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Exponential and logarithm of multivector in low-dimensional ($n = p + q < 3$) Clifford algebras

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Abstract. The aim of the paper is to give a uniform picture of complex, hyperbolic, and quaternion algebras from a perspective of the applied Clifford geometric algebra. Closed form expressions for a multivector exponential and logarithm are presented in real geometric algebras $Cl_{p,q}$ when $n = p + q = 1$ (complex and hyperbolic numbers) and $n = 2$ (Hamilton, split, and conectorine quaternions). Starting from $Cl_{0,1}$ and $Cl_{1,0}$ algebras wherein square of a basis vector is either -1 or $+1$, we have generalized exponential and logarithm formulas to 2D quaternionic algebras $Cl_{0,2}$, $Cl_{1,1}$, and $Cl_{2,0}$. The sectors in the multivector coefficient space, where 2D logarithm exists are found. They are related with a square root of the multivector.

Keywords: Clifford (geometric) algebra, exponential and logarithm of Clifford numbers, quaternions.

1 Introduction

Quaternion algebras find a wide application in graphics, robotics, and control of spatial rotation of solid bodies, including aerospace flight dynamics [4, 10, 13, 14]. During the last ten years, there is a tendency to replace quaternions by multivectors (MVs) of geometric (aka Clifford) algebras (GAs), mainly due to the possibility to carry out calculations in higher dimensional GAs of mixed signatures [9, 11, 15, 17, 18] and, consequently, to employ wider GA capabilities. Of special mention is conformal $Cl_{4,1}$ GA that allows to do complicated graphics in 5D vector space and then transform the graphics to 3D Euclidean space for visualization [21].

In this paper, we investigate low-dimensional algebras from GA perspective, namely, 1D complex and hyperbolic number algebras as well as 2D algebras $Cl_{0,2}$, $Cl_{1,1}$, and

$Cl_{2,0}$ that are isomorphic to quaternionic algebras: the Hamilton quaternion (or briefly the quaternion) [10, 13], coquaternion also known as a split quaternion [7, 20], and conectorine [19]. The properties of the Hamilton quaternion, which is isomorphic to $Cl_{0,2}$, recently have been summarized in a handbook [13]. The coquaternion and conectorine are less known. They are isomorphic to $Cl_{1,1}$ and $Cl_{2,0}$ algebras that are noncommutative too. As we shall see, in all 2D algebras the exponential and logarithm may be treated in a uniform way if they are reformulated in GA terms what, in turn, helps to generalize and better understand known properties as well as to discover new ones, for example, continuous degrees of freedom related to a free vector pointing in an arbitrary direction [3]. Generalized exponential and logarithm formulas and square roots of multivector have been found, including the sectors, where they exist for the first time. The subject considered in this paper is akin to exponential factorization of MV into product of exponentials [12] and square root of MV [1, 5].

In Section 2 the notation and general properties of GA exponential and logarithm functions are introduced. The GA expressions in 1D are presented in Section 3. In Sections 4 and 5, respectively, the exponential and logarithm in 2D are considered. In Addendum (Section 6) the square root of MV is discussed. Finally, in Section 7 the conclusion and short discussion are given.

2 Properties of exponential and logarithm in GA

Let $\mathbf{e}_i$ be the basis vector, and let $\mathbf{e}_{ij} \equiv \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_{ji}$ be the bivector. The latter is the geometric product of two orthogonal basis vectors. Complex and hyperbolic numbers (aka Clifford numbers) in GA [22] are represented by the following MVs:

$$Cl_{0,1}:\quad \mathsf{A} = a_0 + a_1 I,\quad \text{where } I \equiv \mathbf{e}_1 \text{ and } \mathbf{e}_1^2 = -1,$$
$$Cl_{1,0}:\quad \mathsf{A} = a_0 + a_1 I,\quad \text{where } I \equiv \mathbf{e}_1 \text{ and } \mathbf{e}_1^2 = +1,$$

where a_0 and a_1 are the real coefficients, $a_0, a_1 \in \mathbb{R}$. a_0 is called the scalar part of MV, and $a_1 I$ is the pseudoscalar. In 1D GAs the basis vectors coincide with an elementary pseudoscalar I. The squares, $\mathbf{e}_1^2 \equiv \mathrm{i}^2 = -1$ in $Cl_{0,1}$ and $\mathbf{e}_1^2 = 1$ in $Cl_{1,0}$, suggest that we have to do with complex and hyperbolic numbers, respectively.

In 2-dimensional (2D) algebras, there are two basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and a bivector $\mathbf{e}_{12} \equiv I$ (oriented plane). The general Clifford number A is

$$\begin{aligned}
Cl_{0,2}:&\quad \mathsf{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}I,\quad \text{where } \mathbf{e}_1^2 = \mathbf{e}_2^2 = -1,\ I^2 = -1,\\
Cl_{1,1}:&\quad \mathsf{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}I,\quad \text{where } \mathbf{e}_1^2 = -\mathbf{e}_2^2 = -1,\ I^2 = 1,\\
Cl_{2,0}:&\quad \mathsf{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}I,\quad \text{where } \mathbf{e}_1^2 = \mathbf{e}_2^2 = +1,\ I^2 = -1.
\end{aligned} \tag{1}$$

The sum $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ represents a general vector in 2D bivector plane. The basis vectors satisfy $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ (orthogonality) and $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2 \equiv \mathbf{e}_{12}$ (oriented unit plane), where the dot and wedge denote the inner and outer products. $\mathbf{e}_{12}$ plays the role of an elementary pseudoscalar I. The sign of I^2 depends on algebra, Eq. (1). The algebras $Cl_{1,1}$ and $Cl_{2,0}$ are isomorphic under the following exchange of GA basis elements: $\mathbf{e}_1 \leftrightarrow \mathbf{e}_1$, $\mathbf{e}_2 \leftrightarrow \mathbf{e}_{12}$, and $\mathbf{e}_{12} \leftrightarrow \mathbf{e}_2$.

The main involutions, namely, the reversion, inversion, and Clifford conjugation denoted, respectively, by tilde, circumflex, and their combination are defined by the following changes in component signs of MV $\mathsf{A} = a_0 + \mathbf{a} + a_{12}I$:

$$\widetilde{\mathsf{A}} = a_0 + \mathbf{a} - a_{12}I, \qquad \widehat{\mathsf{A}} = a_0 - \mathbf{a} + a_{12}I, \qquad \widetilde{\widehat{\mathsf{A}}} = a_0 - \mathbf{a} - a_{12}I.$$

For complex and hyperbolic numbers, there is only a single involution, $\widehat{\mathsf{A}} = a_0 + a_1\widehat{I} = a_0 - a_1 I$ that usually is denoted by asterisk in physics and engineering and overline in mathematics.

The exponential of MV is another MV that belongs to the same geometric algebra $Cl_{p,q}$. If A and B are MVs, the following properties hold:

$$\exp(\mathsf{A} + \mathsf{B}) = \exp(\mathsf{A})\exp(\mathsf{B}) \quad\Longleftrightarrow\quad \mathsf{AB} = \mathsf{BA},$$
$$\widetilde{\mathrm{e}^{\mathsf{A}}} = \mathrm{e}^{\widetilde{\mathsf{A}}}, \qquad \widehat{\mathrm{e}^{\mathsf{A}}} = \mathrm{e}^{\widehat{\mathsf{A}}}, \qquad \widetilde{\widehat{\mathrm{e}^{\mathsf{A}}}} = \mathrm{e}^{\widetilde{\widehat{\mathsf{A}}}},$$

where e is the base of the natural logarithm. In 1D algebras the first property is always satisfied since the commutation of scalar and vector is satisfied.

The GA exponential can be represented as a power series in a form similar to scalar exponential [16]. In numerical form, i.e., when coefficients at basis elements $\mathbf{e}_1$, $\mathbf{e}_1$, and $\mathbf{e}_{12}$ are real numbers, the exponential can be summed up approximately [3]. To minimize the number of multiplications, it is convenient to rewrite the exponential in a nested form (aka Horner's rule),

$$\mathrm{e}^{\mathsf{A}} = 1 + \frac{\mathsf{A}}{1}\biggl(1 + \frac{\mathsf{A}}{2}\biggl(1 + \frac{\mathsf{A}}{3}(1 + \cdots)\biggr)\biggr), \tag{2}$$

which requires fewer MV products. If numerical coefficients in A are small enough, the exponential e^{A} can be approximated by truncated series (2) to high precision. For examples, we refer to paper [3].

The following properties hold for MV logarithm:

$$\log(\mathsf{AB}) = \log(\mathsf{A}) + \log(\mathsf{B}) \quad\Longleftrightarrow\quad \mathsf{AB} = \mathsf{BA},$$
$$\mathrm{e}^{-\log(\mathsf{A})} = \mathsf{A}^{-1},$$
$$\widetilde{\log(\mathsf{A})} = \log(\widetilde{\mathsf{A}}), \qquad \widehat{\log(\mathsf{A})} = \log(\widehat{\mathsf{A}}), \qquad \widetilde{\widehat{\log(\mathsf{A})}} = \mathrm{e}^{\widetilde{\widehat{\mathsf{A}}}}.$$

When the logarithm of MV exists, it may be approximated by series

$$\log \mathsf{B} = \mathsf{B}\biggl(1 + \mathsf{B}\biggl(-\frac{1}{2} + \mathsf{B}\biggl(\frac{1}{3} + \mathsf{B}\biggl(-\frac{1}{4} + \cdots\biggr)\biggr)\biggr)\biggr), \quad 0 < |\mathsf{B}| < 1.$$

Here $|\mathsf{B}|$ is the determinant norm [3]. If logarithm exists, a series can be summed up but there may be sector(s) in the MV coefficient domain, where the logarithm does not exist at all.

