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Finite-time reliable nonfragile control for fractional-order nonlinear systems with asymmetrical saturation and structured uncertainties

L. Susana Ramya[a], **Rathinasamy Sakthivel**[b,1], **Chao Wang**[c]

[a]Department of Mathematics, Dr. NGP Institute of Technology,
Coimbatore-641 048, India

[b]Department of Applied Mathematics, Bharathiar University,
Coimbatore-641 046, India
krsakthivel@yahoo.com

[c]Department of Mathematics, Yunnan University,
Kunming, Yunnan 65009, China

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Abstract. This paper investigates the finite-time stabilization problem of fractional-order nonlinear differential systems via an asymmetrically saturated reliable control in the sense of Caputo's fractional derivative. In particular, an asymmetrical saturation control problem is converted to a symmetrical saturation control problem by using a linear matrix inequality framework criterion to achieve the essential results. Specifically, in this paper, we obtain two sets of sufficient conditions under different scenarios of structured uncertainty, namely, norm-bounded parametric uncertainty and linear fractional transformation uncertainty. The uncertainty considered in this paper is a combination of polytopic form and structured form. With the help of control theories of fractional-order system and linear matrix inequality technique, some sufficient criteria to ensure reliable finite-time stability of fractional-order differential systems by using the indirect Lyapunov approach are derived. As a final point, the derived criteria are numerically validated by means of examples based on financial fractional-order differential system and permanent magnet synchronous motor chaotic fractional-order differential system.

Keywords: fractional-order differential system, reliable nonfragile controller, finite-time control, asymmetric input saturation, structured uncertainty.

1 Introduction

Integer-order calculus is a classical tool to describe theory of physics. On the other hand, fractional-order calculus possesses infinite memory, and fractional-order variables can refine the performance of the system by rising a unit degree of freedom. Related with traditional integer-order calculus, fractional-order calculus can exactly describe the memory

[1]Corresponding author.

properties and hereditary of numerous equipments. Moreover, fractional-order calculus performs a vital part in various environments, such as in finance, medicine, cardiac tissues, material science, biological models, quantum mechanics, fluid mechanics and viscoelastic systems [9]. In order to exhibit the importance of fractional-order calculus in different areas of real world problems, several works were reported.

A crucial problem, which is dominant in integer-order and fractional-order differential systems, is the appearance of saturation in the control input. Specifically, major two approaches have been established in the studies for dealing with actuator saturations. The first approach is the positive invariance in which the control design works within a domain of linear mode, where saturations never occur. The next approach permits saturations in the controller design while assuring asymptotic stability [17], which gives a bounded ellipsoidal and symmetric stability region that can be established by solving a LMI constraint. One crucial threat in the above discussed approaches is to obtain a sufficient initial states domain, which guarantees asymptotic stability even in the presence of saturations. Lim et al. [11] has addressed the stabilization of fractional-order linear systems by considering the actuator saturation. Shahri et al. [16] applied a state-feedback control strategy to achieve the stability of fractional-order systems with saturation using the Gronwall–Bellman lemma. Song et al. [18] introduced an adaptive fuzzy output-feedback control algorithm for delayed fractional-order systems under saturation. In the existing literatures, the symmetrically saturated constraints are formulated in the form of LMIs. However, in real models, it is important to consider asymmetrically saturated constraints. Indeed, asymmetric saturation function can be approximated as symmetric saturation function with the minimum absolute value of the negative and positive saturation levels as the symmetric saturation level. However, such a manipulation on the saturation level is deemed conservative, and most often it could also result in an unsatisfactory controlled performance, especially, when there is a large difference between the negative and positive saturation levels of the asymmetric saturation function. As an alternative, many attempts were proposed to emphasis the study on LMIs with asymmetric saturations as in [21].

Besides saturation, in engineering systems, the control inputs are often subject to fluctuations within some scopes. The reason for such fluctuations are actuator faults, external disturbance, environment noise and so on. Hence, it is also very important to deal with the oscillation of system variables in order to obtain with the reliable performance in the presence of uncertainties [3]. Various kinds of formulations on uncertainty have been dealt with in the day-to-day research studies, namely, norm-bounded parametric uncertainty [23], linear fractional transformation (LFT) uncertainty [15] and so on. Many vital theories for various fractional-order uncertain differential systems have been developed in [19]. In general, when we design a controller for a system, we always assume that the designed controller gain is precise. However, the actuator degradation or the requirements of readjustment of controller gains during the controller implementation stage in the actual feedback control schemes may cause the fragile disturbances. Therefore, we should design a proper controller to tolerate some level of controller gain variations. This highly-sensitive characteristic of controller is called as fragility or nonresilience [4]. In this paper, we design a nonfragile controller for the considered fractional-order system to ensure the stability against some perturbations and asymmetric saturations [12].

For fractional-order differential systems, many enticing stability results and various kind of control designs have been explored and solved. It is well known that traditional Lyapunov stability always gives more consideration to steady-state behavioral analysis of controller gain dynamics over an infinite-time interval [22]. Also, the finite interval bounds on the dynamical system state trajectories are usually not specified in Lyapunov stability theory. However, almost in all concerned practical problems, the system states are expected not to exceed some bound during some time interval. In this case, we have to correct the values, which are not acceptable to judge if the trajectories of the system states remain inside the prescribed bound within a finite time. Moreover, to handle these transient performance in the dynamical control systems, short-time stability or finite-time stability is designed. As there is a lack in the finite-time stability operative test conditions, the finite-time stability concept has been studied recursively with the aid of linear matrix inequality theory [10, 22]. By utilizing the theories on linear matrix inequalities, some interesting results are established to guarantee the finite-time stabilization of various dynamical control systems including linear systems, nonlinear systems and stochastic systems [6, 14]. However, to date and to the best of our knowledge, the problems of reliable finite-time stability for fractional-order systems with input asymmetric saturation have not been investigated, which motivates the work of this paper.

Though some studies have been presented for fractional-order systems, there remain many disputes in analysis, modeling and designing the control, which needs to be addressed. In spite of the fact that a great amount of research studies have been carried out on stability and stabilization analysis of fractional-order differential systems, there still exists many major, crucial and important issues in design of controllers, which are reliable when subject to asymmetric input saturations in the presence of structured uncertainties. These specifications stimulated us and grabbed our attention resulting to the research study on reliable finite-time control protocol for fractional-order differential systems with asymmetrical saturation and structured uncertainties. The core contributions of this research paper are featured as follows:

- A feedback control design methodology is proposed for fractional-order nonlinear systems with two types of uncertainties, namely, norm-bounded parametric and linear fractional transformation uncertainties to obtain finite-time stability.
- Additionally, we perform an exhaustive study to demonstrate that the proposed design methodology is reliable to variations and asymmetrically saturated constraints in the controller.
- Two real-time validations of fractional-order differential system, namely, financial system and permanent magnet synchronous motor chaotic system are provided.

2 System formulation and preliminaries

Consider the following fractional-order differential system subject to asymmetrically constrained actuator saturation:

$$ {}^{C}D^{\varnothing}x(t) = (A_\eta + \Delta A)x(t) + g\big(x(t), t\big) + B\,\mathrm{sat}_{\underline{u},\overline{u}}\big(u(t)\big), \qquad x(0) = x_0, \quad (1) $$

where the state vector, which belongs to $\mathbb{R}^n$, is denoted as $x(t)$; fractional order $\o = \{\o_1, \o_2, \dots, \o_n\}$ in which $0 < \o_i < 1$ for $i = 1, 2, \dots, n$; $g(x(t), t)$ denotes the nonlinear function, which is bounded and Lipschitz with a Lipschitz constant ϵ_g such that for all $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}$, $g(x_0, t) = 0$, $\|g(x_1(t), t) - g(x_2(t), t)\| \leqslant \epsilon_g \|x_1(t) - x_2(t)\|$; $u(t) \in \mathbb{R}^m$ is the input vector, and $\mathrm{sat}_{\underline{u},\overline{u}}(u(t)) \in \mathbb{R}^m \to \mathbb{R}^m$ represents a saturation function, which is vector valued and asymmetrical in nature. In (1), $B \in \mathbb{R}^{n\times m}$ denotes the input matrix, and $A_\eta \in \mathbb{R}^{n\times n}$ denotes the system state weight matrix with polytopic uncertainty, which belongs to the polyhedral convex bounded domain with N vertices

$$\Omega_\eta \simeq \left\{ A_\eta \;\middle|\; A_\eta = \sum_{i=1}^{N} \eta_i A_i,\ \sum_{i=1}^{N} \eta_i = 1,\ \eta_i \geqslant 0 \right\}, \tag{2}$$

