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Well-posedness and stability for fuzzy fractional differential equations*

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Abstract. In this article, we consider the existence and uniqueness of solutions for a class of initial value problems of fuzzy Caputo–Katugampola fractional differential equations and the stability of the corresponding fuzzy fractional differential equations. The discussions are based on the hyperbolic function, the Banach fixed point theorem and an inequality property. Two examples are given to illustrate the feasibility of our theoretical results.

Keywords: fuzzy fractional differential equations, hyperbolic function, Banach fixed point theorem.

1 Introduction

In this paper, we are concerned with the existence, uniqueness and stability of solutions for a class of initial value problems of fuzzy fractional differential equations (FFDE) of the following form:

$$\begin{aligned} {}^{C}D_{0^+}^{q,p}u(t) &= \lambda u(t) \oplus f\big(t, u(t)\big), \quad t \in (0, T], \\ u(0) &= u_0 \in \mathbb{R}_{\mathcal{F}}, \end{aligned} \tag{1}$$

where ${}^{C}D_{0^+}^{q,p}$ is the fuzzy Caputo–Katugampola fractional generalized Hukuhara derivative of order $q \in (0, 1]$, $p > 0$ is a fixed real number, $\lambda \in \mathbb{R}$, $f : (0, T] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a continuous fuzzy nonlinear mapping, and $\mathbb{R}_{\mathcal{F}}$ is the space of fuzzy numbers.

Fractional-order differential equation can be regarded as a generalization of ordinary integer-order differential equation, and we refer the reader to [13,17,24,26]. However, due

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to errors caused by observations, experiments and maintenance, the variables and parameters that we get are usually fuzzy, incomplete and inaccurate. These uncertainties are introduced into fractional differential equations called fractional fuzzy differential equations.

In recent years, there has been some research on fractional fuzzy differential equations. Except for various numerical solutions, most of the methods transform fractional fuzzy differential equations into fractional fuzzy integral equations and then use nonlinear analysis methods to discuss the qualitative properties of the solutions. In 2010, Agarwal et al. obtained the solution of the initial value problem by studying the corresponding fuzzy integral equation of the initial value problem in [2]. In 2012, Allahviranloo et al. [5] studied the analytical solution to the initial value problem for a class of Riemann–Liouville-type fractional differential equations under the strong generalized Hukuhara differentiability introduced in [8]. Then Allahviranloo et al. studied the initial value problem of the Volterra–Fredholm-type fuzzy integro–differential equation, and established the existence and uniqueness of the solution by using a compact mapping theorem and an iterative method [3]. Recently, Ngo presented results on the existence and uniqueness of solutions for two kinds of fractional fuzzy functional integral equations and fuzzy functional differential equations using the contraction mapping principle and the successive approximation method [11, 12]. For research on solutions of initial boundary value problems for fractional fuzzy differential equations, more information can be found in [1, 4, 6, 13, 22, 27, 32, 35] and the references therein.

The study of Ulam stability can provide an important theoretical basis for the existence and even uniqueness of the solution of the differential equation and it can also provide a reliable theoretical basis for the approximate solution of the corresponding equation. In 1993, Obloza studied the stability of the differential equation in [23]. Miura and others established Ulam stability theory of differential equations in different abstract spaces [18, 19, 31]. In 2013, Rezaei et al. [29] established Hyers–Ulam stability of nth-order linear differential equations with constant coefficients using the Laplace transform method. Mortici et al. [20] studied the general solution of the inhomogeneous Euler equation and the Hyers–Ulam stability on a bounded domain using the integral method. In 2016, Bahyrycz et al. discussed Ulam stability of the generalized Frechet equation in a Banach space using a fixed point theorem in [7]. In 2018, Onitsuka [25] established the Ulam stability of first-order nonhomogeneous linear difference equations.

The purpose of this paper is to introduce fuzzy Caputo–Katugampola fractional differential equations, and discuss the existence, uniqueness and stability of solutions of fuzzy fractional differential equations (1). The structure of the paper is as follows: some preliminaries are given in Section 2. In Section 3, we establish the existence and uniqueness of solutions to problem (1). In Section 4, we discuss the stability of the solution. In Section 5, some examples are given to illustrate the feasibility of the results.

2 Preliminaries

In this section, we briefly introduce some definitions, notations and results related to fuzzy functions, which will be referred to throughout this paper.