In $Cl_{0,1}$ algebra the norm $|\mathsf{B}|$, which is equal to the square root of MV determinant $\mathrm{Det}(\mathsf{B}) = \mathsf{B}\widehat{\mathsf{B}} = b_0^2 + b_1^2 > 0$, is called the magnitude or absolute value of the MV (or magnitude of the complex number in this case). In hyperbolic number theory the similar role is played by product $\mathsf{B}\widehat{\mathsf{B}} = b_0^2 - b_1^2$, which may be positive, negative, or zero. In this case the magnitude called a determinant seminorm (or pseudonorm) $\|\mathsf{B}\| = \sqrt{\operatorname{abs}(\mathsf{B}\widehat{\mathsf{B}})} = \sqrt{\operatorname{abs}(b_0^2 - b_1^2)} \geqslant 0$ is introduced. Note that now the equality sign appears, therefore, the seminorm $\|\mathsf{B}\|$ may be zero even if $\mathsf{B} \neq 0$. The equality sign in case of the norm $|\mathsf{B}|$ would require the MV to nullify.

3 Exponential and logarithm in 1D algebras

One-dimensional GAs are represented by two commutative algebras: the well-known complex number algebra, which is isomorphic to $Cl_{0,1}$, and the hyperbolic number algebra $Cl_{1,0}$ [22]. In Fig. 1 the geometrical properties of both algebras are compared graphically on xy-plane (equivalently on b_0b_1-plane). In Fig. 1(b) the two branches of hyperbola close down at plus/minus infinities [8]. The shaded area in both cases is proportional either to inner φ_c or outer φ_h angle between the center and the point b on circle $y^2 + x^2 = 1$ or hyperbola $y^2 - x^2 = 1$, respectively. If a point b on the circle or hyperbola represents the MV $\mathsf{B} = b_0 + b_1\mathbf{e}_1$, then in GA the quantity $\mathsf{B}\widehat{\mathsf{B}} = b_0^2 + b_1^2 = |\mathsf{B}|^2 > 0$ is the square of norm that graphically represents the sector A_c in Fig. 1(a). Similarly, the sector A_h in Fig. 1(b) represents the seminorm $\|\mathsf{B}\|$ (pseudonorm) that as mentioned may be positive, negative, or zero.

3.1 Exponential and logarithm of MV in $Cl_{0,1}$

In the commutative $Cl_{0,1}$ algebra where $\mathbf{e}_1^2 = -1$, we bave

$$\mathrm{e}^{\mathsf{B}} = \mathrm{e}^{b_0 + b_1\mathbf{e}_1} = \mathrm{e}^{b_0}\mathrm{e}^{b_1\mathbf{e}_1} = \mathrm{e}^{b_0}\bigl(\cos b_1 + \mathbf{e}_1 \sin b_1\bigr), \tag{3}$$

where Euler's rule was used. Presence of trigonometric functions indicates that the exponential in $Cl_{0,1}$ is a periodic function with period $2\pi k$, where $k \in \mathbb{Z}$ is an arbitrary integer. Thus, more generally in $Cl_{0,1}$, we have $\mathrm{e}^{\mathsf{B}} = \mathrm{e}^{b_0 + b_1\mathbf{e}_1 + 2\pi k\mathbf{e}_1}$.

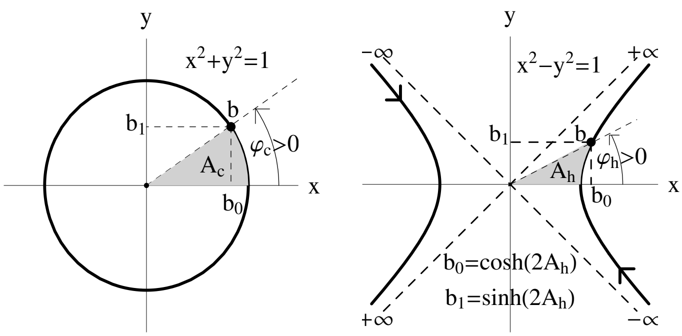
The logarithm of a complex number $z = x + \mathrm{i}y$ is

$$\log z = \log\bigl(r\mathrm{e}^{\mathrm{i}\varphi}\bigr) = \log|z| + \mathrm{i}\varphi = \log\Bigl(\sqrt{x^2 + y^2}\,\Bigr) + \mathrm{i}\varphi,$$

which in $Cl_{0,1}$ algebra notation is

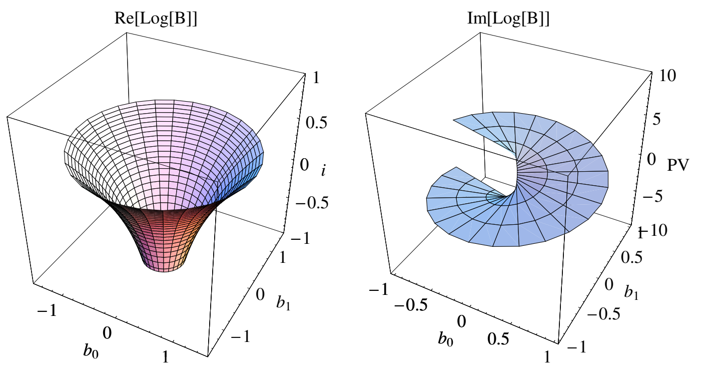
$$\log \mathsf{B} = \log\sqrt{b_0^2 + b_1^2} + \mathbf{e}_1 \arctan\frac{b_1}{b_0} = \log|\mathsf{B}| + \mathbf{e}_1\varphi. \tag{4}$$

The angle $\varphi = \arctan(y/x)$, or $\varphi = \arctan(b_1/b_0)$, is called the argument of logarithm. If $r = (b_0^2 + b_1^2)^{1/2}$ is a constant, then φ may be interpreted as a rotation angle of a vector around coordinate center, Fig. 2(a). To eliminate sign ambiguity between quadrants 1 and 3 (or 2 and 4), the arc tangent of a single argument usually is replaced by double argument arc tangent $\arctan(x, y)$. If signs of x and y are already fixed, then $\arctan(x, y) = \arctan(y/x)$. To include multiple rotations, after every single



(a) (b)

Figure 1. Analogy between unit circle $x^2+y^2 = 1$ and unit hyperbola $x^2-y^2 = 1$. For a circle the coordinates of point b are $x = \cos\varphi_c$ and $y = \sin\varphi_c$, and for a hyperbola, they are $x = \cosh\varphi_h$ and $y = \sinh\varphi_h$. The trigonometric and hyperbolic angles are defined, respectively, by $\varphi_c = \arctan(y/x)$ in the range $[0, 2\pi)$ and $\varphi_h = \operatorname{artanh}(y/x)$ in the range $(-\infty, +\infty)$. For hyperbola, they are limited by asymptotes (dashed lines). The shaded areas A_c and A_h are proportional to trigonometric and hyperbolic angles: $A_c = \varphi_c/2$ and $A_h = \varphi_h/2$. The infinity signs at asymptotes show extreme values of $\varphi_h/2$, where the infinities having opposite signs meet, $-\infty = +\infty$ [8].



(a) (b)

Figure 2. Graphical representation of $Cl_{0,1}$ logarithm: (a) the real part $\operatorname{Re}(\log(\mathsf{B})) = \log(\sqrt{b_0^2 + b_1^2})$ and (b) the principal logarithm $\varphi = \operatorname{Im}(\log(\mathsf{B}))$ in the range $[-\pi, \pi]$ is represented by a single winding on the right panel. At a fixed φ the lines run parallel to horizontal b_0b_1-plane.

rotation, the period 2π is added to φ, so that after k rotations, we have k-windings in Fig. 2(b) and angle $\varphi = \arctan(b_0, b_1) + 2\pi k$, where $k \in \mathbb{Z} = \ldots -2, -1, 0, 1, 2, \ldots$. Similarly, in case of hyperbolic functions, to include the sign of x and y in the quadrants 1–4, one may introduce a double angle hyperbolic tangent[1] $\tanh(x, y) = y/x = \sinh\varphi_h/\cosh\varphi_h = \tanh\varphi_h$. As follows from Fig. 1(b), the range of the hyperbolic

[1]Figure 1(b) represents properties of hyperbola drawn on the Euclidean plane. The properties of hyperbola on sphere and complex cylinder are described in [8]. Note that *Mathematica* computes the area hyperbolic tangent of real or complex argument on a complex closed cylinder.

tangent is $(-1,\dots,1)$ when $-\infty<\varphi_h<\infty$. Then, in the quadrants $1,2,3,4$, we have, respectively, $\tanh(+|x|,+|y|)$, $\tanh(-|x|,+|y|)$, $\tanh(-|x|,-|y|)$, and $\tanh(+|x|,-|y|)$.