where $\Omega_i \simeq A_i$ is the polytope's ith vertex with appropriate dimensions, $i = 1, 2, \dots, N$; time-varying matrix ΔA represents the parameter uncertainty with prior structural information, which will be defined in the later part of this paper. Further, the aspect of each and every component of the asymmetrically saturated vector $\mathrm{sat}_{\underline{u},\overline{u}}(u(t))$ can be modelled by the relation $\mathrm{sat}_{\underline{u},\overline{u}}(u(t)) = [\mathrm{sat}_{\underline{u_1},\overline{u_1}}(u_1(t)),\ \mathrm{sat}_{\underline{u_2},\overline{u_2}}(u_2(t)), \dots, \mathrm{sat}_{\underline{u_m},\overline{u_m}}(u_m(t))]^{\mathrm{T}}$, and for all $j = \{1, 2, \dots, m\}$,

$$\mathrm{sat}_{\underline{u_j},\overline{u_j}}\big(u_j(t)\big) = \begin{cases} \overline{u_j}, & u_j(t) \in (\overline{u_j}, +\infty), \\ u_j(t), & u_j(t) \in [-\underline{u_j}, \overline{u_j}], \\ -\underline{u_j}, & u_j(t) \in (-\infty, \underline{u_j}), \end{cases}$$

where $\overline{u_j} > 0$ and $\underline{u_j} > 0$ represent, respectively, the positive and negative saturation levels, $\underline{u} = [\underline{u_1}, \underline{u_2}, \dots, \underline{u_m}]^{\mathrm{T}}$ and $\overline{u} = [\overline{u_1}, \overline{u_2}, \dots, \overline{u_m}]^{\mathrm{T}}$. Clearly, if $\underline{u_j} = \overline{u_j}$, then $\mathrm{sat}_{\underline{u_j},\overline{u_j}}(u_j(t)) = \mathrm{sat}_{\overline{u_j}}(u_j(t))$ will become a saturation function, which is symmetric in nature. Further, we also assume that $\overline{u_j} > \underline{u_j}$ for all $j = \{1, 2, \dots, m\}$.

At this juncture, to stabilize the saturated system with asymmetrical constraints, we design a state feedback control protocol as

$$u(t) = L(K + \Delta K)x(t) + K_0, \tag{3}$$

where K denotes the controller gain matrix, and ΔK denotes the possible fluctuations in control design. Further, the asymmetrical set $Ł(K)$ will be symmetrized because of the gain matrices $L \in \mathbb{R}^{m\times m}$ and $K_0 \in \mathbb{R}^{m\times m}$. The state space representation of $Ł(K)$ can be given by

$$Ł(K) = \big\{ x(t) \in \mathbb{R}^n \;\big|\; -\Gamma e \leqslant LKx(t) + K_0 \leqslant \Lambda e \big\}, \tag{4}$$

where the diagonal matrices Γ and Λ take the form

$$\Gamma = \begin{bmatrix} \underline{u_1} & 0 & \dots & 0 \\ * & \underline{u_2} & \dots & 0 \\ * & * & \dots & 0 \\ * & * & * & \underline{u_m} \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \overline{u_1} & 0 & \dots & 0 \\ * & \overline{u_2} & \dots & 0 \\ * & * & \dots & 0 \\ * & * & * & \overline{u_m} \end{bmatrix}$$

with $*$ representing the terms induced by symmetry, and $e \in \mathbb{R}^m$ is a vector of the form $e = [1,\ 1,\ \dots,\ 1]^{\mathrm{T}}$. The parameter uncertainty matrix ΔA and control gain fluctuation matrix ΔK are considered to fulfill any one of the understated two cases.

(C1) Norm-bounded parametric uncertainty: Under this case of parametric uncertainty, matrices ΔA and ΔK are considered to satisfy $\|\Delta A\| \leqslant \epsilon_A$ and $\|\Delta K\| \leqslant \epsilon_K$, where ϵ_A and ϵ_K are positive constants.

(C2) Linear fractional transformation (LFT) uncertainty: Under this case LFT uncertainty, matrices ΔA and ΔK are considered to satisfy $\Delta A = D(I - \Delta_1 F)^{-1} \times \Delta_1 E$ and $\Delta K = X(I - \Delta_2 Y)^{-1}\Delta_2 Z$. Here the matrices, which describe the degree of uncertainty, are denoted by D, E, F, X, Y and Z, and the uncertainty Δ_r for $r = 1, 2$ are defined in $\mathcal{B}_{\Delta_r} = \{\Delta_r : \mathbb{R} \to \mathbb{R}^{n\times n},\ \Delta_r \in \nabla_r,\ \|\Delta_r\| \leqslant 1\}$, where $\nabla_r = \mathrm{diag}\{\nabla_{r1}, \nabla_{r2}, \dots, \nabla_{rm}\}$ such that $\|\nabla_{rj}\| \leqslant 1$, $j = 1, 2, \dots, m$.

Remark 1. In case (C2), if we take Δ_r as an unknown time-varying matrix where the elements are Lebesgue measurable and also bounded by $\Delta_r^{\mathrm{T}}(t)\Delta_r(t) \leqslant I$, then it is similar to the case of LFT uncertainty discussed in [14]. Besides, if we consider F and Y as null matrices, then case (C2) deduces to the most familiar norm-bounded uncertainty case. So, case (C2) is a generalization of some special uncertainty cases.

Right away, we are at the juncture to deal with the problem of stabilization of fractional-order differential system (1) by utilizing a state feedback control (3). Therefore, to take over this situation with in a LMI framework, we require the following definitions and lemmas, which are presented here.

Now, we connect the saturation function $\mathrm{sat}_{\underline{u},\overline{u}}(u(t))$ with an asymmetrical saturation function using a normalized saturation symmetrical function. On account of this, we initiate a formal new variable $w_j(t)$ as follows:

$$w_j(t) = u_j(t) - \frac{\overline{u_j} - \underline{u_j}}{2}. \tag{5}$$

Accordingly, the saturation control can be reformulated as

$$\mathrm{sat}_{\underline{u},\overline{u}}\bigl(u_j(t)\bigr) = \mathrm{sat}_{\underline{w},\overline{w}}\bigl(w_j(t)\bigr) + \frac{\overline{u_j} - \underline{u_j}}{2},$$

where for $j = 1, 2, \dots, m$,

$$\mathrm{sat}_{\underline{w},\overline{w}}\bigl(w_j(t)\bigr) = \begin{cases} \frac{\overline{u_j}+\underline{u_j}}{2}, & w_j(t) \geqslant \frac{\overline{u_j}+\underline{u_j}}{2}, \\ w_j(t), & -\frac{\overline{u_j}+\underline{u_j}}{2} < w_j(t) < \frac{\overline{u_j}+\underline{u_j}}{2}, \\ -\frac{\overline{u_j}+\underline{u_j}}{2}, & w_j(t) \leqslant -\frac{\overline{u_j}+\underline{u_j}}{2}. \end{cases}$$

Now, by introducing a second change of variable

$$z_j(t) = w_j(t)\frac{2}{\overline{u_j} + \underline{u_j}} \tag{6}$$

we obtain

$$\operatorname{sat}\bigl(z_j(t)\bigr) = \begin{cases} 1, & z_j(t) \geqslant 1, \\ z_j(t), & -1 \leqslant z_j(t) \leqslant 1, \\ -1, & z_j(t) \leqslant -1, \end{cases} \tag{7}$$

for $j = 1, 2, \dots, m$. The above obtained equation (7) is nothing but the normalized saturation symmetric function. Further, by using equations (5) and (6), it is easy to obtain $u_j(t)$ as $u_j(t) = (\overline{u_j} + \underline{u_j})z_j(t)/2 + (\overline{u_j} - \underline{u_j})/2$. This obtained $u_j(t)$ can be written equivalently in matrix form as $u(t) = (\Lambda + \Gamma)z(t)/2 + (\Lambda - \Gamma)e/2$. Therefore, the asymmetrical saturation $\operatorname{sat}_{\underline{u},\overline{u}}(u(t))$ can be established as a symmetrical saturation function $\operatorname{sat}(z(t))$ by using the relation

$$\operatorname{sat}_{\underline{u},\overline{u}}\bigl(u(t)\bigr) = \frac{\Lambda + \Gamma}{2}\operatorname{sat}\bigl(z(t)\bigr) + \frac{\Lambda - \Gamma}{2}e. \tag{8}$$

Now by using (8) in (1), the fractional-order differential system can be expressed with the symmetrical saturation function $\operatorname{sat}(z(t))$ as

$${}^{C}D^{\phi}x(t) = (A_\eta + \Delta A)x(t) + g\bigl(x(t), t\bigr) + \tilde{B}\operatorname{sat}\bigl(z(t)\bigr) + E\tau(t), \tag{9}$$

where $\tilde{B} = B(\Lambda + \Gamma)/2$, $E = \sqrt{n}B(\Lambda - \Gamma)/2$ and $\tau(t) = e/\sqrt{m}$. Clearly we can observe that $\tau^{\mathrm{T}}(t)\tau(t) = 1$. The obtained system (9) is a fractional-order symmetrically saturated differential system with $\tau(t)$ as its external disturbance. Further, the state feedback control for the fractional-order differential system (9) takes the form $z(t) = (K + \Delta K)x(t)$. Therefore, the asymmetrical set $Ł(K)$ given in (4) has been symmetrized by using the feedback control (3) with $L = (\Lambda + \Gamma)/2$ and $K_0 = (\Lambda - \Gamma)e/2$. For detailed proof, the readers can see [2, Lemma 8.2].

