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Comparative exploration on bifurcation behavior for integer-order and fractional-order delayed BAM neural networks*

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Abstract. In the present study, we deal with the stability and the onset of Hopf bifurcation of two type delayed BAM neural networks (integer-order case and fractional-order case). By virtue of the characteristic equation of the integer-order delayed BAM neural networks and regarding time delay

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as critical parameter, a novel delay-independent condition ensuring the stability and the onset of Hopf bifurcation for the involved integer-order delayed BAM neural networks is built. Taking advantage of Laplace transform, stability theory and Hopf bifurcation knowledge of fractional-order differential equations, a novel delay-independent criterion to maintain the stability and the appearance of Hopf bifurcation for the addressed fractional-order BAM neural networks is established. The investigation indicates the important role of time delay in controlling the stability and Hopf bifurcation of the both type delayed BAM neural networks. By adjusting the value of time delay, we can effectively amplify the stability region and postpone the time of onset of Hopf bifurcation for the fractional-order BAM neural networks. Matlab simulation results are clearly presented to sustain the correctness of analytical results. The derived fruits of this study provide an important theoretical basis in regulating networks.

Keywords: fractional-order BAM neural networks, integer-order delayed BAM neural networks, Hopf bifurcation, stability, bifurcation diagram.

1 Introduction

It is well known that neural networks have been applied in various areas such as image processing, optimization, artificial intelligence, computer version, automatic control and so on [1, 17]. Thus the study on various dynamical behaviors of neural networks is an interesting and important topic in today's world. During the past decades, a great deal of the study achievements on various dynamics (including periodic solution, almost periodic solution, S^p-almost periodic solution, pseudo almost periodic solution, piecewise pseudo almost periodic solution, weight pseudo almost periodic solution, pseudo almost automorphic solution, piecewise asymptotically almost automorphic solution, square-mean almost autorphic solution, Stepanov-like weighted pseudo almost automorphic solution, weight pseudo almost automorphic solutions, anti-periodic solution, weighted pseudo anti-periodic solution, bifurcation, dissipativity, synchronization, global Mittag-Leffler stability, fixed-time stabilization, etc.) have been reported. For example, Abdelaziz and Chérif [1] discussed the piecewise asymptotic almost periodic solutions of fuzzy Cohen–Grossberg neural networks. Aouiti et al. [2] detailedly analyzed the piecewise pseudo almost periodic solution to delayed neutral-type neural networks. Bohner et al. [5] investigated the almost periodic solutions of delayed Cohen–Grossberg neural networks. Huang et al. [12] studied the anti-periodic solutions for cellular neural networks with proportional delay. Zhao et al. [31] established the sufficient condition to ensure the existence, uniqueness and global exponential stability of weighted pseudo almost automorphic solutions for Hopfield neural networks with delays. Dhama1 and Abbas [7] made a systematic analysis on the existence and stability of weighted pseudo almost automorphic solution of dynamic equation. Zhou and Zhao [32] studied the synchronization issue of a class of neural networks with proportional delays. For more detailed contents on these aspects, we refer the readers to [18–20].

However, the involved works above have been restricted to the integer-order delayed differential models. Nowadays many scholars find that fractional calculus has great application prospect in numerous fields such as electromagnetic waves, physics, viscoelasticity, biology, mechanics, neural networks, control science and so on [4, 21]. Lots of researchers hold that fractional calculus can be regarded as a resultful tool to describe the actual problems of natural world since it possesses memory and hereditary properties during the process of development and change of things [10, 14]. Recently, fractional calculus has attracted more and more attention from many scholars. Hopf bifurcation is a vital dynamical property of delayed differential systems. For a long time, a large number of research findings about Hopf bifurcation of integer-order delayed differential models have been published. However, the study on Hopf bifurcation of fractional-order differential systems is very rare. At present, there are some literatures that deal with the Hopf bifurcation of fractional-order differential systems. For instance, Xu et al. [26] considered the effect of multiple time delays on bifurcation for fractional-order neural networks. Huang et al. [16] investigated the stability and Hopf bifurcation for fractional neural networks. Xiao et al. [24] dealt with the PD control technique of Hopf bifurcations for delayed fractional-order small-world networks. In details, one can see [8, 11, 22, 23, 25, 27–30].

Here we would like to point out that few works are concerned with the comparison of Hopf bifurcation for integer-order and fractional-order cases. Time delay plays a vital role in stabilizing system and changing the dynamical behavior of integer-order and fractional-order systems. Fractional-order systems have one more parameter (fractional-order), thus the impact of time delay on stability and Hopf bifurcation for integer-order and fractional-order systems will display different style. How does the effect of time delay on the stability and Hopf bifurcation of integer-order and fractional-order differential systems? Motivated by this idea, in this work, we will focus on the comparative study on bifurcation behavior for integer-order and fractional-order delayed BAM neural networks. In particular, in this work, we will handle the following problems (e.g., the main contribution of this paper): (i) reveal the effect of time delay on stability and Hopf bifurcation of integer-order and fractional-order delayed BAM neural networks; (ii) comparison between the bifurcation point of integer-order delayed BAM neural networks and the bifurcation point of fractional-order delayed BAM neural networks is given.

In this article, we consider the following integer-order BAM neural networks with delay:

$$\begin{aligned}
\dot{w}_1(t) &= -\gamma_1 w_1(t) + h_1\big(w_1(t)\big) + k_1\big(w_4(t-\eta)\big) + k_1\big(w_2(t-\eta)\big),\\
\dot{w}_2(t) &= -\gamma_2 w_2(t) + h_2\big(w_2(t)\big) + k_2\big(w_1(t-\eta)\big) + k_2\big(w_3(t-\eta)\big),\\
\dot{w}_3(t) &= -\gamma_3 w_3(t) + h_3\big(w_3(t)\big) + k_3\big(w_2(t-\eta)\big) + k_3\big(w_4(t-\eta)\big),\\
\dot{w}_4(t) &= -\gamma_4 w_4(t) + h_4\big(w_4(t)\big) + k_4\big(w_3(t-\eta)\big) + k_4\big(w_1(t-\eta)\big)
\end{aligned} \tag{1}$$

and the corresponding fractional-order BAM neural networks with delay:

$$\begin{aligned}
\frac{\mathrm{d}^\sigma w_1(t)}{\mathrm{d}t^\sigma} &= -\gamma_1 w_1(t) + h_1\big(w_1(t)\big) + k_1\big(w_4(t-\eta)\big) + k_1\big(w_2(t-\eta)\big),\\
\frac{\mathrm{d}^\sigma w_2(t)}{\mathrm{d}t^\sigma} &= -\gamma_2 w_2(t) + h_2\big(w_2(t)\big) + k_2\big(w_1(t-\eta)\big) + k_2\big(w_3(t-\eta)\big),
\end{aligned} \tag{2_1}$$

$$\begin{aligned}\frac{\mathrm{d}^\sigma w_3(t)}{\mathrm{d}t^\sigma} &= -\gamma_3 w_3(t) + h_3\big(w_3(t)\big) + k_3\big(w_2(t-\eta)\big) + k_3\big(w_4(t-\eta)\big),\\ \frac{\mathrm{d}^\sigma w_4(t)}{\mathrm{d}t^\sigma} &= -\gamma_4 w_4(t) + h_4\big(w_4(t)\big) + k_4\big(w_3(t-\eta)\big) + k_4\big(w_1(t-\eta)\big),\end{aligned} \tag{2_2}$$

where γ_i ($i = 1, 2, 3, 4$) stands for the internal decay rate, h_i ($i = 1, 2, 3, 4$) stands for the nonlinear feedback function, k_i ($i = 1, 2, 3, 4$) stands for the connection function between two neurons, $\eta \geqslant 0$ stands for the connection delay, $\sigma \in (0, 1]$ is a constant. For more concrete information about system (1), one can see [9].

In order to establish the key conclusions of this paper, we firstly make the following hypothesis:

($\mathcal{Q}_1$) $h_l, k_l \in C^1,\ h_l(0) = 0,\ k_l(0) = 0$ ($l = 1, 2, 3, 4$).

The article is structured as follows. Section 2 presents some elementary knowledge on fractional calculus and integer-order differential equations. Section 3 is concerned with the discussion on the stability and the emergence of Hopf bifurcation of the integer-order delayed BAM neural networks (1). Section 4 deals with the analysis on the stability and the emergence of Hopf bifurcation of the fractional-order delayed BAM neural networks (2). Section 5 carries out computer simulations and bifurcation diagrams to support the rationality of the derived key conclusions. Section 6 finishes our article.

2 Preliminaries

In the part, we list some necessary knowledge on fractional-order dynamical system and integer-order dynamical systems, which will be applied in the following proof.

Lemma 1. (*See* [16].) *Consider the following exponential polynomial*

$$\begin{aligned}&\mathcal{Q}\big(\lambda, \mathrm{e}^{-\lambda\eta_1}, \dots, \mathrm{e}^{-\lambda\eta_m}\big)\\ &\quad= \lambda^n + q_1^{(0)}\lambda^{n-1} + \cdots + q_{n-1}^{(0)}\lambda + q_n^{(0)} + \big[q_1^{(1)}\lambda^{n-1} + \cdots + q_{n-1}^{(1)}\lambda + q_n^{(1)}\big]\mathrm{e}^{-\lambda\eta_1}\\ &\qquad+ \cdots + \big[q_1^{(m)}\lambda^{n-1} + \cdots + q_{n-1}^{(m)}\lambda + q_n^{(m)}\big]\mathrm{e}^{-\lambda\eta_m},\end{aligned}$$

where $\eta_j \geqslant 0$ ($j = 0, 1, 2, \dots, m$) *and* $q_k^{(j)}$ ($j = 0, 1, 2, \dots, m$; $k = 1, 2, \dots, n$) *are constants. If* $(\eta_1, \eta_2, \dots, \eta_m)$ *change, the sum of the orders of* $\mathcal{Q}(\lambda, \mathrm{e}^{-\lambda\eta_1}, \dots, \mathrm{e}^{-\lambda\eta_m})$ *on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

Definition 1. (See [15].) Define Caputo fractional-order derivative as follows:

$$\mathcal{D}^\sigma l(\upsilon) = \frac{1}{\Gamma(n-\sigma)}\int_{\upsilon_0}^{\upsilon} \frac{l^{(n)}(s)}{(\upsilon - s)^{\sigma-n+1}}\,\mathrm{d}s,$$

where $l(\upsilon) \in ([\upsilon_0, \infty), \mathbb{R})$, $\Gamma(s) = \int_0^\infty \upsilon^{s-1}\mathrm{e}^{-\upsilon}\,\mathrm{d}\upsilon$, $\upsilon \geqslant \upsilon_0$, and $n \in \mathbb{Z}^+$, $n - 1 \leqslant \sigma < n$.

