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An upper-lower solution method for the eigenvalue problem of Hadamard-type singular fractional differential equation*

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Abstract. In this paper, we are concerned with the eigenvalue problem of Hadamard-type singular fractional differential equations with multi-point boundary conditions. By constructing the upper and lower solutions of the eigenvalue problem and using the properties of the Green function, the eigenvalue interval of the problem is established via Schauder's fixed point theorem. The main contribution of this work is on tackling the nonlinearity which possesses singularity on some space variables.

Keywords: Hadamard-type fractional differential equation, upper-lower solution method, eigenvalue problem, singularity.

1 Introduction

In this paper, we focus on the existence of positive solutions for the following eigenvalue problem of Hadamard-type singular fractional differential equations with multi-

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point boundary conditions:

$$\begin{aligned} &-{}_HD^{\alpha}x(t) = \lambda f\big(t, x(t)\big), \quad \text{a.e. } t \in (1, \mathrm{e}), \\ &x(1) = 0, \qquad x(\mathrm{e}) = x_0 + \mu \sum_{i=1}^{m} a_i x\big(\varphi(\eta_i)\big), \end{aligned} \tag{1}$$

where $f : [1, \mathrm{e}] \times (0, +\infty)$ is continuous, ${}_HD^{\alpha}$ is the Hadamard fractional derivative of order α with $1 < \alpha < 2$, $1 < \eta_1 < \eta_2 < \cdots < \eta_m < \mathrm{e}$, and the constants x_0, μ and a_i are nonnegative, the function $\varphi : [1, \mathrm{e}] \to [1, \mathrm{e}]$, $\varphi(t) \leqslant t$ is continuous.

In recent years, fractional-order nonlinear problems have attracted the attention of many researchers from mathematics and other applied science due to its wide range of applications in applied mathematics, physics, bioscience, engineering, chemistry, etc. A large number of contributions have been made for fractional differential equations in the sense of the Riemann–Liouville fractional derivative or the Caputo fractional derivative, [1, 4–6, 8, 9, 12–20]. However, the Hadamard-type fractional integral and derivative differ from the Riemann–Liouville and the Caputo fractional derivative since the kernels of the Hadamard-type integral and derivative contain logarithmic functions of arbitrary exponent and thus are regarded as a different kind of weakly singular kernels. Thus it is more difficult to explore the existence of solutions for the Hadamard-type fractional differential equations, [2, 10, 11, 21].

In the recent work [21], by analysing the spectral construct of a linear operator and calculating the fixed point index of the corresponding nonlinear operator, Zhang et al. considered the existence of positive solutions for the following Hadamard-type fractional differential equation:

$$\begin{aligned} &\mathscr{D}_t{}^{\alpha}\mathscr{D}_t{}^{\beta}z(t) = f\big(t, z(t), -\mathscr{D}_t{}^{\beta}z(t)\big), \quad 1 < t < \mathrm{e}, \\ &z(1) = \sigma z(1) = \sigma z(\mathrm{e}) = 0, \\ &\mathscr{D}_t{}^{\beta}z(1) = \sigma\mathscr{D}_t{}^{\beta}z(1) = \sigma\mathscr{D}_t{}^{\beta}z(\mathrm{e}) = 0, \end{aligned} \tag{2}$$

where $2 < \alpha, \beta \leqslant 3$, σ is a differential operator denoted by $t(\mathrm{d}/\mathrm{d}t)$, that is, $\sigma z(t) = t(\mathrm{d}/\mathrm{d}t)z(t)$, $\mathscr{D}_t{}^{\alpha}$ and $\mathscr{D}_t{}^{\beta}$ are the Hadamard fractional derivatives of order α, β, $f \in (1, \mathrm{e}) \times (0, +\infty) \times (0, +\infty)$, $[0, +\infty)$ is a continuous function, and the criteria of the existence of positive solutions were established. Recently, based on Leray–Schauder-type continuation, El-Sayed and Gaafar [3] established the existence of positive solutions to a class of singular nonlinear Hadamard-type fractional differential equations with infinite-point boundary conditions or integral boundary conditions.

However when f possesses singularities on space variables, especially for the eigenvalue problem, few results are established on Hadamard-type fractional differential equations. Inspired by the above works, the aim of this paper is to establish the existence of positive solutions for the eigenvalue problem of the Hadamard-type fractional differential equation (1) when f possesses singularity on space variables.

The rest of this paper is organized as follows. In Section 2, we firstly recall the concepts and properties of Hadamard fractional integral and derivative and then give the logarithmic Green kernel. Our main results are summarized in Section 3.

2 Basic definitions and preliminaries

In this section, we firstly present the definition of Hadamard-type fractional integral and derivatives as given in [7]. Then we give some basic lemmas, which will be used in the rest of the paper.