Now, an ellipsoidal set $\varepsilon_s(P_\eta, \rho)$ is defined by

$$\varepsilon_s(P_\eta, \rho) = \bigl\{x(t) \in \mathbb{R}^n \bigm| x^{\mathrm{T}}(t)x(t) \leqslant 1\bigr\}. \tag{10}$$

Then

$$Ł_s(K) = \bigl\{x(t) \in \mathbb{R}^n \bigm| \bigl|Kx(t)\bigr|_j \leqslant 1,\ j = 1, 2, \dots, m\bigr\} \tag{11}$$

is an equivalent form of $Ł_s(K)$ defined in (4), in such a way, $\varepsilon_s(P_\eta, \rho) \subset Ł_s(K)$. Let us define a matrix D to be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. Suppose that each element of D is labeled as D_s, and each element of $I - D_s$ is taken as the elements of D_s^-. Then the following equation (12) could be obtained by using Lemma 1 in [8]:

$$\operatorname{sat}\bigl(z(t)\bigr) = \sum_{s=1}^{2^m} \delta_s(t)\bigl(D_s z(t) + D_s^- v(t)\bigr), \tag{12}$$

where $z(t) = (K + \Delta K)x(t)$ and $v(t) = Hx(t)$ with $|h_i x(t)| < 1$. If H is an auxiliary control matrix, then h_i is the ith row of H. Henceforward, the problem of stabilization of system (1) with asymmetrical saturated control is converted to the problem of stabilization of system (9) with a symmetrical saturated control. Now, by using (12) in (9), the closed-loop fractional-order system can be expressed in the form

$$\begin{aligned} {}^{C}D^{\emptyset}x(t) &= (A_{\eta} + \Delta A)x(t) + g\big(x(t), t\big) \\ &\quad + \tilde{B}\left(\sum_{s=1}^{2^m} \delta_s(t)(D_sK + D_s\Delta K + D_s^- H)\right)x(t) + E\tau(t) \end{aligned}$$

or in the equivalent form

$$ {}^{C}D^{\emptyset}x(t) = (\mathcal{A} + \Delta\mathcal{A})x(t) + g\big(x(t), t\big) + E\tau(t), \tag{13}$$

where $\mathcal{A} = A_{\eta} + \tilde{B}\sum_{s=1}^{2^m} \delta_s(t)(D_sK + D_s^- H)$ and $\Delta\mathcal{A} = \Delta A + \tilde{B}\sum_{s=1}^{2^m} \delta_s(t)D_s\Delta K$. It is possible to obtain the domain of attraction of the above fractional-order differential system (13) only under the assumption that $\mathcal{A} + \Delta\mathcal{A}$ is Hurwitz. Therefore, we assume that $\mathcal{A} + \Delta\mathcal{A}$ is Hurwitz.

Now let us consider the case when system (1) is subject to actuator faults. Then the controller becomes a reliable controller of the form $z(t) = (\tilde{G}K + \Delta K)x(t)$, where $\tilde{G}$ is the matrix, which represents the actuator fault such that $\tilde{G} = \operatorname{diag}\{g_1, g_2, \dots, g_m\}$. Further, $g_k \in [g_k^l, g_k^u]$, with $g_k^l \geqslant 0$ and $g_k^u \leqslant 1$. Let us define some matrices as follows: $\tilde{G}^u = \operatorname{diag}\{g_1^u, g_2^u, \dots, g_m^u\}$, $\tilde{G}^l = \operatorname{diag}\{g_1^l, g_2^l, \dots, g_m^l\}$, $\tilde{G}_0 = (\tilde{G}^u + \tilde{G}^l)/2$ and $\tilde{G}_1 = (G^u - G^l)/2$. Then $\tilde{G} = \tilde{G}_0 + \tilde{G}_1\Omega$, where $\Omega = \{\omega_1, \omega_2, \dots, \omega_m\} \in \mathbb{R}^{m\times m}$ and $-1 \leqslant \omega_k \leqslant 1$, $k = \{1, 2, \dots, m\}$. Now, by using (12) in (9), the closed-loop fractional-order differential system subject to actuator faults can be expressed as

$$ {}^{C}D^{\emptyset}x(t) = (\mathcal{A}_g + \Delta\mathcal{A})x(t) + g\big(x(t), t\big) + E\tau(t), \tag{14}$$

where $\mathcal{A}_g = A_{\eta} + \tilde{B}\sum_{s=1}^{2^m} \delta_s(t)(D_s\tilde{G}K + D_s^- H)$. Further, for (14), we assume that $\mathcal{A}_g + \Delta\mathcal{A}$ is Hurwitz to obtain the domain of attraction.

3 Finite-time stability of fractional-order differential system subject to norm-bounded parametric uncertainties

The sufficient conditions established in the following Theorem 1 can guarantee finite-time stability of the differential system (13), which is subject to asymmetrically saturated control input. The results are then further advanced to the case when actuator faults affect system (14) in Theorem 2. Assume that the uncertain parameters of the fractional-order differential system and control gain fluctuation fulfil the uncertainty as in case (C1).

Theorem 1. *For given scalars a, α, f_1, T, ϵ_A, ϵ_K, the asymmetrically constraint fractional-order differential system* (13) *is stable within a finite interval of time with respect to* $(f_1, f_2, \mathsf{T}, V)$ *if there exist matrices T and U, matrices $\hat{P}_i$, $\hat{R}$ and $\hat{S}$, which are symmetric* *and positive definite, diagonal matrices $\hat{Q}_i > 0$ and scalars γ, ϵ_a, ϵ_b, ϵ_g such that the following LMI holds for $i = 1, 2, \dots, N$:*

$$\begin{bmatrix} \hat{\Omega}_{1,1} & \hat{R} - a\hat{Q}_i^{\mathrm{T}} & E\hat{R} & \hat{P}_i + \hat{R}A_i^{\mathrm{T}} - \hat{R} & I & \tilde{B}\sum_{s=1}^{2^m}\delta_s D_s & \hat{R} \\ * & -\gamma I & 0 & \hat{Q}_i + \hat{R} & 0 & 0 & 0 \\ * & * & -a\hat{S} & \hat{R}E^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & * & -2\hat{R} & I & \tilde{B}\sum_{s=1}^{2^m}\delta_s D_s & 0 \\ * & * & * & * & -\epsilon_a I & 0 & 0 \\ * & * & * & * & * & -\epsilon_b I & 0 \\ * & * & * & * & * & * & \hat{\Omega}_{7,7} \end{bmatrix} < 0, \tag{15}$$

$$(\lambda_{i2} f_1 + \lambda_3) E_{\o}\bigl(at^{\o}\bigr) < \lambda_{i1} f_2, \tag{16}$$

$$\begin{bmatrix} \frac{-1}{\rho} & U \\ * & -\hat{P}_i \end{bmatrix} \leqslant 0, \tag{17}$$

where $\hat{\Omega}_{1,1} = A_i\hat{R} + \tilde{B}\sum_{s=1}^{2^m}\delta_s(t)(D_s T + D_s^- U) + \alpha\hat{P}_i - a\hat{P}_i$ and $\hat{\Omega}_{7,7} = -(\gamma\epsilon_g^2 + \epsilon_a\epsilon_A^2 + \epsilon_b\epsilon_K^2)^{-1}$. Eventually, K and H can be computed by using the relations $K = T\hat{R}^{-1}$ and $H = U\hat{R}^{-1}$.

