Nonlinear Analysis: Modelling and Control, Vol. 28, No. 1, 20–33
https://doi.org/10.15388/namc.2023.28.29621

Vilnius University Press

Radial symmetry of a relativistic Schrödinger tempered fractional p-Laplacian model with logarithmic nonlinearity*

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Received: June 17, 2022 / **Revised:** October 20, 2022 / **Published online:** December 23, 2022

Abstract. In this paper, by introducing a relativistic Schrödinger tempered fractional p-Laplacian operator $(-\Delta)^{s,m}_{p,\lambda}$, based on the relativistic Schrödinger operator $(-\Delta + m^2)^s$ and the tempered fractional Laplacian $(\Delta + \lambda)^{\beta/2}$, we consider a relativistic Schrödinger tempered fractional p-Laplacian model involving logarithmic nonlinearity. We first establish maximum principle and boundary estimate, which play a very crucial role in the later process. Then we obtain radial symmetry and monotonicity results by using the direct method of moving planes.

Keywords: relativistic Schrödinger tempered fractional p-Laplacian operator, direct method of moving planes, logarithmic nonlinearity, radial symmetry and monotonicity.

1 Introduction

In recent years, the study of the Schrödinger operators have attracted widespread attention from researchers. These operators appear in a variety of different fields, for instance, physics, wireless electronics, telecommunication technology, materials science, mechanics, industrial communication technology, and automation technology. Fall and Felli in [13] introduced the relativistic Schrödinger operator $(-\Delta + m^2)^s$, which is for each

*This work is supported by NSFC (No. 12001344).
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$s \in (0,1)$ and $u(x) \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}&\left(-\Delta + m^2\right)^s u(x)\\&\quad= C_{N,s} m^{(N+2s)/2}\, PV\int\limits_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{(N+2s)/2}} K_{(N+2s)/2}\big(m|x-y|\big)\,\mathrm{d}y\\&\qquad+ m^{2s}u(x),\end{aligned}$$

where PV indicates the Cauchy principal value,

$$C_{N,s} = 2^{1-N/2+s}\pi^{-N/2}\,2^{2s}\frac{s(1-s)}{\Gamma(2-s)},$$

K_τ is the modified Bessel function of the second kind with order τ, meanwhile, they obtained the asymptotics of solutions to relativistic fractional elliptic equations with Hardy-type potentials. The properties of solution for a nonlinear pseudorelativistic Schrödinger equation in $\mathbb{R}^N$ were proved by Ambrosio [1]. Based on a singular homogeneous potential, the essential self-adjointness of a relativistic Schrödinger operator was obtained by Fall and Felli [14]. Dai, Qin, and Wu [10] studied the properties of solutions for several types of equations with respect to the operator $(-\Delta + m^2)^s$ in two kinds of domains, respectively. Some other research results about this operator can be found in the literatures [5, 15, 19, 20] and the references therein.

As we all know, in a β-stable Lévy process, the nonlocal operator fractional Laplacian $(\Delta)^{\beta/2}$ as the infinitesimal generator is used to describe the anomalous dynamics. For Lévy flights, the ξ with finite first moment and η with infinite second moment are independent, leading to infinite propagation speed and the divergent second moments of the distribution of the particles. This causes much difficulty in relating the models to experimental data, especially when analyzing the scaling of the measured moments in time. In order to overcome the shortcoming that it sometimes does not simulate some real physical processes very well, [11] introduced a sufficiently small parameter λ to exponentially temper the isotropic power law measure of the jump length, which generates the tempered fractional Laplacian $(\Delta + \lambda)^{\beta/2}$ as follows:

$$(\Delta + \lambda)^{\beta/2}u(x) = -C_{N,\beta,\lambda}\, PV\int\limits_{\mathbb{R}^N} \frac{u(x)-u(y)}{\mathrm{e}^{\lambda|x-y|}|x-y|^{N+\beta}}\,\mathrm{d}y,$$

where $0 < \beta < 2$, $C_{N,\beta,\lambda} = \Gamma(N/2)/(2\pi^{N/2}|\Gamma(-\beta)|)$. This operator has attracted the attention of many scholars, and many excellent results have emerged [12, 27, 28].

Based on the above work, we consider a relativistic Schrödinger tempered fractional p-Laplacian operator defined by

$$\begin{aligned}&(-\Delta)_{p,\lambda}^{s,m}u(x)\\&\quad= C_{N,sp} m^{(N+sp)/2}\, PV\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]K_{(N+sp)/2}(m|x-y|)}{\mathrm{e}^{\lambda|x-y|}|x-y|^{N+sp}}\,\mathrm{d}y\\&\qquad+ m^{sp}u(x),\end{aligned} \tag{1}$$

where PV represents the Cauchy principal value, λ is a sufficiently small positive constant, $m > 0$ is a constant, K_τ denotes the modified Bessel function of the second kind with order τ having the following property [18]:

$$K_\tau(r) \sim \frac{\sqrt{\pi}}{\sqrt{2}} r^{-1/2} \mathrm{e}^{-r} \quad \text{for } r \to \infty.$$

For integral (1) to make sense, let $u \in C^{1,1}_{\mathrm{loc}} \cap \mathrm{L}_{sp}$ and

$$\mathrm{L}_{sp} = \left\{ u : \mathbb{R}^N \to \mathbb{R} \,\middle|\, \int\limits_{\mathbb{R}^N} \frac{\mathrm{e}^{-|x|}|u(x)|}{1 + |x|^{(N+1)/2+s}} \,\mathrm{d}x < \infty \right\}.$$

It is worth noting that the above operator can degenerate into the following different operators when the parameters take different values.

(i) When $p = 2$, $\lambda = 0$, the relativistic Schrödinger tempered fractional p-Laplacian $(-\Delta)^{s,m}_{p,\lambda}$ becomes the relativistic Schrödinger operator $(-\Delta + m^2)^s$; based on this, when $m \to 0_+$, $(-\Delta + m^2)^s$ turns into the familiar fractional Laplacian $(-\Delta)^s$.

(ii) When $p = 2$, $K_\tau(\cdot) = 1$, $m \to 0_+$, $(-\Delta)^{s,m}_{p,\lambda}$ turns into the tempered fractional Laplacian $(\Delta + \lambda)^{\beta/2}$.