We shall assume that in GA the defining equation of the logarithm is $\log \mathsf{B} = \mathsf{A}$, which takes into account only the principal value (principal logarithm). To include multiple values, we add a free MV F,

$$\log \mathsf{B} = \mathsf{A} + \mathsf{F}, \quad \mathsf{A},\mathsf{B},\mathsf{F} \in Cl_{0,1}, \tag{5}$$

that satisfies $\mathrm{e}^{\mathsf{F}} = 1$. Equation (5) is more general because, as we shall see, it allows to include the multiplicity into GA logarithm in case of higher ($n = 3$) dimensional GAs [2]. Let us apply the described approach to Eq. (4)

$$\log \mathsf{B} = \log|\mathsf{B}| + \mathbf{e}_1 \arctan(b_0, b_1) \equiv \log r + \mathbf{e}_1\varphi,$$

where $|\mathsf{B}| = \sqrt{\mathsf{B}\widehat{\mathsf{B}}} = \sqrt{b_0^2+b_1^2} = r$ is the radius r (magnitude or norm of B), and φ is the angle between the horizontal axis and line that connects the coordinate center with the point b, Fig. 1(a). To include multiplicity in the angle, a free term F is added, $\log \mathsf{B} = \mathsf{A}+\mathsf{F}$. After substitution of $\mathsf{F} = f_0+\mathbf{e}_1 f_1$ into $\mathrm{e}^{\mathsf{F}} = 1$ and using the trigonometric expansion similar to Eq. (3), we find $\mathrm{e}^{f_0}(\cos f_1 + \mathbf{e}_1 \sin f_1) = 1$, the solution of which is $f_0 = 0$ and $f_1 = 2\pi k$, where $k \in \mathbb{Z}$. Thus, the full solution in agreement with the complex function theory can be written

$$Cl_{0,1}\colon \quad \log \mathsf{B} = \log|\mathsf{B}| + \mathbf{e}_1(\varphi + 2\pi k), \quad 0 \leqslant \varphi < 2\pi,\ k \in \mathbb{Z}.$$

At a fixed $r = |\mathsf{B}|$, this equation represents the spiral with period 2π since the argument $(0 \leqslant \varphi < 2\pi)$ increases by 2π after every single winding in the "complex" plane b_0, b_1. The logarithm (4) exists for all values of B. Often it is assumes that the principal logarithm is in the range $-\pi < \varphi < \pi$, the logarithm is

$$Cl_{0,1}\colon \quad \log \mathsf{B} = \begin{cases} \log|\mathsf{B}| + \mathbf{e}_1\varphi & \text{if } b_0 > 0 \text{ and } b_1 \neq 0,\\ \log|\mathsf{B}| + \mathbf{e}_1(\varphi+\pi) & \text{if } b_0 < 0 \text{ and } b_1 > 0,\\ \log|\mathsf{B}| + \mathbf{e}_1(\varphi-\pi) & \text{if } b_0 < 0 \text{ and } b_1 < 0,\\ \hline b_0 & \text{if } b_0 > 0 \text{ and } b_1 = 0,\\ b_0 + \mathbf{e}_1\pi & \text{if } b_0 < 0 \text{ and } b_1 = 0,\\ b_1 + \mathbf{e}_1\pi/2 & \text{if } b_0 = 0 \text{ and } b_1 > 0,\\ b_1 - \mathbf{e}_1\pi/2 & \text{if } b_0 = 0 \text{ and } b_1 < 0. \end{cases} \tag{6}$$

The first three expressions are the main formulas. The remaining represent special cases: they show the behavior of logarithm on the real and imaginary axis. When $b_0 = b_1 = 0$, the logarithm is undefined. The definition given by Eqs. (6) and visualized in Fig. 2(b) frequently is met in applications. It has been accepted in ISO standards such as C programming language and *Mathematica*.

3.2 Exponential and logarithm of MV in $Cl_{1,0}$

For 1D algebras the inverse of MV $\mathsf{B} = b_0 + b_1\mathbf{e}_1$ is

$$Cl_{0,1}\colon\quad \mathsf{B}^{-1} = \frac{\widehat{\mathsf{B}}}{\mathsf{B}\widehat{\mathsf{B}}} = \frac{b_0 - \mathbf{e}_1 b_1}{b_0^2 + b_1^2};\qquad Cl_{1,0}\colon\quad \mathsf{B}^{-1} = \frac{\widehat{\mathsf{B}}}{\mathsf{B}\widehat{\mathsf{B}}} = \frac{b_0 - \mathbf{e}_1 b_1}{b_0^2 - b_1^2} \tag{7}$$

and satisfies $\mathsf{B}^{-1}\mathsf{B} = \mathsf{B}\mathsf{B}^{-1} = 1$. From (7) follows that, in contrast to complex algebra, where each nonzero complex number has its inverse, in $Cl_{1,0}$, zero divisors appear if $b_0^2 = b_1^2$ as shown by dashed lines in a hyperbolic plane in Fig. 3. Since $\mathbf{e}_1^2 > 0$, $Cl_{1,0}$ exponential may be expanded in hyperbolic sine and cosine functions [16],

$$Cl_{1,0}\colon\quad \mathrm{e}^{\mathsf{B}} = \mathrm{e}^{b_0 + b_1\mathbf{e}_1} = \mathrm{e}^{b_0}\mathrm{e}^{b_1\mathbf{e}_1} = \mathrm{e}^{b_0}(\cosh b_1 + \mathbf{e}_1 \sinh b_1), \tag{8}$$

where $b_0, b_1 \in \mathbb{R}$. The hyperbolic functions are monotonic, therefore, the exponential in $Cl_{1,0}$ inherits this property as well. In $Cl_{1,0}$ a logarithm defining equation is $\log \mathsf{B} = \mathsf{A}$, in solution of which the hyperbolic functions and the identity $\cosh^2 x - \sinh^2 x = 1$ are to be used. The following expression for principal logarithm (the first formula) and special case (the second formula) is found:

$$Cl_{1,0}\colon\quad \log \mathsf{B} = \begin{cases} \log\sqrt{b_0^2 - b_1^2} + \mathbf{e}_1 \operatorname{artanh}\frac{b_1}{b_0};\ b_0 > 0 \text{ and } b_0^2 > b_1^2, \\ \frac{1}{2}(\log(0_+) + \log(2b_0)) \pm \mathbf{e}_1 \frac{1}{2}(-\log(0_+) + \log(2b_0);\ b_1 = b_0, \end{cases} \tag{9}$$

where artanh is the area tangent function, $-1 < \operatorname{artanh}(b_1/b_0) < 1$. The scalar part $\log\sqrt{b_0^2 - b_1^2}$ exists if $b_0^2 > b_1^2$. The logarithm has a genuine value if a pair $\{b_0, b_1\}$ is in the shaded sector of Fig. 3. Thus, the existence of both the logarithm and the square root are determined by condition $b_0^2 > b_2^2$. The special case belongs to asymptotes $b_0 = b_1$, where $\log(0_+)$ is the logarithm of a point infinitesimally close to zero. This term vanishes in $\exp(\log \mathsf{B}) = \mathsf{B}$ (see Example 2). The first equation of (9) can be rewritten in hyperbola parameters in Fig. 1(b). Since $\mathsf{B} = r(\cosh\varphi_h + \mathbf{e}_1 \sinh\varphi_h)$, where $r = a_0$ is the radius ($r = 1$ in Fig. 1(b)), we have $b_0 = r\cosh\varphi_h$ and $b_1 = r\sinh\varphi_h$. Since $r = \sqrt{b_0^2 - b_1^2}$

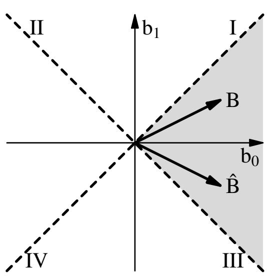


Figure 3. Hyperbolic plane b_0, b_1 that represents $Cl_{1,0}$ algebra. The arrows show MV $\mathsf{B} = b_0 + \mathbf{e}_1 b_1$ and its conjugate $\widehat{\mathsf{B}}$. The dashed lines are asymptotes $b_0^2 - b_1^2 = 0$. The principal square root and logarithm exist in the shaded sector, where $b_0 > b_1$. I–IV are the hyperbolic plane quadrants.

and $\tanh\varphi_h = b_1/b_0$, we have that $\log \mathsf{B} = \log r + \mathbf{e}_1\varphi_h$, which is to be compared with Eq. (4).

To find the free term F, we solve $\exp(\mathsf{F}) = 1$, for this purpose bringing into play Eq. (8),

$$\mathrm{e}^{\mathsf{F}} = \mathrm{e}^{f_0+\mathbf{e}_1 f_1} = \mathrm{e}^{f_0}\big(\cosh|f_1| + \mathbf{e}_1 \sinh|f_1|\big) = 1.$$

This equation can be satisfied if $f_0 = f_1 = 0$. So, in this algebra, we have only the principal logarithm.

Example 1. $Cl_{1,0}$: If $b_0^2 > b_1^2$, $b_0 > 0$, and $\mathsf{B} = 3 \pm 2\mathbf{e}_1$, then

$$\log \mathsf{B} = \log\sqrt{5} \pm \mathbf{e}_1 \operatorname{artanh}\frac{2}{3} = a_0 \pm \mathbf{e}_1 a_1.$$

The exponential of logarithm gives $\exp(\log \mathsf{B}) = \mathsf{B}$. If $b_0^2 > b_1^2$, $b_0 < 0$, and $\mathsf{B}' = -3 \pm 2\mathbf{e}_1$, then

$$\log \mathsf{B}' = \log\sqrt{5} \pm \mathbf{e}_1 \operatorname{artanh}\frac{2}{3} = a_0 \pm \mathbf{e}_1 a_1.$$

The answer is wrong since the initial MV is returned with an opposite sign: $\exp(\log \mathsf{B}') = -\mathsf{B}'$.