We denote the set of all real numbers by $\mathbb{R}$ and the set of all fuzzy numbers on $\mathbb{R}$ is indicated by $\mathbb{R}_{\mathcal{F}}$. A fuzzy number is a mapping $u : \mathbb{R} \to [0,1]$ with the following properties:

(a) u is upper semicontinuous;
(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geqslant \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$, $\lambda \in [0,1]$;
(c) u is normal, i.e., there exists $x_0 \in \mathbb{R}$ for which $u(x_0) = 1$;
(d) $\operatorname{supp} u = \{x \in \mathbb{R} \mid u(x) > 0\}$ is the support of the u, and its closure $\operatorname{cl}(\operatorname{supp} u)$ is compact.

For $\alpha \in (0,1]$ denote $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geqslant \alpha\}$ and $[u]^0 = \operatorname{cl}\{x \in \mathbb{R} \mid u(x) > 0\}$. Then it is well known that the α-level set of u, $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$, is a closed interval for all $\alpha \in [0,1]$, where $\underline{u}$ and $\overline{u}$ represent the upper and lower branches of the fuzzy set $u \in \mathbb{R}_{\mathcal{F}}$, respectively. A fuzzy number function defined on the real set $\mathbb{R}$ and valued in $\mathbb{R}_{\mathcal{F}}$ is called a fuzzy-valued function, that is, $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Let the α-level representation of the fuzzy-valued function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be expressed by $[f(t,u)]^\alpha = [\underline{f}^\alpha(t,u), \overline{f}^\alpha(t,u)]$, $t \in [a,b]$, $\alpha \in [0,1]$.

For $u \in \mathbb{R}_{\mathcal{F}}$, we define the diameter of the α-level set of u as $d([u]^\alpha) = \overline{u}^\alpha - \underline{u}^\alpha$. Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v \oplus w$, then w is called the H-difference of u and v, and it is denoted by $u \ominus v$. In this paper, the sign "$\ominus$" always stands for the H-difference.

The Hausdorff distance between fuzzy numbers is given by $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to [0, +\infty)$,

$$d(u,v) = \sup_{r \in [0,1]} \max\bigl\{\bigl|\underline{u}^\alpha - \underline{v}^\alpha\bigr|, \bigl|\overline{u}^\alpha - \overline{v}^\alpha\bigr|\bigr\}.$$

Then it is easy to see that d is a metric in $\mathbb{R}_{\mathcal{F}}$ and the following properties of the metric d are valid (see [28]):

(i) $d(u \oplus w, v \oplus w) = d(u,v)$ for all $u, v, w \in \mathbb{R}_{\mathcal{F}}$;
(ii) $d(ku, kv) = |k| d(u,v)$, $\quad k \in \mathbb{R}$, $u, v \in \mathbb{R}_{\mathcal{F}}$;
(iii) $d(u \oplus v, w \oplus z) \leqslant d(u,w) + d(v,z)$ for all $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$;
(iv) $(d, \mathbb{R}_{\mathcal{F}})$ is a complete metric space.

For the fuzzy-valued function u, v defined on $[a,b]$, we introduce measure $D(u,v) := \sup_{t \in [a,b]} d(u(t), v(t))$. We say that the fuzzy-valued function f is integrable on $[a,b]$ if the function f is continuous in the metric d and its definite integral exist, and we have

$$\left[\int_a^b f(t,u)\,\mathrm{d}x\right]^\alpha = \left[\int_a^b \underline{f}^\alpha(t,u)\,\mathrm{d}x, \int_a^b \overline{f}^\alpha(t,u)\,\mathrm{d}x\right].$$

Definition 1. (See [20].) The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ (gH-difference for short) is defined as follows:

$$u \ominus_{gH} v = w \quad \Longleftrightarrow \quad \begin{array}{l} \text{(i) } u = v \oplus w, \text{ or} \\ \text{(ii) } v = u + (-1)w. \end{array}$$

A function $u : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is called d-increasing (d-decreasing) on $[a, b]$ if for every $\alpha \in [0, 1]$, the function $t \mapsto d([u(t)]^\alpha)$ is nondecreasing (nonincreasing) on $[a, b]$. If u is d-increasing or d-decreasing on $[a, b]$, then we say that u is d-monotone on $[a, b]$.

Definition 2. (See [9].) Let $u : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $t \in (a, b)$. The fuzzy function u is said to be generalized Hukuhara differentiable (gH-differentiable) at t if there exists an element $u'(t) \in E$ such that

$$u'(t) = \lim_{h\to 0} \frac{u(t + h) \ominus_{gH} u(t)}{h}.$$

Denote by $C([a, b], \mathbb{R}_{\mathcal{F}})$ the set of all continuous fuzzy functions, $AC([a, b], \mathbb{R}_{\mathcal{F}})$ the set of all absolutely continuous fuzzy functions on the interval $[a, b]$ with values in $\mathbb{R}_{\mathcal{F}}$.