(iii) When $\lambda = 0$, $K_\tau(\cdot) = 1$, $m \to 0_+$, $(-\Delta)^{s,m}_{p,\lambda}$ transforms the well-known fractional p-Laplacian $(-\Delta)^s_p$.

Over the past decades, many scholars have done a lot of splendid work on fractional Laplacian, nevertheless, in view of its nonlocality, conventional methods are no longer effective. To surmount the difficulty of nonlocality, an extension method was introduced for transforming the nonlocal problem into a local problem in higher dimensions by Caffarelli and Silvestre [3], which provides a key to solve a class of nonlocal problems; see [2, 4, 8]. An alternative way to overcome the nonlocality is to apply the integral equations method [17, 18, 29].

Jarohs, Weth, Chen, C. Li, and Y. Li jointly introduced the direct method of moving planes, then it was used to work out the different kinds of problems involving various nonlinear operators, there exist some results by applying this efficient and convenient method to research the solutions of fractional Laplacian or fractional p-Laplacian equations and systems; see [6, 7, 9, 16, 21–26].

Enlightened by the brilliant work above, in this paper, we consider the following relativistic Schrödinger tempered fractional p-Laplacian equation with logarithmic nonlinearity:

$$(-\Delta)^{s,m}_{p,\lambda} u(x) = \left[\lg(u(x) + 1)\right]^{p+q}, \tag{2}$$

here $0 < s < 1$, $2 < p < \infty$. At present, as far as we know, the research results of the relativistic Schrödinger tempered fractional p-Laplacian model with logarithmic nonlinearity by using analytical methods have hardly appeared. Next, we obtain the radial symmetry of positive solution of Eq. (2) by the direct method of moving planes in the unit sphere and whole space, respectively. So this is also a new attempt to study the symmetry of the solution of the equation involving this kind operator.

To get our main theorems, we need the following notations and lemmas in preparation, which play a pivotal role in the process of moving planes.

2 Notations and lemmas

We define that

$$\mathscr{T}_\alpha = \{x \in \mathbb{R}^N \mid x_N = \alpha \text{ for some } \alpha \in \mathbb{R}\} \text{ is the moving plane,}$$
$$\Sigma_\alpha = \{x \in \mathbb{R}^N \mid x_N < \alpha\} \text{ is the region to the left of } \mathscr{T}_\alpha,$$
$$x^\alpha = (2\alpha - x_1, x_2, \dots, x_N) \text{ is the reflection of } x = (x_1, x_2, \dots, x_N) \text{ about } \mathscr{T}_\alpha.$$

Meanwhile, we denote

$$u_\alpha(x) = u(x^\alpha), \qquad \mathscr{W}_\alpha(x) = u(x^\alpha) - u(x), \qquad \tilde{\Sigma}_\alpha = \{x \mid x^\alpha \in \Sigma_\alpha\}.$$

Lemma 1. *Let Ω be a bounded region in Σ. Presume $u \in \mathrm{L}_{sp} \cap C^{1,1}_{loc}(\Omega)$. If*

$$\begin{aligned} &(-\Delta)^{s,m}_{p,\lambda} u_\alpha(x) - (-\Delta)^{s,m}_{p,\lambda} u(x) \geqslant 0, \quad x \in \Omega, \\ &\mathscr{W}(x) \geqslant 0, \quad x \in \Sigma \setminus \Omega, \end{aligned} \tag{3}$$

then

$$\mathscr{W}(x) \geqslant 0 \quad \textit{in } \Omega.$$

Go a step further, provided that $\mathscr{W}(x) = 0$ at certain point in Ω. Then

$$\mathscr{W}(x) = 0 \quad \textit{almost everywhere in } \mathbb{R}^N.$$

When Ω is an unbounded region, the following further assumption is required:

$$\varliminf_{|x| \to \infty} \mathscr{W}(x) \geqslant 0.$$

Then the same conclusions hold.

Proof. If $\mathscr{W}(x) \geqslant 0$ in Ω is not true, then there exists a point $\hat{x} \in \Omega$ such that

$$\mathscr{W}(\hat{x}) = \min_{\Omega} \mathscr{W} < 0.$$

To simplify writing, let $\mathfrak{L}(t) = |t|^{p-2}t$. Then $\mathfrak{L}'(t) = (p-1)|t|^{p-2} \geqslant 0$, same as follows:

$$\begin{aligned} &(-\Delta)^{s,m}_{p,\lambda} u_\alpha(\hat{x}) - (-\Delta)^{s,m}_{p,\lambda} u(\hat{x}) \\ &\quad = C_{N,sp} m^{(N+sp)/2} \\ &\qquad \times PV \int\limits_{R^N} \frac{[\mathfrak{L}(u_\alpha(\hat{x}) - u_\alpha(y)) - \mathfrak{L}(u(\hat{x}) - u(y))] K_{(N+sp)/2}(m|\hat{x} - y|)}{\mathrm{e}^{\lambda|\hat{x}-y|} |\hat{x} - y|^{N+sp}} \,\mathrm{d}y \\ &\qquad + m^{sp} \mathscr{W}_\alpha(\hat{x}) \end{aligned}$$