Proof. Equations (10) and (11) imply that $\varepsilon_s(P_\eta, \rho) \subset Ł(K)$. On this basis, in order to accomplish the finite-time stability sufficient conditions for the considered asymmetrically constraint fractional-order differential system (13), we structure a Lyapunov–Krasovskii functional candidate $\mathcal{V}(t, x(t), \eta) \geqslant 0$ as

$$\mathcal{V}\bigl(t, x(t), \eta\bigr) = x^{\mathrm{T}}(t)\mathcal{P}_\eta x(t) + 2\sum_{j=1}^{n} q_{j\eta} \int_0^{x(t)} g_j(s)\,\mathrm{d}s + \tau^{\mathrm{T}}(t) S \tau(t). \tag{18}$$

Further, based on $\tau(t)$ defined in (9) and with the aid of Property 1 in [7], if Γ denotes the gamma function, then the fractional-order derivative of $\mathcal{V}(t, x(t), \eta)$ in the sense of Caputo is given by

$$\begin{aligned} {}^{C}D^{\o}\mathcal{V}\bigl(t, x(t), \eta\bigr) &= {}^{R}D^{\o}\left[x(t)^{\mathrm{T}}\mathcal{P}_\eta x(t) - \sum_{k=1}^{n}\bigl(x(t)^{\mathrm{T}}\mathcal{P}_\eta x(t)\bigr)^{k}(0)\frac{t^k}{k!}\right] \\ &\quad + 2\sum_{j=1}^{n} q_{j\eta} g^{\mathrm{T}}\bigl(x(t), t\bigr)^{C}D^{\o}x(t), \\ {}^{C}D^{\o}\mathcal{V}\bigl(t, x(t), \eta\bigr) &= \bigl[{}^{R}D^{\o}x(t)\bigr]^{\mathrm{T}}\mathcal{P}_\eta x(t) + x^{\mathrm{T}}(t)\mathcal{P}_\eta\bigl[{}^{R}D^{\o}x(t)\bigr] \\ &\quad + \mathcal{P}_\eta \sum_{k=1}^{\infty}\frac{\Gamma(1+\o)}{\Gamma(1+k)\Gamma(1-k+\o)}{}^{R}D^{k}x(t)^{R}D^{\o-k}x(t) \\ &\quad - {}^{R}D^{\o}x^{\mathrm{T}}(0)\mathcal{P}_\eta x(0) + 2\sum_{j=1}^{n} q_{j\eta} g^{\mathrm{T}}(x(t), t)^{C}D^{\o}x(t), \end{aligned}$$

$$
\begin{aligned}
{}^{C}D^{\o}\mathcal{V}\big(t,x(t),\eta\big) &= \big[{}^{R}D^{\o}x(t)\big]^{\mathrm{T}}\mathcal{P}_{\eta}x(t) + x^{\mathrm{T}}(t)\mathcal{P}_{\eta}\big[{}^{R}D^{\o}x(t)\big] \\
&\quad + \mathcal{P}_{\eta}\sum_{k=1}^{\infty}\frac{\Gamma(1+\o)}{\Gamma(1+k)\Gamma(1-k+\o)}\,{}^{R}D^{k}x(t)\,{}^{R}D^{\o-k}x(t) \\
&\quad - \frac{t^{-\o}\mathcal{P}_{\eta}}{\Gamma(1-\o)}x^{\mathrm{T}}(0)x(0) + 2\sum_{j=1}^{n}q_{j\eta}g^{\mathrm{T}}\big(x(t),t\big)\,{}^{C}D^{\o}x(t),
\end{aligned}
$$

where ${}^{R}D^{\o}$ denotes the fractional derivative in the sense of Riemann–Liouville. For the sake of notational ease, let us replace all Riemann–Liouville fractional derivative as Caputo fractional derivative

$$
\begin{aligned}
{}^{C}D^{\o}\mathcal{V}\big(t,x(t),\eta\big) &= \big[{}^{C}D^{\o}x(t)\big]^{\mathrm{T}}\mathcal{P}_{\eta}x(t) + x^{\mathrm{T}}(t)\mathcal{P}_{\eta}\big[{}^{C}D^{\o}x(t)\big] \\
&\quad + \mathcal{P}_{\eta}\chi_{x}(t) - \frac{t^{-\o}\mathcal{P}_{\eta}}{\Gamma(1-\o)}x^{\mathrm{T}}(0)x(0) + 2\sum_{j=1}^{n}q_{j\eta}g^{\mathrm{T}}\big(x(t),t\big)\,{}^{C}D^{\o}x(t),
\end{aligned}
$$

where

$$
\chi_{x}(t) = \sum_{k=1}^{\infty}\frac{\Gamma(1+\o)}{\Gamma(1+k)\Gamma(1-k+\o)}\,{}^{R}D^{k}x(t)\,{}^{R}D^{\o-k}x(t),
$$

and we can consider the boundedness condition $\chi_{x}(t) \leqslant x^{\mathrm{T}}(t)\alpha x(t)$, where $\alpha > 0$ is a constant. For the sake of notational ease, we denote ${}^{C}D^{\o}x(t) = \dot{\tilde{x}}(t)$ and $Q_{\eta} = \sum_{j=1}^{n}q_{j\eta}$. Since $t^{-\o}P/\Gamma(1-\o)x^{\mathrm{T}}(0)x(0) > 0$, we can obtain

$$
{}^{C}D^{\o}\mathcal{V}\big(t,x(t),\eta\big) \leqslant 2x(t)^{\mathrm{T}}P_{\eta}\dot{\tilde{x}}(t) + x^{\mathrm{T}}(t)\alpha\mathcal{P}_{\eta}x(t) + 2g^{\mathrm{T}}\big(x(t),t\big)Q_{\eta}\dot{\tilde{x}}(t). \tag{19}
$$

Further, if $\dot{\tilde{x}}(t) + x(t)$ is taken as $\tilde{x}(t)$, then the following equality is true for any matrix R with respect to the fractional-order differential system (13):

$$
2\tilde{x}^{\mathrm{T}}(t)R\big[(\mathcal{A}+\Delta\mathcal{A})x(t) + g\big(x(t),t\big) + E\tau(t) - \dot{\tilde{x}}(t)\big] = 0,
$$

or, equivalently,

$$
2\tilde{x}^{\mathrm{T}}(t)R\big[\mathcal{A}x(t) + g\big(x(t),t\big) + E\tau(t) - \dot{\tilde{x}}(t)\big] + 2\tilde{x}^{\mathrm{T}}(t)R\Delta\mathcal{A}x(t) = 0. \tag{20}
$$

By using the facts stated in case (C1) and by applying Lemma 2 of [7] for the second term in the above equation, we have

$$
\begin{aligned}
2\tilde{x}^{\mathrm{T}}(t)R\Delta\mathcal{A}x(t) &= 2\tilde{x}^{\mathrm{T}}(t)R\left(\Delta A + \tilde{B}\sum_{s=1}^{2^{m}}\delta_{s}(t)D_{s}\Delta K\right)x(t) \\
&\leqslant \epsilon_{a}x^{\mathrm{T}}(t)\epsilon_{A}^{2}x(t) + \epsilon_{a}^{-1}\tilde{x}^{\mathrm{T}}(t)RR^{\mathrm{T}}\tilde{x}(t) + \epsilon_{b}x^{\mathrm{T}}(t)\epsilon_{K}^{2}x(t) \\
&\quad + \epsilon_{b}^{-1}\tilde{x}^{\mathrm{T}}(t)\left(R\tilde{B}\sum_{s=1}^{2^{m}}\delta_{s}D_{s}\right)\left(R\tilde{B}\sum_{s=1}^{2^{m}}\delta_{s}(t)D_{s}\right)^{\mathrm{T}}\tilde{x}(t).
\end{aligned} \tag{21}
$$

Also, for any scalar $\gamma > 0$, we have

$$\gamma\big[\epsilon_g^2 x^{\mathrm{T}}(t)x(t) - g^{\mathrm{T}}\big(x(t),t\big)g\big(x(t),t\big)\big] \geqslant 0. \tag{22}$$

Now, combining (18)–(22), we have

$$\begin{aligned}
&{}^{C}D^{\phi}\mathcal{V}\big(t,x(t),\eta\big) - a\mathcal{V}\big(t,x(t),\eta\big)\\
&\quad\leqslant x^{\mathrm{T}}(t)\big[2R\mathcal{A} + \gamma\epsilon_g^2 + \epsilon_a\epsilon_A^2 + \epsilon_b\epsilon_K^2\big]x(t) + 2x^{\mathrm{T}}(t)Rg\big(x(t),t\big)\\
&\qquad + 2x^{\mathrm{T}}(t)RE\tau(t) + 2x^{\mathrm{T}}(t)\big[P_\eta + \mathcal{A}^{\mathrm{T}}R^{\mathrm{T}} - R\big]\dot{\tilde{x}}(t)\\
&\qquad + 2g^{\mathrm{T}}\big(x(t),t\big)\big[Q_\eta + R^{\mathrm{T}}\big]\dot{\tilde{x}}(t) - \gamma g^{\mathrm{T}}\big(x(t),t\big)g\big(x(t),t\big)\\
&\qquad + 2\dot{\tilde{x}}^{\mathrm{T}}(t)RE\tau(t) - 2\dot{\tilde{x}}^{\mathrm{T}}(t)R\dot{\tilde{x}}(t) + x^{\mathrm{T}}(t)\alpha P_\eta x(t) + \epsilon_a^{-1}\tilde{x}^{\mathrm{T}}(t)RR^{\mathrm{T}}\tilde{x}(t)\\
&\qquad + \epsilon_b^{-1}\tilde{x}^{\mathrm{T}}(t)\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s D_s\right)\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_s\right)^{\mathrm{T}}\tilde{x}(t)\\
&\qquad - a\left[x^{\mathrm{T}}(t)\mathcal{P}_\eta x(t) + 2\sum_{j=1}^{n}q_{j\eta}\int_0^{x(t)}g_j(s)\,\mathrm{d}s + \tau^{\mathrm{T}}(t)S\tau(t)\right].
\end{aligned}$$