Theorem 1. (*See* [21].) *If* $u \in AC([a, b], \mathbb{R}_{\mathcal{F}})$ *is a* d*-monotone fuzzy function and* $q \in (0, 1)$*, then*

$${}^{C}D^{q,p}_{a^+}u(t) = \frac{p^q}{\Gamma(1-q)} \int\limits_a^t \left(t^p - s^p\right)^{-q} u'(s)\,\mathrm{d}s, \quad t \in (a, b].$$

Now, we consider the fractional hyperbolic functions and their properties that will be used in the next section. The Mittag-Leffler function frequently used in the solution of fractional-order systems (see [15]), is defined as follows:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \qquad E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}.$$

Lemma 1. (*See* [10].) *Set* $\delta > 0$. *Now* $E_\alpha(\cdot)$ *and* $E_{\alpha,\beta}(\cdot)$ *have the following properties*:

(i) *Let* $0 < \alpha < 1$. *Then* $E_\alpha(-\delta t^\alpha) \leqslant 1$ *and* $E_{\alpha,\alpha}(-\delta t^\alpha) \leqslant 1/\Gamma(\alpha)$. *Moreover,* $E_\alpha(0) = 1$ *and* $E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$;

(ii) *Let* $0 < \alpha \leqslant 1$ *and* $\beta < \alpha + 1$. *Then* $E_\alpha(\cdot)$ *and* $E_{\alpha,\beta}(\cdot)$ *are nonnegative. Additionally, put* $t_1 \leqslant t_2$. *Then* $E_\alpha(\delta t_1^\alpha) \leqslant E_\alpha(\delta t_2^\alpha)$ *and* $E_{\alpha,\beta}(\delta t_1^\alpha) \leqslant E_{\alpha,\beta}(\delta t_2^\alpha)$;

(iii) $\int_0^z E_{\alpha,\beta}(t^\alpha)t^{\beta-1}\,\mathrm{d}t = z^\beta E_{\alpha,\beta+1}(z^\alpha)$, $\alpha > 0$, $z \geqslant 0$.

Theorem 2. (*See* [21].) *For* $\lambda > 0$ *and* u *is* d*-increasing, or* $\lambda < 0$ *and* u *is* d*-decreasing,* (*by applying the definition of the Mittag-Leffler function*) *the solution of problem* (1) *is expressed by*

$$u(t) = u_0 E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right) \oplus \frac{1}{p^{q-1}} \int\limits_0^t s^{p-1}\left(t^p - s^p\right)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) \times f\left(s, u(s)\right)\mathrm{d}s,$$

where $t \in [0,T]$ (for the case of $\lambda > 0$ and u is d-increasing), whereas if $\lambda < 0$ and u is d-decreasing, $t \in [0,T]$, then we obtain the solution of problem (1) *is*

$$u(t) = u_0 E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right) \ominus (-1)\frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p - s^p\right)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) \times f\left(s, u(s)\right) \mathrm{d}s.$$

3 Existence and uniqueness results

Let $C[a,b]$ be the space of all continuous fuzzy-valued functions on $[a,b]$. Consider the following assumptions:

(H1) $f : [0,T] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is continuous;
(H2) there exists $L > 0$ such that $d(f(t,u), f(t,v)) \leqslant L d(u,v)$ for $t \in [0,T]$ and $u, v \in \mathbb{R}_{\mathcal{F}}$;
(H3) $L(p^{-1}T^p)^q E_{q,q+1}(|\lambda|(p^{-1}T^p)^q) < 1$.

Theorem 3. *Assume that $\lambda > 0$, u is d-increasing, and conditions* (H1), (H2) *and* (H3) *are satisfied. Then the initial value problem* (1) *has a unique solution in $C[0,T]$.*