Suppose $\alpha \in \mathbb{C}$, $n = [\operatorname{Re}\alpha]$, $\operatorname{Re}\alpha > 0$, and (a, b) is a finite or infinite interval of $\mathbb{R}^+$. The α-order left Hadamard fractional integral is defined by

$$ {}_H I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\ln t - \ln s)^{\alpha-1} x(s) \frac{\mathrm{d}s}{s}, \quad t \in (a, b), $$

and the α left Hadamard fractional derivative is defined by

$$ {}_H D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{\mathrm{d}}{\mathrm{d}t} \right)^n \int_a^t (\ln t - \ln s)^{n-\alpha-1} x(s) \frac{\mathrm{d}s}{s}, \quad t \in (a, b). $$

The relationship between fractional integration and derivative is introduced as follows.

Lemma 1. (*See* [7].). *Suppose $n - 1 < \alpha < n$, $\gamma > 0$.*

(i) *If $1 < \alpha < 2$, then ${}_H D^\alpha x(t) = 0$ if and only if $x(t) = c_1 (\ln t)^{\alpha-1} + c_2 (\ln t)^{\alpha-2}$ for any $c_1, c_2 \in \mathbb{R}$.*

(ii) *The equality ${}_H D^\alpha({}_H I^\alpha x(t)) = x(t)$ holds for every $x \in L^1[1, \mathrm{e}]$.*

(iii) *Let $x \in C[1, \infty) \cap L^1[1, \infty)$. The following formula holds*:

$$ {}_H I^\alpha \big({}_H D^\alpha x(t) \big) = x(t) - \sum_{i=1}^n c_i (\ln t)^{\alpha-i}. $$

(iv) ${}_H I^\alpha({}_H I^\gamma x(t)) = {}_H I^{\alpha+\gamma}(x)(t)$.

Lemma 2. (*See* [3].) *For $g \in L^1[1, \mathrm{e}]$, the boundary value problem*

$$ -{}_H D^\alpha x(t) = \lambda g(t), \quad \text{a.e. } t \in (1, e), $$

subject to the multi-point boundary conditions

$$ x(1) = 0, \qquad x(\mathrm{e}) = x_0 + \mu \sum_{i=1}^m a_i x\big(\varphi(\eta_i)\big) $$

has a unique solution $x \in AC[1, \mathrm{e}]$ if and only if x is a solution of the integral equation

$$ \begin{aligned} x(t) = {}& \lambda \int_1^{\mathrm{e}} G(t, s) g(s) \frac{\mathrm{d}s}{s} + \sum_i^m \frac{\lambda \mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\big(\varphi(\eta_i), s\big) g(s) \frac{\mathrm{d}s}{s} \\ & + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)}, \end{aligned} $$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\ln t)^{\alpha-1}(1-\ln s)^{\alpha-1}-(\ln t-\ln s)^{\alpha-1}, & 1\leqslant s\leqslant t\leqslant \mathrm{e},\\ (\ln t)^{\alpha-1}(1-\ln s)^{\alpha-1}, & 1\leqslant t\leqslant s\leqslant \mathrm{e},\end{cases}$$

and

$$\sigma=\lambda\sum_{i=1}^{m}a_i\bigl(\ln\varphi(\eta_i)\bigr)^{\alpha-1}\neq 1.$$

Lemma 3. (*See* [15]). *Let* $\kappa(t)=t^{\alpha-1}(1-t)$. *The Green's functions* G *has the following properties*:

(i) $G\in C([1,\mathrm{e}]\times[1,\mathrm{e}],\mathbb{R}^+)$.

(ii) *For all* $t,s\in(1,\mathrm{e})$, *the following inequalities hold*:

$$(\alpha-1)\kappa(\ln t)\kappa(1-\ln s)\leqslant\Gamma(\alpha)G(t,s)\leqslant\kappa(\ln t)(1-\ln s)^{\alpha-2}.$$

Definition 1. A continuous function $\psi(t)$ is called a lower solution of (1) if it satisfies

$$-{}_HD^{\alpha}\psi(t)\leqslant\lambda f\bigl(t,\psi(t)\bigr),\quad \text{a.e. } t\in(1,\mathrm{e}),$$
$$\psi(1)\geqslant 0,\qquad \psi(\mathrm{e})\geqslant x_0+\mu\sum_{i=1}^{m}a_i\psi\bigl(\varphi(\eta_i)\bigr).$$

Definition 2. A continuous function $\phi(t)$ is called a upper solution of the eigenvalue problem (1) if it satisfies

$$-{}_HD^{\alpha}\phi(t)\geqslant\lambda f\bigl(t,\phi(t)\bigr),\quad \text{a.e. } t\in(1,\mathrm{e}),$$
$$\phi(1)\leqslant 0,\qquad \phi(\mathrm{e})\leqslant x_0+\mu\sum_{i=1}^{m}a_i\phi\bigl(\varphi(\eta_i)\bigr).$$

We make the following assumptions throughout this paper:

(H1) $f:[1,\mathrm{e}]\times(0,+\infty)\to[0,+\infty)$ is continuous and is nonincreasing in $x>0$;

(H2) For all $r\in(0,1)$, there exists a constant $\epsilon>0$ such that, for any $(t,x)\in[1,\mathrm{e}]\times(0,+\infty)$, $f(t,rx)\leqslant r^{-\epsilon}f(t,x)$.