The Laplace transform of Caputo fractional-order derivative is given by

$$\mathcal{L}\bigl\{\mathcal{D}^{\sigma}g(t);s\bigr\} = s^{\sigma}\mathcal{G}(s) - \sum_{j=0}^{m-1} s^{\sigma-j-1}g^{(j)}(0), \quad m-1 \leqslant \sigma < m \in \mathbb{Z}^{+},$$

where $\mathcal{G}(s) = \mathcal{L}\{g(t)\}$. Especially, if $g^{(j)}(0) = 0$, $j = 1, 2, \dots, m$, then $\mathcal{L}\{\mathcal{D}^{\sigma}g(t);s\} = s^{\sigma}\mathcal{G}(s)$.

Definition 2. (See [3].) $(w_{1*}, w_{2*}, w_{3*}, w_{4*})$ is called an equilibrium point of system (1) (or system (2)), provided that the equations

$$\begin{aligned}
-\gamma_1 w_{1*} + h_1(w_{1*}) + k_1(w_{4*}) + k_1(w_{2*}) &= 0,\\
-\gamma_2 w_{2*} + h_2(w_{2*}) + k_2(w_{1*}) + k_2(w_{3*}) &= 0,\\
-\gamma_3 w_{3*} + h_3(w_{3*}) + k_3(w_{2*}) + k_3(w_{4*}) &= 0,\\
-\gamma_4 w_{4*} + h_4(w_{4*}) + k_4(w_{3*}) + k_4(w_{1*}) &= 0
\end{aligned}$$

are fulfilled.

Lemma 2. (*See* [13].) *Suppose given the fractional-order system*

$$\frac{\mathrm{d}^{\sigma}w(t)}{\mathrm{d}t^{\sigma}} = g\bigl(t, w(t)\bigr), \qquad w(0) = w_0, \tag{3}$$

where $\sigma \in (0,1]$ *and* $g(t, w(t)) : \mathbb{R}^{+} \times \mathbb{R}^{m} \to \mathbb{R}^{m}$. *The equilibrium point of system* (3) *are locally asymptotically stable if all eigenvalues* δ *of the Jacobian matrix* $\partial g(t,w)/\partial w$ *evaluated near the equilibrium point satisfy* $|\arg(\delta)| > \delta\pi/2$.

Lemma 3. (*See* [27].) *Consider the following fractional-order system*:

$$\begin{aligned}
\frac{\mathrm{d}^{\sigma_1}\mathcal{W}_1(t)}{\mathrm{d}t^{\sigma_1}} &= d_{11}\mathcal{W}_1(t-\varrho_{11}) + d_{12}\mathcal{W}_2(t-\varrho_{12}) + \dots + d_{1m}\mathcal{W}_m(t-\varrho_{1m}),\\
\frac{\mathrm{d}^{\sigma_2}\mathcal{W}_2(t)}{\mathrm{d}t^{\sigma_2}} &= d_{21}\mathcal{W}_1(t-\varrho_{21}) + d_{22}\mathcal{W}_2(t-\varrho_{22}) + \dots + d_{2m}\mathcal{W}_m(t-\varrho_{2m}),\\
&\ \ \vdots\\
\frac{\mathrm{d}^{\sigma_m}\mathcal{W}_m(t)}{\mathrm{d}t^{\sigma_m}} &= d_{m1}\mathcal{W}_1(t-\varrho_{m1}) + d_{m2}\mathcal{W}_2(t-\varrho_{m2}) + \dots + d_{mm}\mathcal{W}_m(t-\varrho_{mm}),
\end{aligned} \tag{4}$$

where $0 < \sigma_i < 1$ $(i = 1, 2, \dots, m)$, *the initial values* $\mathcal{W}_i(t) = \psi_i(t) \in C[-\max_{i,l}\varrho_{il}, 0]$, $t \in [-\max_{i,l}\varrho_{il}, 0]$, $i, l = 1, 2, \dots, m$. *Denote*

$$\varDelta(\zeta) = \begin{bmatrix}
\zeta^{\sigma_1} - d_{11}\mathrm{e}^{-\zeta\varrho_{11}} & -d_{12}\mathrm{e}^{-\zeta\varrho_{12}} & \dots & -d_{1m}\mathrm{e}^{-\zeta\varrho_{1m}}\\
-d_{21}\mathrm{e}^{-\zeta\varrho_{12}} & \zeta^{\sigma_2} - d_{22}\mathrm{e}^{-\zeta\varrho_{22}} & \dots & -d_{2m}\mathrm{e}^{-\zeta\varrho_{2m}}\\
\vdots & \vdots & \ddots & \vdots\\
-d_{m1}\mathrm{e}^{-\zeta\varrho_{m1}} & -d_{m2}\mathrm{e}^{-\zeta\varrho_{m2}} & \dots & \zeta^{\sigma_m} - d_{mm}\mathrm{e}^{-\zeta\varrho_{mm}}
\end{bmatrix}.$$

Then the zero solution of Eq. (4) *is Lyapunov asymptotically stable provided that every root of* $\det(\varDelta(\zeta)) = 0$ *possesses negative real parts.*

3 Hopf bifurcation exploration of system (1)

[Hopf bifurcation exploration of system (1)] In this part, we are to explore the stability and the onset of Hopf bifurcation for system (1). In term of $(\mathcal{Q}_1)$, one knows that Eq. (1) has a unique equilibrium $E(0,0,0,0)$. The linear system of Eq. (1) near $E(0,0,0,0)$ takes the form

$$\begin{aligned}
\dot{w}_1(t) &= -\alpha_1 w_1(t) + k_1'(0) w_4(t-\eta) + k_1'(0) w_2(t-\eta),\\
\dot{w}_2(t) &= -\alpha_2 w_2(t) + k_2'(0) w_1(t-\eta) + k_2'(0) w_3(t-\eta),\\
\dot{w}_3(t) &= -\alpha_3 w_3(t) + k_3'(0) w_2(t-\eta) + k_3'(0) w_4(t-\eta),\\
\dot{w}_4(t) &= -\alpha_4 w_4(t) + k_4'(0) w_3(t-\eta) + k_4'(0) w_1(t-\eta),
\end{aligned} \tag{5}$$

where $\alpha_j = \gamma_j - h_j'(0)$ $(j = 1,2,3,4)$. Then the associated characteristic equation of (5) takes the form

$$\det \begin{bmatrix}
\lambda+\alpha_1 & -k_1'(0)\mathrm{e}^{-\lambda\eta} & 0 & -k_1'(0)\mathrm{e}^{-\lambda\eta}\\
-k_2'(0)\mathrm{e}^{-\lambda\eta} & \lambda+\alpha_2 & -k_2'(0)\mathrm{e}^{-\lambda\eta} & 0\\
0 & -k_3'(0)\mathrm{e}^{-\lambda\eta} & \lambda+\alpha_3 & -k_3'(0)\mathrm{e}^{-\lambda\eta}\\
-k_4'(0)\mathrm{e}^{-\lambda\eta} & 0 & -k_4'(0)\mathrm{e}^{-\lambda\eta} & \lambda+\alpha_4
\end{bmatrix} = 0. \tag{6}$$

It follows from (6) that

$$\mathcal{A}_1(s) + \mathcal{A}_2(s)\mathrm{e}^{-2\lambda\eta} + = 0. \tag{7}$$

Here $\mathcal{A}_1(s) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, $\mathcal{A}_2(s) = b_2\lambda^2 + b_1\lambda + b_0$, where

$$\begin{aligned}
a_0 &= \alpha_1\alpha_2\alpha_3\alpha_4,\\
a_1 &= \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4,\\
a_2 &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4,\\
a_3 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\\
b_0 &= \alpha_2\alpha_3 k_1'(0)k_4'(0) - \alpha_1\alpha_4 k_2'(0)k_3'(0)\\
&\quad - \alpha_1\alpha_2 k_3'(0)k_4'(0) - \alpha_3\alpha_4 k_1'(0)k_2'(0),\\
b_1 &= (\alpha_2+\alpha_3)k_1'(0)k_4'(0) - (\alpha_1+\alpha_4)k_2'(0)k_3'(0)\\
&\quad - (\alpha_1+\alpha_2)k_3'(0)k_4'(0) - (\alpha_3+\alpha_4)k_1'(0)k_2'(0),\\
b_2 &= k_1'(0)k_4'(0) - k_2'(0)k_3'(0) - k_3'(0)k_4'(0) - k_1'(0)k_2'(0).
\end{aligned} \tag{8}$$

Now we will discuss the distribution of the roots of Eq. (7). Assume that $\mathrm{i}\psi$ is the root of Eq. (7), then one has

$$\begin{aligned}
&\psi^4 - \mathrm{i}a_3\psi^3 - a_2\psi^2 + \mathrm{i}a_1\psi + a_0\\
&\quad + \left(-b_2\psi^2 + \mathrm{i}b_1\psi + b_0\right)(\cos 2\psi\eta - \mathrm{i}\sin 2\psi\eta) = 0.
\end{aligned} \tag{9}$$

It follows from (9) that

$$\begin{aligned}
(b_0 - b_2\psi^2)\cos 2\psi\eta + b_1\psi\sin 2\psi\eta &= a_2\psi^2 - \psi^4 - a_0,\\
b_1\psi\cos 2\psi\eta - (b_0 - b_2\psi^2)\sin 2\psi\eta &= a_3\psi^3 - a_1\psi.
\end{aligned} \tag{10}$$

By (10) one gets

$$\left(b_0 - b_2\psi^2\right)^2 + (b_1\psi)^2 = \left(a_2\psi^2 - \psi^4 - a_0\right)^2 + \left(a_3\psi^3 - a_1\psi\right)^2,$$

which leads to

$$\psi^8 + r_3\psi^6 + r_2\psi^4 + r_1\psi^2 + r_0 = 0, \tag{11}$$

where

$$\begin{aligned} r_0 &= b_0^2 - a_0^2, \qquad r_1 = 2a_0a_2 - 2b_0b_2 - a_1^2 + b_1^2, \\ r_2 &= b_2^2 - a_2^2 - 2a_0 + 2a_1a_3, \qquad r_3 = 2a_2 - a_3^2. \end{aligned} \tag{12}$$

Let $y = \psi^2$, then Eq. (11) can be rewritten as

$$y^4 + r_3y^3 + r_2y^2 + r_1y + r_0 = 0. \tag{13}$$

Let

$$f(y) = y^4 + r_3y^3 + r_2y^2 + r_1y + r_0.$$

Then

$$\frac{\mathrm{d}f(y)}{\mathrm{d}y} = 4y^3 + 3r_3y^2 + 2r_2y + r_1.$$

Denote

$$4y^3 + 3r_3y^2 + 2r_2y + r_1 = 0. \tag{14}$$

and let $x = y + r_3/4$, then Eq. (14) takes the form

$$x^3 + \vartheta_1x + \vartheta_0 = 0,$$

where

$$\vartheta_1 = \frac{r_2}{2} - \frac{3}{16}r_3^2, \qquad \vartheta_0 = \frac{r_3^3}{32} - \frac{r_2r_3}{8} + \frac{r_1}{4}.$$

Define

$$\begin{aligned} \mathcal{C} &= \left(\frac{\vartheta_0}{2}\right)^2 + \left(\frac{\vartheta_1}{3}\right)^3, \qquad \epsilon = \frac{-1 + \mathrm{i}\sqrt{3}}{2}, \\ x_1 &= \sqrt[3]{-\frac{\vartheta_0}{2} + \sqrt{\mathcal{C}}} + \sqrt[3]{-\frac{\vartheta_0}{2} - \sqrt{\mathcal{C}}}, \\ x_2 &= \sqrt[3]{-\frac{\vartheta_0}{2} + \sqrt{\mathcal{C}}\epsilon} + \sqrt[3]{-\frac{\vartheta_0}{2} - \sqrt{\mathcal{C}}\epsilon^2}, \\ x_3 &= \sqrt[3]{-\frac{\vartheta_0}{2} + \sqrt{\mathcal{C}}\epsilon^2} + \sqrt[3]{-\frac{\vartheta_0}{2} - \sqrt{\mathcal{C}}\epsilon}, \\ y_j &= x_j - \frac{r_3}{4}, \quad j = 1, 2, 3. \end{aligned}$$

According to [6], we get the following conclusions.

