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On a new variant of $\mathcal{F}$-contractive mappings with application to fractional differential equations

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Abstract. The present article intends to prove the existence of best proximity points (pairs) using the notion of measure of noncompactness. We introduce generalized classes of cyclic (noncyclic) $\mathcal{F}$-contractive operators, and then derive best proximity point (pair) results in Banach (strictly convex Banach) spaces. This work includes some of the recent results as corollaries. We apply these conclusions to prove the existence of optimum solutions for a system of Hilfer fractional differential equations.

Keywords: best proximity point, measure of noncompactness, $\mathcal{F}$-contractive operator, fractional differential equation.

1 Introduction and preliminaries

1.1 Measure of noncompactness

We start with listing of some notations and preliminaries that we shall need to express our results. Throughout the paper, we denote $\mathbb{R}$ = the set of real numbers, $\mathbb{N}$ = the set of natural numbers, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. Let $(\mathbb{X}, \|\cdot\|)$ be a real Banach space with zero element θ. By $\mathcal{B}(x, r)$ we denote the closed ball centered at x with radius r. The symbol $\mathcal{B}_r$ stands for the ball $\mathcal{B}(\theta, r)$. If $\mathcal{Z}$ is a nonempty subset of $\mathbb{X}$, then $\overline{\mathcal{Z}}$ and $\overline{\operatorname{conv}}\mathcal{Z}$ denote the closure and closed convex hull of $\mathcal{Z}$, respectively, and $\operatorname{diam}\mathcal{Z}$ as diameter of the set $\mathcal{Z}$. Moreover, let us denote by $\mathfrak{M}_{\mathbb{X}}$ the family of all nonempty and bounded subsets of $\mathbb{X}$ and by $\mathfrak{N}_{\mathbb{X}}$ its subfamily consisting of all relatively compact sets. We also denote $\Lambda(\mathbb{X})$ as a family of all nonempty, bounded, closed and convex subsets of $\mathbb{X}$.

We now recall the concept of measure of noncompactness.

Definition 1. (See [5].) A mapping $\aleph : \mathfrak{M}_{\mathbb{X}} \to \mathbb{R}^+$ is said to be a measure of noncompactness (MNC for brief) in $\mathbb{X}$ if it satisfies the following conditions:

(i) The family $\ker \aleph = \{X \in \mathfrak{M}_{\mathbb{X}}\colon\ \aleph(X) = 0\}$ is nonempty, and $\ker \aleph \subset \mathfrak{N}_{\mathbb{X}}$;
(ii) Monotonicity: $X \subset Y \Rightarrow \aleph(X) \leqslant \aleph(Y)$;
(iii) Invariance under closure: $\aleph(\overline{X}) = \aleph(X)$;
(iv) Invariance under passage to the convex hull: $\aleph(\overline{\mathrm{conv}}X) = \aleph(X)$,
(v) Convexity: $\aleph(\lambda X + (1-\lambda)Y) \leqslant \lambda\aleph(X) + (1-\lambda)\aleph(Y)$ for $\lambda \in [0,1]$;
(vi) $\aleph(X \cup Y) = \max\{\aleph(X), \aleph(Y)\}$, where $X, Y \in \mathfrak{M}_{\mathbb{X}}$;
(vii) Cantor's intersection property: If $\{X_n\}_{n\geqslant 1}$ is a sequence of nonempty, closed sets in $\mathfrak{M}_{\mathbb{X}}$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \dots)$ and $\lim_{n\to\infty} \aleph(X_n) = 0$, then the set $X_\infty := \bigcap_{n=1}^{\infty} X_n$ is nonempty and compact.

The family $\ker \aleph$ defined in axiom (i) is called the kernel of the MNC $\aleph$.

One of the properties of the MNC is $X_\infty \in \ker \aleph$. Indeed, from the inequality $\aleph(X_\infty) \leqslant \aleph(X_n)$ for $n = 1, 2, 3, \dots$, we infer that $\aleph(X_\infty) = 0$.

The well-known measure of noncompactness is due to Kuratowski [15], which is the map $\alpha : \mathfrak{M}_{\mathbb{X}} \to \mathbb{R}^+$ given as

$$\alpha(\mathcal{Q}) = \inf\left\{\epsilon > 0\colon\ \mathcal{Q} \subset \bigcup_{k=1}^{n} S_k,\ S_k \subset E,\ \operatorname{diam} S_k < \epsilon\ (k \in \mathbb{N})\right\}.$$

In 1930, Schauder [20] generalized Brouwer's fixed point theorem to Banach spaces as follows.

Theorem 1. *Let $\mathcal{Z} \in \Lambda(\mathbb{X})$ be a unbounded subset of a Banach space $\mathbb{X}$. Then every compact, continuous map $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ has at least one fixed point.*

We recall that the mapping $T : \mathcal{Z} \to \mathbb{Y}$ is said to be a compact operator if T is continuous and maps bounded sets into relatively compact sets, where $\mathbb{X}$ and $\mathbb{Y}$ are normed linear spaces, and $\mathcal{Z}$ is a subset of $\mathbb{X}$.

In 1955, Darbo [8] used the notion of measure of noncompactness to establish an extension of Schauder's fixed point problem as below.

Theorem 2. *Let $\mathcal{Z} \in \Lambda(\mathbb{X})$ be a subset of a Banach space $\mathbb{X}$, and let $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a continuous and $\aleph$-set contraction operator, that is, there exists a constant $\lambda \in [0,1)$ with*

$$\aleph(\mathcal{T}\mathcal{W}) \leqslant \lambda\aleph(\mathcal{W})$$

for any $\emptyset \neq \mathcal{W} \subset \mathcal{Z}$, where $\aleph$ is an MNC on $\mathbb{X}$. Then $\mathcal{T}$ has a fixed point.

The following well-known theorem was proved in 1967 by Sadovskii [19], it is a generalization of Darbo's fixed point theorem.

Theorem 3. *Let $\mathcal{Z} \in \Lambda(\mathbb{X})$ be a subset of a Banach space $\mathbb{X}$, and let $\mathcal{T}\colon\mathcal{Z} \to \mathcal{Z}$ be a continuous and $\aleph$-condensing operator, that is,*

$$\aleph(\mathcal{W}) > 0 \quad\Longrightarrow\quad \aleph\big(\mathcal{T}(\mathcal{W})\big) < \aleph(\mathcal{W})$$

for any $\emptyset \neq \mathcal{W} \subset \mathcal{Z}$, where $\aleph$ is an MNC on $\mathbb{X}$. Then $\mathcal{T}$ has a fixed point.

1.2 Best proximity theory

It is well understood that a mapping $\mathcal{T}$ on a nonempty subset $\mathcal{A}$ of $\mathbb{X}$ possesses a fixed point if $\mathcal{A} \cap \mathcal{T}(\mathcal{A})$ is nonempty. If $\mathcal{T}$ is fixed point free, then in this case, we intend to find the element ϑ in $\mathcal{A}$ so that ϑ and $\mathcal{T}\vartheta$ have smallest distance. In this case, the point ϑ is a best approximant for $\mathcal{T}$. The credit of pioneering best approximation theory goes to Ky Fan (1969) (refer [6] and references therein for more details of best approximation theory). But the problem arises when $\mathcal{A}$ is mapped into another subset $\mathcal{B}$ of $\mathbb{X}$ by $\mathcal{T}$. In this case the problem is to find a point, which estimates the distance between these two sets $\mathcal{A}$ and $\mathcal{B}$. Such points are known as *best proximity points*.