Remark 1. For $r\geqslant 1$, by (H2), we have the following equivalent conclusion: for any $(t,x)\in[1,\mathrm{e}]\times(0,+\infty)$, $f(t,rx)\geqslant r^{-\epsilon}f(t,x)$.

In fact, for $r\geqslant 1$ and any $(t,x)\in[1,\mathrm{e}]\times(0,+\infty)$, one has $f(t,r\cdot(1/r)x)\leqslant(1/r)^{-\epsilon}f(t,rx)$, that is, $f(t,rx)\geqslant r^{-\epsilon}f(t,x)$.

Lemma 4 [Maximal principle]. *If* $x\in C([0,1],\mathbb{R})$ *satisfies*

$$x(1)=0,\qquad x(\mathrm{e})=x_0+\mu\sum_{i=1}^{m}a_ix\bigl(\varphi(\eta_i)\bigr)$$

and $-{}_HD^{\alpha}x(t)\geqslant 0$ *for any* $t\in[0,1]$, *then* $x(t)\geqslant 0$, $t\in[0,1]$.

Proof. By Lemma 2, the conclusion is obvious, and we thus omit the proof here. □

3 Main results

Let

$$A = \frac{(\alpha - 1)}{\Gamma(\alpha + 2)} + \sum_{i=1}^{m} \frac{\mu a_i(\alpha - 1)\kappa(\ln \varphi(\eta_i))}{(1 - \sigma)\Gamma(\alpha + 2)},$$

then we state our main result as follows.

Theorem 1. *Suppose* (H1) *and* (H2) *hold, and*

(H3) $\inf_{t\in[1,\mathrm{e}]} f(t, 1) > 0$ *and* $0 < \int_1^{\mathrm{e}} (1 - \ln s)^{\alpha-2} f(s, \kappa(\ln s))(\mathrm{d}s/s) < +\infty$.

Then there are constants $0 < \lambda_1 < \lambda^*$ *and* $\rho > 0$ *such that for any* $\lambda \in (\lambda_1, \lambda^*)$*, the eigenvalue problem* (1) *has at least one positive solution* $x(t)$ *satisfying the asymptotic property*

$$\kappa(\ln t) \leqslant x(t) \leqslant \rho(\ln t)^{\alpha-1}.$$

Proof. Firstly, define a function space $E = C[1, \mathrm{e}]$ and a subset Q of E:

$$Q = \bigl\{x(t) \in E \bigm| \exists l_x > 0\colon\ x(t) \geqslant l_x\kappa(\ln t),\ t \in [1, \mathrm{e}]\bigr\}. \tag{3}$$

Obviously, Q is a nonempty since $\kappa(\ln t) \in Q$.

Define an operator T_λ in E:

$$\begin{aligned}(T_\lambda x)(t) = \lambda \int_1^{\mathrm{e}} G(t, s) f\bigl(s, x(s)\bigr)\frac{\mathrm{d}s}{s}\\ + \sum_{i=1}^{m} \frac{\lambda\mu a_i(\ln t)^{\alpha-1}}{(1 - \sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i), s\bigr) f\bigl(s, x(s)\bigr)\frac{\mathrm{d}s}{s} + \frac{x_0(\ln t)^{\alpha-1}}{(1 - \sigma)}.\end{aligned} \tag{4}$$

It follows from Lemma 2 that the fixed point of the operator T_λ is the solution of the eigenvalue problem (1).