Lemma 4. *Equation* (13) *possesses at least one positive root if* $r_0 < 0$*, where* r_0 *is defined by* (12).

Lemma 5. *Let the condition* $r_0 \geqslant 0$ *holds.*

(i) *If* $\mathcal{C} \geqslant 0$*, then Eq.* (14) *possesses positive roots* $\Leftrightarrow y_1 > 0$ *and* $f(y_1) < 0$.

(ii) *If* $\mathcal{C} < 0$*, then Eq.* (14) *possesses positive roots* $\Leftrightarrow$ *there exists at least one* $y_0 \in \{y_1, y_2, y_3\}$ *such that* $y_0 > 0$ *and* $f(y_0) \leqslant 0$.

Assume that Eq. (13) possesses positive roots. Here we suppose that Eq. (13) possesses four positive roots, say y_j^* ($j = 1, 2, 3, 4$). Then Eq. (11) possesses the following four positive roots:

$$\psi_1 = \sqrt{y_1^*}, \qquad \psi_2 = \sqrt{y_2^*}, \qquad \psi_3 = \sqrt{y_3^*}, \qquad \psi_4 = \sqrt{y_4^*}.$$

By (10) one has

$$\eta_j^l = \frac{1}{\psi_j}\left[\arccos\frac{(a_2\psi_j^2 - \psi_j^4 - a_0)(b_0 - b_2\psi_j^2) + (a_3\psi_j^3 - a_1\psi_j)b_1\psi_j}{(b_0 - b_2\psi_j^2)^2 + (b_1\psi_j)^2} + 2l\pi\right],$$

where $j = 1, 2, 3, 4$; $l = 0, 1, 2, \dots$. Define

$$\eta_0 = \eta_{j0}^{(0)} = \min_{1\leqslant j\leqslant 4}\{\eta_j^{(0)}\}, \qquad \psi_0 = \psi_{j0}.$$

Now we make the following assumption:

($\mathcal{Q}_2$) $a_3 > 0$, $a_3(a_2 + b_2) > a_1 + b_1$, $a_0 + b_0 > 0$, $a_3(a_2 + b_2)(a_1 + b_1) > (a_1 + b_1)^2 + a_3^2(a_0 + b_0)$.

Lemma 6. *Assume that condition* ($\mathcal{Q}_2$) *holds.*

(i) *Let one of the three conditions is satisfied*: (a) $r_0 < 0$; (b) $r_0 \geqslant 0$, $\mathcal{C} \geqslant 0$, $y_1 > 0$ *and* $f(y_1) \leqslant 0$; (c) $r_0 \geqslant 0$, $\mathcal{C} < 0$, *and let there exists* $y^* \in \{y_1, y_2, y_3\}$ *such that* $y^* > 0$ *and* $f(y^*) \leqslant 0$. *Then all roots of Eq.* (7) *possesses negative real parts for* $\eta \in [0, \eta_0)$.

(ii) *If conditions* (a)–(c) *of* (i) *do not hold, then all roots of Eq.* (7) *possesses negative real parts for* $\eta \geqslant 0$.

The conclusions of Lemma 6 come from Lemmas 4, 5 and 1. The proof of Lemma 6 is similar to the proof of Lemma 2.4 in Hu and Huang [9]. Here we omit it.

Next, we check the transversality condition for the onset of Hopf bifurcation. The following assumption is made as follows:

($\mathcal{Q}_3$) $\mathcal{L}_{\mathbb{R}}(\psi_0)\mathcal{O}_{\mathbb{R}}(\psi_0) + \mathcal{L}_I(\psi_0)\mathcal{O}_I(\psi_0) \neq 0$,

$$\begin{aligned}
\mathcal{L}_{\mathbb{R}}(\psi_0) &= a_0 - 3a_3\psi_0^2 + b_1\cos 2\psi_0\eta_0 + 2b_2\psi_0\sin 2\psi_0\eta_0,\\
\mathcal{L}_I(\psi_0) &= 2a_2\psi_0 - 4\psi_0^3 + 2b_2\cos 2\psi_0\eta_0 - b_1\sin 2\psi_0\eta_0,\\
\mathcal{O}_{\mathbb{R}}(\psi_0) &= 2\psi_0\big(b_0 - b_2\psi_0^2\big)\sin 2\psi_0\eta_0 - 2b_1\psi_0^2\cos 2\psi_0\eta_0,\\
\mathcal{O}_I(\psi_0) &= 2\psi_0\big(b_0 - b_2\psi_0^2\big)\cos 2\psi_0\eta_0 + 2b_1\psi_0^2\sin 2\psi_0\eta_0.
\end{aligned}$$

Lemma 7. *Assume that* $s(\eta) = \alpha(\eta) + \mathrm{i}\beta(\eta)$ *is the root of* (7) *at* $\eta = \eta_0$ *and* $\alpha(\eta_0) = 0$, $\beta(\eta_0) = \psi_0$, *then* $\mathrm{Re}\,\mathrm{d}s\mathrm{d}\eta|_{\eta=\eta_0,\,\psi=\psi_0} \neq 0$.

Proof. According to (7), one gets

$$\begin{aligned}&\big(4\lambda^3+3a_3\lambda^2+2a_2\lambda+a_1\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\eta}+(2b_2\lambda+b_1)\mathrm{e}^{-2\lambda\eta}\\&\quad-\mathrm{e}^{-2\lambda\eta}\bigg(\frac{\mathrm{d}\lambda}{\mathrm{d}\eta}\eta+\lambda\bigg)\big(b_2\lambda^2+b_1\lambda+b_0\big)=0.\end{aligned}$$

Then

$$\bigg[\frac{\mathrm{d}\lambda}{\mathrm{d}\eta}\bigg]^{-1}=\frac{\mathcal{L}(\lambda)}{\mathcal{O}(\lambda)}-\frac{\eta}{\lambda},$$

where

$$\begin{aligned}\mathcal{L}(\lambda)&=\big(4\lambda^3+3a_3\lambda^2+2a_2\lambda+a_1\big)+(2b_2\lambda+b_1)\mathrm{e}^{-2\lambda\eta},\\\mathcal{O}(\lambda)&=2\lambda\mathrm{e}^{-2\lambda\eta}\big(b_2\lambda^2+b_1\lambda+b_0\big).\end{aligned}$$

By $(\mathcal{Q}_3)$ one has

$$\mathrm{Re}\bigg\{\bigg[\frac{\mathrm{d}s}{\mathrm{d}\eta}\bigg]^{-1}\bigg\}\bigg|_{\eta=\eta_0,\,\psi=\psi_0}=\frac{\mathcal{L}_{\mathbb{R}}(\psi_0)\mathcal{O}_{\mathbb{R}}(\psi_0)+\mathcal{L}_I(\psi_0)\mathcal{O}_I(\psi_0)}{(\mathcal{O}_{\mathbb{R}}(\psi_0))^2+(\mathcal{O}_I(\psi_0))^2}\neq 0.\qquad\square$$

Based on Lemmas 6 and 7, the following result can be established.

Theorem 1. *For system* (1)*, assume that conditions* $(\mathcal{Q}_1)$ *and* $(\mathcal{Q}_2)$ *hold. Then the following conclusions hold true*:

- (i) *Let one of the three conditions*: (a) $r_0<0$; (b) $r_0\geqslant 0$*,* $\mathcal{C}\geqslant 0$*,* $y_1>0$ *and* $f(y_1)\leqslant 0$; (c) $r_0\geqslant 0,\mathcal{C}<0$*, and let there exists* $y^*\in\{y_1,y_2,y_3\}$ *such that* $y^*>0$ *and* $f(y^*)\leqslant 0$ *do not hold. Then the zero equilibrium point* $E(0,0,0,0)$ *is asymptotically stable for all* $\eta\geqslant 0$*;*
- (ii) *If one of the conditions* (a), (b) *and* (c) *is fulfilled, then the zero equilibrium point* $E(0,0,0,0)$ *is asymptotically stable for* $\eta\in[0,\eta_0)$.
- (iii) *If conditions* $(\mathcal{Q}_3)$ *and* (ii) *hold, then a Hopf bifurcation will happen around the zero equilibrium point* $E(0,0,0,0)$.