Alternatively, the above obtained inequality can be written in an explanatory form as

$$\begin{aligned}
&{}^{C}D^{\phi}\mathcal{V}\big(t,x(t),\eta\big) - a\mathcal{V}\big(t,x(t),\eta\big)\\
&\quad\leqslant x^{\mathrm{T}}(t)\big[2R\mathcal{A} + \gamma\epsilon_g^2 + \epsilon_a\epsilon_A^2 + \epsilon_b\epsilon_K^2 + \alpha P_\eta - aP_\eta\big]x(t)\\
&\qquad + 2x^{\mathrm{T}}(t)Rg\big(x(t),t\big) + 2x^{\mathrm{T}}(t)RE\tau(t) + 2x^{\mathrm{T}}(t)\big[P_\eta + \mathcal{A}^{\mathrm{T}}R^{\mathrm{T}} - R\big]\dot{\tilde{x}}(t)\\
&\qquad + 2g^{\mathrm{T}}\big(x(t),t\big)\big[Q_\eta + R^{\mathrm{T}}\big]\dot{\tilde{x}}(t) - \gamma g^{\mathrm{T}}\big(x(t),t\big)g\big(x(t),t\big) - a\tau^{\mathrm{T}}(t)S\tau(t)\\
&\qquad + 2\dot{\tilde{x}}^{\mathrm{T}}(t)RE\tau(t) - 2\dot{\tilde{x}}^{\mathrm{T}}(t)R\dot{\tilde{x}}(t) - 2g^{\mathrm{T}}(x(t),t)aQ_\eta x(t) + \epsilon_a^{-1}\tilde{x}^{\mathrm{T}}(t)RR^{\mathrm{T}}\tilde{x}(t)\\
&\qquad + \epsilon_b^{-1}\tilde{x}^{\mathrm{T}}(t)\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s D_s\right)\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_s\right)^{\mathrm{T}}\tilde{x}(t)\\
&\quad\leqslant \xi^{\mathrm{T}}(t)\varTheta_\eta\xi(t) + \epsilon_a^{-1}\tilde{x}^{\mathrm{T}}(t)RR^{\mathrm{T}}\tilde{x}(t)\\
&\qquad + \epsilon_b^{-1}\tilde{x}^{\mathrm{T}}(t)\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s D_s\right)\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_s\right)^{\mathrm{T}}\tilde{x}(t),
\end{aligned}\tag{23}$$

where $\xi^{\mathrm{T}}(t) = [x^{\mathrm{T}}(t),\, g^{\mathrm{T}}(x(t),t),\, \tau^{\mathrm{T}}(t),\, \dot{\tilde{x}}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and

$$\varTheta_\eta = \begin{bmatrix} \varTheta_{(1,1),\eta} & R - aQ_\eta^{\mathrm{T}} & RE & P_\eta + \mathcal{A}^{\mathrm{T}}R^{\mathrm{T}} - R\\ * & -\gamma I & 0 & Q_\eta + R^{\mathrm{T}}\\ * & * & -aS & E^{\mathrm{T}}R^{\mathrm{T}}\\ * & * & * & -2R \end{bmatrix}$$

with $\Theta_{(1,1),\eta} = 2R\mathcal{A} + \gamma\epsilon_g^2 + \epsilon_a\epsilon_A^2 + \epsilon_b\epsilon_K^2 + \alpha P_\eta - aP_\eta$. Then, using the convexity defined in (2), it is easy to get

$$\Theta_\eta = \sum_{i=1}^{N} \eta_i \Theta_i \quad \forall \sum_{i=1}^{N} \eta_i = 1,\ \eta_i \geqslant 0.$$

Then (23) becomes

$$\begin{aligned} &{}^C D^{\text{ø}} \mathcal{V}\big(t, x(t), \eta\big) - a\mathcal{V}\big(t, x(t), \eta\big) \\ &\quad \leqslant \sum_{i=1}^{N} \eta_i \Bigg[\xi^{\mathrm{T}}(t)\Theta_i \xi(t) + \epsilon_a^{-1} \tilde{x}^{\mathrm{T}}(t) R R^{\mathrm{T}} \tilde{x}(t) \\ &\qquad + \epsilon_b^{-1} \tilde{x}^{\mathrm{T}}(t) \Bigg(R\tilde{B} \sum_{s=1}^{2^m} \delta_s D_s \Bigg) \Bigg(R\tilde{B} \sum_{s=1}^{2^m} \delta_s(t) D_s \Bigg)^{\mathrm{T}} \tilde{x}(t) \Bigg], \end{aligned} \tag{24}$$

where

$$\Theta_i = \begin{bmatrix} \Theta_{(1,1),i} & R - aQ_i & RE & P_i + \mathcal{A}^{\mathrm{T}} R - R \\ * & -\gamma I & 0 & Q_i + R \\ * & * & -aS & E^{\mathrm{T}} R \\ * & * & * & -2R \end{bmatrix}$$

with $\Theta_{(1,1),i} = 2R\mathcal{A} + \gamma\epsilon_g^2 + \epsilon_a\epsilon_A^2 + \epsilon_b\epsilon_K^2 + \alpha P_i - aP_i$. By using Schur complement and $\mathcal{A} = A_\eta + \tilde{B}\sum_{s=1}^{2^m} \delta_s(t)(D_s K + D_s^- H)$, it can be observed that right side of (24) is equivalent to $\sum_{i=1}^{N} \eta_i \Omega_i$, where

$$\Omega_i = \begin{bmatrix} \Omega_{(1,1),i} & R - aQ_i & RE & P_i + A_\eta^{\mathrm{T}} R - R & R & R\tilde{B}\sum_{s=1}^{2^m} \delta_s D_s \\ * & -\gamma I & 0 & Q_i + R & 0 & 0 \\ * & * & -aS & E^{\mathrm{T}} R & 0 & 0 \\ * & * & * & -2R & R & R\tilde{B}\sum_{s=1}^{2^m} \delta_s D_s \\ * & * & * & * & -\epsilon_a I & 0 \\ * & * & * & * & * & -\epsilon_b I \end{bmatrix}$$

with $\Omega_{(1,1),i} = RA_\eta + R\tilde{B}\sum_{s=1}^{2^m} \delta_s(t)(D_s K + D_s^- H) + \gamma\epsilon_g^2 + \epsilon_a\epsilon_A^2 + \epsilon_b\epsilon_K^2 + \alpha P_i - aP_i$. Then, by pre- and post-multiplying Ω_i by $\operatorname{diag}\{R^{-1}, R^{-1}, R^{-1}, R^{-1}, I, I\}$ and by denoting $R^{-1} = \hat{R}$, $\hat{R}P_\eta\hat{R} = \hat{P}_\eta, \hat{R}Q_\eta\hat{R} = \hat{Q}_\eta$ and $\hat{R}S\hat{R} = \hat{S}$, it is easy to get that Ω_i and the left side of equation (15) are equivalent. Therefore, if (15) is satisfied, then ${}^C D^{\text{ø}} \mathcal{V}(t, x(t), \eta) - a\mathcal{V}(t, x(t), \eta) < 0$.