Let us take two nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{X}$. It is to be assume that a pair $(\mathcal{A}, \mathcal{B})$ satisfies a property if $\mathcal{A}$ and $\mathcal{B}$ individually satisfy that property. For example, we say a pair $(\mathcal{A}, \mathcal{B})$ is compact if and only if $\mathcal{A}$ and $\mathcal{B}$ are compact. For the pair $(\mathcal{A}, \mathcal{B})$, we will define

$$\mathcal{A}_0 = \bigl\{p \in \mathcal{A}\colon \exists q' \in \mathcal{B} \bigm| \|p - q'\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})\bigr\},$$
$$\mathcal{B}_0 = \bigl\{q \in \mathcal{B}\colon \exists p' \in \mathcal{A} \bigm| \|p' - q\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})\bigr\}.$$

It is worth noticing that the pair $(\mathcal{A}_0, \mathcal{B}_0)$ may be empty, but in particular, if $(\mathcal{A}, \mathcal{B})$ is a nonempty, convex and weakly compact pair in $\mathbb{X}$, then $(\mathcal{A}_0, \mathcal{B}_0)$ is also nonempty, convex and weakly compact. If $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{B}_0 = \mathcal{B}$, then the pair $(\mathcal{A}, \mathcal{B})$ is called *proximinal*.

A mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is called *cyclic* if $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{A}$, and if $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{A}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{B}$, then $\mathcal{T}$ is *noncyclic*. $\mathcal{T}$ is called *relatively nonexpansive* if it satisfies $\|\mathcal{T}p - \mathcal{T}q\| \leqslant \|p - q\|$ whenever $p \in \mathcal{A}$ and $q \in \mathcal{B}$. In special case, if $\mathcal{A} = \mathcal{B}$, then $\mathcal{T}$ is called nonexpansive self-mapping. We consider a best proximity point for a cyclic mapping T, which is defined as a point $\varrho \in \mathcal{A} \cup \mathcal{B}$ satisfying

$$\|\varrho - \mathcal{T}\varrho\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}) = \inf\bigl\{\|p - q\|\colon p \in \mathcal{A},\ q \in \mathcal{B}\bigr\}.$$

In case of a noncyclic mapping $\mathcal{T}$, we consider existence of a pair $(q, p) \in (\mathcal{A}, \mathcal{B})$ for which $q = \mathcal{T}q$, $p = \mathcal{T}p$ and $\|q - p\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. Such pairs are called *best proximity pairs*.

Eldred et al. in [9] coined the idea of cyclic (noncyclic) relatively nonexpansive mappings and obtained best proximity point (pair) results. In doing so, they have used the concept, which is called as proximal normal structure (in short, PNS). In 2017, Gabeleh [11] proved that every convex and compact (nonempty) pair in a Banach space has PNS by using a concept of proximal diametral sequences. Considering this fact, Gabeleh obtains following result. Recall that the compactness of $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ means that $(\overline{\mathcal{T}(\mathcal{A})}, \overline{\mathcal{T}(\mathcal{B})})$ is compact.

Theorem 4. (*See* [12].) *Let $\mathbb{X}$ be a Banach space, and let $(\mathcal{A}, \mathcal{B}) \in \Lambda(\mathcal{X}) \times \Lambda(\mathcal{X})$. Assume that $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a relatively nonexpansive cyclic mapping, then $\mathcal{T}$ has a best proximity point, provided $\mathcal{T}$ is compact and $\mathcal{A}_0 \neq \emptyset$.*

Before stating the result for noncyclic mappings, let us recall a mathematical concept of strict convexity of Banach spaces. A Banach space $\mathbb{X}$ is strictly convex if for $p, q, x \in \mathbb{X}$ and $M > 0$,

$$\left[\|p - x\| \leqslant M,\ \|q - x\| \leqslant M,\ p \neq q\right] \quad \Longrightarrow \quad \left\|\frac{p + q}{2} - x\right\| < M,$$

holds. The L^p space ($1 < p < \infty$) and Hilbert spaces are examples of strictly convex Banach spaces.

Theorem 5. (*See* [12].) *Let $\mathbb{X}$ be a strictly convex Banach space, and let $(\mathcal{A}, \mathcal{B}) \in \Lambda(\mathcal{X}) \times \Lambda(\mathcal{X})$. Assume that $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a relatively nonexpansive noncyclic mapping. If $\mathcal{T}$ is compact and $\mathcal{A}_0 \neq \emptyset$, then $\mathcal{T}$ has a best proximity pair*

Recently, several works appeared (see [12–14, 16, 18, 21]) in which best proximity point (pair) results are obtained using measure of noncompactness.

1.3 Concepts from fractional calculus

We present some concepts and outcomes from fractional calculus, which will be used in application part of this article. Let $-\infty < a < b < \infty$. Let $C[a, b]$ denotes the space of all continuous functions on $[a, b]$. We denote by $L^m(a, b)$, $m \geqslant 1$, the spaces of Lebesgue-integrable functions on (a, b). See [10] for more details on fractional calculus.

The left-sided Riemann–Liouville fractional integrals and derivatives are defined as follows.

Definition 2. Let $f \in L^1(a, b)$. The integral

$$I^p_{a^+} f(x) = \frac{1}{\Gamma(p)} \int\limits_a^x (x - s)^{p-1} f(s)\, \mathrm{d}s, \quad x > a,\ p > 0,$$

is called left-sided Riemann–Liouville fractional integral of order p of the function f.

Definition 3. The left-sided Riemann–Liouville fractional derivative of order p of f is defined as the following expression:

$$D^p_{a^+} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} I^p_{a^+} f(x), \quad x > a,\ 0 < p < 1,$$

provided the right-hand side exists.

We have following results for above power functions.

Lemma 1. *For $x > a$, we have*

$$\left[I^p_{a^+}(t - a)^{q-1}\right](x) = \frac{\Gamma(q)}{\Gamma(p + q)}(x - a)^{p+q-1}, \quad p \geqslant 0,\ q > 0,$$

$$\left[D^p_{a^+}(t - a)^{p-1}\right](x) = 0, \quad 0 < p < 1.$$

Lemma 2. *For $p, q \geqslant 0$ and $f \in L^1(a, b)$, we have*

$$I^p_{a^+} I^q_{a^+} f(x) = I^{p+q}_{a^+} f(x), \quad \text{a.e. } x \in [a, b].$$

Definition 4. (See [10].) The left-sided Hilfer fractional derivative operator of order $0 < p < 1$ and type $0 \leqslant q \leqslant 1$ is defined by

$$D^{p,q}_{a^+} = I^{q(1-p)}_{a^+} D I^{(1-p)(1-q)}_{a^+}, \quad D = \frac{\mathrm{d}}{\mathrm{d}x}.$$

Remark 1. The Hilfer derivative is considered as an interpolator between the Riemann–Liouville and Caputo derivative since

$$D^{p,q}_{a^+} = \begin{cases} D^p_{a^+}, & q = 0, \\ I^{1-p}_{a^+} D, & q = 1. \end{cases}$$

The differential equations with fractional derivatives gain a lot of importance in recent years. For proving existence of solutions for such equations, the fixed point theory and the concept of measure of noncompactness is of immense importance. For more applications of fixed point theorems and MNC, we refer the readers to following works [1–3, 22] and references therein.

In this article, we first present the results proving existence of best proximity points (pairs) for some new variants of $\mathcal{F}$-contractive mappings. These conclusions extend some of recent results in the literature. As an application, we prove existence of optimum solutions for the differential equations of arbitrary fractional order involving the left-sided Hilfer fractional differential operator.

2 Main results

We start with defining the following notion introduced in [17, 24].

Definition 5. Let $\mathcal{F}$ be a family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that:

(F1) F is strictly increasing;
(F2) for each sequences $\{\xi_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \xi_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\xi_n) = -\infty$.