Proof. Consider the operator $A_1 : C[0,T] \to C[0,T]$ given by

$$A_1 u(t) = u_0 E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right) \oplus \frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p - s^p\right)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) \times f\left(s, u(s)\right) \mathrm{d}s,$$

where $t \in [0,T]$, and it is easy to see that u is a solution to the initial value problem (1) if and only if $u = A_1 u$. From Lemma 1 we have

$$\begin{aligned}
&d\left(A_1 u(t), A_1 v(t)\right) \\
&\quad = d\left(\frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p - s^p\right)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) f\left(s, u(s)\right) \mathrm{d}s, \right. \\
&\qquad\qquad \left. \frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p - s^p\right)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) f\left(s, v(s)\right) \mathrm{d}s\right)^q \\
&\quad \leqslant \frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p - s^p\right)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) d\left(f\left(s, u(s)\right), f\left(s, v(s)\right)\right) \mathrm{d}s \\
&\quad \leqslant L\frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p - s^p\right)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) d\left(u(s), v(s)\right) \mathrm{d}s
\end{aligned}$$

$$\begin{aligned}&\leqslant LD(u,v)\frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p-s^p\right)^{q-1}E_{q,q}\left(\lambda\left(\frac{t^p-s^p}{p}\right)^q\right)\mathrm{d}s\\&=L(p^{-1}t^p)^qE_{q,q+1}\left(\lambda\left(p^{-1}t^p\right)^q\right)D(u,v)\end{aligned}$$

for $u,v\in\mathbb{R}_{\mathcal{F}}$ and for each $t\in[0,T]$, which implies that

$$D(A_1u,A_1v)\leqslant L\left(p^{-1}T^p\right)^qE_{q,q+1}\left(|\lambda|\left(p^{-1}T^p\right)^q\right)D(u,v).$$

Therefore, the Banach contraction mapping principle guarantees that $u=A_1u$ has a unique fixed point $u^*\in C[0,T]$, so there is a unique solution to problem (1). The proof is completed. □

Theorem 4. *Assume that $\lambda<0$, u is d-decreasing, conditions* (H1)–(H3) *and the following condition are satisfied*:

(H4) *for any $t\in(0,T]$,*

$$E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right)\underline{u_0}^\alpha\oplus\frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p-s^p\right)^{q-1}E_{q,q}\left(\lambda\left(\frac{t^p-s^p}{p}\right)^q\right)\times\overline{f\left(s,u(s)\right)}^\alpha\,\mathrm{d}s$$

is nonincreasing in α,

$$E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right)\overline{u_0}^\alpha\oplus\frac{1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p-s^p\right)^{q-1}E_{q,q}\left(\lambda\left(\frac{t^p-s^p}{p}\right)^q\right)\times\underline{f\left(s,u(s)\right)}^\alpha\,\mathrm{d}s$$

is nonincreasing in α, for any $\alpha\in[0,1]$ and $t\in(0,T]$,

$$\begin{aligned}&\int_0^t s^{p-1}\left(t^p-s^p\right)^{q-1}E_{q,q}\left(\lambda\left(\frac{t^p-s^p}{p}\right)^q\right)d\left[f\left(s,u(s)\right)\right]^\alpha\,\mathrm{d}s\\&\qquad\leqslant p^{q-1}E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right)d[u_0]^\alpha.\end{aligned}$$

Then the initial value problem (1) *has a unique solution in $C[0,T]$.*

Proof. Consider the operator $A_2:C[0,T]\to C[0,T]$ given by

$$A_2u(t)=u_0E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right)\ominus\frac{-1}{p^{q-1}}\int_0^t s^{p-1}\left(t^p-s^p\right)^{q-1}E_{q,q}\left(\lambda\left(\frac{t^p-s^p}{p}\right)^q\right)\times f\left(s,u(s)\right)\mathrm{d}s,$$

where $t \in [0,T]$. It is easy to see that u is a solution to the initial value problem (1) if and only if $u = A_1 u$. From Lemma 1 we have

$$\begin{aligned}
&d\big(A_2u(t), A_2v(t)\big) \\
&\quad\leqslant \frac{1}{p^{q-1}}\int_0^t s^{p-1}\big(t^p - s^p\big)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) d\big(f\big(s,u(s)\big), f\big(s,v(s)\big)\big)\,\mathrm{d}s \\
&\quad\leqslant LD(u,v)\frac{1}{p^{q-1}}\int_0^t s^{p-1}\big(t^p - s^p\big)^{q-1} E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right)\mathrm{d}s \\
&\quad\leqslant LD(u,v)\frac{1}{p^{q-1}}\int_0^t s^{p-1}\big(t^p - s^p\big)^{q-1} E_{q,q}\left(|\lambda|\left(\frac{t^p - s^p}{p}\right)^q\right)\mathrm{d}s \\
&\quad= L\big(p^{-1}t^p\big)^q E_{q,q+1}\big(|\lambda|\big(p^{-1}t^p\big)^q\big) D(u,v)
\end{aligned}$$

for $u, v \in \mathbb{R}_{\mathcal{F}}$ and for each $t \in [0,T]$, which implies that

$$D(A_2u, A_2v) \leqslant L\big(p^{-1}T^p\big)^q E_{q,q+1}\big(|\lambda|\big(p^{-1}T^p\big)^q\big) D(u,v).$$

Therefore, the Banach contraction mapping principle guarantees that $u = A_2u$ has a unique fixed point $u^* \in C[0,T]$, so there is a unique solution to problem (1). The proof is completed. $\square$

4 Stability results

Motivated by E_α-Ulam-type stability concepts of fractional differential equations (see [33]) and Ulam-type stability concepts of fuzzy differential equations (see [34]), we introduce some new E_q-Ulam-type stability concepts of fuzzy fractional differential equations.