The above inequality indicates that ${}^C D^{\text{ø}} \mathcal{V}(t, x(t), \eta) + J(t, x(t), \eta) = a\mathcal{V}(t, x(t), \eta)$ for some function $J(t, x(t), \eta) > 0$. By using the theory of the Laplace transform, we have $s^{\text{ø}}\mathcal{V}(s, x(s), \eta) - \mathcal{V}(0, x(0), \eta)s^{\text{ø}-1} + J(s, x(s), \eta) = a\mathcal{V}(s, x(s), \eta)$. Consequently, $\mathcal{V}(s, x(s), \eta) = (\mathcal{V}(0, x(0), \eta)s^{\text{ø}-1} - J(s, x(s), \eta))/(s^{\text{ø}} - a)$. Now by using the theory of inverse Laplace transform, we can arrive at an equation as follows:

$$\mathcal{V}\big(t, x(t), \eta\big) = \mathcal{V}\big(0, x(0), \eta\big) E_{\o}\big(at^{\o}\big) - \int_0^t J(\mu)\big[(t-\mu)^{\o-1} E_{\o,\o}\big(a(t-\mu)^{\o}\big)\big]\,\mathrm{d}\mu.$$

Since $(t-\mu)^{\o-1}$ and $E_{\o,\o}(a(t-\mu)^{\o})$ are nonnegative functions, from the above equation we obtain

$$\mathcal{V}\big(t, x(t), \eta\big) \leqslant \mathcal{V}\big(0, x(0), \eta\big) E_{\o}\big(at^{\o}\big). \tag{25}$$

By using convexity definition and defining $\tilde{P}_i = (1/\sqrt{V})P_i(1/\sqrt{V})$, $\tilde{S} = (1/\sqrt{V})S \times (1/\sqrt{V})$, $\lambda_{i1} = \lambda_{\min}(\tilde{P}_i)$, $\lambda_{i2} = \lambda_{\max}(\tilde{P}_i)$ and $\lambda_3 = \lambda_{\max}(S)$, we obtain

$$\begin{aligned}
&\mathcal{V}\big(t, x(t), \eta\big) \\
&\quad = x^{\mathrm{T}}(t)\sqrt{V}\tilde{P}_i\sqrt{V}x(t) + \tau^{\mathrm{T}}(t)\sqrt{V}\tilde{S}\sqrt{V}\tau(t) \\
&\quad \geqslant +2\sum_{j=1}^{n} q_{j\eta} \int_0^{\dot{\bar{x}}(t)} g_j(s)\,\mathrm{d}s\,\lambda_{\min}(\tilde{P}_i)x^{\mathrm{T}}(t)Vx(t) = \lambda_{i1}x^{\mathrm{T}}(t)Vx(t),
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\mathcal{V}(0, x(0), \eta)E_{\o}\big(at^{\o}\big) \\
&\quad = \left(x^{\mathrm{T}}(0)P_i x(0) + \tau^{\mathrm{T}}(0)S\tau(0) + 2\sum_{j=1}^{n} q_{j\eta}\int_0^{\dot{\bar{x}}(0)} g_j(s)\,\mathrm{d}s\right)E_{\o}\big(at^{\o}\big) \\
&\quad \leqslant \lambda_{\max}(\tilde{P}_i)x^{\mathrm{T}}(0)Vx(0) + \lambda_{\max}(S)\tau^{\mathrm{T}}(0)\tau(0)E_{\o}\big(at^{\o}\big).
\end{aligned}$$

If $x^{\mathrm{T}}(0)Vx(0) \leqslant f_1$, the above expression can be written in a simplified form as

$$\mathcal{V}\big(0, x(0), \eta\big)E_{\o}\big(at^{\o}\big) \leqslant (\lambda_{i2}f_1 + \lambda_3)E_{\o}\big(at^{\o}\big). \tag{27}$$

Combining (25), (26) and (27), we can derive that

$$\begin{aligned}
\lambda_{i1}x(t)^{\mathrm{T}}Vx(t) &\leqslant \mathcal{V}\big(t, x(t), \eta\big) \leqslant \mathcal{V}\big(0, x(0), \eta\big)E_{\o}\big(at^{\o}\big) \\
&\leqslant (\lambda_{i2}f_1 + \lambda_3)E_{\o}\big(at^{\o}\big).
\end{aligned}$$

Hence, $x(t)^{\mathrm{T}}Vx(t) < (\lambda_{i2}f_1 + \lambda_3)E_{\o}(at^{\o})/\lambda_{i1}$. Therefore, if (16) holds, then $x(t)^{\mathrm{T}}Vx(t) < f_2$ for all $t \in [0, \mathcal{T}]$. Thus, for all $x_0 \in \varepsilon_s(P_\eta, \rho)$, it follows that system (13) is finite-time bounded according to Definitions 3 and 4 in [1].

Furthermore, $\rho h_i P_\eta^{-1} h_i^{\mathrm{T}} \leqslant 1$, $i = 1, 2, \dots, m$, is the equivalent form of $\varepsilon_s(P_\eta, \rho) \subset \mathcal{L}(K)$, where h_i is the ith row of H. $\rho h_i P_\eta^{-1} h_i^{\mathrm{T}} \leqslant 1$, $i = 1, 2, \dots, m$, can be alternatively written by using the Schur complement lemma in the form of matrix as

$$\begin{bmatrix} \frac{-1}{\rho} & H \\ * & -P_\eta \end{bmatrix} \leqslant 0. \tag{28}$$

Further, by using the change of variables as stated in the previous section and convexity definition, then pre- and post-multiplying (28) by $\operatorname{diag}\{I, \hat{R}\}$, we obtain (17). Hence, the proof is completed. □

In the forthcoming corollary, we establish the finite-time stability of fractional-order differential system (13) when controller is affected by a symmetric saturation. As discussed in the previous section, if $\underline{u_j} = \overline{u_j}$, then $\mathrm{sat}_{\underline{u_j},\overline{u_j}}(u_j(t))$ is converted to a symmetric saturation function $\mathrm{sat}_{\overline{u_j}}(u_j(t))$ with its saturation level $\overline{u_j}$. Under such a case,

$$\Lambda = \Gamma = \begin{bmatrix} \overline{u_1} & 0 & \dots & 0 \\ * & \overline{u_2} & \dots & 0 \\ * & * & \dots & 0 \\ * & * & * & \overline{u_m} \end{bmatrix}, \quad \tilde{B} = B\Gamma \text{ and } E = 0.$$

Corollary 1. *For given scalars a, α, f_1, T, ϵ_A, ϵ_K, the fractional-order differential system* (13) *under symmetrically saturated controller achieves stability within a finite interval of time with respect to $(f_1, f_2, \mathsf{T}, V)$ if there exist matrices T and U, symmetric matrices $\hat{P}_i > 0$ and $\hat{R} > 0$, diagonal matrices $\hat{Q}_i > 0$ and scalars γ, ϵ_a, ϵ_b, ϵ_g such that for $i = 1, 2, \dots, N$,* (16), (17) *and the following LMI hold*:

$$\begin{bmatrix} \hat{\Omega}_{s1,s1} & \hat{R} - a\hat{Q}_i^{\mathrm{T}} & \hat{P}_i + \hat{R}A^{\mathrm{T}} - \hat{R} & I & B\Gamma\sum_{s=1}^{2^m}\delta_s D_s & \hat{R} \\ * & -\gamma I & \hat{Q}_i + \hat{R} & 0 & 0 & 0 \\ * & * & \hat{R}E^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -2\hat{R} & I & B\Gamma\sum_{s=1}^{2^m}\delta_s D_s & 0 \\ * & * & * & -\epsilon_a I & 0 & 0 \\ * & * & * & * & -\epsilon_b I & 0 \\ * & * & * & * & * & \hat{\Omega}_{s7,s7} \end{bmatrix} < 0,$$

where $\hat{\Omega}_{s1,s1} = A_i\hat{R} + B\Gamma\sum_{s=1}^{2^m}\delta_s(t)(D_sT + D_s^{-}U) + \alpha\hat{P}_i - a\hat{P}_i$ and $\hat{\Omega}_{s7,s7} = -(\gamma\epsilon_g^2 + \epsilon_a\epsilon_A^2 + \epsilon_b\epsilon_K^2)^{-1}$. Further, the desired control gain can be calculated by using the relation $K = T\hat{R}^{-1}$ and $H = U\hat{R}^{-1}$.

Proof. Assume that $\varepsilon_s(P_\eta, \rho) \subset Ł(K)$ holds to establish the stability criterion within a finite interval of time for (13) under symmetrically saturated control design. Now, we frame a Lyapunov–Krasovskii functional candidate $\mathcal{V}(t, x(t), \eta) \geqslant 0$ such that

$$\mathcal{V}\big(t, x(t), \eta\big) = x^{\mathrm{T}}(t)\mathcal{P}_\eta x(t) + 2\sum_{j=1}^{n} q_{j\eta} \int_0^{\dot{x}(t)} g_j(s)\,\mathrm{d}s.$$

Now by following the same proof as in Theorem 1 with the above given $\mathcal{V}(t, x(t), \eta)$, the set of sufficient conditions given in the statement of this corollary can be readily obtained by putting $\tilde{B} = B\Gamma$ and matrix $E = 0$. □

As a sequel, in the subsequent theorem, we will discuss the reliable finite-time stabilization of the uncertain fractional-order differential system (14) with known or fixed actuator faults.