Moreover, $\varPi$ denotes the set of all mappings $\tau : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\liminf_{t\to s^+} \tau(t) > 0 \quad \text{for all } s \in \mathbb{R}^+.$$

We refer the interested readers to the chapter [23] for review of class of F-contractive conditions. The authors give fixed point existence result established by using such contraction condition together with measure of noncompactness. Moreover, the applicability of these results in the theory of functional equations is discussed.

We define a new notion of cyclic (noncyclic) contractive operator using these two classes of functions. Throughout this section, $\aleph$ is an MNC on $\mathbb{X}$, and $(\mathcal{A}, \mathcal{B}) \in \varLambda(\mathcal{X}) \times \varLambda(\mathcal{X})$.

Definition 6. An operator $\mathcal{T}:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ is said to be cyclic (noncyclic) $\mathcal{F}-\varPi-\varphi$-contractive if there exist $F\in\mathcal{F}$, $\tau\in\varPi$ and a lower semi-continuous function $\varphi:[0,\infty)\to[0,\infty)$ such that $\min\{\aleph(\mathcal{T}\mathcal{K}_1),\aleph(\mathcal{T}\mathcal{K}_2)\}>0$ implies

$$\begin{aligned}&\tau\bigl(\aleph(\mathcal{K}_1\cup\mathcal{K}_2)\bigr)+F\bigl(\aleph(\mathcal{T}\mathcal{K}_1\cup\mathcal{T}\mathcal{K}_2)+\varphi\bigl(\aleph(\mathcal{T}\mathcal{K}_1\cup\mathcal{T}\mathcal{K}_2)\bigr)\bigr)\\&\quad\leqslant F\bigl(\aleph(\mathcal{K}_1\cup\mathcal{K}_2)+\varphi\bigl(\aleph(\mathcal{K}_1\cup\mathcal{K}_2)\bigr)\bigr)\end{aligned}$$

for every proximinal and $\mathcal{T}$ invariant pair $\varLambda(\mathbb{X})\times\varLambda(\mathbb{X})\ni(\mathcal{K}_1,\mathcal{K}_2)\subseteq(\mathcal{A},\mathcal{B})$ with $\operatorname{dist}(\mathcal{K}_1,\mathcal{K}_2)=\operatorname{dist}(\mathcal{A},\mathcal{B})$.

If $\varphi=0$, then the operator $\mathcal{T}$ is called a cyclic (noncyclic) $\mathcal{F}-\varPi$-contractive operator. We now state the first main existence result.

Theorem 6. *Let $\mathbb{X}$ be a Banach space, and let $\mathcal{T}:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ be a relatively nonexpansive cyclic $\mathcal{F}-\varPi-\varphi$-contractive operator. If $\mathcal{A}_0\neq\emptyset$, then $\mathcal{T}$ has a best proximity point.*

Proof. Note that $(\mathcal{A}_0,\mathcal{B}_0)\in\varLambda(\mathbb{X})\times\varLambda(\mathbb{X})$ is proximinal. Also if $p\in\mathcal{A}_0$, there exists an element $q\in\mathcal{B}_0$ such that $\|p-q\|=\operatorname{dist}(\mathcal{A},\mathcal{B})$. Since $\mathcal{T}$ is relatively nonexpansive,

$$\|\mathcal{T}p-\mathcal{T}q\|\leqslant\|p-q\|=\operatorname{dist}(\mathcal{A},\mathcal{B}),$$

which gives $\mathcal{T}p\in\mathcal{B}_0$, that is, $\mathcal{T}(\mathcal{A}_0)\subseteq\mathcal{B}_0$. Similarly, $\mathcal{T}(\mathcal{B}_0)\subseteq\mathcal{A}_0$, and so $\mathcal{T}$ is cyclic on $\mathcal{A}_0\cup\mathcal{B}_0$.

We start with assumption $\mathcal{P}_0=\mathcal{A}_0$ and $\mathcal{Q}_0=\mathcal{B}_0$ and define a sequence pair $\{(\mathcal{P}_n,\mathcal{Q}_n)\}$ as $\mathcal{P}_n=\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{P}_{n-1}))$ and $\mathcal{Q}_n=\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_{n-1}))$ for all $n\geqslant1$. We claim that

$$\mathcal{P}_{n+1}\subseteq\mathcal{Q}_n\subseteq\mathcal{P}_{n-1}\quad\forall n\in\mathbb{N}.$$

We have $\mathcal{Q}_1=\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_0))=\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{B}_0))\subseteq\overline{\operatorname{conv}}\mathcal{A}_0=\mathcal{A}_0=\mathcal{P}_0$. Therefore, $\mathcal{T}(\mathcal{Q}_1)\subseteq\mathcal{T}(\mathcal{P}_0)$. So $\mathcal{Q}_2=\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_1))\subseteq\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{P}_0))=\mathcal{P}_1$. Continuing this pattern, we get $\mathcal{Q}_n\subseteq\mathcal{P}_{n-1}$ by using induction. Similarly, we can see that $\mathcal{P}_{n+1}\subseteq\mathcal{Q}_n$ for all $n\in\mathbb{N}$. Thus $\mathcal{P}_{n+2}\subseteq\mathcal{Q}_{n+1}\subseteq\mathcal{P}_n\subseteq\mathcal{Q}_{n-1}$ for all $n\in\mathbb{N}$. Hence, we get a decreasing sequence $\{(\mathcal{P}_{2n},\mathcal{Q}_{2n})\}$ of nonempty, closed and convex pairs in $\mathcal{A}_0\times\mathcal{B}_0$. Moreover, $\mathcal{T}(\mathcal{Q}_{2n})\subseteq\mathcal{T}(\mathcal{P}_{2n-1})\subseteq\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{P}_{2n-1}))=\mathcal{P}_{2n}$ and $\mathcal{T}(\mathcal{P}_{2n})\subseteq\mathcal{T}(\mathcal{Q}_{2n-1})\subseteq\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_{2n-1}))=\mathcal{Q}_{2n}$. Therefore for all $n\in\mathbb{N}$, the pair $(\mathcal{P}_{2n},\mathcal{Q}_{2n})$ is $\mathcal{T}$-invariant. By a similar manner we can see that $(\mathcal{P}_{2n-1},\mathcal{Q}_{2n-1})$ is also $\mathcal{T}$-invariant for all $n\in\mathbb{N}$.

Besides, if $(\nu,\vartheta)\in\mathcal{A}_0\times\mathcal{B}_0$ is such that $\|\nu-\vartheta\|=\operatorname{dist}(\mathcal{A},\mathcal{B})$, then $(\mathcal{T}^{2n}\nu,\mathcal{T}^{2n}\vartheta)\in\mathcal{P}_{2n}\times\mathcal{Q}_{2n}$ and

$$\operatorname{dist}(\mathcal{P}_{2n},\mathcal{Q}_{2n})\leqslant\bigl\|\mathcal{T}^{2n}\nu-\mathcal{T}^{2n}\vartheta\bigr\|\leqslant\|\nu-\vartheta\|=\operatorname{dist}(\mathcal{A},\mathcal{B}).$$

Next, we show that the pair $(\mathcal{P}_n,\mathcal{Q}_n)$ is proximinal using mathematical induction. Obviously, for $n=0$, the pair $(\mathcal{P}_0,\mathcal{Q}_0)$ is proximinal. Suppose that $(\mathcal{P}_k,\mathcal{Q}_k)$ is proximinal. We show that $(\mathcal{P}_{k+1},\mathcal{Q}_{k+1})$ is also proximinal. Let x be an arbitrary member in $\mathcal{P}_{k+1}=\overline{\operatorname{conv}}(\mathcal{T}(\mathcal{P}_k))$. Then it is represented as $x=\sum_{l=1}^m\lambda_l\mathcal{T}(x_l)$ with $x_l\in\mathcal{P}_k$, $m\in\mathbb{N}$, $\lambda_l\geqslant0$

and $\sum_{l=1}^{m} \lambda_l = 1$. Due to proximinality of the pair $(\mathcal{P}_k, \mathcal{Q}_k)$, there exists $y_l \in \mathcal{Q}_k$ for $1 \leqslant l \leqslant m$ such that $\|x_l - y_l\| = \operatorname{dist}(\mathcal{P}_k, \mathcal{Q}_k) = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. Take $y = \sum_{l=1}^{m} \lambda_l \mathcal{T}(y_l)$. Then $y \in \overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_k)) = \mathcal{Q}_{k+1}$ and

$$\|x - y\| = \left\| \sum_{l=1}^{m} \lambda_l \mathcal{T}(x_l) - \sum_{l=1}^{m} \lambda_l \mathcal{T}(y_l) \right\| \leqslant \sum_{l=1}^{m} \lambda_l \|x_l - y_l\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}).$$

This means that the pair $(\mathcal{P}_{k+1}, \mathcal{Q}_{k+1})$ is proximinal, and induction does the rest to prove that $(\mathcal{P}_n, \mathcal{Q}_n)$ is proximinal for all $n \in \mathbb{N}$.