4 Hopf bifurcation exploration of system (2)

[Hopf bifurcation exploration of system (2)] In this section, we are to analyze the stability and the existence of Hopf bifurcation of model (2). In term of $(\mathcal{Q}_1)$, it is easy to see that Eq. (2) has a unique equilibrium $E(0,0,0,0)$. The linear system of Eq. (2) around $E(0,0,0,0)$ is given by

$$\begin{aligned}\frac{\mathrm{d}^\sigma w_1(t)}{\mathrm{d}t^\sigma}&=-\alpha_1w_1(t)+k_1'(0)w_4(t-\eta)+k_1'(0)w_2(t-\eta),\\\frac{\mathrm{d}^\sigma w_2(t)}{\mathrm{d}t^\sigma}&=-\alpha_2w_2(t)+k_2'(0)w_1(t-\eta)+k_2'(0)w_3(t-\eta),\\\frac{\mathrm{d}^\sigma w_3(t)}{\mathrm{d}t^\sigma}&=-\alpha_3w_3(t)+k_3'(0)w_2(t-\eta)+k_3'(0)w_4(t-\eta),\\\frac{\mathrm{d}^\sigma w_4(t)}{\mathrm{d}t^\sigma}&=-\alpha_4w_4(t)+k_4'(0)w_3(t-\eta)+k_4'(0)w_1(t-\eta),\end{aligned}\tag{15}$$

where $\alpha_j = \gamma_j - h'_j(0)$ $(j = 1, 2, 3, 4)$. The associated characteristic equation of (15) takes the form

$$\det \begin{bmatrix} s^\sigma + \alpha_1 & -k'_1(0)\mathrm{e}^{-s\eta} & 0 & -k'_1(0)\mathrm{e}^{-s\eta} \\ -k'_2(0)\mathrm{e}^{-s\eta} & s^\sigma + \alpha_2 & -k'_2(0)\mathrm{e}^{-s\eta} & 0 \\ 0 & -k'_3(0)\mathrm{e}^{-s\eta} & s^\sigma + \alpha_3 & -k'_3(0)\mathrm{e}^{-s\eta} \\ -k'_4(0)\mathrm{e}^{-s\eta} & 0 & -k'_4(0)\mathrm{e}^{-s\eta} & s^\sigma + \alpha_4 \end{bmatrix} = 0. \tag{16}$$

It follows from (16) that

$$\mathcal{B}_1(s) + \mathcal{B}_2(s)\mathrm{e}^{-2s\eta} = 0. \tag{17}$$

Here

$$\mathcal{B}_1(s) = s^{4\sigma} + a_3 s^{3\sigma} + a_2 s^{2\sigma} + a_1 s^\sigma + a_0, \qquad \mathcal{B}_2(s) = b_2 s^{2\sigma} + b_1 s^\sigma + b_0,$$

where a_i $(i = 0, 1, 2, 3)$ and b_j $(j = 0, 1)$ are defined by (8).

Suppose that $s = \mathrm{i}\varrho = \varrho(\cos(\pi/2) + \mathrm{i}\sin(\pi/2))$ is a root of (17) and $\mathcal{B}_{iR}(s)$ and $\mathcal{B}_{iI}(s)$ $(i = 1, 2)$ stand for the real parts and imaginary parts of $\mathcal{B}_i(s)$ $(i = 1, 2)$, respectively. Then we have

$$\begin{aligned} \mathcal{B}_{2\mathbb{R}}(\varrho)\cos 2\varrho\eta + \mathcal{B}_{2I}(\varrho)\sin 2\varrho\eta &= -\mathcal{B}_{1\mathbb{R}}(\varrho), \\ \mathcal{B}_{2I}(\varrho)\cos 2\varrho\eta - \mathcal{B}_{2\mathbb{R}}(\varrho)\sin 2\varrho\eta &= -\mathcal{B}_{1I}(\varrho), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mathcal{B}_{1\mathbb{R}}(\varrho) &= \varrho^{4\sigma}\cos 2\sigma\pi + a_3\varrho^{3\sigma}\cos\frac{3\sigma\pi}{2} + a_2\varrho^{2\sigma}\cos\sigma\pi + a_1\varrho^\sigma\cos\frac{\sigma\pi}{2} + a_0, \\ \mathcal{B}_{1I}(\varrho) &= \varrho^{4\sigma}\sin 2\sigma\pi + a_3\varrho^{3\sigma}\sin\frac{3\sigma\pi}{2} a_2\varrho^{2\sigma}\sin\sigma\pi + a_1\varrho^\sigma\sin\frac{\sigma\pi}{2}, \\ \mathcal{B}_{2\mathbb{R}}(\varrho) &= b_2\varrho^{2\sigma}\cos\sigma\pi + b_1\varrho^\sigma\cos\frac{\sigma\pi}{2} + b_0, \\ \mathcal{B}_{2I}(\varrho) &= b_2\varrho^{2\sigma}\sin\sigma\pi + b_1\varrho^\sigma\sin\frac{\sigma\pi}{2}. \end{aligned} \tag{19}$$

By (18) one gets

$$\begin{aligned} \cos 2\varrho\eta &= -\frac{\mathcal{B}_{1\mathbb{R}}(\varrho)\mathcal{B}_{2\mathbb{R}}(\varrho) + \mathcal{B}_{1I}(\varrho)\mathcal{B}_{2I}(\varrho)}{\mathcal{B}_{2\mathbb{R}}^2(\varrho) + \mathcal{B}_{2I}^2(\varrho)}, \\ \sin 2\varrho\eta &= \frac{\mathcal{B}_{1I}(\varrho)\mathcal{B}_{2\mathbb{R}}(\varrho) - \mathcal{B}_{1\mathbb{R}}(\varrho)\mathcal{B}_{2I}(\varrho)}{\mathcal{B}_{2\mathbb{R}}^2(\varrho) + \mathcal{B}_{2I}^2(\varrho)}. \end{aligned} \tag{20}$$

Let

$$\begin{aligned} &\beta_1 = \cos 2\sigma\pi, \qquad \beta_2 = a_3\cos\frac{3\sigma\pi}{2}, \qquad \beta_3 = a_2\cos\sigma\pi, \\ &\beta_4 = a_1\cos\frac{\sigma\pi}{2}, \qquad \beta_5 = a_0 \qquad \beta_6 = \sin 2\sigma\pi, \\ &\beta_7 = a_3\sin\frac{3\sigma\pi}{2}, \qquad \beta_8 = a_2\sin\sigma\pi, \qquad \beta_9 = a_1\sin\frac{\sigma\pi}{2}, \end{aligned}$$

$$\beta_{10} = b_2 \cos \sigma\pi, \qquad \beta_{11} = b_1 \cos \frac{\sigma\pi}{2}, \qquad \beta_{12} = b_0,$$
$$\beta_{13} = b_2 \sin \sigma\pi, \qquad \beta_{14} = b_1 \sin \frac{\sigma\pi}{2}.$$

Then (19) can be rewritten as

$$\begin{aligned} \mathcal{B}_{1\mathbb{R}}(\varrho) &= \beta_1 \varrho^{4\sigma} + \beta_2 \varrho^{3\sigma} + \beta_3 \varrho^{2\sigma} + \beta_4 \varrho^{\sigma} + \beta_5, \\ \mathcal{B}_{1I}(\varrho) &= \beta_6 \varrho^{4\sigma} + \beta_7 \varrho^{3\sigma} + \beta_8 \varrho^{2\sigma} + \beta_9 \varrho^{\sigma}, \\ \mathcal{B}_{2\mathbb{R}}(\varrho) &= \beta_{10} \varrho^{2\sigma} + \beta_{11} \varrho^{\sigma} + \beta_{12}, \\ \mathcal{B}_{2I}(\varrho) &= \beta_{13} \varrho^{2\sigma} + \beta_{14}. \end{aligned} \tag{21}$$

In view of (20) and (21), one gets

$$\begin{aligned} &\big[\mathcal{B}_{1\mathbb{R}}(\varrho)\mathcal{B}_{2\mathbb{R}}(\varrho) + \mathcal{B}_{1I}(\varrho)\mathcal{B}_{2I}(\varrho)\big]^2 + \big[\mathcal{B}_{1I}(\varrho)\mathcal{B}_{2\mathbb{R}}(\varrho) - \mathcal{B}_{1\mathbb{R}}(\varrho)\mathcal{B}_{2I}(\varrho)\big]^2 \\ &\quad = \big[\mathcal{B}_{2\mathbb{R}}^2(\varrho) + \mathcal{B}_{2I}^2(\varrho)\big]^2. \end{aligned} \tag{22}$$

Since

$$\begin{aligned} &\mathcal{B}_{1\mathbb{R}}(\varrho)\mathcal{B}_{2\mathbb{R}}(\varrho) + \mathcal{B}_{1I}(\varrho)\mathcal{B}_{2I}(\varrho) \\ &\quad = \mu_1 \varrho^{6\sigma} + \mu_2 \varrho^{5\sigma} + \mu_3 \varrho^{4\sigma} + \mu_4 \varrho^{3\sigma} + \mu_5 \varrho^{2\sigma} + \mu_6 \varrho^{\sigma} + \mu_7, \\ &\mathcal{B}_{1I}(\varrho)\mathcal{B}_{2\mathbb{R}}(\varrho) - \mathcal{B}_{1\mathbb{R}}(\varrho)\mathcal{B}_{2I}(\varrho) \\ &\quad = \nu_1 \varrho^{6\sigma} + \nu_2 \varrho^{5\sigma} + \nu_3 \varrho^{4\sigma} + \nu_4 \varrho^{3\sigma} + \nu_5 \varrho^{2\sigma} + \nu_6 \varrho^{\sigma} + \nu_7, \\ &\mathcal{B}_{2\mathbb{R}}^2(\varrho) + \mathcal{B}_{2I}^2(\varrho) = \varsigma_1 \varrho^{4\sigma} + \varsigma_2 \varrho^{3\sigma} + \varsigma_3 \varrho^{2\sigma} + \varsigma_4 \varrho^{\sigma} + \varsigma_5, \end{aligned} \tag{23}$$

where

$$\begin{aligned} &\mu_1 = \beta_1\beta_{10} + \beta_3\beta_{13}, \qquad \mu_2 = \beta_1\beta_{11} + \beta_2\beta_{10} + \beta_7\beta_{13}, \\ &\mu_3 = \beta_1\beta_{12} + \beta_2\beta_{11} + \beta_3\beta_{10} + \beta_6\beta_{14} + \beta_8\beta_{13}, \\ &\mu_4 = \beta_2\beta_{12} + \beta_3\beta_{11} + \beta_4\beta_{10} + \beta_7\beta_{14} + \beta_9\beta_{13}, \\ &\mu_5 = \beta_3\beta_{12} + \beta_4\beta_{11} + \beta_5\beta_{10} + \beta_8\beta_{14}, \\ &\mu_6 = \beta_4\beta_{12} + \beta_5\beta_{11} + \beta_9\beta_{14}, \qquad \mu_7 = \beta_5\beta_{12} \end{aligned}$$

and

$$\begin{aligned} &\nu_1 = \beta_6\beta_{10} - \beta_1\beta_{13}, \qquad \nu_2 = \beta_6\beta_{11} + \beta_7\beta_{10} - \beta_2\beta_{13}, \\ &\nu_3 = \beta_6\beta_{12} + \beta_7\beta_{11} + \beta_8\beta_{10} - \beta_1\beta_{14} - \beta_3\beta_{13}, \\ &\nu_4 = \beta_7\beta_{12} + \beta_8\beta_{11} + \beta_9\beta_{10} - \beta_2\beta_{14} - \beta_4\beta_{13}, \\ &\nu_5 = \beta_8\beta_{12} + \beta_9\beta_{11} - \beta_3\beta_{14} + \beta_5\beta_{13}, \\ &\nu_6 = \beta_9\beta_{12} - \beta_4\beta_{14}, \qquad \nu_7 = -\beta_5\beta_{14}, \\ &\varsigma_1 = \beta_{10}^2 + \beta_{13}^2, \qquad \varsigma_2 = 2\beta_{10}\beta_{11}, \\ &\varsigma_3 = \beta_{11}^2 + 2\beta_{10}\beta_{12} + 2\beta_{13}\beta_{14}, \\ &\varsigma_4 = 2\beta_{11}\beta_{12}, \qquad \varsigma_5 = \beta_{12}^2 + \beta_{14}^2. \end{aligned}$$