Example 2. $Cl_{1,0}$: $b_0 = b_1$, $\mathsf{B} = 2 + 2\mathbf{e}_1$. In this case the second formula of (9) should be used:

$$\log \mathsf{B} = \frac{1}{2}(\log 4 + \log 0_+) + \frac{1}{2}\mathbf{e}_1(\log 4 - \log 0_+).$$

Then $\exp(\log \mathsf{B}) = 4(1+\mathbf{e}_1)/2 + \mathrm{e}^{\log 0_+}(-1+\mathbf{e}_1)/2 \to 2(1+\mathbf{e}_1)$, which in the limit $\log 0_+ \to -\infty$ gives $\mathsf{B} = 2 + 2\mathbf{e}_1$ that represents a point on the asymptote.

4 Exponential and logarithm in 2D algebras

4.1 Quaternionic "vector"

The following defining equations for exponential and logarithm in 2D GAs are used, $\exp \mathsf{B} = \mathsf{A}$ and $\log \mathsf{B} = \mathsf{A}$, where MVs A and B belong to the same algebra. It is convenient to introduce base-free MVs $A' = \mathbf{a} + b_{12}\mathbf{e}_{12}$ and $B' = \mathbf{b} + b_{12}\mathbf{e}_{12}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbf{e}_{12}$-plane. By analogy to Hamilton quaternion theory [10, 13], in the following, we shall treat the quantity A' as a 3D "vector". Introduction of such a "vector" appears very helpful in calculating the exponential as well as logarithm in all 2D algebras. Thus, a full MV in 2D algebras may be represented as a sum of scalar and "vector":

$$\mathsf{A} = a_0 + A', \quad A' = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}\mathbf{e}_{12}.$$

In $Cl_{0,2}$ the squares of all three basis elements satisfy $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_{12}^2 = -1$ and $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_{12} = -1$. Similarly, in the Hamilton quaternion algebra [10, 13] a set of three imaginary units $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ satisfy $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$ and $\mathrm{ijk} = -1$. In 3D Euclidean space the quaternionic vector is defined by $\mathbf{v} = a_1\mathrm{i} + a_2\mathrm{j} + a_{12}\mathrm{k}$, the square of which is a negative number. The same property is satisfied by "vector", $(A')^2 \equiv A'^2 = -a_1^2 - a_2^2 - a_{12}^2 < 0$, where a_1, a_2, and a_{12} are the real numbers. Thus, the MV A' is equivalent to quaternion vector, and A' may be treated exactly in the same way as the Hamilton vector $\mathbf{v} = a_1\mathrm{i} + a_2\mathrm{j} + a_{12}\mathrm{k}$. For $Cl_{0,2}$ "vector" A', the norm is defined by $|A'| = \sqrt{-A'^2}$.

Because $Cl_{1,1}$ and $Cl_{2,0}$ are not division algebras, i.e., in these algebras, not every MV has inverse, for these algebras, we have different cases. Now $A'^2 \equiv (A')^2$ may be either positive or negative, or even zero. The first (positive) case, as we shall see, is related to hyperbolic functions, while the second is related to trigonometric functions. Both cases will be investigated separately in $Cl_{1,1}$ and $Cl_{2,0}$ algebras. The seminorm of "vector" A' is defined by $\|A'\| = \sqrt{\mathrm{abs}(A'^2)} \geqslant 0$.

4.2 Exponentials of MV in 2D algebras

In Table 1, two-dimensional exponentials in expanded form including the case of null MV (when $\mathsf{B}^2 = 0$) are summarized. The structure of the formulas reminds de Moivre's–Euler's rules. For $Cl_{0,2}$, only trigonometric functions appear. For algebras $Cl_{1,1}$ and $Cl_{2,0}$, also hyperbolic functions appear if $B'^2 > 0$. In case of $Cl_{0,2}$, which is isomorphic to Hamilton quaternion, the exponential formula can be found easily if the property $(B'/|B'|)^2 = -1$ is taken into account. Since $B'/|B'|$ behaves like an imaginary unit, we can write at once

$$\mathrm{e}^{B'} = \cos|B'| + \frac{B'}{|B'|}\sin|B'|.$$

Then the exponential of $\mathsf{B} = b_0 + B'$ is

$$Cl_{0,2}\colon\quad \mathrm{e}^{\mathsf{B}} = \mathrm{e}^{b_0 + B'} = \mathrm{e}^{b_0}\mathrm{e}^{B'} = \mathrm{e}^{b_0}\biggl(\cos|B'| + \frac{B'}{|B'|}\sin|B'|\biggr).$$

In the remaining algebras the different normalization must be used. The square of a normalized "vector" now is $(B'/\|B'\|)^2 = \pm 1$, and apart from trigonometric, in addition, hyperbolic functions for plus sign appear,

$$Cl_{1,1}, Cl_{2,0}\colon\quad \mathrm{e}^{\mathsf{B}} = \mathrm{e}^{b_0}\mathrm{e}^{B'} = \begin{cases} \mathrm{e}^{b_0}\bigl(\cos\|B'\| + \frac{B'}{\|B'\|}\sin\|B'\|\bigr), & B'^2 < 0,\\ \mathrm{e}^{b_0}\bigl(\cosh\|B'\| + \frac{B'}{\|B'\|}\sinh\|B'\|\bigr), & B'^2 > 0. \end{cases}$$

Thus, in $Cl_{1,1}$ and $Cl_{2,0}$ algebras depending on sign of B'^2 and coefficient values in the seminorm, the exponentials may be expanded either in trigonometric or in hyperbolic functions and, as a result, may be periodic or monotonic. Finally, in Table 1 the exponential $\mathrm{e}^{\mathsf{B}} = \mathrm{e}^{b_0}(1 + B')$ comes from the null MV, the square of which nullifies $B'^2 = 0$ and yields a linearly dependence on B'. Recently, we have found [3] that in three-dimensional GAs (and probably in higher dimensional spaces) the entanglement or mixing of vector and bivector components may take place, so that in the expanded form the exponential loses de Moivre's–Euler's formula structure. The latter is regained if both the vector and bivector lie in the same plane. This is in agreement with the present 2D formulas, where the vector and bivector are always in $\mathbf{e}_1\mathbf{e}_2$-plane.

Table 1. Exponentials of general MV $\mathsf{B} = b_0 + B' = b_0 + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_{12}\mathbf{e}_{12}$ in 2D GAs. $|B'| = \sqrt{-B'^2}$ and $\|B'\| = \sqrt{\mathrm{abs}(B'^2)}$ are real numbers that represent the norm and seminorm, respectively. The trigonometric functions appear when $B'^2 < 0$, while hyperbolic when $B'^2 > 0$. Zero values of the seminorm corresponds to $\lim_{x\to 0}\sin(x)/x = \lim_{x\to 0}\sinh(x)/x = 1$.

	$\exp(\mathsf{B}) = \exp(b_0 + B')$	$-B' = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_{12}\mathbf{e}_{12}$
$Cl_{0,2}$	$\mathrm{e}^{b_0}(\cos\lvert B'\rvert + \frac{B'}{\lvert B'\rvert}\sin\lvert B'\rvert)$	$B'^2 = b_1^2 + b_2^2 + b_{12}^2 > 0$
	e^{b_0}	$B' = 0$
$Cl_{1,1}$	$\mathrm{e}^{b_0}(\cosh\lVert B'\rVert + \frac{B'}{\lVert B'\rVert}\sinh\lVert B'\rVert)$	$B'^2 = b_1^2 - b_2^2 + b_{12}^2 > 0$
	$\mathrm{e}^{b_0}(1 + B')$	$B'^2 = b_1^2 - b_2^2 + b_{12}^2 = 0$
	$\mathrm{e}^{b_0}(\cos\lVert B'\rVert + \frac{B'}{\lVert B'\rVert}\sin\lVert B'\rVert)$	$B'^2 = b_1^2 - b_2^2 + b_{12}^2 < 0$
$Cl_{2,0}$	$\mathrm{e}^{b_0}(\cosh\lVert B'\rVert + \frac{B'}{\lVert B'\rVert}\sinh\lVert B'\rVert)$	$B'^2 = b_1^2 + b_2^2 - b_{12}^2 > 0$
	$\mathrm{e}^{b_0}(1 + B')$	$B'^2 = b_1^2 + b_2^2 - b_{12}^2 = 0$
	$\mathrm{e}^{b_0}(\cos\lVert B'\rVert + \frac{B'}{\lVert B'\rVert}\sin\lVert B'\rVert)$	$B'^2 = b_1^2 + b_2^2 - b_{12}^2 < 0$

Example 3.