Theorem 2. *Consider the actuator fault matrix $\tilde{G}$ to be known. The fractional-order system differential* (14) *achieves stability within a finite interval of time with respect to $(f_1, f_2, \mathsf{T}, V)$ for all $x_0 \in \varepsilon_s(P_\eta, \rho)$ if there exist matrices T and U, symmetric matrices*

$\hat{P}_i > 0$, $\hat{R} > 0$ *and* $\hat{S} > 0$, *diagonal matrices* $\hat{Q}_i > 0$ *and scalars* γ, ϵ_a, ϵ_b, ϵ_g *such that inequalities* (16), (17) *and the following LMI hold*:

$$\begin{bmatrix} \overline{\Omega}_{y1,y1} & \hat{R}-a\hat{Q}_i^{\mathrm{T}} & E\hat{R} & \hat{P}_i+\hat{R}A_i^{\mathrm{T}}-\hat{R} & I & \tilde{B}\sum_{s=1}^{2^m}\delta_s D_s & \hat{R} \\ * & -\gamma I & 0 & \hat{Q}_i+\hat{R} & 0 & 0 & 0 \\ * & * & -a\hat{S} & \hat{R}E^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & * & -2\hat{R} & I & \tilde{B}\sum_{s=1}^{2^m}\delta_s D_s & 0 \\ * & * & * & * & -\epsilon_a I & 0 & 0 \\ * & * & * & * & * & -\epsilon_b I & 0 \\ * & * & * & * & * & * & \hat{\Omega}_{7,7} \end{bmatrix} < 0,$$

for $i = 1, 2, \ldots, N$, *where* $\overline{\Omega}_{y1,y1} = A_i\hat{R} + \tilde{B}\sum_{s=1}^{2^m}\delta_s(t)(D_s\tilde{G}T + D_s^{-}U) + \alpha\hat{P}_i - a\hat{P}_i$ *and* $\hat{\Omega}_{7,7} = -(\gamma\epsilon_g^2 + \epsilon_a\epsilon_A^2 + \epsilon_b\epsilon_K^2)^{-1}$. *Further, the desired control gain can be computed by the relations* $K = T\hat{R}^{-1}$ *and* $H = U\hat{R}^{-1}$, *and all the other parameters remain same as defined in Theorem* 1.

Proof. By considering the similar lines as in Theorem 1 for system (14), the set of sufficient conditions stated in this theorem can be easily derived. □

4 Finite-time stabilization of fractional-order system subject to LFT uncertainty

Now let us assume that the uncertain parameters of the system and the control parameters satisfy case (C2). The conditions, which can guarantee stabilization within a finite interval of time for system (13) with asymmetrically saturated controller, are established in Theorem 3. Further, in Theorem 4 they results further extended to the case when system (14) is affected by reliable asymmetrically saturated controller.

Theorem 3. *For given scalar diagonal matrices* $G, M > 0$, *the fractional-order differential system* (13) *achieves finite-time stability for all* $x_0 \in \varepsilon_s(P_\eta, \rho)$ *if there exist positive definite symmetric matrices* $\hat{P}_i$, $\hat{R}$ *and* $\hat{S}$, *diagonal matrices* $\hat{Q}_i > 0$, *matrices* T *and* U *and constants* γ, ϵ_g *such that inequalities* (16), (17) *and the LMI given below hold*:

$$\begin{bmatrix} \tilde{\Omega}_{b1,b1} & \hat{R}-a\hat{Q}_i^{\mathrm{T}} & E\hat{R} & \hat{P}_i+\hat{R}A_i^{\mathrm{T}}-\hat{R} & \tilde{\Omega}_{b1,b5} & \tilde{\Omega}_{b1,b6} & \hat{R} & \hat{R}E^{\mathrm{T}} & \hat{R}Z^{\mathrm{T}} \\ * & -\gamma I & 0 & \hat{Q}_i+\hat{R} & 0 & 0 & 0 & 0 & 0 \\ * & * & -a\hat{S} & \hat{R}E^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -2\hat{R} & \tilde{\Omega}_{b4,b5} & \tilde{\Omega}_{b4,b6} & 0 & 0 & 0 \\ * & * & * & * & \tilde{\Omega}_{b5,b5} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\Omega}_{b6,b6} & 0 & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Omega}_{b7,b7} & 0 & 0 \\ * & * & * & * & * & * & * & -G^{-1} & 0 \\ * & * & * & * & * & * & * & 0 & -M^{-1} \end{bmatrix} < 0, \quad (29)$$

for $i = 1, 2, \ldots, N$, *where* $\tilde{\Omega}_{b1,b1} = A_i\hat{R} + \tilde{B}\sum_{s=1}^{2^m}\delta_s(t)(D_sT + D_s^{-}U) + \alpha\hat{P}_i - a\hat{P}_i$, $\tilde{\Omega}_{b1,b5} = \tilde{\Omega}_{b4,b5} = D + \hat{R}E^{\mathrm{T}}GF$, $\tilde{\Omega}_{b5,b5} = FGF^{\mathrm{T}} - G$, $\tilde{\Omega}_{b1,b6} = \tilde{\Omega}_{b4,b6} =$

$\tilde{B}\sum_{s=1}^{2^m}\delta_s D_s X + \hat{R}Z^{\mathrm{T}}MY$, $\tilde{\Omega}_{b6,b6} = YMY^{\mathrm{T}} - M$ *and* $\tilde{\Omega}_{b7,b7} = -(\gamma\epsilon_g^2)^{-1}$. *Further, the required control gain is computed by means of the relations* $K = T\hat{R}^{-1}$, $H = U\hat{R}^{-1}$ *with the other parameters same as in Theorem* 1.

Proof. Under case (C2), the matrices ΔA and ΔK are subject to LFT uncertainty. Therefore, equation (21) gets transformed into the form

$$\begin{aligned}
2\tilde{x}^{\mathrm{T}}(t)R\Delta\mathcal{A}x(t) &= 2\tilde{x}^{\mathrm{T}}(t)R\left(\Delta A + \tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_s\Delta K\right)x(t)\\
&\leqslant 2\tilde{x}^{\mathrm{T}}(t)RD(I-\Delta_1 F)^{-1}\Delta_1 E\tilde{x}(t)\\
&\quad + 2\tilde{x}^{\mathrm{T}}(t)R\left(\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_s\right)X(I-\Delta_2 Y)^{-1}\Delta_2 Z\tilde{x}(t).
\end{aligned}$$

Now according to Lemma 2.9 in [13], for $\mathcal{G} = G - FGF^{\mathrm{T}} > 0$ and $\mathcal{M} = T - YMY^{\mathrm{T}} > 0$, where $G > 0$ and $M > 0$ are scalar diagonal matrices, we get

$$\begin{aligned}
&2\tilde{x}^{\mathrm{T}}(t)R\Delta\mathcal{A}x(t)\\
&\quad\leqslant x^{\mathrm{T}}(t)E^{\mathrm{T}}GEx(t) + \tilde{x}^{\mathrm{T}}(t)\left[\left(RD + E^{\mathrm{T}}GF\right)\mathcal{G}^{-1}\left(RD + E^{\mathrm{T}}GF\right)^{\mathrm{T}}\right]\tilde{x}(t)\\
&\qquad + x^{\mathrm{T}}(t)Z^{\mathrm{T}}MZx(t) + \tilde{x}^{\mathrm{T}}(t)\left[\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_sX + Z^{\mathrm{T}}MY\right)\mathcal{M}^{-1}\right.\\
&\qquad \left.\times\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_sX + Z^{\mathrm{T}}MY\right)^{\mathrm{T}}\right]\tilde{x}(t).
\end{aligned}\tag{30}$$