By (23) it follows from (22) that

$$\vartheta_1\varrho^{12\sigma}+\vartheta_2\varrho^{11\sigma}+\vartheta_3\varrho^{10\sigma}+\vartheta_4\varrho^{9\sigma}+\vartheta_5\varrho^{8\sigma}+\vartheta_6\varrho^{7\sigma}+\vartheta_7\varrho^{6\sigma}$$
$$+\vartheta_8\varrho^{5\sigma}+\vartheta_9\varrho^{4\sigma}+\vartheta_{10}\varrho^{3\sigma}+\vartheta_{11}\varrho^{2\sigma}+\vartheta_{12}\varrho^{\sigma}+\vartheta_{13}=0, \tag{24}$$

where

$$\begin{aligned}
\vartheta_1&=\mu_1^2+\nu_1^2,\qquad \vartheta_2=2\mu_1\mu_2+2\nu_1\nu_2,\qquad \vartheta_3=\mu_2^2+2\mu_1\mu_3+\nu_2^2+2\nu_1\nu_3,\\
\vartheta_4&=2\mu_1\mu_4+2\mu_2\mu_3+2\nu_1\nu_4+2\nu_2\nu_3,\\
\vartheta_5&=\mu_3^2+2\mu_2\mu_4+2\mu_2\mu_5+\nu_3^2+2\nu_2\nu_4+2\nu_2\nu_5-\varsigma_1^2,\\
\vartheta_6&=2\mu_1\mu_6+2\mu_2\mu_5+2\mu_3\mu_4-2\varsigma_1\varsigma_2,\\
\vartheta_7&=\mu_4^2+2\mu_1\mu_7+2\mu_2\mu_6+2\mu_3\mu_5+\nu_4^2+2\nu_1\nu_7+2\nu_2\nu_6+2\nu_3\nu_5-2\varsigma_1\varsigma_3-\varsigma_2^2,\\
\vartheta_8&=2\mu_2\mu_7+2\mu_3\mu_6+2\mu_4\mu_5+2\nu_2\nu_7+2\nu_3\nu_6+2\nu_4\nu_5-2\varsigma_1\varsigma_4-2\varsigma_2\varsigma_3,\\
\vartheta_9&=\mu_5^2+2\mu_3\mu_7+2\mu_4\mu_6+\nu_5^2+2\nu_3\nu_7+2\nu_4\nu_6-\varsigma_3^2-2\varsigma_1\varsigma_5-2\varsigma_2\varsigma_4,\\
\vartheta_{10}&=2\mu_4\mu_7+2\mu_5\mu_6+2\nu_4\nu_7+2\nu_5\nu_6-2\varsigma_2\varsigma_5-2\varsigma_3\varsigma_4,\\
\vartheta_{11}&=\mu_6^2+2\mu_5\mu_7+\nu_6^2+2\nu_5\nu_7-\varsigma_4^2-2\varsigma_3\varsigma_5,\\
\vartheta_{12}&=2\mu_6\mu_7+2\nu_6\nu_7-2\varsigma_4\varsigma_5,\qquad \vartheta_{13}=\mu_7^2+\nu_7^2-\varsigma_5^2.
\end{aligned}$$

Set

$$\mathcal{M}(\varrho)=\vartheta_1\varrho^{12\sigma}+\vartheta_2\varrho^{11\sigma}+\vartheta_3\varrho^{10\sigma}+\vartheta_4\varrho^{9\sigma}+\vartheta_5\varrho^{8\sigma}+\vartheta_6\varrho^{7\sigma}+\vartheta_7\varrho^{6\sigma}$$
$$+\vartheta_8\varrho^{5\sigma}+\vartheta_9\varrho^{4\sigma}+\vartheta_{10}\varrho^{3\sigma}+\vartheta_{11}\varrho^{2\sigma}+\vartheta_{12}\varrho^{\sigma}+\vartheta_{13} \tag{25}$$

and

$$\mathcal{N}(\xi)=\vartheta_1\xi^{12}+\vartheta_2\xi^{11}+\vartheta_3\xi^{10}+\vartheta_4\xi^{9}+\vartheta_5\xi^{8}+\vartheta_6\xi^{7}+\vartheta_7\xi^{6}$$
$$+\vartheta_8\xi^{5}+\vartheta_9\xi^{4}+\vartheta_{10}\xi^{3}+\vartheta_{11}\xi^{2}+\vartheta_{12}\xi+\vartheta_{13}.$$

Lemma 8.

(i) *In addition to the condition $a_0+b_0\neq 0$, if $\vartheta_l>0$ ($l=1,2,\dots,13$), then Eq.* (17) *possesses no the root with zero real parts.*

(ii) *If $\vartheta_{13}>0$ and there exists $\xi_0>0$, which satisfies $\mathcal{N}(\xi_0)<0$, then Eq.* (17) *possesses at least two pairs of purely imaginary roots.*

Proof. (i) In term of (25), one has

$$\frac{\mathrm{d}\mathcal{M}(\varrho)}{\mathrm{d}\varrho}=12\sigma\vartheta_1\varrho^{12\sigma-1}+11\sigma\vartheta_2\varrho^{11\sigma-1}+10\sigma\vartheta_3\varrho^{10\sigma-1}+9\sigma\vartheta_4\varrho^{9\sigma-1}$$
$$+8\sigma\vartheta_5\varrho^{8\sigma-1}+7\sigma\vartheta_6\varrho^{7\sigma-1}+6\sigma\vartheta_7\varrho^{6\sigma-1}+5\sigma\vartheta_8\varrho^{5\sigma-1}$$
$$+4\sigma\vartheta_9\varrho^{4\sigma-1}+3\sigma\vartheta_{10}\varrho^{3\sigma-1}+2\sigma\vartheta_{11}\varrho^{2\sigma-1}+\sigma\vartheta_{12}\varrho^{\sigma-1}.$$

By $\vartheta_l>0$ ($l=1,2,\dots,12$) one gets $\mathrm{d}\mathcal{M}(\varrho)/\mathrm{d}\varrho>0$ for all $\varrho>0$. Noting that $\mathcal{M}(0)=\vartheta_{13}>0$, one obtains that Eq. (24) has no positive real root. In addition, by $a_0+b_0\neq 0$ one knows that $s=0$ is not the root of (17). This completes the proof of (i).

(ii) By $\mathcal{N}(0) = \vartheta_{13} > 0$, $\mathcal{N}(\xi_0) < 0$ ($\xi_0 > 0$) and $\lim_{\varepsilon\to+\infty} N(\varepsilon)/\mathrm{d}\varepsilon = +\infty$ one can find $\xi_1 \in (0, \xi_0)$ and $\xi_2 \in (\xi_0, +\infty)$, which satisfy $\mathcal{N}(\xi_1) = \mathcal{N}(\xi_2) = 0$. Then Eq. (24) possesses at least two positive real roots. Therefore (17) possesses at least two pairs of purely imaginary roots. The proof of (ii) is finished. □

Assume that Eq. (24) has twelve positive real roots ϱ_j ($j = 1, 2, \dots, 12$). It follows from (20) that

$$\eta_j^l = \frac{1}{2\varrho_k}\left[\arccos\left(-\frac{\mathcal{B}_{1\mathbb{R}}(\varrho_k)\mathcal{B}_{2\mathbb{R}}(\varrho_k) + \mathcal{B}_{1I}(\varrho_k)\mathcal{B}_{2I}(\varrho_k)}{\mathcal{B}_{2\mathbb{R}}^2(\varrho_k) + \mathcal{B}_{2I}^2(\varrho_k)}\right) + 2l\pi\right],$$

where $l = 0, 1, 2, \dots$, $j = 1, 2, \dots, 12$. Denote

$$\eta_0 = \min_{j=1,2,\dots,12}\{\eta_j^0\}, \qquad \varrho_0 = \varrho|_{\eta=\eta_0}.$$

In the sequel, we make the following hypothesis:

($\mathcal{Q}_4$) $\mathcal{W}_{11}\mathcal{W}_{21} + \mathcal{W}_{12}\mathcal{W}_{22} > 0$, where

$$\begin{aligned}
\mathcal{W}_{11} &= 4\sigma\varrho_0^{4\sigma-1}\cos\frac{(4\sigma-1)\pi}{2} + 3\sigma a_3\varrho_0^{3\sigma-1}\cos\frac{(3\sigma-1)\pi}{2} \\
&\quad + 2\sigma a_2\varrho_0^{2\sigma-1}\cos\frac{(2\sigma-1)\pi}{2} + \sigma a_1\varrho_0^{\sigma-1}\cos\frac{(\sigma-1)\pi}{2} \\
&\quad + \left[2\sigma b_2\varrho_0^{2\sigma-1}\cos\frac{(2\sigma-1)\pi}{2} + \sigma b_1\varrho_0^{\sigma-1}\cos\frac{(\sigma-1)\pi}{2}\right]\cos\varrho_0\eta_0 \\
&\quad + \left[2\sigma b_2\varrho_0^{2\sigma-1}\sin\frac{(2\sigma-1)\pi}{2} + \sigma b_1\varrho_0^{\sigma-1}\sin\frac{(\sigma-1)\pi}{2}\right]\sin\varrho_0\eta_0, \\
\mathcal{W}_{12} &= 4\sigma\varrho_0^{4\sigma-1}\sin\frac{(4\sigma-1)\pi}{2} + 3\sigma a_3\varrho_0^{3\sigma-1}\sin\frac{(3\sigma-1)\pi}{2} \\
&\quad + 2\sigma a_2\varrho_0^{2\sigma-1}\sin\frac{(2\sigma-1)\pi}{2} + \sigma a_1\varrho_0^{\sigma-1}\sin\frac{(\sigma-1)\pi}{2} \\
&\quad - \left[2\sigma b_2\varrho_0^{2\sigma-1}\cos\frac{(2\sigma-1)\pi}{2} + \sigma b_1\varrho_0^{\sigma-1}\cos\frac{(\sigma-1)\pi}{2}\right]\sin\varrho_0\eta_0 \\
&\quad + \left[2\sigma b_2\varrho_0^{2\sigma-1}\sin\frac{(2\sigma-1)\pi}{2} + \sigma b_1\varrho_0^{\sigma-1}\sin\frac{(\sigma-1)\pi}{2}\right]\cos\varrho_0\eta_0, \\
\mathcal{W}_{21} &= 2\left(b_2\varrho_0^{2\sigma}\cos\sigma\pi + b_1\varrho_0^{\sigma}\cos\frac{\sigma\pi}{2} + b_0\right)\varrho_0\sin 2\varrho_0\eta_0 \\
&\quad - 2\left(b_2\varrho_0^{2\sigma}\sin\sigma\pi + b_1\varrho_0^{\sigma}\sin\frac{\sigma\pi}{2}\right)\varrho_0\cos 2\varrho_0\eta_0, \\
\mathcal{W}_{22} &= 2\left(b_2\varrho_0^{2\sigma}\cos\sigma\pi + b_1\varrho_0^{\sigma}\cos\frac{\sigma\pi}{2} + b_0\right)\varrho_0\cos 2\varrho_0\eta_0 \\
&\quad + 2\left(b_2\varrho_0^{2\sigma}\sin\sigma\pi + b_1\varrho_0^{\sigma}\sin\frac{\sigma\pi}{2}\right)\varrho_0\sin 2\varrho_0\eta_0.
\end{aligned}$$