$$\mathsf{B} = 2 + 5\mathbf{e}_1 - 4\mathbf{e}_2 - 7\mathbf{e}_{12} = 2 + B',$$

$$Cl_{0,2}\colon \quad \exp\mathsf{B} = \mathrm{e}^2\biggl(\cos\sqrt{90} + \frac{5\mathbf{e}_1 - 4\mathbf{e}_2 - 7\mathbf{e}_{12}}{\sqrt{90}}\sin\sqrt{90}\biggr),$$

$$B'^2 = -90, \qquad |B'| = \sqrt{90},$$

$$Cl_{1,1}\colon \quad \exp\mathsf{B} = \mathrm{e}^2\biggl(\cosh\sqrt{58} + \frac{5\mathbf{e}_1 - 4\mathbf{e}_2 - 7\mathbf{e}_{12}}{\sqrt{58}}\sinh\sqrt{58}\biggr),$$

$$B'^2 = 58, \qquad \|B'\| = \sqrt{58},$$

$$Cl_{2,0}\colon \quad \exp\mathsf{B} = \mathrm{e}^2\biggl(\cos\sqrt{8} + \frac{5\mathbf{e}_1 - 4\mathbf{e}_2 - 7\mathbf{e}_{12}}{\sqrt{8}}\sin\sqrt{8}\biggr),$$

$$B'^2 = -8, \qquad \|B'\| = \sqrt{8}.$$

4.3 Products of exponentials

Using Table 1, it is easy to calculate the geometric product of two exponentials. For example, for trigonometric functions in $Cl_{0,2}$ when $B'^2 < 0$, we find

$$\begin{aligned}
\mathrm{e}^{\mathsf{A}}\mathrm{e}^{\mathsf{B}} &= \mathrm{e}^{a_0 + A'}\mathrm{e}^{b_0 + B'} \\
&= \mathrm{e}^{a_0 + b_0}\biggl(\cos|A'|\cos|B'| + \frac{\langle A'B'\rangle_0}{|A'||B'|}\sin|A'|\sin|B'|\biggr) \\
&\quad + \mathrm{e}^{a_0 + b_0}\biggl(\frac{A'}{|A'|}\sin|A'|\cos|B'| + \frac{B'}{|B'|}\sin|B'|\cos|A'| \\
&\quad + \frac{1}{2}\frac{[A', B']}{|A'||B'|}\sin|A'|\sin|B'|\biggr).
\end{aligned} \tag{10}$$

When $A' = B'$, the commutator $[A', B'] = 0$, and Eq. (10) reduces to double 2A argument exponential. For remaining algebras, the norm should be replaced by seminorm. Below, particular cases follow from (10).

Case 1. Product of vectorial exponentials. If A and B represent vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$, then

$$\begin{aligned}\mathrm{e}^{\mathbf{a}}\mathrm{e}^{\mathbf{b}} &= \cos|\mathbf{a}|\cos|\mathbf{b}| - \cos\theta\sin|\mathbf{a}|\sin|\mathbf{b}| \\ &\quad + \frac{\mathbf{a}}{|\mathbf{a}|}\sin|\mathbf{a}|\cos|\mathbf{b}| + \frac{\mathbf{b}}{|\mathbf{b}|}\sin|\mathbf{b}|\cos|\mathbf{a}| \\ &\quad + \mathbf{e}_{12}\sin\theta\sin|\mathbf{a}|\sin|\mathbf{b}|,\end{aligned}$$

where θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$.

Case 2. Product of bivectorial exponentials. If A and B are simple bivectors $\mathcal{A} = a_{12}\mathbf{e}_{12}$, $\mathcal{B} = b_{12}\mathbf{e}_{12}$, then

$$\mathrm{e}^{\mathcal{A}}\mathrm{e}^{\mathcal{B}} = \mathrm{e}^{\mathcal{A}+\mathcal{B}} = \cos|\mathcal{A}+\mathcal{B}| + \mathbf{e}_{12}\sin|\mathcal{A}+\mathcal{B}|,$$

where $|\mathcal{A}+\mathcal{B}| = \sqrt{(\mathcal{A}+\mathcal{B})\widetilde{(\mathcal{A}+\mathcal{B})}}$. Since $\mathcal{A}+\mathcal{B}$ is the bivector and $(\mathcal{A}+\mathcal{B})^2 < 0$, this formula follows directly.

Case 3. Product of vector and bivector exponentials:

$$\mathrm{e}^{\mathbf{a}}\mathrm{e}^{\mathcal{B}} = \left(\cos|\mathbf{a}| + \frac{\mathbf{a}}{|\mathbf{a}|}\sin|\mathbf{a}|\right)\bigl(\cos|\mathcal{B}| + \mathbf{e}_{12}\sin|\mathcal{B}|\bigr).$$

Case 4. The commutator also vanishes if the coefficients satisfy $a_1b_{12} = b_1a_{12}$, $a_2b_{12} = b_2a_{12}$, and $a_2b_1 = b_2a_1$. Since in this case $[\mathsf{A}, \mathsf{B}] = [A', B']$, we have $\mathrm{e}^{\mathsf{A}}\mathrm{e}^{\mathsf{B}} = \mathrm{e}^{\mathsf{A}+\mathsf{B}}$ and Eq. (10) reduced to

$$\mathrm{e}^{\mathsf{A}}\mathrm{e}^{\mathsf{B}} = \mathrm{e}^{\mathsf{A}+\mathsf{B}} = \mathrm{e}^{\mathsf{B}}\mathrm{e}^{\mathsf{A}} = \mathrm{e}^{a_0+b_0}\left(\cos|A'+B'| + \frac{A'+B'}{|A'+B'|}\sin|A'+B'|\right).$$

When $B'^2 > 0$, similar formulas exist for hyperbolic functions.

5 Logarithm of MV in 2D algebras

The approach to commutative algebras in Section 3.1 here is generalized to 2D algebras. The "vector" property $B'^2 \gtreqless 0$ allows to get 2-dimensional logarithm formulas that are very similar to those found in 1D case but with basis vector $\mathbf{e}_1$ replaced by unit multivector $B'/|B'|$ or $B'/\|B'\|$.

5.1 $Cl_{0,2}$ algebra

In this algebra, according to Table 1, the norm (or magnitude) of B' is

$$|B'| = \sqrt{-B'^2} = \sqrt{B'\widetilde{\widehat{B'}}} = \sqrt{b_1^2 + b_2^2 + b_{12}^2}.$$

The logarithm defining equation is $\log \mathsf{B} = \mathsf{A}$, where B is a given MV, and coefficients of A are to be determined. Since $B'^2 < 0$, the exponential of logarithm can be expanded by trigonometric functions

$$\mathrm{e}^{\log \mathsf{B}} = \mathrm{e}^{\mathsf{A}} = \mathrm{e}^{a_0 + A'} = \mathrm{e}^{a_0}\left(\cos|A'| + \frac{A'}{|A'|}\sin|A'|\right),$$

from which we write the following relation between "'vectors"' B' and A':

$$b_0 + B' = \mathrm{e}^{a_0}\left(\cos|A'| + \frac{A'}{|A'|}\sin|A'|\right). \tag{11}$$

Equation (11) can be rewritten as a system of two equations

$$b_0 = \mathrm{e}^{a_0}\cos|A'|, \qquad B' = \mathrm{e}^{a_0}\frac{A'}{|A'|}\sin|A'|, \tag{12}$$

where the second equation, in fact, represents three scalar equations. System (12) can be solved with respect to a_0 and A' in the following way. After squaring both sides of (12) and noting that in $Cl_{0,2}$, $B'^2 = -|B'|^2$, we have

$$b_0^2 = \mathrm{e}^{2a_0}\cos^2|A'|, \qquad |B'|^2 = \mathrm{e}^{2a_0}\sin^2|A'|. \tag{13}$$

The sum gives $b_0^2 + |B'|^2 = \mathrm{e}^{2a_0}$, from which and $b_0^2 + |B'|^2 = |\mathsf{B}|^2$ follows

$$a_0 = \log|\mathsf{B}|. \tag{14}$$

The ratio of equations in (13) gives $|B'|/b_0 = \tan|A'|$. The inverse of the latter is

$$|A'| = \arctan\frac{|B'|}{b_0}. \tag{15}$$

To express the "vector" A' in terms of B', the second equation in (12) is divided by the first,

$$\frac{B'}{b_0} = \frac{A'}{|A'|}\tan|A'|. \tag{16}$$

As from (15) follows $\tan|A'| = |B'|/b_0$, therefore, Eq. (16) reduces to

$$\frac{B'}{b_0} = \frac{A'}{|A'|}\frac{|B'|}{b_0},$$

from which the property $B'/|B'| = A'/|A'|$, i.e., B' and A' are parallel in ijk-space, follows. The latter along with (14) allow to get

$$\log \mathsf{B} = a_0 + A' = \log|\mathsf{B}| + |A'|\frac{B'}{|B'|}.$$

Finally, the needed generic logarithm formula is

$$Cl_{0,2}\colon \quad \log \mathsf{B} = \log|\mathsf{B}| + \frac{B'}{|B'|}\left(\arctan\frac{|B'|}{b_0}\right), \quad B'^2 < 0. \tag{17}$$