Let $\varepsilon > 0$ and $\phi : [0,T] \to \mathbb{R}_+$ be a continuous function. We consider the equation

$$^{C}D_{0+}^{q,p}x(t) = \lambda x(t) \oplus f\big(t, x(t)\big), \quad t \in (0,T], \tag{2}$$

and the associated three inequalities

$$d\big(^{C}D_{0+}^{q,p}y(t), \lambda y(t) \oplus f\big(t, y(t)\big)\big) \leqslant \varepsilon, \quad t \in [0,T], \tag{3}$$
$$d\big(^{C}D_{0+}^{q,p}y(t), \lambda y(t) \oplus f\big(t, y(t)\big)\big) \leqslant \phi(t), \quad t \in [0,T], \tag{4}$$
$$d\big(^{C}D_{0+}^{q,p}y(t), \lambda y(t) \oplus f\big(t, y(t)\big)\big) \leqslant \varepsilon\phi(t), \quad t \in [0,T]. \tag{5}$$

Definition 3. Equation (2) is E_q-Ulam–Hyers stable if there exists $c > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C[0,T]$ to inequality (3), there exists a solution $x \in C[0,T]$ to Eq. (2) with

$$d\big(y(t), x(t)\big) \leqslant cE_q\big(\gamma_f t^q\big)\varepsilon, \quad \gamma_f \geqslant 0,\ t \in [0,T].$$

Definition 4. Equation (2) is generalized E_q-Ulam–Hyers stable if there exists a continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\theta(0) = 0$ such that for each solution $y \in C[0,T]$ to inequality (3), there exists a solution $x \in C[0,T]$ to Eq. (2) with

$$d\bigl(y(t), x(t)\bigr) \leqslant \theta(\varepsilon) E_q\bigl(\gamma_f t^q\bigr), \quad \gamma_f \geqslant 0,\ t \in [0,T].$$

Definition 5. Equation (2) is E_q-Ulam–Hyers–Rassias stable with respect to ϕ if there exists $c_\phi > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C[0,T]$ to inequality (5) there exists a solution $x \in C[0,T]$ to Eq. (2) with

$$d\bigl(y(t), x(t)\bigr) \leqslant c_\phi \varepsilon \phi(t) E_q\bigl(\gamma_f t^q\bigr), \quad \gamma_f \geqslant 0,\ t \in [0,T].$$

Definition 6. Equation (2) is generalized E_q-Ulam–Hyers–Rassias stable with respect to ϕ if there exists $c_\phi > 0$ such that for each solution $y \in C[0,T]$ to inequality (4) there exists a solution $x \in C[0,T]$ to Eq. (2) with

$$d\bigl(y(t), x(t)\bigr) \leqslant c_\phi \phi(t) E_q\bigl(\gamma_f t^q\bigr), \quad \gamma_f \geqslant 0,\ t \in [0,T].$$

Lemma 2. *A function $y \in C[0,T]$ is a solution of inequality* (5) *with*

(H5) ${}^{C}D_{0+}^{q,p} y(t) \ominus [\lambda y(t) \oplus f(t, y(t))]$ *exists in $\mathbb{R}_{\mathcal{F}}$ for all $t \in [0,T]$ if and only if there exists a function $g \in C[0,T]$ (which depends on y) such that*:

- (i) $d(g(t), \widehat{0}) \leqslant \varepsilon\phi(t)$, $t \in (0,T]$;
- (ii) ${}^{C}D_{0+}^{q,p} y(t) = \lambda y(t) \oplus f(t, y(t)) \oplus g(t)$, $t \in (0,T]$.

