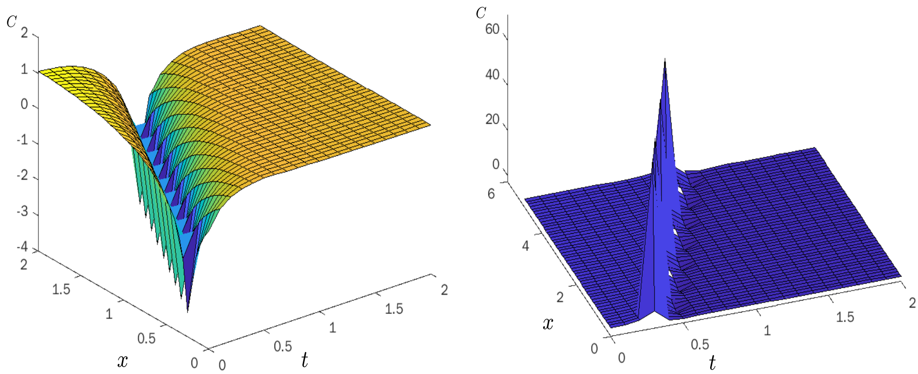

Figure 1. Surface plot of c_1 with $\theta = 0.5$, $\omega = 2$ and c_2 with $\theta = 0.02$, $\omega = 7$.

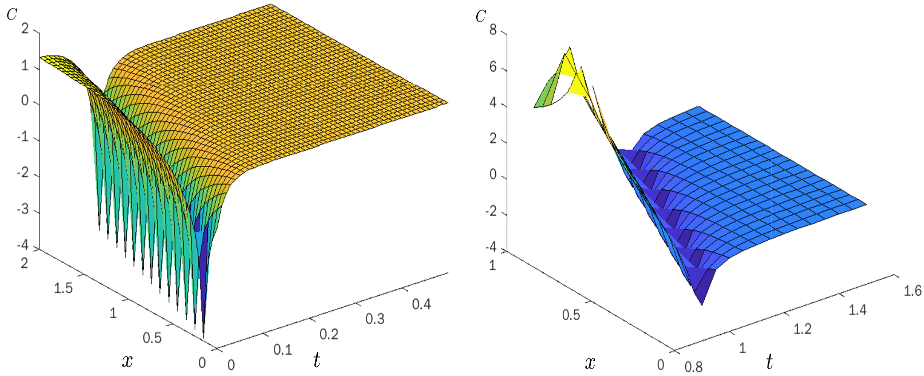

Figure 2. Surface plot of c_1 with $\theta = 1, \omega = 15$ and c_2 with $\theta = 0.7, \omega = 3$.

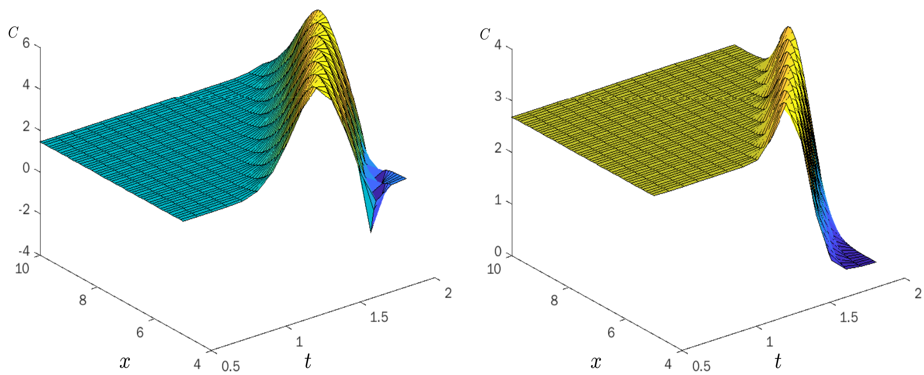

Figure 3. Surface plot of c_3 with $\theta = 0.02, p = 3, q = 5, \omega = 5$.

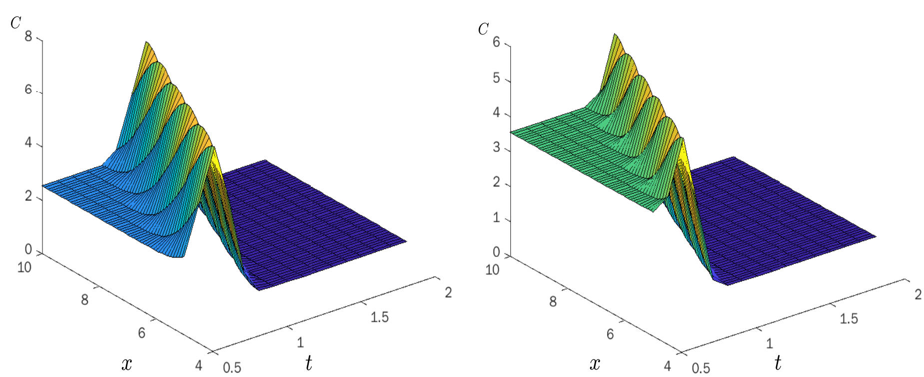

Figure 4. Surface plot of c_3 with $\theta = 0.6, p = 3, q = 5, \omega = 10$.