It is worth noticing that if $\max\{\aleph(\mathcal{P}_{2n_0}), \aleph(\mathcal{Q}_{2n_0})\} = 0$ for some $n_0 \in \mathbb{N}$, then the relatively nonexpansive mapping $\mathcal{T} : \mathcal{P}_{2n_0} \cup \mathcal{Q}_{2n_0} \to \mathcal{P}_{2n_0} \cup \mathcal{Q}_{2n_0}$ is compact, and the result follows from Theorem 4.

So we assume $\max\{\aleph(\mathcal{P}_n), \aleph(\mathcal{Q}_n)\} > 0$ for all $n \in \mathbb{N}$. Since $\tau \in \Pi$, there exists $r > 0$ and $k \in \mathbb{N}$ such that $\tau(\aleph(P_{2n} \cup Q_{2n})) \geqslant r$ for every $n > k$. As $\mathcal{T}$ is $\mathcal{F}$–Π–φ-contractive operator, we have

$$\begin{aligned}
&\tau\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr) + F\bigl(\aleph(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}) + \varphi\bigl(\aleph(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1})\bigr)\bigr) \\
&\quad = \tau\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr) \\
&\qquad + F\bigl(\max\bigl\{\aleph(\mathcal{P}_{2n+1}), \aleph(\mathcal{Q}_{2n+1})\bigr\} + \varphi\bigl(\max\bigl\{\aleph(\mathcal{P}_{2n+1}), \aleph(\mathcal{Q}_{2n+1})\bigr\}\bigr)\bigr) \\
&\quad = \tau\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr) \\
&\qquad + F\bigl(\max\bigl\{\aleph\bigl(\overline{\operatorname{conv}}\bigl(\mathcal{T}(\mathcal{P}_{2n})\bigr)\bigr), \aleph\bigl(\overline{\operatorname{conv}}\bigl(\mathcal{T}(\mathcal{Q}_{2n})\bigr)\bigr)\bigr\} \\
&\qquad + \varphi\bigl(\max\bigl\{\aleph\bigl(\overline{\operatorname{conv}}\bigl(\mathcal{T}(\mathcal{P}_{2n})\bigr)\bigr), \aleph\bigl(\overline{\operatorname{conv}}\bigl(\mathcal{T}(\mathcal{Q}_{2n})\bigr)\bigr)\bigr\}\bigr)\bigr) \\
&\quad = \tau\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr) \\
&\qquad + F\bigl(\max\bigl\{\aleph\bigl(\mathcal{T}(\mathcal{P}_{2n})\bigr), \aleph\bigl(\mathcal{T}(\mathcal{Q}_{2n})\bigr)\bigr\} + \varphi\bigl(\max\bigl\{\aleph\bigl(\mathcal{T}(\mathcal{P}_{2n})\bigr), \aleph\bigl(\mathcal{T}(\mathcal{Q}_{2n})\bigr)\bigr\}\bigr)\bigr) \\
&\quad = \tau\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr) + F\bigl(\aleph\bigl(\mathcal{T}(\mathcal{P}_{2n}) \cup \mathcal{T}(\mathcal{Q}_{2n})\bigr) + \varphi\bigl(\aleph\bigl(\mathcal{T}(\mathcal{P}_{2n}) \cup \mathcal{T}(\mathcal{Q}_{2n})\bigr)\bigr)\bigr) \\
&\quad \leqslant F\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}) + \varphi\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr)\bigr).
\end{aligned}$$

For all $n > k$, we deduce that

$$\begin{aligned}
&F\bigl(\aleph(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}) + \varphi\bigl(\aleph(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1})\bigr)\bigr) \\
&\quad \leqslant F\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}) + \varphi\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr)\bigr) - \tau\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr) \\
&\quad \leqslant F\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}) + \varphi\bigl(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\bigr)\bigr) - r \\
&\quad \leqslant \cdots \\
&\quad \leqslant F\bigl(\aleph(\mathcal{P}_0 \cup \mathcal{Q}_0) + \varphi\bigl(\aleph(\mathcal{P}_0 \cup \mathcal{Q}_0)\bigr)\bigr) - 2(n-k)r,
\end{aligned}$$

that is,

$$\begin{aligned}
&F\bigl(\aleph(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}) + \varphi\bigl(\aleph(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1})\bigr)\bigr) \\
&\quad \leqslant F\bigl(\aleph(\mathcal{P}_0 \cup \mathcal{Q}_0) + \varphi\bigl(\aleph(\mathcal{P}_0 \cup \mathcal{Q}_0)\bigr)\bigr) - 2(n-k)r \quad \text{for all } n > k.
\end{aligned}$$

Therefore, $F(\aleph(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}) + \varphi(\aleph(\mathcal{P}_{2n+1} \cup \mathcal{Q}_{2n+1}))) \to -\infty$ as $n \to \infty$, and by (F2) we must have

$$\lim_{n\to\infty} \aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}) = \lim_{n\to\infty} \varphi\big(\aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n})\big) = 0.$$

That is, $\lim_{n\to\infty} \aleph(\mathcal{P}_{2n} \cup \mathcal{Q}_{2n}) = \max\{\lim_{n\to\infty} \aleph(\mathcal{P}_{2n}), \lim_{n\to\infty} \aleph(\mathcal{Q}_{2n})\} = 0$. Now, let $\mathcal{P}_\infty := \cap_{n=0}^{\infty} \mathcal{P}_{2n}$ and $\mathcal{Q}_\infty := \cap_{n=0}^{\infty} \mathcal{Q}_{2n}$. Using property (vii) of Definition 1, the pair $(\mathcal{P}_\infty, \mathcal{Q}_\infty)$ is nonempty, convex, compact and $\mathcal{T}$-invariant with $\operatorname{dist}(\mathcal{P}_\infty, \mathcal{Q}_\infty) = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. Therefore, $\mathcal{T}$ admits a best proximity point in $\mathcal{P}_\infty \cup \mathcal{Q}_\infty$, and this completes the proof. □

If we put $\varphi = 0$ in Theorem 6, then we have following result for $\mathcal{F}{-}\varPi$-contractive mapping.

Corollary 1. *Let $\mathbb{X}$ be a Banach space, and let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a relatively nonexpansive cyclic $\mathcal{F}{-}\varPi$-contractive operator. If $\mathcal{A}_0 \neq \emptyset$, then $\mathcal{T}$ has a best proximity point.*

Corollary 2. *Let $\mathbb{X}$ be a Banach space, and let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a relatively nonexpansive cyclic operator, which satisfies*

$$\aleph(\mathcal{T}\mathcal{K}_1 \cup \mathcal{T}\mathcal{K}_2) + \varphi\big(\aleph(\mathcal{T}\mathcal{K}_1 \cup \mathcal{T}\mathcal{K}_2)\big) \leqslant \mathrm{e}^{-k}\big[\aleph(\mathcal{K}_1 \cup \mathcal{K}_2) + \varphi\big(\aleph(\mathcal{K}_1 \cup \mathcal{K}_2)\big)\big].$$

If $\mathcal{A}_0 \neq \emptyset$, then $\mathcal{T}$ has a best proximity point.