In what follows, we prove that the operator T_λ is well defined and $T_\lambda(Q) \subset Q$. To do this, for any $x^* \in Q$, it follows from the definition of Q that there exists a positive number $l^*_{x^*}$ such that $x^*(t) \geqslant l^*_{x^*}\kappa(\ln t)$ for any $t \in [1, \mathrm{e}]$. Choose $l_{x^*} = \min\{1/2, l^*_{x^*}\}$, then we have $x^*(t) \geqslant l_{x^*}\kappa(\ln t)$ for any $t \in [1, \mathrm{e}]$. So by Lemma 3, (H2) and (H3), we gets

$$\begin{aligned}&(T_\lambda x^*)(t)\\ &\quad\leqslant \frac{\lambda\kappa(\ln t)}{\Gamma(\alpha)} \int_1^{\mathrm{e}} (1 - \ln s)^{\alpha-2} f\bigl(s, x^*(s)\bigr)\frac{\mathrm{d}s}{s}\\ &\qquad + \sum_{i=1}^{m} \frac{\lambda\mu a_i(\ln t)^{\alpha-1}\kappa(\ln \varphi(\eta_i))}{(1 - \sigma)\Gamma(\alpha)} \int_1^{\mathrm{e}} (1 - \ln s)^{\alpha-2} f\bigl(s, x^*(s)\bigr)\frac{\mathrm{d}s}{s}\\ &\qquad + \frac{x_0(\ln t)^{\alpha-1}}{(1 - \sigma)}\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{\lambda\kappa(\ln t)}{\Gamma(\alpha)}\int_1^{\mathrm{e}}(1-\ln s)^{\alpha-2}f\bigl(s,l_{x^*}\kappa(\ln s)\bigr)\frac{\mathrm{d}s}{s}\\
&\quad+\sum_{i=1}^m\frac{\lambda\mu a_i(\ln t)^{\alpha-1}\kappa(\ln\varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)}\int_1^{\mathrm{e}}(1-\ln s)^{\alpha-2}f\bigl(s,l_{x^*}\kappa(\ln s)\bigr)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{x_0(\ln t)^{\alpha-1}}{(1-\sigma)}\\
&\leqslant\left(\frac{\lambda {l_{x^*}}^{-\epsilon}}{\Gamma(\alpha)}+\sum_{i=1}^m\frac{\lambda\mu a_i {l_{x^*}}^{-\epsilon}\kappa(\ln\varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)}\right)\int_1^{\mathrm{e}}(1-\ln s)^{\alpha-2}f\bigl(s,\kappa(\ln s)\bigr)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{x_0}{(1-\sigma)}\\
&<+\infty. \qquad (5)
\end{aligned}$$

Next, take $B=\max\{2,\max_{t\in[1,\mathrm{e}]}x^*(t)\}$, then it follows from Lemma 3 and (H2) that

$$\begin{aligned}
&(T_\lambda x^*)(t)\\
&\quad\geqslant\frac{\lambda(\alpha-1)}{\Gamma(\alpha)}\kappa(\ln t)\int_1^{\mathrm{e}}\kappa(1-\ln s)f\bigl(s,x^*(s)\bigr)\frac{\mathrm{d}s}{s}\\
&\qquad+\sum_{i=1}^m\frac{\lambda\mu(\alpha-1)a_i\kappa(\ln t)\kappa(\varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)}\int_1^{\mathrm{e}}\kappa(1-\ln s)f\bigl(s,x^*(s)\bigr)\frac{\mathrm{d}s}{s}\\
&\quad\geqslant\frac{\lambda(\alpha-1)}{\Gamma(\alpha)}\kappa(\ln t)\int_1^{\mathrm{e}}\kappa(1-\ln s)f(s,B)\frac{\mathrm{d}s}{s}\\
&\qquad+\sum_{i=1}^m\frac{\lambda\mu(\alpha-1)a_i\kappa(\ln t)\kappa(\varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)}\int_1^{\mathrm{e}}\kappa(1-\ln s)f(s,B)\frac{\mathrm{d}s}{s}\\
&\quad\geqslant\frac{\lambda(\alpha-1)B^{-\epsilon}}{\Gamma(\alpha)}\kappa(\ln t)\int_1^{\mathrm{e}}\kappa(1-\ln s)f(s,1)\frac{\mathrm{d}s}{s} \qquad (6)\\
&\qquad+\sum_{i=1}^m\frac{\lambda\mu(\alpha-1)B^{-\epsilon}a_i\kappa(\ln t)\kappa(\varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)}\int_1^{\mathrm{e}}\kappa(1-\ln s)f(s,1)\frac{\mathrm{d}s}{s}\\
&\quad\geqslant\left[\frac{\lambda(\alpha-1)B^{-\epsilon}}{\Gamma(\alpha)}\left(1+\sum_{i=1}^m\frac{\mu a_i\kappa(\varphi(\eta_i))}{(1-\sigma)}\right)\int_1^{\mathrm{e}}\kappa(1-\ln s)f(s,1)\frac{\mathrm{d}s}{s}\right]\kappa(\ln t). \qquad (7)
\end{aligned}$$

(5) and (7) indicate that T_λ is well defined and $T_\lambda(Q)\subset Q$.