To logarithm (17) we may add a free MV $\mathsf{F} = 2\pi k\hat{F}'$, where $\hat{F}'$ plays the role of imaginary unit, $(\hat{F}')^2 = -1$. In addition, it satisfies $\exp(2\pi k\hat{F}') = 1$ and $|\hat{F}'| = 1$. As we shall see, the free MV takes into account the multivaluedness of arc tangent. Then $\log \mathsf{B} = \mathsf{A} + \mathsf{F} = \mathsf{A} \pm 2\pi k\hat{F}'$. Since $\mathsf{F} = f_0 + f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_{12}\mathbf{e}_{12} = f_0 + F'$ and $|F'| = \sqrt{-F'^2} = (F'\widetilde{\widehat{F'}})^{1/2} = \sqrt{f_1^2 + f_2^2 + f_{12}^2}$, we have

$$\mathrm{e}^{\mathsf{F}} = \mathrm{e}^{f_0}\left(\cos|F'| + \frac{F'}{|F'|}\sin|F'|\right) = 1,$$

which is satisfied if $f_0 = 0$ and $|F'| = \sqrt{f_1^2 + f_2^2 + f_{12}^2} = 2\pi k$, $k \in \mathbb{Z}$. Thus, in $Cl_{0,2}$, we have that the free MV is

$$\mathsf{F} = 2\pi k\hat{F}' = 2\pi k\frac{f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_{12}\mathbf{e}_{12}}{\sqrt{f_1^2 + f_2^2 + f_{12}^2}} = 2\pi k\frac{F'}{|F'|}$$

that represents all possible "vectors", the ends of which lie on a sphere of radius equal to 1 in the 3D anti-Euclidean ijk-space. When $k = 0$, $|F'| = 0$ and $\mathsf{F} = 0$ because $f_0 = 0$. Thus, we conclude that the generic solution of equation $\log \mathsf{B} = \mathsf{A}$ represents the principal value of argument $\varphi = \arctan(|B'|/b_0) = \arctan(\sqrt{b_1^2 + b_2^2 + b_3^2}/b_0)$ in the range $0 \leqslant \varphi < 2\pi$ if $k = 0$. When multivaluedness is included, the Hamilton quaternion logarithm takes the form

$$\log \mathsf{B} + \mathsf{F} = \log|\mathsf{B}| + \frac{B'}{|B'|}\left(\arctan\frac{|B'|}{b_0} + 2\pi k\right), \quad B'^2 < 0, \quad k \in \mathbb{Z}, \tag{18}$$

$|\mathsf{B}| = (b_0^2 + |B'|^2)^{1/2}$. Formula (18) satisfies $\mathrm{e}^{\log \mathsf{B} + \mathsf{F}} = \mathsf{B}$ for all integers k. Thus, in $Cl_{0,2}$ the free MV is $\mathsf{F} = F' = (B'/|B'|)2\pi k$, where the "vector" $(B'/|B'|)$ plays the role of an imaginary unit (compare with Eq. (4)). When $k = 0$, we return back to the principal logarithm. Thus, after replacement of the arc tangent by a double-argument arc tangent in order to take account of all four quadrants correctly, the generic formula (18) with special cases included becomes

$$Cl_{0,2}\colon \ \log \mathsf{B} = \begin{cases} \log|\mathsf{B}| + (\arctan(b_0, |B'|) + 2\pi k)\frac{B'}{|B'|}, & |B'| \neq 0, \\ \log(b_0) + 2\pi k\,\hat{F}', & (|B'| = 0) \wedge (b_0 > 0), \\ \log(-b_0) + \pi(2k+1)\,\hat{F}', & (|B'| = 0) \wedge (b_0 < 0). \end{cases} \tag{19}$$

To summarize, we have shown that, similar to complex number logarithm, the Hamilton number logarithm is a multivalued function too.

Example 4. $Cl_{0,2}$: $\mathsf{B} = 2 + 4\mathbf{e}_1 - 5\mathbf{e}_2 - \mathbf{e}_{12}$, $|\mathsf{B}| = \sqrt{46}$, $B'^2 = -42 < 0$, $|B'| = \sqrt{42}$. The principal logarithm is

$$\log \mathsf{B} = \log\sqrt{46} + \arctan\left(\frac{\sqrt{42}}{2}\right)\frac{4\mathbf{e}_1 - 5\mathbf{e}_2 - \mathbf{e}_{12}}{\sqrt{42}}$$
$$\approx 1.914 + 4.662\mathbf{e}_1 - 5.826\mathbf{e}_2 - 1.165\mathbf{e}_{12}.$$

After exponentiation $\mathrm{e}^{\log \mathsf{B}}$, we recover the initial MV B.

5.2 $Cl_{1,1}$ and $Cl_{2,0}$ algebras when $B'^2 \leqslant 0$

When $B'^2 = -B'\widetilde{\widehat{B'}} < 0$, the exponentials for both $Cl_{1,1}$ and $Cl_{2,0}$ algebra, are expressed in trigonometric functions in the same way as for $Cl_{0,2}$ but the norm replaced by seminorm (see Table 1). The free MV $\mathsf{F} = f_0 + \hat{F}'$ also satisfies the condition $\mathrm{e}^{2\pi k \mathsf{F}} = 1$, from which we have $f_0 = 0$ and

$$\begin{aligned} Cl_{1,1}\colon \quad \hat{F}' &= f_1\mathbf{e}_1 + f_{12}\mathbf{e}_{12} + \sqrt{1 + f_1^2 + f_{12}^2}\,\mathbf{e}_2, \\ Cl_{2,0}\colon \quad \hat{F}' &= f_1\mathbf{e}_1 + f_2\mathbf{e}_2 - \sqrt{1 + f_1^2 + f_2^2}\,\mathbf{e}_{12} \end{aligned} \tag{20}$$

with properties $\|\hat{F}'\| = 1$ and $\hat{F}'^2 = -1$.

Generic logarithm for both $Cl_{1,1}$ and $Cl_{2,0}$, are given by equations similar to (19) but with $\hat{F}'$ replaced by (20) and norm replaced by seminorm,

$$Cl_{1,1}, Cl_{2,0}\colon \quad \log(\mathsf{B}) = \begin{cases} \log(\|\mathsf{B}\|) + (\arctan(b_0, \|B'\|) + 2\pi k)\frac{B'}{\|B'\|}, \\ \quad (B'^2 < 0), \\ \log(b_0) + \frac{B'}{b_0}, & (B'^2 = 0) \wedge (b_0 > 0), \\ \log(-b_0) + \pi(2k+1)\hat{F}', & (B' = 0) \wedge (b_0 < 0). \end{cases} \tag{21}$$

The last two equations represent special cases. Now the logarithm of MV exists when $\|B'\| \neq 0$ and $\|B'\| = 0$. More specific cases are presented in Section 5.4.

Example 5. $Cl_{1,1}$: $\mathsf{B} = 2 + 4\mathbf{e}_1 - 5\mathbf{e}_2 - \mathbf{e}_{12}$, $B'^2 = -8$, $\|\mathsf{B}\| = \sqrt{12}$, $\|B'\| = \sqrt{8}$. The answer

$$\log \mathsf{B} = \frac{1}{2}\log(12) + \frac{1}{2\sqrt{2}}(\arctan\sqrt{2} + 2\pi k)(4\mathbf{e}_1 - 5\mathbf{e}_2 - \mathbf{e}_{12})$$

satisfies $\mathrm{e}^{\log \mathsf{B}} = \mathsf{B}$.

Example 6. $Cl_{2,0}$: $\mathsf{B} = 2 + 5\mathbf{e}_1 - 4\mathbf{e}_2 - 7\mathbf{e}_{12}$, $B'^2 = -8$, $\|B'\| = \sqrt{8}$, $\|\mathsf{B}\| = \sqrt{12}$. The answer

$$\log \mathsf{B} = \frac{\log(12)}{2} + \frac{1}{2\sqrt{2}}(\arctan\sqrt{2} + 2\pi k)(5\mathbf{e}_1 - 4\mathbf{e}_2 - 7\mathbf{e}_{12}).$$

satisfies $\mathrm{e}^{\log \mathsf{B}} = \mathsf{B}$.

Example 7. $Cl_{2,0}$: $\mathsf{B} = 2 + 3\mathbf{e}_1 - 4\mathbf{e}_2 - 5\mathbf{e}_{12}$, $B'^2 = 0$, $\|B'\| = 0$, $\|\mathsf{B}\| = 2$. The answer

$$\log \mathsf{B} = \log b_0 + \frac{B'}{b_0} = \log 2 + \frac{1}{2}(3\mathbf{e}_1 - 4\mathbf{e}_2 - 5\mathbf{e}_{12}).$$

satisfies $\mathrm{e}^{\log \mathsf{B}} = \mathsf{B}$.

Example 8. $Cl_{2,0}$: $\mathsf{B} = -2$, $B'^2 = 0$, $\|B'\| = 0$.

$$\begin{aligned}\log \mathsf{B} &= \log(-b_0) + \pi(2k+1)\hat{F}' \\ &= \log 2 + \pi(2k+1)\bigl(f_1\mathbf{e}_1 + f_2\mathbf{e}_2 - \sqrt{1 + f_1^2 + f_2^2}\,\bigr)\mathbf{e}_{12}.\end{aligned}$$

The answer satisfies $\mathrm{e}^{\log \mathsf{B}} = -2$.