Proof. If we set $\tau(t) = k$ and $F(t) = \ln(t)$, then the proof follows from Theorem 6. □

It is noteworthy here that if we consider $\varphi = 0$ in above corollary, then we get a particular case of Darbo-type best proximity point theorem.

The second existence result is for relatively nonexpansive noncyclic $\mathcal{F}{-}\varPi{-}\varphi$-contractive operator.

Theorem 7. *Let $\mathbb{X}$ be a strictly convex Banach space, and let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a relatively nonexpansive noncyclic $\mathcal{F}{-}\varPi{-}\varphi$-contractive operator. If $\mathcal{A}_0$ is nonempty, then $\mathcal{T}$ has a best proximity pair.*

Proof. Let $(p, q) \in \mathcal{A}_0 \times \mathcal{B}_0$ be such that $\|p - q\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. Since $\mathcal{T}$ is relatively nonexpansive noncyclic mapping,

$$\|\mathcal{T}p - \mathcal{T}q\| \leqslant \|p - q\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}),$$

which gives $\mathcal{T}p \in \mathcal{A}_0$, that is, $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{A}_0$. Similarly, $\mathcal{T}(\mathcal{B}_0) \subseteq \mathcal{B}_0$ and so $\mathcal{T}$ is noncyclic on $\mathcal{A}_0 \cup \mathcal{B}_0$.

Let us define a pair $(\mathcal{P}_n, \mathcal{Q}_n)$ as $\mathcal{P}_n = \overline{\operatorname{conv}}(\mathcal{T}(\mathcal{P}_{n-1}))$ and $\mathcal{Q}_n = \overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_{n-1}))$, $n \geqslant 1$ with $\mathcal{P}_0 = \mathcal{A}_0$ and $\mathcal{Q}_0 = \mathcal{B}_0$. We have that $\mathcal{Q}_1 = \overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_0)) = \overline{\operatorname{conv}}(\mathcal{T}(\mathcal{B}_0)) \subseteq \mathcal{B}_0 = \mathcal{Q}_0$. Therefore, $\mathcal{T}(\mathcal{Q}_1) \subseteq \mathcal{T}(\mathcal{Q}_0)$. Thus $\mathcal{Q}_2 = \overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_1)) \subseteq \overline{\operatorname{conv}}(\mathcal{T}(\mathcal{Q}_0)) = \mathcal{Q}_1$. Continuing this pattern, we get $\mathcal{Q}_n \subseteq \mathcal{Q}_{n-1}$ by using induction. Similarly, we can see that $\mathcal{P}_n \subseteq \mathcal{P}_{n-1}$ for all $n \in \mathbb{N}$. Hence we get a decreasing sequence $\{(\mathcal{P}_n, \mathcal{Q}_n)\}$ of nonempty, closed and convex pairs in $\mathcal{A}_0 \times \mathcal{B}_0$. Also, $\mathcal{T}(\mathcal{Q}_n) \subseteq \mathcal{T}(\mathcal{Q}_{n-1}) \subseteq \overline{\text{conv}}(\mathcal{T}(\mathcal{Q}_{n-1})) = \mathcal{Q}_n$ and $\mathcal{T}(\mathcal{P}_n) \subseteq \mathcal{T}(\mathcal{P}_{n-1}) \subseteq \overline{\text{conv}}(\mathcal{T}(\mathcal{P}_{n-1})) = \mathcal{P}_n$. Therefore, for all $n \in \mathbb{N}$, the pair $(\mathcal{P}_n, \mathcal{Q}_n)$ is $\mathcal{T}$-invariant. From the proof of Theorem 6 we have $(\mathcal{P}_n, \mathcal{Q}_n)$ is a proximinal pair such that $\operatorname{dist}(\mathcal{P}_n, \mathcal{Q}_n) = \operatorname{dist}(\mathcal{A}, \mathcal{B})$ for all $n \in \mathbb{N} \cup \{0\}$.

Following the proof of Theorem 6, if $\max\{\aleph(\mathcal{P}_{n_0}), \aleph(\mathcal{Q}_{n_0})\} = 0$ for some $n_0 \in \mathbb{N}$, then the relatively nonexpansive mapping $\mathcal{T} : \mathcal{P}_{n_0} \cup \mathcal{Q}_{n_0} \to \mathcal{P}_{n_0} \cup \mathcal{Q}_{n_0}$ is compact, and the result follows from Theorem 5.

So we assume that $\max\{\aleph(\mathcal{P}_n), \aleph(\mathcal{Q}_n)\} > 0$ for all $n \in \mathbb{N}$. In view of the fact that $\tau \in \varPi$, there exist $r > 0$ and $k \in \mathbb{N}$ such that $\tau(\aleph(P_n \cup Q_n)) \geqslant r$ for every $n \geqslant k$. Since $\mathcal{T}$ is $\mathcal{F}-\varPi-\varphi$-contractive operator,

$$
\begin{aligned}
&\tau\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big) + F\big(\aleph(\mathcal{P}_{n+1} \cup \mathcal{Q}_{n+1}) + \varphi\big(\aleph(\mathcal{P}_{n+1} \cup \mathcal{Q}_{n+1})\big)\big) \\
&\quad = \tau\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big) \\
&\qquad + F\big(\max\big\{\aleph(\mathcal{P}_{n+1}), \aleph(\mathcal{Q}_{n+1})\big\} + \varphi\big(\max\big\{\aleph(\mathcal{P}_{n+1}), \aleph(\mathcal{Q}_{n+1})\big\}\big)\big) \\
&\quad = \tau\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big) \\
&\qquad + F\big(\max\big\{\aleph\big(\overline{\text{conv}}\big(\mathcal{T}(\mathcal{P}_n)\big)\big), \aleph\big(\overline{\text{conv}}\big(\mathcal{T}(\mathcal{Q}_n)\big)\big)\big\} \\
&\qquad + \varphi\big(\max\big\{\aleph\big(\overline{\text{conv}}\big(\mathcal{T}(\mathcal{P}_n)\big)\big), \aleph\big(\overline{\text{conv}}\big(\mathcal{T}(\mathcal{Q}_n)\big)\big)\big\}\big)\big) \\
&\quad = \tau\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big) \\
&\qquad + F\big(\max\big\{\aleph\big(\mathcal{T}(\mathcal{P}_n)\big), \aleph\big(\mathcal{T}(\mathcal{Q}_n)\big)\big\} + \varphi\big(\max\big\{\aleph\big(\mathcal{T}(\mathcal{P}_n)\big), \aleph\big(\mathcal{T}(\mathcal{Q}_n)\big)\big\}\big)\big) \\
&\quad = \tau\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big) + F\big(\aleph\big(\mathcal{T}(\mathcal{P}_n) \cup \mathcal{T}(\mathcal{Q}_n)\big) + \varphi\big(\aleph\big(\mathcal{T}(\mathcal{P}_n) \cup \mathcal{T}(\mathcal{Q}_n)\big)\big)\big) \\
&\quad \leqslant F\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n) + \varphi\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big)\big).
\end{aligned}
$$