Now we shall try to construct the upper and lower solutions of the eigenvalue problem (1). As the operator T_λ is decreasing on x, let

$$L(t) = \int_1^{\mathrm{e}} G(t,s) f\bigl(s, \kappa(\ln s)\bigr) \frac{\mathrm{d}s}{s} + \sum_{i=1}^{m} \frac{\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i), s\bigr) f\bigl(s, \kappa(\ln s)\bigr) \frac{\mathrm{d}s}{s},$$

then, similar to 7, for all $t \in [1, \mathrm{e}]$, one gets

$$L(t) \geqslant \left[\frac{(\alpha-1)}{\Gamma(\alpha)} \left(1 + \sum_{i=1}^{m} \frac{\mu a_i \kappa(\varphi(\eta_i))}{(1-\sigma)} \right) \int_1^{\mathrm{e}} \kappa(1-\ln s) f\bigl(s, \kappa(\ln s)\bigr) \frac{\mathrm{d}s}{s} \right] \kappa(\ln t),$$

i.e.,

$$\lambda_1 L(t) \geqslant \kappa(\ln t) \quad \forall t \in [1, \mathrm{e}],$$

where

$$\lambda_1 = \frac{\Gamma(\alpha)}{(\alpha-1)(1+\sum_{i=1}^{m} \frac{\mu a_i \kappa(\varphi(\eta_i))}{(1-\sigma)}) \int_1^{\mathrm{e}} \kappa(1-\ln s) f(s, \kappa(\ln s)) \frac{\mathrm{d}s}{s}}.$$

On the other hand, notice that $f(t,x)$ is decreasing in $x > 0$, thus, for any $\lambda > \lambda_1$, it follows from Lemma 3 and (H3) that

$$\begin{aligned}
&\int_1^{\mathrm{e}} G(t,s) f\bigl(s, \lambda L(s)\bigr) \frac{\mathrm{d}s}{s} + \sum_{i=1}^{m} \frac{\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i), s\bigr) f\bigl(s, \lambda L(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \\
&\leqslant \int_1^{\mathrm{e}} G(t,s) f\bigl(s, \lambda_1 L(s)\bigr) \frac{\mathrm{d}s}{s} + \sum_{i=1}^{m} \frac{\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i), s\bigr) f\bigl(s, \lambda_1 L(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \\
&\leqslant \int_1^{\mathrm{e}} G(t,s) f\bigl(s, \kappa(\ln s)\bigr) \frac{\mathrm{d}s}{s} + \sum_{i=1}^{m} \frac{\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i), s\bigr) f\bigl(s, \kappa(\ln s)\bigr) \frac{\mathrm{d}s}{s} \\
&+ \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \\
&\leqslant \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\mu a_i \kappa(\ln \varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)} \right) \int_1^{\mathrm{e}} (1-\ln s)^{\alpha-2} f\bigl(s, \kappa(\ln s)\bigr) \frac{\mathrm{d}s}{s} + \frac{x_0}{(1-\sigma)} \\
&< +\infty.
\end{aligned}$$

Now take $C = \max\{2, \max_{t\in[1,\mathrm{e}]} L(t)\}$ and

$$\lambda^* > \max\left\{1, \lambda_1, \left[\frac{C^\epsilon}{A \inf_{s\in[1,\mathrm{e}]} f(s,1)}\right]^{1/(-\epsilon+1)}\right\}.$$

Let

$$R(t) = \lambda^* \int_1^{\mathrm{e}} G(t,s) f\bigl(s, \lambda^* L(s)\bigr) \frac{\mathrm{d}s}{s} + \lambda^* \sum_{i=1}^m \frac{\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i), s\bigr) f\bigl(s, \lambda^* L(s)\bigr) \frac{\mathrm{d}s}{s} + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)}.$$

By (H2), for any $t \in [1, \mathrm{e}]$, we have

$$\lambda^* f\bigl(s, \lambda^* L(s)\bigr) \geqslant (\lambda^*)^{-\epsilon+1} f(s, C) \geqslant (\lambda^*)^{-\epsilon+1} C^{-\epsilon} f(s,1) \geqslant A^{-1}.$$

Thus it follows from Lemma 3 that

$$\begin{aligned} R(t) &\geqslant \frac{(\alpha-1)A^{-1}}{\Gamma(\alpha)} \kappa(\ln t) \int_1^{\mathrm{e}} \kappa(1-\ln s) \frac{\mathrm{d}s}{s} \\ &\quad + \sum_{i=1}^m \frac{\mu a_i (\alpha-1) A^{-1} \kappa(\ln \varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)} \kappa(\ln t) \int_1^{\mathrm{e}} \kappa(1-\ln s) \frac{\mathrm{d}s}{s} \\ &= \left(\frac{(\alpha-1)A^{-1}}{\Gamma(\alpha+2)} + \sum_{i=1}^m \frac{\mu a_i (\alpha-1) A^{-1} \kappa(\ln \varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha+2)} \right) \kappa(\ln t) \\ &\geqslant \kappa(\ln t) \quad \forall\, t \in [1, \mathrm{e}]. \end{aligned}$$