Lemma 9. *Suppose that* $s(\eta) = \rho(\eta) + \mathrm{i}\kappa(\eta)$ *is the root of Eq.* (17) *at* $\eta = \eta_0$*, which satisfies* $\rho(\eta_0) = 0$, $\kappa(\eta_0) = \varrho_0$. *Then one has* $\mathrm{Re}[\mathrm{d}s/\mathrm{d}\eta]|_{\eta=\eta_0,\,\varrho=\varrho_0} > 0$.

Proof. It follows from Eq. (17) that

$$\begin{aligned}&\left[4\sigma s^{4\sigma-1} + 3\sigma a_3 s^{3\sigma-1} + 2\sigma a_2 s^{2\sigma-1} + \sigma a_1 s^{\sigma-1}\right]\frac{\mathrm{d}s}{\mathrm{d}\eta}\\ &\quad + \left[2\sigma b_2 s^{2\sigma-1} + \sigma b_1 s^{\sigma-1}\right]\mathrm{e}^{-2s\eta}\frac{\mathrm{d}s}{\mathrm{d}\eta}\\ &\quad - 2\mathrm{e}^{-2s\eta}\left(\frac{\mathrm{d}s}{\mathrm{d}\eta}\eta + s\right)\left(b_2 s^{2\sigma} + b_1 s^{\sigma} + b_0\right) = 0.\end{aligned}$$

Then

$$\left[\frac{\mathrm{d}s}{\mathrm{d}\eta}\right]^{-1} = \frac{\mathcal{W}_1(s)}{\mathcal{W}_2(s)} - \frac{\eta}{s},$$

where

$$\begin{aligned}\mathcal{W}_1(s) &= 4\sigma s^{4\sigma-1} + 3\sigma a_3 s^{3\sigma-1} + 2\sigma a_2 s^{2\sigma-1} + \sigma a_1 s^{\sigma-1}\\ &\quad + \left[2\sigma b_2 s^{2\sigma-1} + \sigma b_1 s^{\sigma-1}\right]\mathrm{e}^{-2s\eta},\\ \mathcal{W}_2(s) &= 2s\mathrm{e}^{-2s\eta}\left(b_2 s^{2\sigma} + b_1 s^{\sigma} + b_0\right).\end{aligned}$$

Thus

$$\mathrm{Re}\left\{\frac{\mathrm{d}s}{\mathrm{d}\eta}\right\}\bigg|_{\eta=\eta_0,\,\varrho=\varrho_0} = \mathrm{Re}\left\{\frac{W_1(s)}{W_2(s)}\right\}\bigg|_{\eta=\eta_0,\,\varrho=\varrho_0} = \frac{\mathcal{W}_{11}\mathcal{W}_{21} + \mathcal{W}_{12}\mathcal{W}_{22}}{\mathcal{W}_{21}^2 + \mathcal{W}_{22}^2}.$$

In view of $(\mathcal{Q}_4)$, we have

$$\mathrm{Re}\left\{\left[\frac{\mathrm{d}s}{\mathrm{d}\eta}\right]^{-1}\right\}\bigg|_{\eta=\eta_0,\,\varrho=\varrho_0} > 0.$$

The proof of Lemma 9 finishes. □

Lemma 10. *If* $\eta = 0$ *and* $(\mathcal{Q}_2)$ *holds true, then system* (2) *is locally asymptotically stable.*

Proof. When $\eta = 0$, then (17) takes the form

$$\lambda^4 + a_3\lambda^3 + (a_2 + b_2)\lambda^2 + (a_1 + b_1)\lambda + a_0 + b_0 = 0. \tag{26}$$

By $(\mathcal{Q}_2)$ one knows that all roots λ_i of (26) satisfy $|\arg(\lambda_i)| > \sigma\pi/2$ $(i = 1, 2, 3, 4)$. So we know that Lemma 10 is correct. The proof ends. □

Based on the investigation above, we have the following conclusion.

Theorem 2. *Let hypotheses* $(\mathcal{Q}_1)$*–*$(\mathcal{Q}_4)$ *hold. Then the zero equilibrium point* $E(0,0,0,0)$ *of system* (2) *is locally asymptotically stable if* η *fall into the interval* $[0, \eta_0)$ *and a Hopf bifurcation will take place in the vicinity of* $E(0,0,0,0)$ *when* $\eta = \eta_0$.

5 Two examples

Example 1. Give the system as follows:

$$\begin{aligned}\dot{w}_1(t) &= -0.5w_1(t) - 0.5\tanh\big(w_1(t)\big) + 0.4\tanh\big(w_4(t-\eta)\big)\\ &\quad + 0.4\tanh(w_2(t-\eta)),\\ \dot{w}_2(t) &= -1.5w_2(t) + 0.5\tanh\big(w_2(t)\big) - 1.2\tanh\big(w_1(t-\eta)\big)\\ &\quad - 1.2\tanh(w_3(t-\eta)),\\ \dot{w}_3(t) &= -0.6w_3(t) - 0.4\tanh\big(w_3(t)\big) + 0.6\tanh\big(w_2(t-\eta)\big)\\ &\quad + 0.6\tanh(w_4(t-\eta)),\\ \dot{w}_4(t) &= -1.8w_4(t) + 0.8\tanh\big(w_4(t)\big) + 0.8\tanh\big(w_3(t-\eta)\big)\\ &\quad - \tanh\big(w_1(t-\eta)\big).\end{aligned} \tag{27}$$

Obviously, system (27) has the zero equilibrium point $E(0,0,0,0)$. With the aid of Matlab software, one gets $\psi_0 = 0.2160$ and $\eta_0 = 0.228$. By algebraic operations with Matlab 7.0 (since the complexity of expression, we can compute the values of different expressions by performing multiple hybrid operations) we can check that conditions (ii), (iii) in Theorem 1 are satisfied.

If $\eta \in [0, 0.228)$, the zero equilibrium point $E(0,0,0,0)$ of system (27) is locally asymptotically stable. In this situation, let $\eta = 0.19 < \eta_0 = 0.228$. The computer simulation diagram are displayed in Fig. 1. Figure 1 indicates that when the time delay η is less than the critical value $\eta_0 = 0.228$, then all the states of the neurons of neural networks (27) will be tardily close to zero.

If $\eta \in [0.8616, +\infty)$, then system (27) loses its stability, and a Hopf bifurcation emerges. In this situation, we choose $\eta = 0.4$. The computer simulation diagrams are displayed in Fig. 2. Figure 2 implies that when the time delay η is greater than the critical value $\eta_0 = 0.228$, then all the states of the neurons will maintain periodic motion around the zero equilibrium point $E(0,0,0,0)$, i.e., a Hopf bifurcation takes place around the zero equilibrium point $E(0,0,0,0)$. In order to explain this fact intuitively, we give the bifurcation diagram Fig. 3 of system (27).

Figure 3 has revealed the relation of η–w_1, η–w_2, η–w_3, η–w_4, respectively. Clearly, from Fig. 3 one can easily know that the bifurcation point of system (27) is 0.228. Moreover, the relationship of ψ_0 and η_0 is also displayed in Table 1.

Table 1. The quantitative relationship of ψ_0 and η_0 of system (27).

ψ_0	η_0	ψ_0	η_0
0.1070	0.1052	0.5601	0.7345
0.1952	0.2033	0.6372	0.8755
0.2160	0.2280	0.6509	0.9018
0.3915	0.4622	0.7648	1.1334
0.5128	0.6531		