Thus, for all $n > k$, we obtain

$$
\begin{aligned}
&F\big(\aleph(\mathcal{P}_{n+1} \cup \mathcal{Q}_{n+1}) + \varphi\big(\aleph(\mathcal{P}_{n+1} \cup \mathcal{Q}_{n+1})\big)\big) \\
&\quad \leqslant F\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n) + \varphi\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big)\big) - \tau\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big) \\
&\quad \leqslant F\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n) + \varphi\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big)\big) - r \\
&\quad \leqslant \cdots \\
&\quad \leqslant F\big(\aleph(\mathcal{P}_0 \cup \mathcal{Q}_0) + \varphi\big(\aleph(\mathcal{P}_0 \cup \mathcal{Q}_0)\big)\big) - (n-k)r,
\end{aligned}
$$

that is,

$$
\begin{aligned}
&F\big(\aleph(\mathcal{P}_{n+1} \cup \mathcal{Q}_{n+1}) + \varphi(\aleph(\mathcal{P}_{n+1} \cup \mathcal{Q}_{n+1}))\big) \\
&\quad \leqslant F\big(\aleph(\mathcal{P}_0 \cup \mathcal{Q}_0) + \varphi\big(\aleph(\mathcal{P}_0 \cup \mathcal{Q}_0)\big)\big) - (n-k)r \quad \text{for all } n > k.
\end{aligned}
$$

This implies that $F(\aleph(\mathcal{P}_{n+1} \cup \mathcal{Q}_{n+1}), \varphi(\aleph(\mathcal{P}_{n+1} \cup \mathcal{Q}_{n+1}))) \to -\infty$ as $n \to \infty$, and by (F2) we have

$$
\lim_{n\to\infty} \aleph(\mathcal{P}_n \cup \mathcal{Q}_n) = \lim_{n\to\infty} \varphi\big(\aleph(\mathcal{P}_n \cup \mathcal{Q}_n)\big) = 0.
$$

Thereby, $\lim_{n\to\infty} \aleph(\mathcal{P}_n \cup \mathcal{Q}_n) = \max\{\lim_{n\to\infty} \aleph(\mathcal{P}_n), \lim_{n\to\infty} \aleph(\mathcal{Q}_n)\} = 0$. Now, let $\mathcal{P}_\infty := \cap_{n=0}^{\infty} \mathcal{P}_n$ and $\mathcal{Q}_\infty := \cap_{n=0}^{\infty} \mathcal{Q}_n$. Using property (vii) of Definition 1, the pair $(\mathcal{P}_\infty, \mathcal{Q}_\infty)$ is nonempty, convex, compact and $\mathcal{T}$-invariant with $\operatorname{dist}(\mathcal{P}_\infty, \mathcal{Q}_\infty) = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. Therefore, $\mathcal{T}$ has a best proximity pair. □

If we set $\varphi = 0$ in Theorem 7, then we have the following result for $\mathcal{F}-\varPi$-contractive mapping.

Corollary 3. *Let $\mathbb{X}$ be a strictly convex Banach space, and let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a relatively nonexpansive noncyclic $\mathcal{F}-\varPi$-contractive operator. If $\mathcal{A}_0$ is nonempty, then $\mathcal{T}$ has a best proximity pair.*

Corollary 4. *Let $\mathbb{X}$ be a strictly convex Banach space, and let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a relatively nonexpansive noncyclic operator, which satisfies*

$$\aleph(\mathcal{T}\mathcal{K}_1 \cup \mathcal{T}\mathcal{K}_2) + \varphi\big(\aleph(\mathcal{T}\mathcal{K}_1 \cup \mathcal{T}\mathcal{K}_2)\big) \leqslant \mathrm{e}^{-k}\big[\aleph(\mathcal{K}_1 \cup \mathcal{K}_2) + \varphi\big(\aleph(\mathcal{K}_1 \cup \mathcal{K}_2)\big)\big].$$

If $\mathcal{A}_0 \neq \emptyset$, then $\mathcal{T}$ has a best proximity pair.

Proof. If we set $\tau(t) = k$ and $F(t) = \ln t$, then the proof follows from Theorem 7. □

It is noteworthy here that if we consider $\varphi = 0$ in above corollary, then we get a particular case of Darbo-type best proximity pair theorem.

3 Application

In this section, we establish the existence of an optimal solution of the following problem involving systems of Hilfer fractional differential equations with initial conditions.

Let K and γ be positive real numbers, $\mathcal{I} = [0, K]$, and let $(E, \|\cdot\|)$ be a Banach space. Let $B_1 = B(\alpha_0, \gamma)$ and $B_2 = B(\beta_0, \gamma)$ be closed balls in E, where $\alpha_0, \beta_0 \in E$.

We consider the following system of Hilfer fractional differential equation of arbitrary order with initial conditions:

$$\begin{aligned} &D_{0^+}^{\nu;\mu} x(t) = u\big(t, x(t)\big), \quad t \in (0, K], \\ &I_{0^+}^{(1-\nu)(1-\mu)} x(0) = \alpha_0, \end{aligned} \tag{1}$$

$$\begin{aligned} &D_{0^+}^{\nu;\mu} y(t) = v\big(t, y(t)\big), \quad t \in (0, K], \\ &I_{0^+}^{(1-\nu)(1-\mu)} y(0) = \beta_0, \end{aligned} \tag{2}$$

where $D_{0^+}^{\nu;\mu}$ is the left-sided Hilfer fractional differential operator, $0 \leqslant \nu \leqslant 1$, $0 < \mu < 1$; the state $x(\cdot)$ takes the values from Banach space E; $u : \mathcal{I} \times B_1 \to E$ and $v : \mathcal{I} \times B_2 \to E$ are given mappings satisfying some assumptions. The following result establishes the equivalence of (1) with the integral equation.

Lemma 3. (*See* [10].) *The initial value problem* (1) *is equivalent to the following integral equation*:

$$x(t) = \frac{\alpha_0}{\Gamma(\nu(1-\mu)+\mu)} t^{(\nu-1)(1-\mu)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} u\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in \mathcal{I}.$$

Let $\mathcal{J} \subseteq \mathcal{I}$, and let $S = C(\mathcal{J}, E)$ be a Banach space of continuous mappings from $\mathcal{J}$ into E endowed with supremum norm. Let

$$S_1 = \big\{x \in C(\mathcal{J}, B_1)\colon\ I^{(1-\nu)(1-\mu)} x(0) = \alpha_0\big\},$$
$$S_2 = \big\{y \in C(\mathcal{J}, B_2)\colon\ I^{(1-\nu)(1-\mu)} x(0) = \beta_0\big\}.$$

So (S_1, S_2) is a nonempty, bounded, closed and convex pair in $S \times S$. Now, for every $\phi \in S_1$ and $\psi \in S_2$, we have $\|\phi - \psi\| = \sup \|\phi(s) - \psi(s)\| \geqslant \|\alpha_0 - \beta_0\|$. Therefore $\operatorname{dist}(S_1, S_2) = \|\alpha_0 - \beta_0\|$, which ensures that $(S_1)_0$ is nonempty. Now, let us define the operator $T : S_1 \cup S_2 \to S$ as follows:

$$Tx(t) = \begin{cases} \frac{\beta_0\, t^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} u(s, x(s))\,\mathrm{d}s, & x \in S_1, \\ \frac{\alpha_0\, t^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} v(s, x(s))\,\mathrm{d}s, & x \in S_2. \end{cases} \tag{3}$$

Lemma 4. *The operator $T : S_1 \cup S_2 \to S$ defined by* (3) *is cyclic if u and v are bounded and continuous such that $u, v \in L^1(0, K)$.*

Proof. Let $x \in S_1$ and set $p = \mu + \nu - \mu\nu$. We have

$$Tx(t) = \frac{\beta_0}{\Gamma(p)} t^{(p-1)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} u\big(s, x(s)\big)\,\mathrm{d}s.$$