Let

$$\phi(t) = \lambda^* L(t) + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)}, \qquad \psi(t) = R(t),$$

then by Lemma 2, for any $t \in [1, \mathrm{e}]$, we have

$$\begin{aligned} &\phi(t) = \lambda^* L(t) + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \geqslant \kappa(\ln t), \\ &\phi(1) = 0, \qquad \phi(\mathrm{e}) = x_0 + \mu \sum_{i=1}^m a_i \phi\bigl(\varphi(\eta_i)\bigr), \end{aligned} \tag{8}$$

$$\begin{aligned} &\psi(t) = R(t) \geqslant \kappa(\ln t), \quad t \in [1, \mathrm{e}], \\ &\psi(1) = 0, \qquad \psi(\mathrm{e}) = x_0 + \mu \sum_{i=1}^m a_i \psi\bigl(\varphi(\eta_i)\bigr). \end{aligned} \tag{9}$$

It follows from (8) and (9) that $\phi(t), \psi(t) \in Q$ and

$$\kappa(\ln t) \leqslant \psi(t) = T_{\lambda^*}\big(\lambda^* L(t)\big), \quad \kappa(\ln t) \leqslant \phi(t) \quad \forall\, t \in [1, \mathrm{e}], \tag{10}$$

which implies

$$\begin{aligned}
\psi(t) &= T_{\lambda^*}\big(\lambda^* L(t)\big) \\
&= \lambda^* \int_1^{\mathrm{e}} G(t,s) f\big(s, \lambda^* L(s)\big) \frac{\mathrm{d}s}{s} \\
&\quad + \lambda^* \sum_{i=1}^{m} \frac{\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\big(\varphi(\eta_i), s\big) f\big(s, \lambda^* L(s)\big) \frac{\mathrm{d}s}{s} + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \\
&\leqslant \lambda^* \int_1^{\mathrm{e}} G(t,s) f\big(s, \lambda_1 L(s)\big) \frac{\mathrm{d}s}{s} \\
&\quad + \lambda^* \sum_{i=1}^{m} \frac{\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\big(\varphi(\eta_i), s\big) f\big(s, \lambda_1 L(s)\big) \frac{\mathrm{d}s}{s} + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \\
&\leqslant \lambda^* \int_1^{\mathrm{e}} G(t,s) f\big(s, \kappa(\ln s)\big) \frac{\mathrm{d}s}{s} \\
&\quad + \lambda^* \sum_{i=1}^{m} \frac{\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\big(\varphi(\eta_i), s\big) f\big(s, \kappa(\ln s)\big) \frac{\mathrm{d}s}{s} + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \\
&= \phi(t) \quad \forall t \in [1, \mathrm{e}].
\end{aligned} \tag{11}$$

Thus, by (10), (11), we have

$$\begin{aligned}
HD^{\alpha}\psi(t) + \lambda^* f\big(t, \psi(t)\big) &\geqslant {}_H D^{\alpha}\big(T_{\lambda^*}\big(\lambda^* L(t)\big)\big) + \lambda^* f\big(t, \phi(t)\big) \\
&= -\lambda^* f\big(t, \lambda^* L(t)\big) + \lambda^* f\big(t, \phi(t)\big) \geqslant 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
{}_H D^{\alpha}\phi(t) + \lambda^* f\big(t, \phi(t)\big) &\leqslant {}_H D^{\alpha}\left(\lambda^* L(t) + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)}\right) + \lambda^* f\big(t, \phi(t)\big) \\
&= {}_H D^{\alpha}\big(T_{\lambda^*}\kappa(\ln t)\big) + \lambda^* f\big(t, \phi(t)\big) \\
&= -\lambda^* f\big(t, \kappa(\ln t)\big) + \lambda^* f\big(t, \phi(t)\big) \\
&\leqslant -\lambda^* f\big(t, \kappa(\ln t)\big) + \lambda^* f\big(t, \kappa(\ln t)\big) = 0.
\end{aligned} \tag{13}$$

(8) and (9) imply that ϕ, ψ satisfy the boundary value conditions of the eigenvalue problem (1). Thus it follows from (11)–(13) that $\psi(t), \phi(t)$ are upper and lower solutions of the eigenvalue problem (1) when $\lambda = \lambda^*$ and $\psi(t), \phi(t) \in Q$.