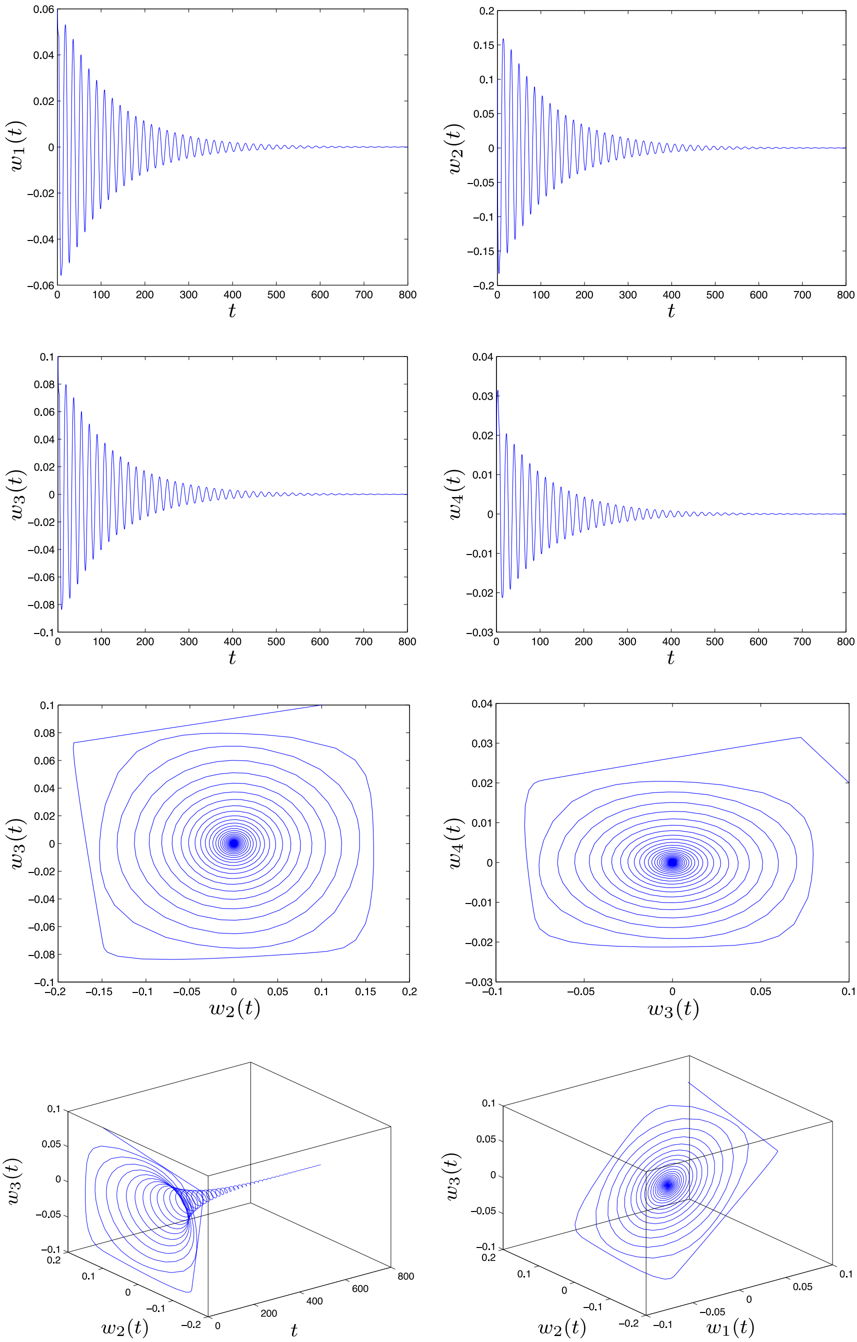

Figure 1. Simulation results for system (27) when $\eta = 0.19 < \eta_0 = 0.228$.

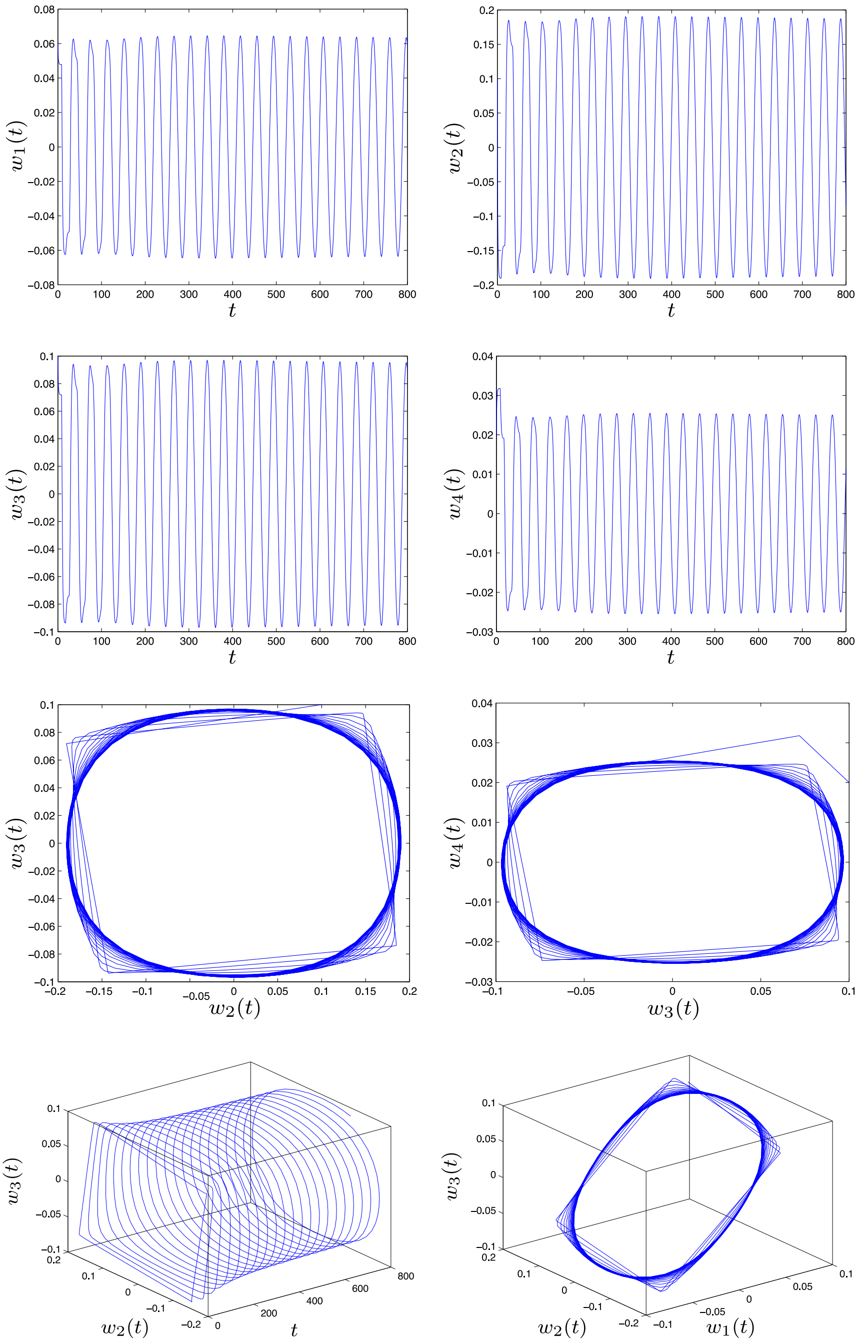

Figure 2. Simulation results for system (27) when $\eta = 0.4 > \eta_0 = 0.228$.

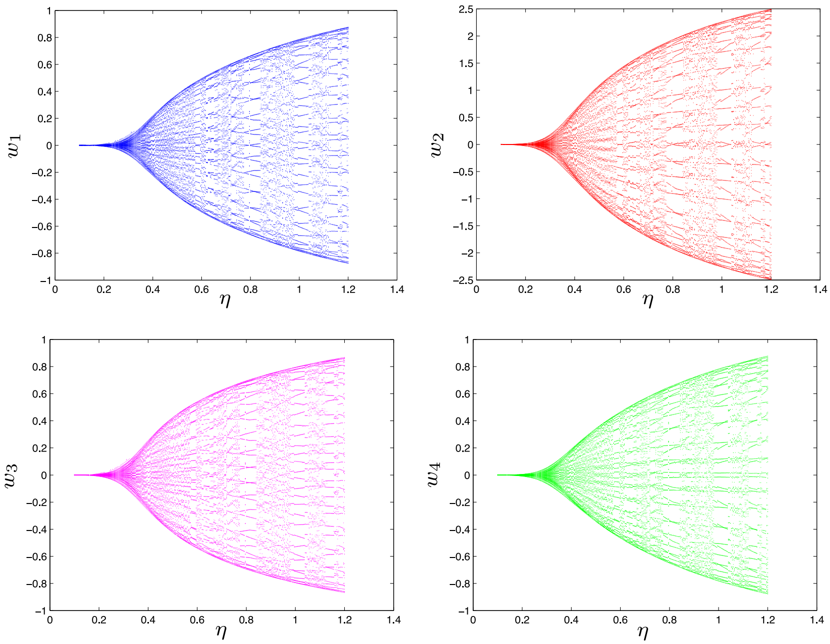

Figure 3. Bifurcation diagram for system (27): η versus w_1 (blue), η versus w_2 (red), η versus w_3 (magenta), η versus w_4 (green).

Example 2. Give the system as follows:

$$\begin{aligned}\frac{\mathrm{d}^\sigma u_1(t)}{\mathrm{d}t^\sigma} &= -0.5w_1(t) - 0.5\tanh\bigl(w_1(t)\bigr) + 0.4\tanh\bigl(w_4(t-\eta)\bigr)\\&\quad + 0.4\tanh\bigl(w_2(t-\eta)\bigr),\\\frac{\mathrm{d}^\sigma u_1(t)}{\mathrm{d}t^\sigma} &= -1.5w_2(t) + 0.5\tanh\bigl(w_2(t)\bigr) - 1.2\tanh\bigl(w_1(t-\eta)\bigr)\\&\quad - 1.2\tanh\bigl(w_3(t-\eta)\bigr),\\\frac{\mathrm{d}^\sigma u_1(t)}{\mathrm{d}t^\sigma} &= -0.6w_3(t) - 0.4\tanh\bigl(w_3(t)\bigr) + 0.6\tanh\bigl(w_2(t-\eta)\bigr)\\&\quad + 0.6\tanh\bigl(w_4(t-\eta)\bigr),\\\frac{\mathrm{d}^\sigma u_1(t)}{\mathrm{d}t^\sigma} &= -1.8w_4(t) + 0.8\tanh\bigl(w_4(t)\bigr) + 0.8\tanh\bigl(w_3(t-\eta)\bigr)\\&\quad - \tanh\bigl(w_1(t-\eta)\bigr).\end{aligned} \tag{28}$$

Clearly, system (28) has zero equilibrium point $E(0,0,0,0)$. Let $\sigma = 0.86$. Using Matlab software, one gets $\varrho_0 = 0.7319$ and $\eta_0 = 0.512$. By algebraic operations with Matlab 7.0 (since the complexity of expression, we can compute the values of different expressions by performing multiple hybrid operations) one can check that all the assumptions of Theorem 2 are fulfilled. In fact, we have explained the implication of Figs. 4–6 (see pp. 1050–1051).