$$
\begin{aligned}
&= C_{N,sp} m^{(N+sp)/2} \\
&\quad \times PV\biggl\{ \int\limits_{\Sigma} \frac{[\mathfrak{L}(u_\alpha(\hat{x}) - u_\alpha(y)) - \mathfrak{L}(u(\hat{x}) - u(y))] K_{(N+sp)/2}(m|\hat{x} - y|)}{\mathrm{e}^{\lambda|\hat{x}-y|}|\hat{x} - y|^{N+sp}}\,\mathrm{d}y \\
&\quad + \int\limits_{\Sigma} \frac{[\mathfrak{L}(u_\alpha(\hat{x}) - u(y)) - \mathfrak{L}(u(\hat{x}) - u_\alpha(y))] K_{(N+sp)/2}(m|\hat{x} - y^\alpha|)}{\mathrm{e}^{\lambda|\hat{x}-y^\alpha|}|\hat{x} - y^\alpha|^{N+sp}}\,\mathrm{d}y \biggr\} \\
&\quad + m^{sp} \mathscr{W}_\alpha(\hat{x}) \\
&= C_{N,sp} m^{(N+sp)/2} \\
&\quad \times PV\biggl\{ \int\limits_{\Sigma} \biggl[\frac{1}{\mathrm{e}^{\lambda|\hat{x}-y|}|\hat{x} - y|^{N+sp}} - \frac{1}{\mathrm{e}^{\lambda|\hat{x}-y^\alpha|}|\hat{x} - y^\alpha|^{N+sp}} \biggr] \\
&\qquad\qquad \times \bigl[\mathfrak{L}\bigl(u_\alpha(\hat{x}) - u_\alpha(y)\bigr) - \mathfrak{L}\bigl(u(\hat{x}) - u(y)\bigr)\bigr] K_{(N+sp)/2}\bigl(m|\hat{x} - y|\bigr)\,\mathrm{d}y \\
&\quad + \int\limits_{\Sigma} \frac{\mathfrak{L}(u_\alpha(\hat{x}) - u_\alpha(y)) - \mathfrak{L}(u(\hat{x}) - u(y)) + \mathfrak{L}(u_\alpha(\hat{x}) - u(y)) - \mathfrak{L}(u(\hat{x}) - u_\alpha(y))}{\mathrm{e}^{\lambda|\hat{x}-y^\alpha|}|\hat{x} - y^\alpha|^{N+sp}} \\
&\qquad \times K_{(N+sp)/2}(m|\hat{x} - y^\alpha|)\,\mathrm{d}y \\
&\quad + \int\limits_{\Sigma} \frac{[\mathfrak{L}(u_\alpha(\hat{x}) - u_\alpha(y)) - \mathfrak{L}(u(\hat{x}) - u(y))]}{\mathrm{e}^{\lambda|\hat{x}-y^\alpha|}|\hat{x} - y^\alpha|^{N+sp}} \\
&\qquad \times \bigl[K_{(N+sp)/2}\bigl(m|\hat{x} - y|\bigr) - K_{(N+sp)/2}\bigl(m|\hat{x} - y^\alpha|\bigr)\bigr]\,\mathrm{d}y \biggr\} \\
&\quad + m^{sp} \mathscr{W}_\alpha(\hat{x}) \\
&= C_{N,sp} m^{(N+sp)/2}\, PV\{H_1 + H_2 + H_3\} + m^{sp} \mathscr{W}_\alpha(\hat{x}). \qquad (4)
\end{aligned}
$$

To estimate H_1, we observe the fact

$$
\frac{1}{\mathrm{e}^{\lambda|\hat{x}-y|}|\hat{x} - y|^{N+sp}} > \frac{1}{\mathrm{e}^{\lambda|\hat{x}-y^\alpha|}|\hat{x} - y^\alpha|^{N+sp}} \quad \forall x, y \in \Sigma
$$

due to

$$
\bigl[u_\alpha(\hat{x}) - u_\alpha(y)\bigr] - \bigl[u(\hat{x}) - u(y)\bigr] = \mathscr{W}(\hat{x}) - \mathscr{W}(y) \leqslant 0 \quad \text{but} \not\equiv 0.
$$

In view of the strict monotonicity of $\mathfrak{L}$ and $K_{(N+sp)/2}(m|\hat{x} - y|) > 0$, we have

$$
\bigl[\mathfrak{L}\bigl(u_\alpha(\hat{x}) - u_\alpha(y)\bigr) - \mathfrak{L}\bigl(u(\hat{x}) - u(y)\bigr)\bigr] K_{(N+sp)/2}\bigl(m|\hat{x} - y|\bigr) \leqslant 0 \quad \text{but } \not\equiv 0.
$$

Therefore,

$$
H_1 < 0. \qquad (5)
$$

To evaluate H_2, by applying the mean value theorem, we obtain

$$H_2 = \int\limits_{\Sigma} \frac{[\mathfrak{L}(u_\alpha(\hat{x}) - u_\alpha(y)) - \mathfrak{L}(u(\hat{x}) - u(y)) + \mathfrak{L}(u_\alpha(\hat{x}) - u(y)) - \mathfrak{L}(u(\hat{x}) - u_\alpha(y))]}{\mathrm{e}^{\lambda|\hat{x}-y^\alpha|}|\hat{x} - y^\alpha|^{N+sp}} \times K_{(N+sp)/2}(m|\hat{x} - y^\alpha|)\,\mathrm{d}y]$$

$$= \mathscr{W}(\hat{x}) \int\limits_{\Sigma} \frac{[\mathfrak{L}'(\xi(y)) + \mathfrak{L}'(\eta(y))]}{\mathrm{e}^{\lambda|\hat{x}-y^\alpha|}|\hat{x} - y^\alpha|^{N+sp}}\,\mathrm{d}y \leqslant 0. \tag{6}$$

As for H_3, by reason of the fact

$$\bigl[u_\alpha(\hat{x}) - u_\alpha(y)\bigr] - \bigl[u(\hat{x}) - u(y)\bigr] = \mathscr{W}(\hat{x}) - \mathscr{W}(y) \leqslant 0 \quad \text{but } \not\equiv 0,$$
$$|\hat{x} - y| < \bigl|\hat{x} - y^\alpha\bigr|$$

and the monotonicity of $\mathfrak{L}$ and $K_{(N+sp)/2}(\cdot)$, we have

$$H_3 < 0. \tag{7}$$

Combining (4), (5), (6), and (7), one can deduce

$$(-\Delta)^{s,m}_{p,\lambda} u_\alpha(\hat{x}) - (-\Delta)^{s,m}_{p,\lambda} u(\hat{x}) < 0.$$

This inequality is in contradiction with the first condition in (3), therefore

$$\mathscr{W}(\hat{x}) \geqslant 0.$$

In case of $\mathscr{W}(x) = 0$ at some point $x \in \Sigma$, equivalently, x is the minimum point of $\mathscr{W}$ in Σ, so, $H_2 = 0$ and $H_3 = 0$. Now, in the light of the first inequality in (3), we get $H_1 \geqslant 0$, which stands for

$$\bigl[\mathfrak{L}\bigl(u_\alpha(\hat{x}) - u_\alpha(y)\bigr) - \mathfrak{L}\bigl(u(\hat{x}) - u(y)\bigr)\bigr] K_{(N+sp)/2}\bigl(m|\hat{x} - y|\bigr) \geqslant 0.$$