Note one can have similar results for inequations (3) *and* (4) *and we omit them here.*

Proof. The sufficiency is obvious and we only prove the necessity. Let

$$g(t) = {}^{C}D_{0+}^{q,p} y(t) \ominus \bigl[\lambda y(t) \oplus f\bigl(t, y(t)\bigr)\bigr], \quad t \in [0,T].$$

Then we get (ii). Additionally, due to

$$\begin{aligned} d\bigl({}^{C}D_{0+}^{q,p} y(t), \lambda y(t) \oplus f\bigl(t, y(t)\bigr)\bigr) &= d\bigl({}^{C}D_{0+}^{q,p} y(t) \ominus \bigl[\lambda y(t) \oplus f\bigl(t, y(t)\bigr)\bigr], \hat{0}\bigr) \\ &= d\bigl(g(t), \hat{0}\bigr) \end{aligned}$$

and inequality (5), we can see (i) holds. The proof is completed. □

Lemma 3. *Let y be a solution of inequality* (5) *with $y(0) = y_0$. Assume that condition* (H5) *is satisfied. Then y satisfies the integral inequality*

$$d\bigl(y(t), B_1(f,t)\bigr) \leqslant \varepsilon p^{1-q} E_{q,q}\bigl(|\lambda| \bigl(p^{-1} t^p\bigr)^q\bigr) \int_0^t \frac{\phi(s)}{(t^p - s^p)^{1-q} s^{1-p}}\, \mathrm{d}s,$$

if $\lambda > 0$ and y is d-increasing, where $t \in [0,T]$ and

$$\begin{aligned} B_1(f,t) := E_{q,1}\biggl(\lambda\biggl(\frac{t^p}{p}\biggr)^q\biggr) y_0 \oplus \frac{1}{p^{q-1}} \int_0^t s^{p-1}\bigl(t^p - s^p\bigr)^{q-1} E_{q,q}\biggl(\lambda\biggl(\frac{t^p - s^p}{p}\biggr)^q\biggr) \\ \times f\bigl(s, y(s)\bigr)\, \mathrm{d}s. \end{aligned}$$

Proof. From Lemma 2 we see y satisfies

$$\begin{aligned} {}^{C}D_{0+}^{q,p}y(t) &= \lambda y(t) \oplus f\big(t, y(t)\big) \oplus g(t), \quad t \in [0, T], \\ y(0) &= y_0 \in \mathbb{R}_{\mathcal{F}}, \end{aligned} \tag{6}$$

if $\lambda > 0$ and y is d-increasing, noticing y is a solution to problem (6), we have

$$y(t) = E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right)y_0 \oplus \frac{1}{p^{q-1}}\int_0^t \frac{E_{q,q}(\lambda(\frac{t^p-s^p}{p})^q)[f(s, y(s)) \oplus g(s)]}{s^{1-p}(t^p - s^p)^{1-q}}\,\mathrm{d}s.$$

Let

$$C_1(g, t) := \frac{1}{p^{q-1}}\int_0^t \frac{E_{q,q}(\lambda(\frac{t^p-s^p}{p})^q)g(s)}{s^{1-p}(t^p - s^p)^{1-q}}\,\mathrm{d}s.$$

Then

$$\begin{aligned} y(t) &= E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right)y_0 \\ &\quad \oplus \frac{1}{p^{q-1}}\left[\int_0^t \frac{E_{q,q}(\lambda(\frac{t^p-s^p}{p})^q)f(s, y(s))}{s^{1-p}(t^p - s^p)^{1-q}}\,\mathrm{d}s \oplus \int_0^t \frac{E_{q,q}(\lambda(\frac{t^p-s^p}{p})^q)g(s)}{s^{1-p}(t^p - s^p)^{1-q}}\,\mathrm{d}s\right] \\ &= B_1(f, t) \oplus C_1(g, t). \end{aligned}$$

From this it follows that

$$\begin{aligned} &d\big(y(t), B_1(f, t)\big) \\ &\quad = d\big(y(t) \oplus C_1(g, t),\ B_1(f, t) \oplus C_1(g, t)\big) = d\big(y(t) \oplus C_1(g, t),\ y(t)\big) \\ &\quad = d\big(C_1(g, t), \hat{0}\big) \leqslant \frac{1}{p^{q-1}}\int_0^t \frac{E_{q,q}(\lambda(\frac{t^p-s^p}{p})^q)d(g(s), \hat{0})}{s^{1-p}\big(t^p - s^p\big)^{1-q}}\,\mathrm{d}s \\ &\quad \leqslant \varepsilon p^{1-q}E_{q,q}\big(|\lambda|\big(p^{-1}t^p\big)^q\big)\int_0^t \frac{\phi(s)}{s^{1-p}(t^p - s^p)^{1-q}}\,\mathrm{d}s. \end{aligned}$$ □