5.3 $Cl_{1,1}$ and $Cl_{2,0}$ algebras when $B'^2 > 0$

When $B'^2 > 0$, calculations proceed in a similar way. Therefore, only intermediate results are put down briefly. Let the general MV be $\mathsf{B} = b_0 + B'$. The seminorm in $Cl_{1,1}$ is $\|B'\| = \sqrt{b_1^2 - b_2^2 + b_{12}^2}$, and in $Cl_{2,0}$ it is $\|B'\| = \sqrt{b_1^2 + b_2^2 - b_{12}^2}$, where expression under the root should be positive. As in previous case, the defining equation is $\log \mathsf{B} = \mathsf{A}$, where $\mathsf{A} = a_0 + A'$. Since the square of A' now is positive scalar, $A'^2 > 0$, in agreement with the Table 1, the exponential is expanded in hyperbolic functions,

$$\mathrm{e}^{\log \mathsf{B}} = \mathrm{e}^{a_0 + A'} = \mathrm{e}^{a_0}\biggl(\cosh\|A'\| + \frac{A'}{\|A'\|}\sinh\|A'\|\biggr).$$

Thus, we have the following relation between "vectors" B' and A':

$$b_0 + B' = \mathrm{e}^{a_0}\biggl(\cosh\|A'\| + \frac{A'}{\|A'\|}\sinh\|A'\|\biggr)$$

that may be rewritten as a system of equations

$$b_0 = \mathrm{e}^{a_0}\cosh\|A'\|, \qquad B' = \mathrm{e}^{a_0}\frac{A'}{\|A'\|}\sinh\|A'\|. \tag{22}$$

Squaring of Eqs. (22) and the property $B'^2 = \|B'\|^2$ give

$$b_0^2 = \mathrm{e}^{2a_0}\cosh^2\|A'\|, \qquad \|B'\|^2 = \mathrm{e}^{2a_0}\sinh^2\|A'\|. \tag{23}$$

Now, applying the property $\cosh^2 A' - \sinh^2 A' = 1$, the difference of equations in (23) yields the scalar equation $b_0^2 - \|B'\|^2 = \mathrm{e}^{2a_0}$, from which and the relation $\|\mathsf{B}\|^2 = b_0^2 - \|B'\|^2$ follows $a_0 = \log\|\mathsf{B}\|$. Also, from Eq. (23) we have the ratio $\|B'\|/b_0 = \tanh\|A'\|$, from which we find equation analogous Eq. (15), $\|A'\| = \operatorname{artanh}(\|B'\|/b_0)$, where artanh is the area hyperbolic tangent. To express A' in terms of B', we divide equations in (22),

$$\frac{B'}{b_0} = \frac{A'}{\|A'\|}\tanh\|A'\|. \tag{24}$$

Since $\tanh\|A'\| = \|B'\|/b_0$, Eq. (24) can be reduced to

$$\frac{B'}{b_0} = \frac{A'}{\|A'\|}\frac{|B'|}{b_0},$$

from which the property $B'/\|B'\| = A'/\|A'\|$ follows. The latter allows to get the required formula for the principal logarithm,

$$Cl_{1,1}, Cl_{2,0}\colon\quad \log \mathsf{B} = \log\|\mathsf{B}\| + \operatorname{artanh}\biggl(\frac{\|B'\|}{b_0}\biggr)\frac{B'}{\|B'\|},\quad \bigl(B'^2 > 0\bigr)\wedge(b_0 \neq 0).$$

To this formula we should add a free MV $\mathsf{F} = f_0 + F' = f_0 + f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_{12}\mathbf{e}_{12}$ that satisfies $\mathrm{e}^{\mathsf{F}} = 1$, or

$$\mathrm{e}^{\mathsf{F}} = \mathrm{e}^{f_0}\mathrm{e}^{F'} = \mathrm{e}^{f_0}\biggl(\cosh\|F'\| + \frac{F'}{\|F'\|}\sinh\|F'\|\biggr) = 1.$$

The solution of this MV equation (equivalently of four scalar equations) is $f_0 = 0$, and $\|F'\| = \sqrt{f_1^2 - f_2^2 + f_{12}^2} = 0$ for $Cl_{1,1}$ and $\|F'\| = \sqrt{f_1^2 + f_2^2 - f_{12}^2} = 0$ for $Cl_{2,0}$. From this we conclude that $\mathsf{F}' = 0$. Thus, in the case $B^2 > 0$ the principal logarithm is the only solution.

5.4 $Cl_{1,1}$ and $Cl_{2,0}$ algebras: Summary

Taking into account the generic and special cases, finally, we can write

$$Cl_{1,1}, Cl_{2,0}\colon\quad \log \mathsf{B} = \begin{cases} \log\|\mathsf{B}\| + (\arctan(b_0, \|B'\|) + 2\pi k)\frac{B'}{\|B'\|}, \\ \quad (B'^2 < 0), \\ \log\|\mathsf{B}\| + \operatorname{arctanh}(\frac{\|B'\|}{b_0})\frac{B'}{\|B'\|}, \\ \quad (B'^2 > 0)\wedge(b_0 > 0)\wedge(b_0^2 - B'^2 > 0), \\ \log(b_0) + \frac{1}{2}\log(0_+)(1 - \frac{B'}{\|B'\|}) + \frac{1}{2}\log(2)(1 + \frac{B'}{\|B'\|}), \\ \quad (B'^2 > 0)\wedge(b_0 > 0)\wedge(\|\mathsf{B}\| = 0), \\ \log(b_0) + \frac{B'}{b_0},\quad (B'^2 = 0)\wedge(b_0 > 0), \\ \log(-b_0) + (\pi + 2\pi k)\hat{F}',\quad (B' = 0)\wedge(b_0 \leqslant 0), \\ \varnothing,\quad \text{no solution}, \\ \quad (B'^2 > 0)\wedge((b_0 \leqslant 0)\vee(b_0^2 - B'^2 \leqslant 0)), \end{cases} \tag{25}$$

where $\|\mathsf{B}\| = \sqrt{\operatorname{abs}(b_0^2 - B'^2)}$ and $\|B'\| = \sqrt{\operatorname{abs}(B'^2)}$. Explicit form of a free MV $\hat{F}'$ is algebra dependent and is given in (20).

In conclusion, we shall remark that in $Cl_{0,2}$ algebra the GA logarithm is defined for all MVs. However, in the remaining 2D algebras, we have to satisfy the conditions for coefficients for a logarithm to exist. Thus, in these algebras, there are sectors in a domain of argument, where logarithm does not exist at all.

Example 9. $Cl_{2,0}$, line 2 in (25). Case $(B'^2 > 0) \wedge (b_0 > 0) \wedge (b_0^2 - B'^2 > 0)$, $\mathsf{B} = 2+B' = 2+5\mathbf{e}_1-\mathbf{e}_2-5\mathbf{e}_{12}$, $B'^2 = 1$, $\|\mathsf{B}\| = \sqrt{3}$, $\|B'\| = 1$, $\|\mathsf{B}\|^2 = b_0^2 - B'^2 = 3$. The answer:

$$\log \mathsf{B} = \frac{1}{2}\log 3 + \operatorname{arctanh}\left(\frac{1}{2}\right)(5\mathbf{e}_1 - 5\mathbf{e}_{12} - \mathbf{e}_2).$$

Example 10. $Cl_{2,0}$, line 3 in (25). Case $(B'^2 > 0) \wedge (b_0 > 0) \wedge (b_0^2 - B'^2 = 0)$, $\mathsf{B} = 9 - 9\mathbf{e}_1 + 8\mathbf{e}_2 + 8\mathbf{e}_{12}$, $B'^2 = 81$, $\|B'\| = 9$, $\|\mathsf{B}\|^2 = b_0^2 - B'^2 = 0$. The answer:

$$\log \mathsf{B} = \frac{1}{18}\log\left(\frac{2}{0_+}\right)(-9\mathbf{e}_1 + 8\mathbf{e}_2 + 8\mathbf{e}_{12}) + \frac{1}{2}\log(2 + 0_+) + \log(9),$$

where 0_+ is infinitesimally small positive number. $\lim_{0_+\to 0} \mathrm{e}^{\log \mathsf{B}} = \mathsf{B}$.

Example 11. $Cl_{2,0}$, line 4 in (25). Case $(\mathsf{B}'^2 = 0)\wedge(b_0 > 0)$, $\mathsf{B} = 2+3\mathbf{e}_1-4\mathbf{e}_2-5\mathbf{e}_{12} = 2 + B'$, $\|\mathsf{B}\| = 2$, $B'^2 = 0$, $\|B'\| = 0$. The answer:

$$\log \mathsf{B} = \log 2 + \frac{B'}{b_0} = \log 2 + \frac{1}{2}(3\mathbf{e}_1 - 4\mathbf{e}_2 - 5\mathbf{e}_{12}).$$

Example 12. $Cl_{2,0}$, line 5 in (25). Case $(\mathsf{B}' = 0) \wedge (b_0 \leqslant 0)$. The logarithm of MV $\mathsf{B} = -2$ is

$$\log \mathsf{B} = \log(2) + (\pi + 2\pi k)\bigl(f_1\mathbf{e}_1 - \mathbf{e}_{12}\sqrt{1 + f_1^2 + f_2^2} + f_2\mathbf{e}_2\bigr)$$

After exponentiation, $\hat{F}'$ simplifies out, and we get $\mathrm{e}^{\log \mathsf{B}} = -2$.

Example 13. $Cl_{1,1}$, line 6 in (25). $\mathsf{B} = 2+5\mathbf{e}_1-4\mathbf{e}_2-7\mathbf{e}_{12}$, $B'^2 = 58$, $b_0^2 - B'^2 = -54$. Logarithm does not exist since, under the condition $B'^2 > 0$, solution exists only when $b_0^2 - B'^2 > 0$ (case 2 of (25)). Here we have $b_0^2 - B'^2 < 0$, and therefore, by line 6 of (25) the solution set is empty.