Considering the monotonicity of $\mathfrak{L}$ and the fact of $K_{(N+sp)/2}(\cdot) > 0$, we have

$$\bigl[u_\alpha(\hat{x}) - u_\alpha(y)\bigr] - \bigl[u(\hat{x}) - u(y)\bigr]$$
$$= \mathscr{W}(x) - \mathscr{W}(y) = -\mathscr{W}(y) \geqslant 0 \quad \text{for almost all } y \in \Sigma.$$

Consequently,

$$\mathscr{W}(y) = 0 \quad \text{almost everywhere in } \Sigma,$$

besides, since $\mathscr{W}(x)$ is antisymmetric with respect to x, $\mathscr{W}(y) = 0$ is established almost everywhere in $\mathbb{R}^N$. If Ω is unbounded, in this case, in view of assumption

$$\varliminf_{|x|\to\infty} \mathscr{W}(x) \geqslant 0,$$

if

$$\mathscr{W}(x) \geqslant 0, \quad x \in \Sigma,$$

does not hold, there is a point $x \in \Sigma$ such that x is the negative minimum point of $\mathscr{W}(x)$. Being similar to the above discussion, one can find a contradiction. The proof is completed. □

Lemma 2. *If $\mathscr{W}_{\alpha_0} > 0$ for $x \in \Sigma_{\alpha_0}$, there exist two sequences $\{\alpha_k\}$ ($\alpha_k \searrow \alpha_0$) and $\{x^k\} \in \Sigma_{\alpha_k}$ such that*

$$\mathscr{W}_{\alpha_k}\big(x^k\big) = \min_{\Sigma_{\alpha_k}} \mathscr{W}_{\alpha_k} \leqslant 0 \quad \textit{and} \quad x^k \to x^0 \in \partial\Sigma_{\alpha_0}.$$

Define

$$\iota_k = \operatorname{dist}\big(x^k, \partial\Sigma_{\alpha_k}\big) \equiv \big|\alpha_k - x_N^k\big|.$$

Then

$$\overline{\lim_{\iota_k \to 0}} \frac{(-\Delta)_{p,\lambda}^{s,m} u_{\alpha_k}(x^k) - (-\Delta)_{p,\lambda}^{s,m} u(x^k)}{\iota_k} < 0.$$

Proof. Similar to the calculation in (4), we can get

$$\begin{aligned}
&\frac{(-\Delta)_{p,\lambda}^{s,m} u_{\alpha_k}(x^k) - (-\Delta)_{p,\lambda}^{s,m} u(x^k)}{\iota_k} \\
&\quad = \frac{C_{N,sp} m^{(N+sp)/2}}{\iota_k} \\
&\qquad \times PV \int_{\Sigma_{\alpha_k}} \left(\frac{1}{\mathrm{e}^{\lambda|x^k - y|}|x^k - y|^{N+sp}} - \frac{1}{\mathrm{e}^{\lambda|x^k - y^{\alpha_k}|}|x^k - y^{\alpha_k}|^{N+sp}} \right) \\
&\qquad\qquad \times \big[\mathfrak{L}\big(u_\alpha\big(x^k\big) - u_\alpha(y)\big) - \mathfrak{L}\big(u\big(x^k\big) - u(y)\big)\big] K_{(N+sp)/2}\big(m\big|x^k - y\big|\big)\,\mathrm{d}y \\
&\qquad + \frac{C_{N,sp} m^{(N+sp)/2}}{\iota_k} \mathscr{W}_{\alpha_k}\big(x^k\big) \\
&\qquad \times PV \int_{\Sigma_{\alpha_k}} \frac{[\mathfrak{L}'(\xi(y)) + \mathfrak{L}'(\eta(y))] K_{(N+sp)/2}(m|x^k - y^{\alpha_k}|)}{\mathrm{e}^{\lambda|x^k - y^{\alpha_k}|}|x^k - y^{\alpha_k}|^{N+sp}}\,\mathrm{d}y \\
&\qquad + \frac{C_{N,sp} m^{(N+sp)/2}}{\iota_k} \\
&\qquad \times PV \int_{\Sigma} \frac{[\mathfrak{L}(u_\alpha(x^k) - u_\alpha(y)) - \mathfrak{L}(u(x^k) - u(y))]}{\mathrm{e}^{\lambda|x^k - y^\alpha|}|x^k - y^\alpha|^{N+sp}} \\
&\qquad\qquad \times \big[K_{(N+sp)/2}\big(m\big|x^k - y\big|\big) - K_{(N+sp)/2}\big(m\big|x^k - y^\alpha\big|\big)\big]\,\mathrm{d}y \\
&\qquad + \frac{m^{sp} \mathscr{W}_\alpha(x^k)}{\iota_k} \\
&\quad = C_{N,sp} m^{(N+sp)/2}\, PV\{H_{1k} + H_{2k} + H_{3k}\} + \frac{m^{sp} \mathscr{W}_\alpha(x^k)}{\iota_k}.
\end{aligned}$$

Apparently,

$$H_{2k} \leqslant 0 \quad \text{and} \quad H_{3k} \leqslant 0.$$

As for H_{1k}, on the grounds of the mean value theorem, we have

$$\begin{aligned}
&\frac{1}{\iota_k}\left[\frac{1}{\mathrm{e}^{\lambda|x^k-y|}|x^k-y|^{N+sp}}-\frac{1}{\mathrm{e}^{\lambda|x^k-y^{\alpha_k}|}|x^k-y^{\alpha_k}|^{N+sp}}\right]\\
&\quad=\frac{2(N+sp+\lambda|\varphi_k(y)|)(\alpha_k-y_n)}{\mathrm{e}^{\lambda|\varphi_k(y)|}|\varphi_k(y)|^{N+sp+2}}\\
&\quad\to\frac{2(N+sp+\lambda|\varphi_0(y)|)(\alpha_0-y_n)}{\mathrm{e}^{\lambda|\varphi_0(y)|}|\varphi_0(y)|^{N+sp+2}}>0,\quad k\to\infty,
\end{aligned}$$

where $|x^k-y|\leqslant|\varphi_k(y)|\leqslant|x^k-y^k|$, $|x^0-y|\leqslant|\varphi_0(y)|\leqslant|x^0-y^0|$. Due to the monotonicity of $\mathfrak{L}$ and $K_{(N+sp)/2}(\cdot)>0$,

$$\left[u_{\alpha_0}(x^0)-u_{\alpha_0}(y)\right]-\left[u(x^0)-u(y)\right]=\mathscr{W}_{\alpha_0}(x^0)-\mathscr{W}_{\alpha_0}(y)<0.$$

We have

$$\begin{aligned}
&\left[\mathfrak{L}(u_\alpha(x^k)-u_\alpha(y))-\mathfrak{L}(u(x^k)-u(y))\right]K_{(N+sp)/2}(m|x^k-y|)\\
&\quad\to\left[\mathfrak{L}(u_{\alpha_0}(x^0)-u_{\alpha_0}(y))-\mathfrak{L}(u(x^0)-u(y))\right]K_{(N+sp)/2}(m|x^0-y|)<0
\end{aligned}$$

for all $y\in\Sigma_{\alpha_0}$ when $k\to\infty$.