Lemma 4. *Let y be a solution of inequality* (5) *with $y(0) = y_0$. Assume that condition* (H5) *is satisfied. Then y satisfies integral inequality*

$$d\big(y(t), B_2(f, t)\big) \leqslant \varepsilon p^{1-q}E_{q,q}\big(|\lambda|\big(p^{-1}t^p\big)^q\big)\int_0^t \frac{\phi(s)}{(t^p - s^p)^{1-q}s^{1-p}}\,\mathrm{d}s$$

if $\lambda < 0$ and y is d-decreasing, where $t \in [0, T]$ and

$$\begin{aligned} B_2(f, t) := E_{q,1}\left(\lambda\left(\frac{t^p}{p}\right)^q\right)y_0 \ominus \frac{-1}{p^{q-1}}\int_0^t &s^{p-1}\big(t^p - s^p\big)^{q-1}E_{q,q}\left(\lambda\left(\frac{t^p - s^p}{p}\right)^q\right) \\ &\times f\big(s, y(s)\big)\,\mathrm{d}s. \end{aligned}$$

Proof. The proof of Lemma 4 is similar to Lemma 3, so we omit it here. □

Remark 1. One can have similar results to Lemmas 3, 4 inequalities (3) and (4)

Theorem 5. *Assume that* $\lambda > 0$ *and* u *is* d*-increasing, conditions* (H1)–(H3) *and*

(H6) *there exists a nonnegative, nondecreasing and continuous function* ϕ *such that*

$$p^{1-q}E_{q,q}\big(|\lambda|(p^{-1}t^p)^q\big)\int_0^t \frac{\phi(s)}{s^{1-p}\big(t^p-s^p\big)^{1-q}}\,\mathrm{d}s \leqslant C_\phi\phi(t), \quad t\in[0,T],$$

holds.

Suppose also that a function $y\in C[0,T]$ *satisfies inequality* (5) *and condition* (H5) *holds. Then Eq.* (2) *is* E_q*-Ulam–Hyers–Rassias stable.*

Proof. Let x be a solution to problem (1), and denote y as a solution to inequality (5) with $y(0)=u_0$. According to Lemma 3, we have

$$\begin{aligned} d\big(y(t),B_1(f,t)\big) &\leqslant \varepsilon p^{1-q}E_{q,q}\big(|\lambda|(p^{-1}t^p)^q\big)\int_0^t \frac{\phi(s)}{(t^p-s^p)^{1-q}s^{1-p}}\,\mathrm{d}s \\ &\leqslant C_\phi\varepsilon\phi(t). \end{aligned}$$

where $t\in[0,T]$, From condition (H6) it follows that

$$\begin{aligned} d\big(y(t),x(t)\big) &\leqslant d\big(y(t),B_1(f,t)\big)+d\big(B_1(f,t),x(t)\big) \\ &\leqslant C_\phi\varepsilon\phi(t)+\frac{1}{p^{q-1}}\int_0^t \frac{E_{q,q}(\lambda(\frac{t^p-s^p}{p})^q)d(f(s,y(s)),f(s,x(s)))}{s^{1-p}(t^p-s^p)^{1-q}}\,\mathrm{d}s \\ &\leqslant C_\phi\varepsilon\phi(t)+\frac{L}{p^{q-1}}\int_0^t \frac{E_{q,q}(\lambda(\frac{t^p-s^p}{p})^q)d(y(s),x(s))}{s^{1-p}(t^p-s^p)^{1-q}}\,\mathrm{d}s \\ &\leqslant C_\phi\varepsilon\phi(t)+\frac{L}{p^{q-1}}E_{q,q}\left(\lambda\left(\frac{T^p}{p}\right)^q\right)\int_0^t \frac{d(y(s),x(s))}{s^{1-p}(t^p-s^p)^{1-q}}\,\mathrm{d}s, \end{aligned}$$

by the generalized Gröwall inequality (see [30, 34]), and we obtain

$$d(y(t),x(t))\leqslant C_\phi\varepsilon\phi(t)E_q\left(\frac{L\Gamma(q)E_{q,q}(|\lambda|(p^{-1}T^p)^q)t^q}{p^q}\right).$$

Therefore Eq. (2) is E_q-Ulam–Hyers–Rassias stable according to Definition 5. The proof is completed. □

Remark 2. Under the assumptions of Theorem 5, we consider Eq. (2) and inequality (4). One can verify that Eq. (2) is generalized E_q-Ulam–Hyers–Rassias stable according to Definition 6. Under the assumptions except (H6) of Theorem 5, we consider Eq. (2) and inequality (3). One can show that Eq. (2) is E_q-Ulam–Hyers stable and generalized E_q-Ulam–Hyers stable according to Definitions 3 and 4, respectively.