Applying $I_{0^+}^{1-p}$ on both sides and applying Lemma 1, we get

$$\begin{aligned} I_{0^+}^{1-p} Tx(t) &= \frac{\beta_0}{\Gamma(p)} I_{0^+}^{1-p}\, t^{(p-1)} + I_{0^+}^{1-p} \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} u\big(s, x(s)\big)\,\mathrm{d}s \\ &= \frac{\beta_0}{\Gamma(p)} \frac{\Gamma(p)}{\Gamma((1-p)+(p-1))} t^{(1-p)+(p-1)} + I_{0^+}^{1-p} I_{0^+}^{\mu} u\big(s, x(s)\big)(t) \\ &= \beta_0 t^0 + \big[I^{1-\nu(1-\mu)} 0^+ u\big(s, x(s)\big)\big](t) \\ &= \beta_0 + \big[I^{1-\nu(1-\mu)} 0^+ u\big(s, x(s)\big)\big](t). \end{aligned}$$

Here $[I^{1-\nu(1-\mu)} 0^+ u(s, x(s))](t) \to 0$ as $t \to 0$ by Lemma 2. Therefore $I_{0^+}^{1-p} Tx(0) = \beta_0$, which means $Tx(t) \in S_2$. Similarly, one can show that $Tx(t) \in S_1$ if $x \in S_2$. Thus T is cyclic operator. □

We say that $z \in S_1 \cup S_2$ is an optimal solution for system (1) and (2), provided that $\|z - Tz\| = \operatorname{dist}(S_1, S_2)$, that is, z is a best proximity point of the operator T defined in (3).

Assumptions. We consider the following hypotheses to prove the existence of optimal solutions to the differential equations.

(A1) Let $\aleph$ be any MNC. For any bounded pair $(N_1, N_2) \subseteq (B_1, B_2)$, there exist $F \in \mathcal{F}$, a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\tau > 0$ such that $\aleph(u(J \times N_2))$, $\aleph(v(J \times N_1)) > 0$ implies

$$\aleph\big(f(\mathcal{J} \times N_1) \cup g(\mathcal{J} \times N_2)\big) < \frac{\Gamma(\mu)}{\tau^{\mu-1}} \aleph(N_1 \cup N_2)$$

and

$$\begin{aligned}&\tau + F\big(\aleph\big(u(J \times N_2) \cup v(J \times N_1)\big) + \varphi\big(\aleph\big(u(J \times N_2) \cup v(J \times N_1)\big)\big)\big)\\&\quad \leqslant F\big(\aleph(N_1 \cup N_2) + \varphi\big(\aleph(N_1 \cup N_2)\big)\big).\end{aligned}$$

(A2) For all $(x, y) \in S_1 \times S_2$,

$$\big\|f\big(t, x(t)\big) - g\big(t, y(t)\big)\big\| \leqslant \frac{\Gamma(\mu+1)}{\tau^{\mu}} \left(\big\|x(t) - y(t)\big\| - \frac{\tau^{p-1}}{p} \|\beta_0 - \alpha_0\|\right).$$

The following result is the mean-value theorem for fractional differential, which we have rewritten according to our notations.

Theorem 8. (*See* [7].) *Let $\mathcal{I}$, u, $q > 0$ and E be given as above. Let u be integrable on $\mathcal{I}$, and let m and M be the infimum and supremum of u, respectively, on $\mathcal{I}$. Then there exists a point ζ in $\mathcal{I}$ such that*

$$\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u\big(s, x(s)\big)\, \mathrm{d}s = \frac{t^{q-1}}{\Gamma(q)} u\big(\zeta, x(\zeta)\big).$$

Then we give the following result.

Theorem 9. *Under notations defined above, the hypotheses of Lemma* 4 *and assumptions* (A1) *and* (A2)*, the system of Hilfer fractional differential equation* (1)–(2) *has an optimal solution.*

Proof. It is clear that system (1)–(2) has an optimal solution if the operator T defined in (3) has a best proximity point.

From Lemma 4, T is a cyclic operator. It follows trivially that $T(S_1)$ is a bounded subset of S_2. We prove that $T(S_1)$ is also an equicontinuous subset of S_2. For $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in S_1$, we observe that

$$\begin{aligned}&\big\|Tx(t_1) - Tx(t_2)\big\|\\&\quad = \left\| \frac{\beta_0}{\Gamma(p)} t_2^{(p-1)} + \frac{1}{\Gamma(\mu)} \int_0^{t_2} (t_2 - s)^{\mu-1} u\big(s, x(s)\big)\, \mathrm{d}s - \frac{\beta_0}{\Gamma(p)} t_1^{(p-1)} \right.\\&\qquad \left. - \frac{1}{\Gamma(\mu)} \int_0^{t_1} (t_1 - s)^{\mu-1} u\big(s, x(s)\big)\, \mathrm{d}s \right\|\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{\beta_0}{\Gamma(p)} \left(t_2^{p-1} - t_1^{p-1}\right) + \frac{1}{\Gamma(\mu)} \int\limits_0^{t_1} \left((t_2 - s)^{\mu-1} - (t_1 - s)^{\mu-1}\right) u\big(s, x(s)\big) \,\mathrm{d}s \right. \\
&\quad \left. + \frac{1}{\Gamma(\mu)} \int\limits_{t_1}^{t_2} (t_2 - s)^{\mu-1} u\big(s, x(s)\big) \,\mathrm{d}s \right| \\
&\leqslant \frac{\beta_0}{\Gamma(p)} \left|t_2^{p-1} - t_1^{p-1}\right| + \frac{M}{\mu} \left| \int\limits_0^{t_1} \left[(t_2 - s)^{\mu-1} - (t_1 - s)^{\mu-1}\right] \mathrm{d}s \right| \\
&\quad + \frac{M}{\mu} \left| \int\limits_{t_1}^{t_2} \left[(t_2 - s)^{\mu-1}\right] \mathrm{d}s \right|.
\end{aligned}
$$

As $t_2 \to t_1$, right-hand side tends to 0. Thus $\|Tx(t_2) - Tx(t_1)\| \to 0$ as $t_2 \to t_1$. Thus $T(S_1)$ is equicontinuous. With the similar argument, we can prove that $T(S_2)$ is bounded and equicontinuous subset of S_1. Thus the application of Arzela–Ascoli theorem concludes that (S_1, S_2) is relatively compact.

Next, we show that T is relatively nonexpansive. For any $(x, y) \in S_1 \times S_2$, we have

$$
\begin{aligned}
&\big\|Tx(t) - Ty(t)\big\| \\
&= \left\| \frac{\beta_0}{\Gamma(p)} t^{(p-1)} + \frac{1}{\Gamma(\mu)} \int\limits_0^{t} (t - s)^{\mu-1} u\big(s, x(s)\big) \,\mathrm{d}s \right. \\
&\quad \left. - \left(\frac{\alpha_0}{\Gamma(p)} t^{(p-1)} + \frac{1}{\Gamma(\mu)} \int\limits_0^{t} (t - s)^{\mu-1} v\big(s, x(s)\big) \,\mathrm{d}s \right) \right\| \\
&= \frac{t^{(p-1)}}{\Gamma(p)} (\beta_0 - \alpha_0) + \left| \frac{1}{\Gamma(\mu)} \int\limits_0^{t} (t - s)^{\mu-1} \big[u\big(s, x(s)\big) - v\big(s, y(s)\big)\big] \,\mathrm{d}s \right| \\
&\leqslant \frac{\tau^{(p-1)}}{\Gamma(p)} \|\beta_0 - \alpha_0\| \\
&\quad + \frac{1}{\Gamma(\mu)} \int\limits_0^{t} (t - s)^{\mu-1} \frac{\Gamma(\mu+1)}{\tau^{\mu}} \left[\big\|x(s) - y(s)\big\| - \frac{\tau^{p-1}}{\Gamma(p)} \|\beta_0 - \alpha_0\| \right] \mathrm{d}s \quad \text{(by (A2))} \\
&\leqslant \frac{\tau^{(p-1)}}{\Gamma(p)} \|\beta_0 - \alpha_0\| + \frac{1}{\Gamma(\mu)} \frac{\tau^{\mu}}{\mu} \frac{\Gamma(\mu+1)}{\tau^{\mu}} \left[\|x(s) - y(s)\| - \frac{\tau^{p-1}}{\Gamma(p)} \|\beta_0 - \alpha_0\| \right] \\
&= \|x - y\|,
\end{aligned}
$$

and thereby, $\|Tx - Ty\| \leqslant \|x - y\|$. Therefore T is relatively nonexpansive.