Hence

$$\overline{\lim_{\iota_k\to0}}\frac{(-\Delta)_{p,\lambda}^{s,m}u_{\alpha_k}(x^k)-(-\Delta)_{p,\lambda}^{s,m}u(x^k)}{\iota_k}<0.\qquad\square$$

3 Main results

In this part, we give two theorems, which describe the radial symmetry and monotonicity of positive solution in $B_1(0)$ and $\mathbb{R}^N$, respectively. The two theorems are also based on the previous two lemmas by means of the moving plane method.

We first consider the following problem in a unit ball:

$$\begin{aligned}
&(-\Delta)_{p,\lambda}^{s,m}u(x)=\left[\lg(u(x)+1)\right]^{p+q},\quad x\in B_1(0),\\
&u(x)=0,\quad x\notin B_1(0).
\end{aligned}\tag{8}$$

Let us call

$$\Omega_\alpha=\Sigma_\alpha\cap B_1(0),\qquad S=\Sigma_\alpha\setminus\Omega_\alpha.$$

Theorem 1. *If $u(x)\in C_{\mathrm{loc}}^{1,1}(B_1(0))\cap C(\overline{B_1(0)})$ is a positive solution of (8) with $p+q\geqslant1$, then $u(x)$ will be radially symmetric and monotone decreasing with respect to the origin.*

Proof. In Ω_α, we have

$$(-\Delta)_{p,\lambda}^{s,m}u_\alpha(x)-(-\Delta)_{p,\lambda}^{s,m}u(x)\geqslant\frac{(p+q)[\lg(u(x)+1)]^{p+q-1}}{u(x)+1}\mathscr{W}_\alpha(x).\tag{9}$$

Step 1. In this step, we prove that when $\alpha > -1$ and approaches -1 sufficiently,

$$\mathscr{W}_\alpha(x) \geqslant 0 \quad \forall x \in \Omega_\alpha. \tag{10}$$

If not, there is a point $\tilde{x} \in \Omega_\alpha$ such that

$$\mathscr{W}_\alpha(\tilde{x}) = \min_{\Omega_\alpha} \mathscr{W}_\alpha = \min_{\Sigma_\alpha} \mathscr{W}_\alpha < 0.$$

$$\begin{aligned}
&(-\Delta)_{p,\lambda}^{s,m} u_\alpha(\tilde{x}) - (-\Delta)_{p,\lambda}^{s,m} u(\tilde{x}) \\
&\quad = C_{N,sp} m^{(N+sp)/2} \\
&\qquad \times PV\biggl\{ \int\limits_{\Sigma} \biggl(\frac{1}{\mathrm{e}^{\lambda|\tilde{x}-y|}|\tilde{x}-y|^{N+sp}} - \frac{1}{\mathrm{e}^{\lambda|\tilde{x}-y^\alpha|}|\tilde{x}-y^\alpha|^{N+sp}} \biggr) \\
&\qquad\qquad \times \bigl[\mathfrak{L}\bigl(u_\alpha(\tilde{x}) - u_\alpha(y)\bigr) - \mathfrak{L}\bigl(u(\tilde{x}) - u(y)\bigr)\bigr] K_{(N+sp)/2}\bigl(m|\tilde{x}-y|\bigr)\,\mathrm{d}y \\
&\qquad + \mathscr{W}_\alpha(\tilde{x}) \int\limits_{\Sigma} \frac{[\mathfrak{L}'(\xi(y)) + \mathfrak{L}'(\eta(y))] K_{(N+sp)/2}(m|\tilde{x}-y^\alpha|)}{\mathrm{e}^{\lambda|\tilde{x}-y^\alpha|}|\tilde{x}-y^\alpha|^{N+sp}}\,\mathrm{d}y \\
&\qquad + \int\limits_{\Sigma} \frac{[\mathfrak{L}(u_\alpha(\tilde{x}) - u_\alpha(y)) - \mathfrak{L}(u(\tilde{x}) - u(y))]}{\mathrm{e}^{\lambda|\tilde{x}-y^\alpha|}|\tilde{x}-y^\alpha|^{N+sp}} \\
&\qquad\qquad \times \bigl[K_{(N+sp)/2}\bigl(m|\tilde{x}-y|\bigr) - K_{(N+sp)/2}\bigl(m|\tilde{x}-y^\alpha|\bigr)\bigr]\,\mathrm{d}y \biggr\} \\
&\qquad + m^{sp} \mathscr{W}_\alpha(\tilde{x}) \\
&\quad \leqslant C_{N,sp} m^{(N+sp)/2} \mathscr{W}_\alpha(\tilde{x}) \int\limits_{\Sigma_\alpha} \frac{[\mathfrak{L}'(\xi(y)) + \mathfrak{L}'(\eta(y))] K_{(N+sp)/2}(m|\tilde{x}-y^\alpha|)}{\mathrm{e}^{\lambda|\tilde{x}-y^\alpha|}|\tilde{x}-y^\alpha|^{N+sp}}\,\mathrm{d}y \\
&\quad = C_{N,sp} m^{(N+sp)/2} \mathscr{W}_\alpha(\tilde{x}) J,
\end{aligned}$$

where

$$\begin{aligned}
\xi(y) &\in \bigl(u_\alpha(\tilde{x}) - u_\alpha(y),\ u(\tilde{x}) - u_\alpha(y)\bigr), \\
\eta(y) &\in \bigl(u_\alpha(\tilde{x}) - u(y),\ u(\tilde{x}) - u(y)\bigr).
\end{aligned}$$