Next, construct a function F:

$$F(y) = \begin{cases} f(t, \phi(t)), & y < \psi(t), \\ f(t, y(t)), & \psi(t) \leqslant y \leqslant \phi(t), \\ f(t, \psi(t)), & y > \phi(t). \end{cases} \tag{14}$$

For any $\lambda \in (\lambda_1, \lambda^*)$, consider the following modified eigenvalue problem:

$$\begin{aligned} &-{}_H D^\alpha y(t) = \lambda F(y), \quad \text{a.e. } t \in (1, \mathrm{e}), \\ &y(1) = 0, \qquad y(\mathrm{e}) = x_0 + \mu \sum_{i=1}^m a_i y\bigl(\varphi(\eta_i)\bigr). \end{aligned} \tag{15}$$

We define an operator $\mathfrak{A}_\lambda$ in E:

$$\begin{aligned} (\mathfrak{A}_\lambda y)(t) = {} & \lambda \int_1^{\mathrm{e}} G(t,s) F\bigl(y(s)\bigr) \frac{\mathrm{d}s}{s} + \sum_{i=1}^m \frac{\lambda \mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i), s\bigr) F\bigl(y(s)\bigr) \frac{\mathrm{d}s}{s} \\ & + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \quad \forall y \in E. \end{aligned}$$

It follows from the assumption that $F : [0, +\infty) \to [0, +\infty)$ is continuous. Thus it is clear that a fixed point of the operator $\mathfrak{A}_\lambda$ is a solution of the modified eigenvalue problem (15).

For all $y \in E$, it follows from Lemma 3, (14) and $\psi(t) \geqslant \kappa(\ln t)$ that

$$\begin{aligned} (\mathfrak{A}_\lambda y)(t) = {} & \lambda \int_1^{\mathrm{e}} G(t,s) F\bigl(y(s)\bigr) \frac{\mathrm{d}s}{s} + \sum_{i=1}^m \frac{\lambda \mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i), s\bigr) F\bigl(y(s)\bigr) \frac{\mathrm{d}s}{s} \\ & + \frac{x_0 (\ln t)^{\alpha-1}}{(1-\sigma)} \\ \leqslant {} & \lambda^* \biggl(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\mu a_i \kappa(\ln \varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)} \biggr) \int_1^{\mathrm{e}} (1 - \ln s)^{\alpha-2} F\bigl(y(s)\bigr) \frac{\mathrm{d}s}{s} + \frac{x_0}{(1-\sigma)} \\ \leqslant {} & \lambda^* \biggl(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\mu a_i \kappa(\ln \varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)} \biggr) \int_1^{\mathrm{e}} (1 - \ln s)^{\alpha-2} f\bigl(s, \psi(s)\bigr) \frac{\mathrm{d}s}{s} + \frac{x_0}{(1-\sigma)} \\ \leqslant {} & \lambda^* \biggl(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\mu a_i \kappa(\ln \varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)} \biggr) \int_1^{\mathrm{e}} (1 - \ln s)^{\alpha-2} f\bigl(s, \kappa(\ln s)\bigr) \frac{\mathrm{d}s}{s} \\ & + \frac{x_0}{(1-\sigma)} < +\infty. \end{aligned}$$

So $\mathfrak{A}_\lambda$ is bounded. It is easy to see that $\mathfrak{A}_\lambda : E \to E$ is continuous from the continuity of $F(y)$ and $G(t,s)$.

On the other hand, for any $\Omega \subset E$ bounded, since $G(t,s)$ is uniformly continuous on $[1,\mathrm{e}]$, we know that $\mathfrak{A}_\lambda(\Omega)$ is equicontinuous. Thus the Arzela–Ascoli theorem implies that $\mathfrak{A}_\lambda : E \to E$ is completely continuous. It follows from the Schauder fixed point theorem that $\mathfrak{A}_\lambda$ has at least one fixed point y such that $y = \mathfrak{A}_\lambda y$.

Now we show

$$\psi(t) \leqslant y(t) \leqslant \phi(t), \quad t \in [1,\mathrm{e}].$$

To do this, let $w(t) = \phi(t) - y(t)$, $t \in [1,\mathrm{e}]$. Since $\phi(t)$ is the upper solution of the eigenvalue problem (1) and y is a fixed point of $\mathfrak{A}_\lambda$, we have

$$w(1) = 0, \qquad w(\mathrm{e}) = x_0 + \mu \sum_{i=1}^{m} a_i w\bigl(\varphi(\eta_i)\bigr). \tag{16}$$