The knowledge of logarithm and exponential provides a possibility to calculate the square root of a MV by formula $\sqrt{\mathsf{B}} = \pm\exp(\log(\mathsf{B})/2)$. For example, for $\mathsf{B} = 2 - \mathbf{e}_1 + 2\mathbf{e}_{12}$ in $Cl_{2,0}$, we have $B'^2 = -3$ and $\|\mathsf{B}\| = \sqrt{7}$. Then, using the formula (21), one finds

$$\log \mathsf{B} = \frac{1}{2}\log 7 - \frac{1}{\sqrt{3}}(\mathbf{e}_1 - 2\mathbf{e}_{12})\arctan\frac{1}{\sqrt{3}},$$

and after multiplication by $1/2$ and application of exponential, one obtains the root

$$\sqrt{\mathsf{B}} = \frac{2 + \sqrt{7} - \mathbf{e}_1 + 2\mathbf{e}_{12}}{\sqrt{2(2 + \sqrt{7})}}.$$

In this way calculated root coincides with a root formula (27) in the next section. It is easy to verify that the geometric product $\sqrt{\mathsf{B}}\sqrt{\mathsf{B}}$ simplifies to the initial B. The formula $\sqrt{\mathsf{B}} = \pm\exp(\log(\mathsf{B})/2)$ gives only two (plus/minus) roots. If B is a unity, $\mathsf{B} = 1$, then the square root exists for all algebras since $\sqrt{1} = \pm\exp(\log(1)/2) = \pm 1$. From works [1, 5] we know that in the Clifford number algebra, in general, the square root of MV is a multivalued function. Below in Section 6, such (isolated) roots are presented for $\sqrt{+1}$ and $\sqrt{-1}$. The multiple roots also have been found for a general MV in $Cl_{1,1}$ (see the next section), where up to four roots may happen simultaneously. In higher dimensional GAs the number of roots may be even larger [1].

6 Addendum: Formulas for square roots of MV

Below the square roots $\sqrt{\mathsf{B}}$ for 1D and 2D algebras that may be useful in practice are presented. They were calculated from the defining equation $\mathsf{B} = \mathsf{A}^2$, which is equivalent to a system of two in 2D or four 2D real coupled equations.

For $Cl_{0,1}$, there are two (plus and minus) roots of $\mathsf{B} = b_0 + \mathbf{e}_1 b_1$,

$$Cl_{0,1}:\quad \sqrt{\mathsf{B}} = \pm\frac{\sqrt{b_0^2 + b_1^3} + (b_0 + b_1\mathbf{e}_1)}{\sqrt{2}\sqrt{b_0 + \sqrt{b_0^2 + b_1^2}}} = \pm\frac{|\mathsf{B}| + \mathsf{B}}{\sqrt{2}\sqrt{\langle\mathsf{B}\rangle_0 + |\mathsf{B}|}}, \tag{26}$$

where $|\mathsf{B}| = \sqrt{\mathsf{B}\widehat{\mathsf{B}}}$ is the magnitude (norm), and $\langle\mathsf{B}\rangle_0 = b_0$ is the scalar part of MV. The roots (26) exists for all MVs $\mathsf{B} \neq 0$.

For $Cl_{1,0}$ algebra, in general, there are four roots of $\mathsf{B} = b_0 + \mathbf{e}_1 b_1$,

$$Cl_{1,0}:\quad \sqrt{\mathsf{B}} = \left\{\pm\frac{-\sqrt{b_0^2 - b_1^2} + b_0 + b_1\mathbf{e}_1}{\sqrt{2}\sqrt{b_0 - \sqrt{b_0^2 - b_1^2}}},\ \pm\frac{\sqrt{b_0^2 - b_1^2} + b_0 + b_1\mathbf{e}_1}{\sqrt{2}\sqrt{b_0 + \sqrt{b_0^2 - b_1^2}}}\right\}.$$

If $b_0 > b_1$, all roots are different. If $b_0 = b_1$, only two distinct roots remain.

In 2D algebras the MV is $\mathsf{B} = b_0 + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_{12}\mathbf{e}_{12}$. The square root has the same form for all algebras:

$$Cl_{0,2},\ Cl_{1,1},\ Cl_{2,0}:\quad \sqrt{\mathsf{B}} = \pm\frac{b_0 + \sqrt{\operatorname{Det}\mathsf{B}} + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_{12}\mathbf{e}_{12}}{\sqrt{2}\sqrt{b_0 + \sqrt{\operatorname{Det}\mathsf{B}}}}. \tag{27}$$

For individual algebras, the determinant is

$$\operatorname{Det}\mathsf{B} = \mathsf{B}\widetilde{\widehat{\mathsf{B}}} = \begin{cases} b_0^2 + b_1^2 + b_2^2 + b_{12}^2 & \text{for } Cl_{0,2},\\ b_0^2 - b_1^2 + b_2^2 - b_{12}^2 & \text{for } Cl_{1,1},\\ b_0^2 - b_1^2 - b_2^2 + b_{12}^2 & \text{for } Cl_{2,0}. \end{cases}$$

For Hamilton quaternion, the square root exists for all MVs. For $Cl_{1,1}$ and $Cl_{2,0}$ algebras, the roots exist when the determinant is positive or zero. In $Cl_{1,1}$ an additional plus/minus roots may appear if $\operatorname{Det}\mathsf{B} > b_0 > 0$ [7],

$$\sqrt{\mathsf{B}} = \pm\frac{-b_0 + \sqrt{\operatorname{Det}\mathsf{B}} - (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_{12}\mathbf{e}_{12})}{\sqrt{2}\sqrt{-b_0 + \sqrt{\operatorname{Det}\mathsf{B}}}}.$$

Square roots of $+1$ *in 2D*. The square roots of $\mathsf{B} = +1$ in 2D are

$$\sqrt{+1} = \begin{cases} Cl_{0,2}\colon & \{\pm 1\}, \\ Cl_{1,1}\colon & \{\pm 1, \mathbf{e}_1, \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm \mathbf{e}_{12})\} \\ & \text{and } \pm(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 \pm \sqrt{1 + c_1^2 + c_2^2}\,\mathbf{e}_{12}), \\ Cl_{2,0}\colon & \{\pm 1, \mathbf{e}_1, \mathbf{e}_2, \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm \mathbf{e}_2)\} \\ & \text{and } \pm(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 \pm \sqrt{-1 + c_1^2 + c_2^2}\,\mathbf{e}_{12}). \end{cases}$$

In case of $Cl_{2,0}$ and $Cl_{1,1}$ the coefficients c_1 and c_2 are arbitrary and may be considered as free parameters. Their range is determined by expression under the square root, which must be positive. Thus, in this case, we have two types of roots: the isolated roots and the continuum of roots determined by free parameters c_i.

Square roots of -1 *in 2D*.The square roots of $\mathsf{B} = -1$ in 2D are

$$\sqrt{-1} = \begin{cases} Cl_{0,2}\colon & \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}, \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm \mathbf{e}_2), \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm \mathbf{e}_{12}), \frac{1}{\sqrt{2}}(\mathbf{e}_2 \pm \mathbf{e}_{12})\} \\ & \text{and } \pm(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 \pm \sqrt{1 - c_1^2 - c_2^2}\,\mathbf{e}_{12}), \\ Cl_{1,1}\colon & \{\mathbf{e}_2\} \text{ and } \pm(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 \pm \sqrt{-1 - c_1^2 + c_2^2}\,\mathbf{e}_{12}), \\ Cl_{2,0}\colon & \{\mathbf{e}_{12}\} \text{ and } \pm(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 \pm \sqrt{1 + c_1^2 + c_2^2}\,\mathbf{e}_{12}). \end{cases}$$

7 Conclusions and discussion

The paper presents exponential and logarithm functions of multivector argument for $Cl_{0,1}$, $Cl_{1,0}$ (1-dimensional commutative) and $Cl_{0,2}$, $Cl_{1,1}$, $Cl_{2,0}$ (2-dimensional noncommutative) Clifford geometric algebras (GAs). The well-known approach to Hamilton quaternion identified by three imaginaries $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ was generalized and adapted, specifically, the imaginaries have been replaced by 2D unit multivectors, the squares of which are equal to ± 1, and which have been constructed from Clifford basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}$.

The 2D basis-free exponentials in this approach assume a form of either Euler's or de Moivre's rules. The principal logarithm was determined as an inverse of respective exponential. 2π-multiplicity in the logarithm was included by adding a free MV F that satisfies the condition $\mathrm{e}^{\mathsf{F}} = 1$. The obtained formulas for exponential and logarithm can be applied to quaternions too since $Cl_{0,2}$, $Cl_{1,1}$, and $Cl_{2,0}$ algebras are isomorphic to, correspondingly, the Hamilton quaternion, coquaternion, and conectorine [19],

Since in GA the nth root of MV is $(\mathsf{A})^{1/n} = \exp(\log(\mathsf{A})/n)$, the obtained exponential and logarithm formulas may be applied to extract the nth root from general MV. However, this exp-log formula allows to calculate no more then two (plus/minus) square roots. Workable formulas for square roots are presented. In particular, using the defining equation, we have found explicit formulas for square roots in 1D and 2D Clifford algebras and the sectors of their existence in MV coefficient space. In the space of MV coefficients the sectors, where the roots do not exist, the logarithm does not exist as well. Also, multiple square roots (up to four) have been found in $Cl_{1,0}$ and $Cl_{1,1}$ algebras.

The presented results may be useful in applied GAs, especially in dealing with GA differential equations, the solutions of which are expressed through GA exponentials [3, 6]. Finally, the low-dimensional Clifford algebras may be helpful in doing calculations in higher dimensional algebras as well because the former are subalgebras of the latter. Also, it is expected that the described in the paper approach may be adapted to higher grade Clifford algebras.

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