With the help of Lemma 3.1 in [6], considering $u(y) = 0$ in the region of S, we can get

$$\begin{aligned}
J &\geqslant \int\limits_{\Sigma_\alpha} \frac{c_1 u^{p-2}(\tilde{x}) K_{(N+sp)/2}(m|\tilde{x}-y^\alpha|)}{\mathrm{e}^{\lambda|\tilde{x}-y^\alpha|}|\tilde{x}-y^\alpha|^{N+sp}}\,\mathrm{d}y \\
&= c_1 \int\limits_{S} \frac{u^{p-2}(\tilde{x}) K_{(N+sp)/2}(m|\tilde{x}-y^\alpha|)}{\mathrm{e}^{\lambda|\tilde{x}-y^\alpha|}|\tilde{x}-y^\alpha|^{N+sp}}\,\mathrm{d}y \\
&\geqslant \frac{C u^{p-2}(\tilde{x})}{\mathrm{e}^{\lambda\delta} m^{(N+sp)/2} \delta^{(N+3sp)/2}},
\end{aligned}$$

where $\delta = (\alpha + 1)$ is the width of Ω_α in x_N-direction, c_1 and C are positive constants.

For δ sufficiently small,

$$\begin{aligned}
&(-\Delta)_{p,\lambda}^{\beta/2,m} u_\alpha(\tilde{x}) - (-\Delta)_{p,\lambda}^{\beta/2,m} u(\tilde{x}) - \frac{(p+q)[\lg(\xi(\tilde{x})+1)]^{p+q-1}}{\xi(\tilde{x})+1} \mathscr{W}_\alpha(\tilde{x}) \\
&\quad \leqslant C_{N,sp} \mathscr{W}_\alpha(\tilde{x}) \left[\frac{C u^{p-2}(\tilde{x})}{\mathrm{e}^{\lambda\delta} m^{(N+sp)/2} \delta^{(N+3sp)/2}} - \frac{(p+q)[\lg(u(\tilde{x})+1)]^{p+q-1}}{u(\tilde{x})+1} \right] \\
&\quad = C_{N,sp} \mathscr{W}_\alpha(\tilde{x}) \\
&\qquad \times \left[\frac{C u^{p-2}(\tilde{x})[u(\tilde{x})+1] - (p+q)\mathrm{e}^{\lambda\delta} m^{(N+sp)/2} \delta^{(N+3sp)/2} [\lg(u(\tilde{x})+1)]^{p+q-1}}{\mathrm{e}^{\lambda\delta} m^{(N+sp)/2} \delta^{(N+3sp)/2} [u(\tilde{x})+1]} \right] \\
&\quad < 0,
\end{aligned}$$

where C is a positive constant, which contradicts with (9). As a result, when α approaches -1 enough, (10) must be established.

Step 2. In this step, we will use the initial point provided in Step 1 as a starting point to gradually move the plane from left to right. As long as (10) is true, we can keep moving the plane all the way to the limit position, which is defined $\mathscr{T}_{\alpha_0}$, here

$$\alpha_0 = \sup\{\alpha \leqslant 0 \mid \mathscr{W}_\mu \geqslant 0,\ x \in \Omega_\mu,\ \mu \leqslant \alpha\}.$$

Next, we show that $\alpha_0 = 0$ to obtain the radial symmetry of $u(x)$ of (8) with respect to the origin. If not, $\alpha < 0$ because of $\mathscr{W}_{\alpha_0} \not\equiv 0$. By Lemma 1 we get

$$\mathscr{W}_{\alpha_0} > 0, \quad x \in \Omega_{\alpha_0}.$$

From the definition of α_0 there are sequences $\{\alpha_k\}$ $(\alpha_k \searrow \alpha_0)$ and $\{x^k\} \in \Omega_{\alpha_k}$, which meet

$$\mathscr{W}_{\alpha_k}(x^k) = \min_{\Sigma_{\alpha_k}} \mathscr{W}_{\alpha_k} < 0 \quad \text{and} \quad \nabla \mathscr{W}_{\alpha_k}(x^k) = 0.$$

There exists a subsequence $\{x^k\} \to x^0$. In addition, since $\mathscr{W}_\alpha(x)$ and $\nabla \mathscr{W}_\alpha(x)$ are continuous with respect to α and x, one gets

$$\mathscr{W}_{\alpha_0}(x^0) \leqslant 0, \quad x^0 \in \partial \Sigma_{\alpha_0}, \qquad \nabla \mathscr{W}_{\alpha_0}(x^0) = 0.$$

For $k \to \infty$,

$$\frac{\mathscr{W}_{\alpha_0}(x^0)}{\iota_k} \to 0.$$

However, from another perspective, from Lemma 2 the limit of the above expression is less than zero. Consequently,

$$\alpha_0 = 0, \qquad \mathscr{W}_0(x) \geqslant 0, \quad x \in \Omega_0.$$

Due to the arbitrariness of the direction of x_N, we can draw a conclusion that $u(x)$ is radially symmetric and monotone decreasing about the origin. □

Theorem 2. *Assume that* $u(x) \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N) \cap \mathrm{L}_{sp}$ *is a positive solution of* (2) *with* $p+q<1$ *and*

$$u(x) \sim \mathrm{e}^{|x|^{-\gamma(x)}}, \quad |x| \to \infty, \tag{11}$$

where

$$\gamma(x) < \min\left\{\frac{2sp+1}{2(p+q-1)}, \log|x|\frac{p-1}{\lambda+m} - 1\right\}. \tag{12}$$

Then $u(x)$ *will be radially symmetric and monotone decreasing about some point in* $\mathbb{R}^N$*.*

Proof. According (2),

$$(-\Delta)^{s,m}_{p,\lambda} u_\alpha(x) - (-\Delta)^{s,m}_{p,\lambda} u(x) = \frac{(p+q)[\lg(\xi(x)+1)]^{p+q-1}}{\xi(x)+1}\mathscr{W}_\alpha(x),$$

where $\xi(x)$ is between $u(x)$ and $u_\alpha(x)$.