It follows from the definition of F, (10) and (11) that

$$f\bigl(t,\phi(t)\bigr) \leqslant F\bigl(y(t)\bigr) \leqslant f\bigl(t,\psi(t)\bigr) \leqslant f\bigl(t,\kappa(\ln t)\bigr) \quad \forall y \in E,\ \forall t \in [1,\mathrm{e}], \tag{17}$$

i.e.,

$${}_HD^\alpha w(t) ={}_H D^\alpha \phi(t) -{}_H D^\alpha y(t) = -\lambda^* f\bigl(t,\kappa(\ln t)\bigr) + \lambda F\bigl(y(t)\bigr) \leqslant 0, \tag{18}$$

which implies that $-{}_HD^\alpha w(t) \geqslant 0$. It follows from Lemma 4 that $w(t) \geqslant 0$, that is, $y(t) \leqslant \phi(t)$ on $[0,1]$. By the same way, we have $y(t) \geqslant \psi(t)$ on $[0,1]$, thus we get

$$\psi(t) \leqslant y(t) \leqslant \phi(t), \quad t \in [1,\mathrm{e}]. \tag{19}$$

By (14), we have $F(y(t)) = f(t,y(t))$, $t \in [1,\mathrm{e}]$. Consequently, $y(t)$ is a positive solution of the eigenvalue problem (1).

Finally, we prove the asymptotic properties of solutions. Firstly, from (19) we get

$$y(t) \geqslant \psi(t) \geqslant \kappa(\ln t). \tag{20}$$

On the other hand, it follows from (20) and Lemma 3 that

$$\begin{aligned}
y(t) &= \lambda \int_1^{\mathrm{e}} G(t,s) f\bigl(s,y(s)\bigr)\frac{\mathrm{d}s}{s} + \sum_{i=1}^{m} \frac{\lambda\mu a_i (\ln t)^{\alpha-1}}{(1-\sigma)} \int_1^{\mathrm{e}} G\bigl(\varphi(\eta_i),s\bigr) f\bigl(s,y(s)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{x_0(\ln t)^{\alpha-1}}{(1-\sigma)} \\
&\leqslant \left[\lambda^*\left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\mu a_i \kappa(\ln\varphi(\eta_i))}{(1-\sigma)\Gamma(\alpha)}\right) \int_1^{\mathrm{e}} (1-\ln s)^{\alpha-2} f\bigl(s,\kappa(\ln s)\bigr)\frac{\mathrm{d}s}{s} + \frac{x_0}{(1-\sigma)}\right] \\
&\quad \times (\ln t)^{\alpha-1} \\
&= \rho(\ln t)^{\alpha-1}.
\end{aligned}$$

Thus we get the asymptotic properties of solutions $\kappa(\ln t) \leqslant y(t) \leqslant \rho(\ln t)^{\alpha-1}$. $\square$

Example. Consider the following singular eigenvalue problem:

$$\begin{aligned} &-{}_HD^{3/2}x(t) = \lambda(1-\ln t)^2 x^{-2/3}(t), \quad \text{a.e. } t \in (1,\mathrm{e}), \\ &x(1) = 0, \qquad x(\mathrm{e}) = \frac{1}{2} + 2x\biggl(\varphi\biggl(\frac{3}{2}\biggr)\biggr) + x\biggl(\varphi\biggl(\frac{5}{2}\biggr)\biggr), \end{aligned} \tag{21}$$

where $\varphi(t) = t^{1/2}$.

Proof. Let $\alpha = 3/2$, $\mu = 1$, $\eta_1 = 3/2$, $\eta_2 = 5/2$,

$$f(t,x) = (1-\ln t)^2 x^{-2/3}(t),$$

then (H1) holds, and for all $r \in (0,1)$ and for any $(t,x) \in [1,\mathrm{e}] \times (0,+\infty)$,

$$f(t,rx) = r^{-2/3}(1-\ln t)^2 x^{-2/3} \leqslant r^{-2/3} f(t,x),$$

which implies that (H2) also holds.

Also, by direct calculation, we have $\inf_{t\in[1,\mathrm{e}]} f(t,1) = 1 > 0$,

$$\begin{aligned} 0 &< \int_1^{\mathrm{e}} (1-\ln s)^{\alpha-2} f\bigl(s,\kappa(\ln s)\bigr)\frac{\mathrm{d}s}{s} = \int_1^{\mathrm{e}} (1-\ln s)^{-1/2}(1-\ln s)^2 \kappa^{-2/3}(s)\frac{\mathrm{d}s}{s} \\ &\leqslant \int_1^{\mathrm{e}} (1-\ln s)^{-1/6} \ln^{-1/3}(s)\frac{\mathrm{d}s}{s} < +\infty. \end{aligned}$$

Hence (H3) holds. Hence, by Theorem 1, there are two constants $0 < \lambda_1 < \lambda^*$ such that for any $\lambda \in (\lambda_1, \lambda^*)$, the singular eigenvalue problem (21) has at least one positive solution $x(t)$, and there exists a constant $\rho > 0$ such that

$$\ln^{1/2}(t)(1-\ln t) \leqslant x(t) \leqslant \rho \ln^{1/2}(t). \qquad \square$$

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