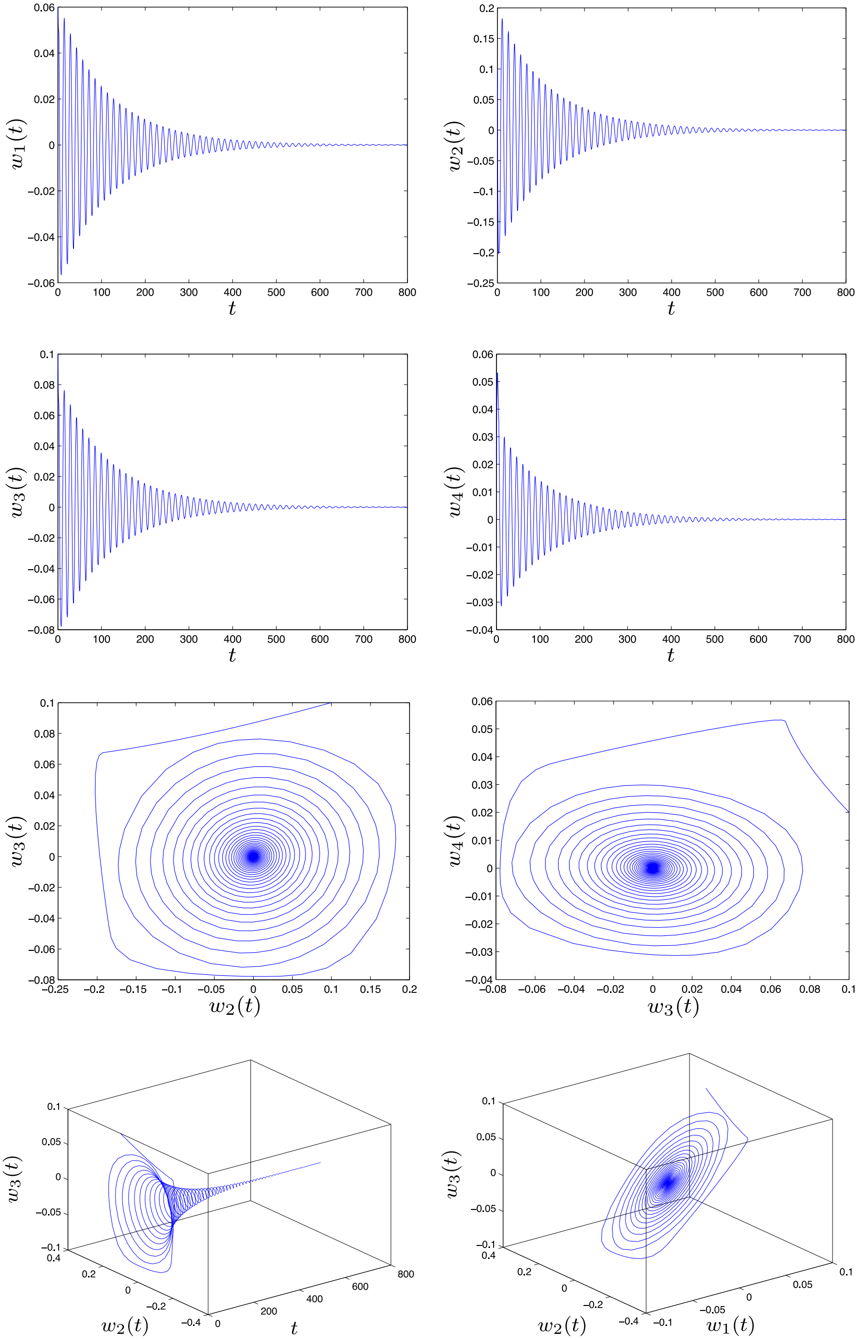

Figure 4. Simulation results for system (28) when $\eta = 0.48 < \eta_0 = 0.512$.

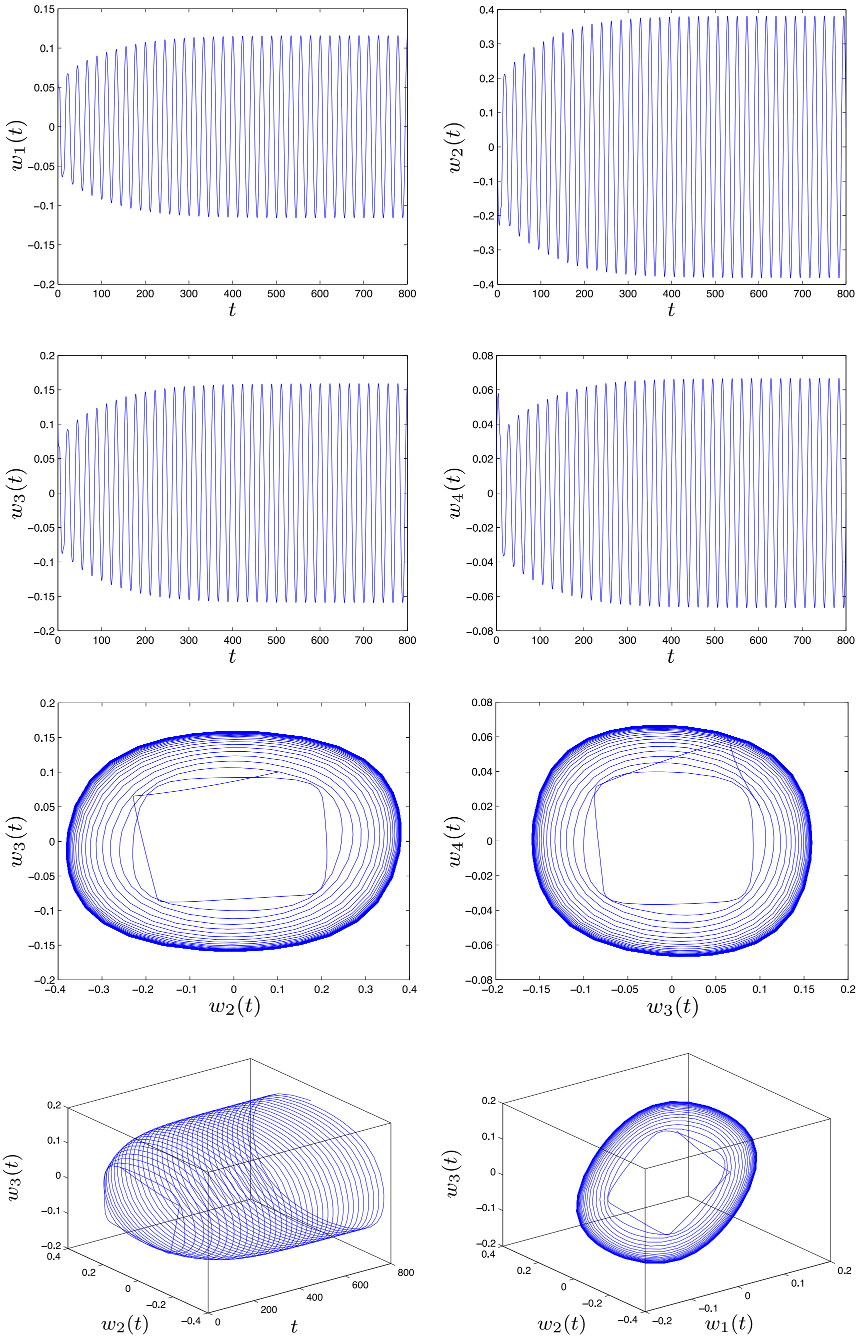

Figure 5. Simulation results for system (28) when $\eta = 0.68 > \eta_0 = 0.512$.

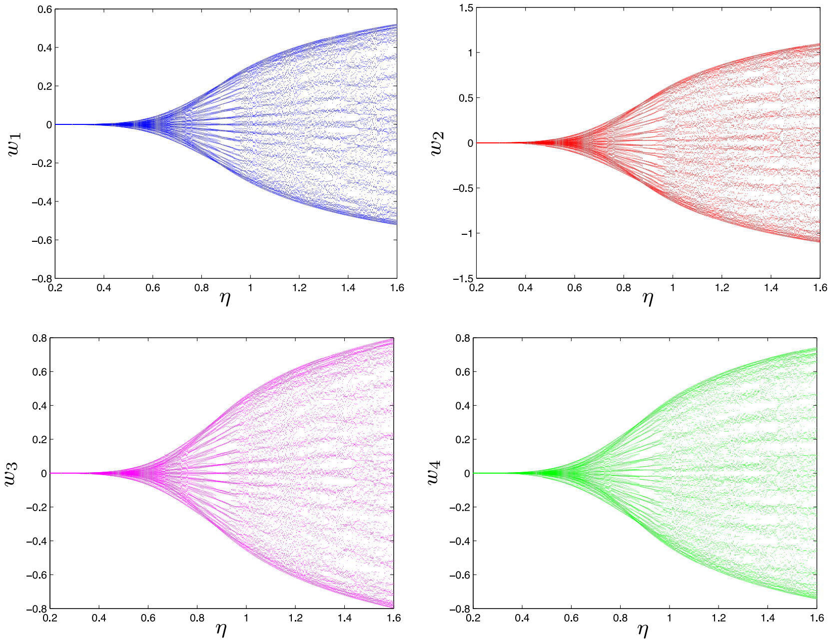

Figure 6. Bifurcation diagram for system (28): η versus w_1 (blue), η versus w_2 (red), η versus w_3 (magenta), η versus w_4 (green).

Table 2. The quantitative relationship of ϱ_0 and η_0 of system (28).

ϱ_0	η_0	ϱ_0	η_0
0.3859	0.2352	1.1703	0.9674
0.5869	0.3877	1.2860	1.1089
0.7319	0.5120	1.4719	1.3561
0.9327	0.7051	1.6075	1.5521
1.0626	0.8438		

When $\eta \in [0, 0.512)$, the zero equilibrium point $E(0,0,0,0)$ of system (28) is locally asymptotically stable. In this situation, let $\eta = 0.48 < \eta_0 = 0.512$. The computer simulation diagrams are displayed in Fig. 4. Figure 4 indicates that when the time delay η is less than the critical value $\eta_0 = 0.512$, then all the states of the neurons of neural networks (28) will be tardily close to zero.

If $\eta \in [0.512, +\infty)$, then system (28) loses its stability and a Hopf bifurcation emerges. In this situation, we choose $\eta = 0.68$. The computer simulation diagrams are displayed in Fig. 5. Figure 5 implies that when the time delay η is greater than the critical value $\eta_0 = 0.512$, then all the states of the neurons will maintain periodic motion around the zero equilibrium point $E(0,0,0,0)$, i.e., a Hopf bifurcation takes place around the zero equilibrium point $E(0,0,0,0)$. In order to explain this fact intuitively, we give the bifurcation diagram Fig. 6 of system (28).

Figure 6 has revealed the relation of η–w_1, η–w_2, η–w_3, η–w_4, respectively. Clearly, from Fig. 6 one can easily know that the bifurcation point of system (28) is 0.512. Moreover, the relationship of ψ_0 and η_0 is also given in Table 2.

6 Conclusions

The research on the effect of time delay on the stability and Hopf bifurcation of delayed differential systems (including integer-order and fractional-order) is an important topic in differential dynamical systems. In this paper, we have investigated the effect of time delay on the stability and Hopf bifurcation of integer-order and fractional-order delayed BAM neural networks. With the help of stability theory and Hopf bifurcation theory of integer-order and fractional-order delayed differential equations, we have established two sets of sufficient conditions to guarantee the stability and the appearance of Hopf bifurcation of the involved integer-order and fractional-order delayed BAM neural networks. Meanwhile, the different impact of time delay on the stability and the appearance of Hopf bifurcation of integer-order and fractional-order delayed BAM neural networks has been revealed. The comparative study on bifurcation behavior for integer-order and fractional-order delayed BAM neural networks shows that under a suitable condition, we can enlarge the stability region and delay the time of the appearance of Hopf bifurcation by fractional-order delayed BAM neural networks. The established theoretical results possess great theoretical guiding significance to design and control the network. We will consider the influence of leakage delay of integer-order and fractional-order BAM neural networks with leakage delays in near future.

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