Step 1. In this step, we show that

$$\mathscr{W}_\alpha(x) \geqslant 0 \quad \text{for } x \in \Sigma_\alpha,\ \alpha \to -\infty. \tag{13}$$

If (13) is not true, there is a point where the following inequality must be true:

$$\mathscr{W}_\alpha(\grave{x}) = \min_{\Sigma_\alpha} \mathscr{W}_\alpha < 0.$$

Let $M = |\grave{x}|$, we pick a point $x_M \in \Sigma_\alpha$, which meets

$$B_M(x_M) \subset \Sigma_\alpha \quad \text{and} \quad |x_M| = TM,$$

here T is a large enough number such that when $y \in B_M(x_M)$. According to (11), the following inequality holds:

$$u(y) \leqslant \frac{A_1}{\mathrm{e}^{T^{-\gamma(M)}M^{-\gamma(M)}}} \leqslant \frac{A_2}{\mathrm{e}^{M^{-\gamma(M)}}} \leqslant u(x),$$

where A_1 and A_2 are positive constants.

Similar to the calculation process of (4), in line with condition (11), we have

$$\begin{aligned}
&(-\Delta)^{s,m}_{p,\lambda} u_\alpha(\grave{x}) - (-\Delta)^{s,m}_{p,\lambda} u(\grave{x}) \\
&\quad\leqslant C_{N,sp} m^{(N+sp)/2} \mathscr{W}_\alpha(\grave{x}) \int\limits_{\Sigma_\alpha} \frac{[\mathfrak{L}'(\xi(y)) + \mathfrak{L}'(\eta(y))] K_{(N+sp)/2}(m|\grave{x} - y^\alpha|)}{\mathrm{e}^{\lambda|\grave{x}-y^\alpha|}|\grave{x} - y^\alpha|^{N+sp}}\,\mathrm{d}y \\
&\quad\leqslant C_{N,sp} m^{(N+sp)/2} \mathscr{W}_\alpha(\grave{x}) \int\limits_{B_M(x_M)} \frac{[\mathfrak{L}'(\xi(y)) + \mathfrak{L}'(\eta(y))] K_{(N+sp)/2}(m|\grave{x} - y^\alpha|)}{\mathrm{e}^{\lambda|\grave{x}-y^\alpha|}|\grave{x} - y^\alpha|^{N+sp}}\,\mathrm{d}y \\
&\quad= C_{N,sp} \frac{\sqrt{\pi}}{\sqrt{2}} m^{(N+sp-1)/2} \mathscr{W}_\alpha(\grave{x}) \int\limits_{B_M(x_M)} \frac{u^{p-2}(\grave{x})}{\mathrm{e}^{(\lambda+m)|\grave{x}-y^\alpha|}|\grave{x} - y^\alpha|^{N+sp+1/2}}\,\mathrm{d}y \\
&\quad\leqslant \frac{C_0}{M^{sp+1/2}\mathrm{e}^{(\lambda+m)M-(p-2)M^{-\gamma(x)}}} \mathscr{W}_\alpha(\grave{x}),
\end{aligned} \tag{14}$$

here

$$\xi(y) \in \big(u_\alpha(\grave{x}) - u_\alpha(y),\ u(\grave{x}) - u_\alpha(y)\big),$$
$$\eta(y) \in \big(u_\alpha(\grave{x}) - u(y),\ u(\grave{x}) - u(y)\big),$$

and C_0 is a positive constant. On the other side,

$$(-\Delta)_{p,\lambda}^{s,m} u_\alpha(\grave{x}) - (-\Delta)_{p,\lambda}^{s,m} u(\grave{x}) \geqslant \frac{(p+q)[\lg(u(\grave{x})+1)]^{p+q-1}}{u(\grave{x})+1} \mathscr{W}_\alpha(\grave{x}). \tag{15}$$

According to (14) and (15), we have

$$\frac{C_0}{M^{sp+1/2} \mathrm{e}^{(\lambda+m)M-(p-2)M^{\gamma(x)}}} \leqslant \frac{C_1}{M^{(p+q-1)\gamma(x)} \mathrm{e}^{M^{-\gamma(x)}}},$$

which is inconsistent with condition (12). As a result, (13) must be true.

Step 2. Based on the starting point provided in Step 1, we move the plane from left to right to $\mathscr{T}_{\alpha_0}$, which is defined

$$\alpha_0 = \sup\{\alpha \mid \mathscr{W}_\mu(x) \geqslant 0,\ x \in \Sigma_\mu,\ \mu \leqslant \alpha\}.$$

Next, we confirm the truth of

$$\mathscr{W}_{\alpha_0} \equiv 0, \quad x \in \Sigma_{\alpha_0}.$$

If not, we have

$$\mathscr{W}_{\alpha_0} > 0, \quad x \in \Sigma_{\alpha_0}.$$

According to the definition of α_0, there exist two sequences $\{\alpha_k\} \searrow \alpha_0$ and $\{x^k\} \in \Sigma_{\alpha_k}$, which satisfy

$$\mathscr{W}_{\alpha_k}\big(x^k\big) = \min_{\Sigma_{\alpha_k}} \mathscr{W}_{\alpha_k} < 0 \quad \text{and} \quad \nabla \mathscr{W}_{\alpha_k}\big(x^k\big) = 0.$$

In addition, from (11) we know that the sequence $\{x^k\}$ is bounded, and

$$\mathscr{W}_{\alpha_0}\big(x^0\big) \leqslant 0, \quad \nabla \mathscr{W}_{\alpha_0}\big(x^0\big) = 0, \quad x^0 \in \partial\Sigma_{\alpha_0}.$$

Then we can get the following relation naturally:

$$\frac{\mathscr{W}_{\alpha_0}(x^0)}{\iota_k} \to 0, \quad k \to \infty,$$

which contradicts with Lemma 2, hence $\mathscr{W}_{\alpha_0} \equiv 0$ holds. Considering the arbitrariness of direction x_N, we can get that $u(x)$ is radially symmetric at certain point in $\mathbb{R}^N$. □