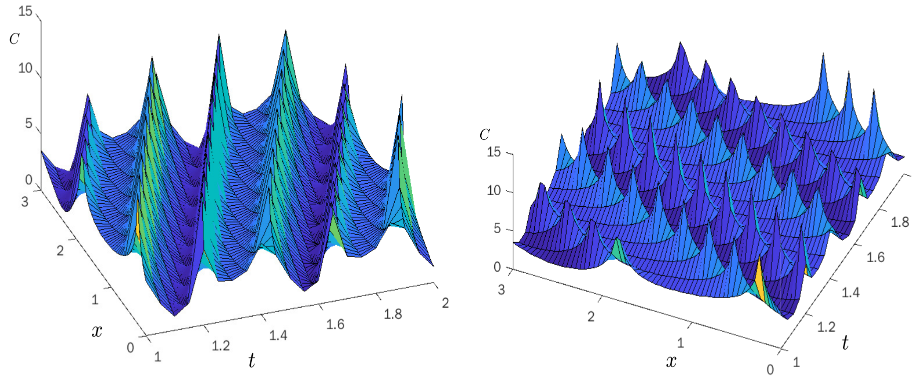

Figure 5. Surface plot of c_4 with $\theta = 0.02$, $p = 1$, $q = 2$, $\omega = 7$.

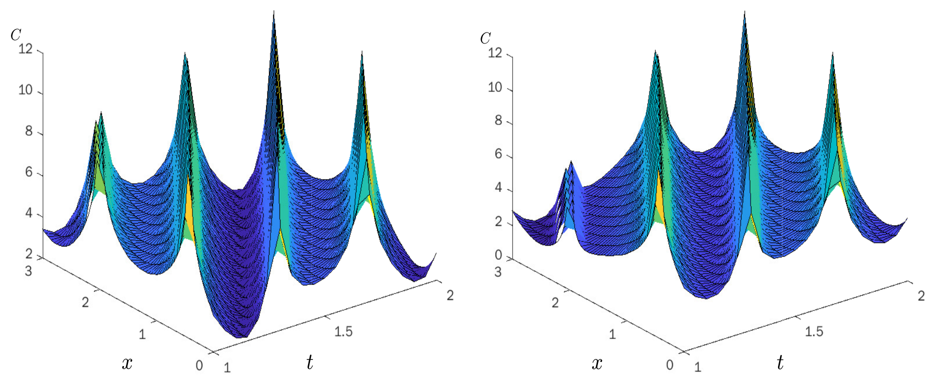

Figure 6. Surface plot of c_4 with $\theta = 0$, $p = 3$, $q = 1$, $\omega = 4$.

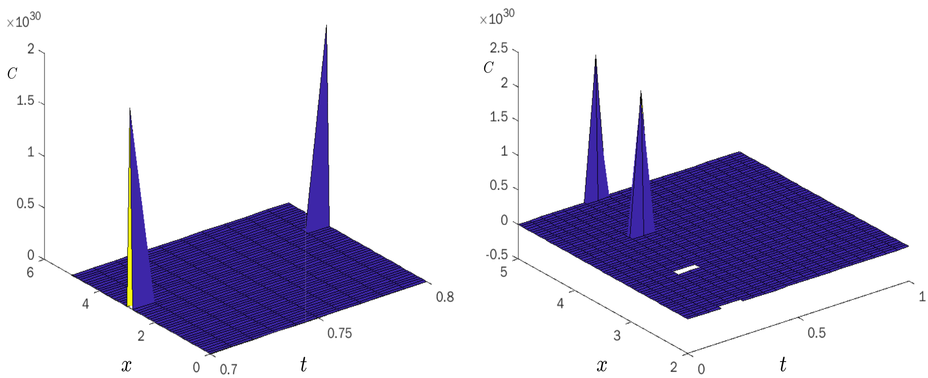

Figure 7. Surface plot of c_5 with $\theta = 0.02$, $p = 1$, $q = 0.5$, $\omega = 7$, $\theta = 0.9$, $p = 5$, $q = 9$, $\omega = 20$.

Remark 4. The first surface plot of solution c_3 expresses the graph having the plus sign in both Fig. 3 and 4. Second graphical representation of solution c_3 having negative sign assuming different value of parameters. Similar is the case for c_4 solution in Figs. 5 and 6.

7 Conclusion

In this study, we investigate the soliton solutions of biofilm model, basically, the bistable Allen–Cahn equation with quartic potential. The solution $c(x, y, t)$ physically expressed the density of microbes forming the biofilm on the rectangular region. Along with the exact solution, we also apply the existence theory to represent the interval in which the solution exists with the help of Schauder fixed point theorem. The different type of hyperbolic, trigonometric, and rational soliton solutions are gained. In the biofilm model, different parameters belong to certain spaces and give the exact solutions by applying the technique of new extended direct algebraic method, exact solutions of biofilm model are highlighted along with their restriction. For more physical understanding, the 3D plots of the solutions are presented. These plots show the singular dark bright solitons.

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