At last, let $(K_1, K_2) \subseteq (S_1, S_2)$ be nonempty, closed, convex and proximinal pair, which is T-invariant and such that $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(S_1, S_2) (= \|\alpha_0 - \beta_0\|)$. By using a generalized version of Arzela–Ascoli theorem(see Ambrosetti [4]) and assumption (A1) we get

$$
\begin{aligned}
&\tau + \mathcal{F}\big(\aleph(T(K_1) \cup T(K_2)) + \varphi\big(\aleph\big(T(K_1) \cup T(K_2)\big)\big)\big)\\
&\quad = \tau + \mathcal{F}\big(\max\big\{\aleph(T(K_1)), \aleph\big(T(K_2)\big)\big\} + \varphi\big(\max\big\{\aleph\big(T(K_1)\big), \aleph\big(T(K_2)\big)\big\}\big)\big)\\
&\quad \leqslant \tau + \mathcal{F}\Big(\max\Big\{\sup_{t\in\mathcal{J}}\big\{\aleph\big(\big\{Tx(t)\colon x \in K_1\big\}\big)\big\}, \sup_{t\in\mathcal{J}}\big\{\aleph\big(\big\{Ty(t)\colon y \in K_2\big\}\big)\big\}\Big\}\\
&\qquad + \varphi\Big(\max\Big\{\sup_{t\in\mathcal{J}}\big\{\aleph\big(\big\{Tx(t)\colon x \in K_1\big\}\big)\big\}, \sup_{t\in\mathcal{J}}\big\{\aleph\big(\big\{Ty(t)\colon y \in K_2\big\}\big)\big\}\Big\}\Big)\Big)\\
&\quad \leqslant \tau + \mathcal{F}\left(\max\left\{\sup_{t\in\mathcal{J}}\left\{\aleph\left(\left\{\frac{\beta_0 t^{p-1}}{\Gamma(p)} + \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{q-1} f\big(s, x(s)\big)\,\mathrm{d}s\colon x \in K_1\right\}\right)\right\},\right.\right.\\
&\qquad \left.\sup_{t\in\mathcal{J}}\left\{\aleph\left(\left\{\frac{\alpha_0 t^{p-1}}{\Gamma(p)} + \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{q-1} g\big(s, y(s)\big)\,\mathrm{d}s\colon y \in K_2\right\}\right)\right\}\right\}\\
&\qquad + \varphi\left(\max\left\{\sup_{t\in\mathcal{J}}\left\{\aleph\left(\left\{\frac{\beta_0 t^{p-1}}{\Gamma(p)} + \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{q-1} f\big(s, x(s)\big)\,\mathrm{d}s\colon x \in K_1\right\}\right)\right\},\right.\right.\\
&\qquad \left.\left.\left.\sup_{t\in\mathcal{J}}\left\{\aleph\left(\left\{\frac{\alpha_0 t^{p-1}}{\Gamma(p)} + \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{q-1} g\big(s, y(s)\big)\,\mathrm{d}s\colon y \in K_2\right\}\right)\right\}\right\}\right)\right).
\end{aligned}
$$

So, in view of Theorem 8, it follows that

$$
\begin{aligned}
&\tau + F\big(\aleph\big(T(K_1) \cup T(K_2)\big) + \varphi\big(\aleph\big(T(K_1) \cup T(K_2)\big)\big)\big)\\
&\quad \leqslant \tau + F\left(\max\left\{\sup_{t\in\mathcal{J}}\left\{\aleph\left(\left\{\frac{\beta_0\ t^{p-1}}{\Gamma(p)} + \frac{t^{\mu-1}}{\Gamma(\mu)}\overline{\operatorname{conv}}\big(\big\{f\big(\sigma, x(\sigma)\big)\colon \sigma \in J\big\}\big)\right\}\right)\right\},\right.\right.\\
&\qquad \left.\sup_{t\in\mathcal{J}}\left\{\aleph\left(\left\{\frac{\alpha_0 t^{p-1}}{\Gamma(p)} + \frac{t^{\mu-1}}{\Gamma(\mu)}\overline{\operatorname{conv}}\big(\big\{g\big(\sigma, x(\sigma)\big)\colon \sigma \in J\big\}\big)\right\}\right)\right\}\right\}\\
&\qquad + \varphi\left(\max\left\{\sup_{t\in\mathcal{J}}\left\{\aleph\left(\left\{\frac{\beta_0 t^{p-1}}{\Gamma(p)} + \frac{t^{\mu-1}}{\Gamma(\mu)}\overline{\operatorname{conv}}\big(\big\{f\big(\sigma, x(\sigma)\big)\colon \sigma \in J\big\}\big)\right\}\right)\right\},\right.\right.\\
&\qquad \left.\left.\left.\sup_{t\in\mathcal{J}}\left\{\aleph\left(\left\{\frac{\alpha_0 t^{p-1}}{\Gamma(p)} + \frac{t^{\mu-1}}{\Gamma(\mu)}\overline{\operatorname{conv}}\big(\big\{g\big(\sigma, x(\sigma)\big)\colon \sigma \in J\big\}\big)\right\}\right)\right\}\right\}\right)\right)\\
&\quad \leqslant \tau + F\left(\max\left\{\frac{t^{\mu-1}}{\Gamma(\mu)}\aleph\big(f(\mathcal{J} \times N_1)\big), \frac{t^{\mu-1}}{\Gamma(\mu)}\aleph\big(g(\mathcal{J} \times N_2)\big)\right\}\right.\\
&\qquad \left. + \varphi\left(\max\left\{\frac{t^{\mu-1}}{\Gamma(\mu)}\aleph\big(f(\mathcal{J} \times N_1)\big), \frac{t^{\mu-1}}{\Gamma(\mu)}\aleph\big(g(\mathcal{J} \times N_2)\big)\right\}\right)\right)\\
&\quad \leqslant \tau + F\left(\frac{t^{\mu-1}}{\Gamma(\mu)}\aleph\big(f(\mathcal{J} \times N_1) \cup g(\mathcal{J} \times N_2)\big)\right.\\
&\qquad \left. + \varphi\left(\frac{t^{\mu-1}}{\Gamma(\mu)}\aleph\big(f(\mathcal{J} \times N_1) \cup g(\mathcal{J} \times N_2)\big)\right)\right)
\end{aligned}
$$

$$\begin{aligned}&\leqslant \tau + F\big(\aleph\big(f(\mathcal{J}\times N_1)\cup g(\mathcal{J}\times N_2)\big) + \varphi\big(\aleph\big(f(\mathcal{J}\times N_1)\cup g(\mathcal{J}\times N_2)\big)\big)\big)\\ &\leqslant F\big(\mu(N_1\cup N_2) + \varphi\big(\mu(N_1\cup N_2)\big)\big).\end{aligned}$$

Therefore, we conclude that T satisfies all the hypotheses of Theorem 6, and so the operator T has a best proximity point $z \in S_1 \cup S_2$, which is an optimal solution for system (1) and (2). □

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