References

1. V. Ambrosio, Ground states solutions for a non-linear equation involving a pseudo-relativistic Schrödinger operators, *J. Math. Phys.*, **57**(5):395–414, 2016, https://doi.org/10.1063/1.4949352.
2. C. Brandle, E. Colorado, A de Pablo, U. Sanchez, A concave-convex ellipotic problem involving the fractional Laplacian, *Proc. R. Soc. Edinb., Sect. A, Math.*, **143**:39–71, 2013, https://doi.org/10.1017/S0308210511000175.
3. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Partial Differ. Equations*, **32**(8):1245–1260, 2007, https://doi.org/10.1080/03605300600987306.
4. L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, *Commun. Pure Appl. Math.*, **62**:597–638, 2009, https://doi.org/10.1002/CPA.20274.
5. R. Carmona, W. Master, Relativistic Schrödinger operators: Asymptotic behavior of the eigenfunctions, *J. Funct. Anal.*, **91**:117–142, 1990, https://doi.org/10.1016/0022-1236(90)90049-Q.
6. W. Chen, C. Li, Maximum principles for the fractional p-Laplacian and symmetry of solutions, *Adv. Math.*, **335**:735–758, 2018, https://doi.org/10.1016/j.aim.2018.07.016.
7. W. Chen, C. Li, Y. Li, A direct method of moving planes for the fractional Laplacian, *Adv. Math.*, **308**:404–437, 2017, https://doi.org/10.1016/j.aim.2016.11.038.
8. W. Chen, J. Zhu, Indefinite fractional elliptic problem and Liouville theorems, *J. Differ. Equations*, **260**:4758–4785, 2016, https://doi.org/10.1016/j.jde.2015.11.029.
9. Y. Chen, B. Liu, Symmetry and non-existence of positive solutions for fractional p-Laplacian systems, *Nonlinear Anal., Theory Methods Appl.*, **183**:303–322, 2019, https://doi.org/10.1016/j.na.2019.02.023.
10. W. Dai, G. Qin, D. Wu, Direct methods for pseudo-relativistic Schrödinger operators, *J. Geom. Anal.*, **31**:5555–5618, 2021, https://doi.org/10.1007/s12220-020-00492-1.
11. W. Deng, B. Li, W. Tian, P. Zhang, Boundary problems for the fractional and tempered fractional operators, *Multiscale Model. Simul.*, **16**(1):125–149, 2018, https://doi.org/10.1137/17M1116222.
12. S. Duo, Y. Zhang, Numerical approximations for the tempered fractional Laplacian: Error analysis and applications, *J. Sci. Comput.*, **81**(1):569–593, 2019, https://doi.org/10.1007/s10915-019-01029-7.
13. M. Fall, V. Felli, Unique continuation properties for relativistic Schrödinger operators with a singular potential, *Discrete Contin. Dyn. Syst.*, **35**(12), 2013, https://doi.org/10.3934/dcds.2015.35.5827.
14. M. Fall, V. Felli, Sharp essential self-adjointness of relativistic Schrödinger operators with a singular potential, *J. Funct. Anal.*, **267**:1851–1877, 2014, https://doi.org/10.1016/j.jfa.2014.06.010.

15. Y. Guo, S. Peng, A direct method of moving planes for fully nonlinear nonlocal operators and applications, *Discrete Contin. Dyn. Syst., Ser. S*, **14**(6):1871–1897, 2021, https://doi.org/10.3934/dcdss.2020462.
16. P. Le, V. Ho, Fractional p-laplacian problems with negative powers in a ball or an exterior domain, *J. Pseudo-Differ. Oper. Appl.*, **11**(2):1–15, 2020, https://doi.org/10.1007/s11868-019-00307-0.
17. Y. Lei, C. Li, C. Ma, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy-Littlewood-Sobolev system of integral equations, *Calc. Var. Partial Differ. Equ.*, **45**:43–61, 2012, https://doi.org/10.1007/s00526-011-0450-7.
18. G. Lu, J. Zhu, Symmetry and regularity of extremals of an integral equation related to the Hardy-Sobolev inequality, *Calc. Var. Partial Differ. Equ.*, **42**:563–577, 2011, https://doi.org/10.1007/s00526-011-0398-7.
19. V. Maz'ya, I. Verbitsky, The form boundedness criterion for the relativistic Schrödinger operator, *Ann. Inst. Fourier*, **54**(2):317–339, 2004, https://doi.org/10.5802/aif.2020.
20. M. Ryznar, Estimates of green function for relativistic α-stable process, *Potential Anal.*, **17**:1–23, 2002, https://doi.org/10.1023/A:1015231913916.
21. G. Wang, X. Ren, Radial symmetry of standing waves for nonlinear fractional Laplacian Hardy–Schrödinger systems, *Appl. Math. Lett.*, **110**:106560, 2020, https://doi.org/10.1016/j.aml.2020.106560.
22. G. Wang, X. Ren, Z. Bai, W. Hou, Radial symmetry of standing waves for nonlinear fractional Hardy–Schrödinger equation, *Appl. Math. Lett.*, **96**:131–137, 2019, https://doi.org/10.1016/j.aml.2019.04.024.
23. L. Wu, P. Niu, Symmetry and nonexistence of positive solutions to fractional p-Laplacian equations, *Discrete Contin. Dyn. Syst.*, **39**(3):1573–1583, 2019, https://doi.org/10.3934/dcds.2019069.
24. L. Zhang, B. Ahmad, G. Wang, X. Ren, Radial symmetry of solution for fractional p-laplacian system, *Nonlinear Anal., Theory Methods Appl.*, **196**:111801, 2020, https://doi.org/10.1016/j.na.2020.111801.
25. L. Zhang, W. Hou, Standing waves of nonlinear fractional p-Laplacian Schrödinger equation involving logarithmic nonlinearity, *Appl. Math. Lett.*, **102**:106149, 2020, https://doi.org/10.1016/j.aml.2019.106149.
26. L. Zhang, Q. Liu, Radial symmetry and monotonicity of the positive solutions for k-Hessian equations, *Appl. Math. Lett.*, **135**:108383, 2022, https://doi.org/10.1016/j.aml.2022.108383.
27. Z. Zhang, W. Deng, H. Fan, Finite difference schemes for the tempered fractional Laplacian, *Numer. Math., Theory Methods Appl.*, **12**(2), 2018, https://doi.org/10.4208/nmtma.OA-2017-0141.
28. Z. Zhang, W. Deng, GE. Karniadakis, A Riesz basis Galerkin method for the tempered fractional Laplacian, *SIAM J. Numer. Anal.*, **56**(5):3010–3039, 2018, https://doi.org/10.1137/17M1151791.
29. R. Zhuo, W. Chen, X. Cui, Z. Yuan, Symmetry and non-existence of solutions for a nonlinear system involving the fractional Laplacian, *Discrete Contin. Dyn. Syst.*, **36**:1125–1141, 2016, https://doi.org/10.3934/dcds.2016.36.1125.