Theorem 6. *Assume that $\lambda < 0$ and u is d-decreasing, conditions* (H1)–(H3) *are satisfied. Suppose also that a function $y \in C[0, T]$ satisfies inequality* (5) *and* (H5), (H6) *hold. Then Eq.* (2) *is E_q-Ulam–Hyers–Rassias stable.*

Proof. The proof of is similar to Theorem 5, so we omit it here. □

Remark 3. Under the assumptions of Theorem 6, we consider Eq. (2) and inequality (4). One can verify that Eq. (2) is generalized E_q-Ulam–Hyers–Rassias stable according to Definition 6. Under the assumptions except (H6) of Theorem 5, we consider Eq. (2) and inequality (3). One can show that Eq. (2) is E_q-Ulam–Hyers stable and generalized E_q-Ulam–Hyers stable according to Definitions 3 and 4, respectively.

5 Examples

Example 1. Consider the following initial value problem for the fuzzy fractional differential equation

$$\begin{aligned} {}^{C}D_{0^+}^{1/2,1}u(t) &= u(t)\left(\frac{1}{2}\sin t \oplus 1\right) \oplus \mathrm{e}^t A, \quad t \in (0, \pi], \\ u(0) &= \hat{0} \in \mathbb{R}_{\mathcal{F}}. \end{aligned} \tag{7}$$

where $\lambda = 1, T = \pi, p = 1, f(t, u(t)) = (1/2)u(t)\sin t \oplus \mathrm{e}^t A$, and $A = (1, 2, 3) \in \mathbb{R}_{\mathcal{F}}$ is a symmetric triangular fuzzy number. Take $L = 1/2$, clearly, conditions (H1)–(H3) hold, then according to Theorem 3, problem (7) has a unique solution.

Assume that a fuzzy-valued function $u : (0, \pi] \to \mathbb{R}_{\mathcal{F}}$ satisfies

$$d\left({}^{C}D_{0^+}^{1/2,1}u(t),\, u(t)\left(\frac{1}{2}\sin t + 1\right) \oplus \mathrm{e}^t A\right) \leqslant \varepsilon t.$$

Take $\phi(t) = t$ and $C_\phi = (4\pi^{1/2}/3)E_{1/2,1/2}(\pi^{1/2})$. Then we have

$$E_{1/2,1/2}(t^{1/2}) \int_0^t \frac{\phi(s)}{s^{1-p}(t^p - s^p)^{1-q}}\,\mathrm{d}s = \frac{4t^{3/2}}{3} E_{1/2,1/2}(t^{1/2}) \leqslant C_\phi \phi(t),$$

which means condition (H6) holds. Thus Eq. (7) is E_q-Ulam–Hyers–Rassias stable according to Theorem 5.

Example 2. Consider the following initial value problem for fuzzy fractional differential equation:

$$\begin{aligned} {}^{C}D^{1/2,1}_{0^+}u(t) &= -u(t) + t + 1, \quad t \in (0, \pi], \\ u(0) &= u_0 \in \mathbb{R}_{\mathcal{F}}. \end{aligned} \tag{8}$$

where $\lambda = -1$, $p = 1$, $f(t, u(t)) = t + 1$, and $u_0 = (1, 2, 3) \in \mathbb{R}_{\mathcal{F}}$ is a symmetric triangular fuzzy number. Take $L = 1/2$, obviously, conditions (H1)–(H3) holds, then according to Theorem 4, problem (8) has a unique solution.

Assume that a fuzzy-valued function $u : (0, \pi] \to \mathbb{R}_{\mathcal{F}}$ satisfies

$$d\bigl({}^{C}D^{1/2,1}_{0^+}u(t),\, -u(t) + t + 1\bigr) \leqslant \varepsilon t.$$

Take $\phi(t) = t$ and $C_\phi = (4\pi^{1/2}/3)E_{1/2,1/2}(\pi^{1/2})$. Then we have

$$E_{1/2,1/2}\bigl(t^{1/2}\bigr)\int\limits_0^t \frac{\phi(s)}{s^{1-p}(t^p - s^p)^{1-q}}\,\mathrm{d}s = \frac{4t^{3/2}}{3}E_{1/2,1/2}\bigl(t^{1/2}\bigr) \leqslant C_\phi\phi(t),$$

which means condition (H5) holds, thus Eq. (8) is E_q-Ulam–Hyers–Rassias stable according to Theorem 6.

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