Then, by combining (19), (20), (22) and (30), we obtain

$$\begin{aligned}
&{}^{C}D^{\phi}\mathcal{V}\big(t,x(t),\eta\big) - a\mathcal{V}\big(t,x(t),\eta\big)\\
&\quad\leqslant \xi^{\mathrm{T}}(t)\tilde{\Theta}_\eta\xi(t) + \tilde{x}^{\mathrm{T}}(t)\left[\left(RD + E^{\mathrm{T}}GF\right)\mathcal{G}^{-1}\left(RD + E^{\mathrm{T}}GF\right)^{\mathrm{T}}\right]\tilde{x}(t)\\
&\qquad + \tilde{x}^{\mathrm{T}}(t)\left[\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_sX + Z^{\mathrm{T}}MY\right)\right.\\
&\qquad \left.\times\mathcal{M}^{-1}\left(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_sX + Z^{\mathrm{T}}MY\right)^{\mathrm{T}}\right]\tilde{x}(t),
\end{aligned}$$

where

$$\tilde{\Theta}_\eta = \begin{bmatrix}
\tilde{\Theta}_{(1,1),\eta} & R - aQ_\eta^{\mathrm{T}} & RE & P_\eta + \mathcal{A}^{\mathrm{T}}R^{\mathrm{T}} - R\\
* & -\gamma I & 0 & Q_\eta + R^{\mathrm{T}}\\
* & * & -aS & E^{\mathrm{T}}R^{\mathrm{T}}\\
* & * & * & -2R
\end{bmatrix}$$

with $\tilde{\Theta}_{(1,1),\eta} = 2R\mathcal{A} + \gamma\epsilon_g^2 + E^{\mathrm{T}}GE + Z^{\mathrm{T}}MZ + \alpha P_\eta - aP_\eta$. Then, using the convexity defined in (2), it is easy to get

$$\begin{aligned}
&{}^{C}D^{\phi}\mathcal{V}\bigl(t, x(t), \eta\bigr) - a\mathcal{V}\bigl(t, x(t), \eta\bigr) \\
&\quad \leqslant \sum_{i=1}^{N} \eta_i \Biggl\{ \xi^{\mathrm{T}}(t)\tilde{\Theta}_i\xi(t) + \tilde{x}^{\mathrm{T}}(t)\bigl[\bigl(RD + E^{\mathrm{T}}GF\bigr)\mathcal{G}^{-1}\bigl(RD + E^{\mathrm{T}}GF\bigr)^{\mathrm{T}}\bigr]\tilde{x}(t) \\
&\qquad + \tilde{x}^{\mathrm{T}}(t)\Biggl[\Biggl(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_sX + Z^{\mathrm{T}}MY\Biggr)\mathcal{M}^{-1} \\
&\qquad \times \Biggl(R\tilde{B}\sum_{s=1}^{2^m}\delta_s(t)D_sX + Z^{\mathrm{T}}MY\Biggr)^{\mathrm{T}}\Biggr]\tilde{x}(t)\Biggr\},
\end{aligned}$$

where

$$\tilde{\Theta}_i = \begin{bmatrix} \tilde{\Theta}_{(1,1),i} & R - aQ_i^{\mathrm{T}} & RE & P_i + \mathcal{A}^{\mathrm{T}}R^{\mathrm{T}} - R \\ * & -\gamma I & 0 & Q_i + R^{\mathrm{T}} \\ * & * & -aS & E^{\mathrm{T}}R^{\mathrm{T}} \\ * & * & * & -2R \end{bmatrix}$$

with $\tilde{\Theta}_{(1,1),i} = 2R\mathcal{A} + \gamma\epsilon_g^2 + E^{\mathrm{T}}GE + Z^{\mathrm{T}}MZ + \alpha P_i - aP_i$. Thus, by following the same steps as in Theorem 1 and by using Schur complement, we can readily obtain LMI (29). Therefore, the proof is completed. □

Next, we will establish about the reliable finite-time stabilization of (14) with known or fixed actuator faults against LFT uncertainty.

Theorem 4. *The fractional-order differential system* (14) *achieves stability within a finite interval of time with respect to* $(f_1, f_2, \mathsf{T}, V)$ *for all* $x_0 \in \varepsilon_s(P_\eta, \rho)$ *if there exist symmetric matrices* $\hat{P}_i > 0$, $\hat{R} > 0$ *and* $\hat{S} > 0$, *matrices* T *and* U, *diagonal matrices* $\hat{Q}_i > 0$ *and constants* γ, ϵ_a, ϵ_b, ϵ_g *such that inequalities* (16), (17) *together with the subsequent LMI hold*:

$$\begin{bmatrix}
\underline{\Omega}_{1,1} & \hat{R} - a\hat{Q}_i^{\mathrm{T}} & E\hat{R} & \hat{P}_i + \hat{R}A^{\mathrm{T}} - \hat{R} & \underline{\Omega}_{1,5} & \underline{\Omega}_{1,6} & \hat{R} & \hat{R}E^{\mathrm{T}} & \hat{R}Z^{\mathrm{T}} \\
* & -\gamma I & 0 & \hat{Q}_i + \hat{R} & 0 & 0 & 0 & 0 & 0 \\
* & * & -a\hat{S} & \hat{R}E^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -2\hat{R} & \underline{\Omega}_{4,5} & \underline{\Omega}_{4,6} & 0 & 0 & 0 \\
* & * & * & * & \underline{\Omega}_{5,5} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \underline{\Omega}_{6,6} & 0 & 0 & 0 \\
* & * & * & * & * & * & \underline{\Omega}_{7,7} & 0 & 0 \\
* & * & * & * & * & * & * & -G^{-1} & 0 \\
* & * & * & * & * & * & * & 0 & -M^{-1}
\end{bmatrix} < 0,$$

for $i = 1, 2, \ldots, N$, *where* $\underline{\Omega}_{1,1} = A_i\hat{R} + \tilde{B}\sum_{s=1}^{2^m}\delta_s(t)(D_s\tilde{G}T + D_s^{-}U) + \alpha\hat{P}_i - a\hat{P}_i$, $\underline{\Omega}_{1,5} = \underline{\Omega}_{4,5} = D + \hat{R}E^{\mathrm{T}}GF$, $\underline{\Omega}_{5,5} = FGF^{\mathrm{T}} - G$, $\underline{\Omega}_{1,6} = \underline{\Omega}_{4,6} = \tilde{B}\sum_{s=1}^{2^m}\delta_sD_s\tilde{G}X + \hat{R}Z^{\mathrm{T}}MY$, $\underline{\Omega}_{6,6} = YMY^{\mathrm{T}} - T$ *and* $\underline{\Omega}_{7,7} = -(\gamma\epsilon_g^2)^{-1}$. *The parameters of the controller are determined by means of the relations* $K = T\hat{R}^{-1}$ *and* $H = U\hat{R}^{-1}$, *while the other parameters remain same as in Theorem* 3.

Proof. Following the similar procedure as in Theorem 3, for system (14), the sufficient conditions stated in this theorem can be obtained easily. □

Remark 2. In the above theorems, we have obtained the finite-time stability of the closed-loop system (13) and (14). We can equivalently write the unsymmetrically saturated system (1) in closed loop under the symmetrical form (13) developed in Section 2 and derive the sufficient conditions of stabilizability by using LMIs. Let ${}^{C}D^{\o}\tilde{x}(t) = {}^{C}D^{\o}x(t) + \zeta$, where $\zeta = (A_\eta + \Delta A)^{-1}K_0$. Now by following the same lines as in the proof of Theorem 8.1 in [2], the unsymmetrically saturated closed-loop system converges toward $-\zeta$.

5 Simulation results

This section provides two numerical models to exhibit clearly the usefulness of the results accomplished in this paper. In Example 1, we validate the results obtained in Section 3 by considering a financial fractional-order differential system as in [9], and in Example 2, the results of Section 4 are validated by considering a permanent magnet synchronous motor chaotic fractional-order differential system as in [20].

Example 1. In this example, we will validate the proposed results of Section 3 by considering a financial system, which is described by nonlinear fractional differential equations. The financial fractional-order differential system is described in the following form [5,9]:

$$\begin{aligned} {}^{C}D^{\o_1}x_1(t) &= x_3(t) + (x_2(t) - \hat{a})x_1(t), \\ {}^{C}D^{\o_2}x_2(t) &= 1 - \hat{b}x_2(t) - x_1(t)x_1(t), \\ {}^{C}D^{\o_3}x_3(t) &= -x_1(t) - \hat{c}x_3(t), \end{aligned} \tag{31}$$

where the fractional order of the system is denoted by $\o = (\o_1, \o_2, \o_3)$. The state variable $x_1(t)$ is the interest rate, $x_2(t)$ is the investment demand and $x_3(t)$ is the price index. Further, $\hat{a}$ denotes the saving amount, $\hat{b}$ denotes the cost per investment and c denotes the elasticity of demand of commercial market. The parameters of the above financial fractional-order differential system are taken as $a = 1.0$, $b = 0.1$, $c = 1.0$, fractional orders $\o_1 = 0.95$, $\o_2 = 0.9$, $\o_3 = 0.8$ with step time $h = 0.04166$, which is one hour sampling approximately, and end time as 200 days. With these system state parameters, the system exhibits a chaotic behavior with initial conditions $x(0) = (x_1(0), x_2(0), x_3(0))^{\mathrm{T}} = (1, -1, 1)^{\mathrm{T}}$ as the only point of equilibrium. The abstract form of the dynamics of financial fractional-order differential system (31) can be condensed as

$${}^{C}D^{\o}x(t) = Ax(t) + g\bigl(x(t), t\bigr), \tag{32}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is the system state. Also, from the equations (31) and (32) we deduce

$$A = \begin{bmatrix} -\hat{a} & 0 & 1 \\ 0 & -\hat{b} & 0 \\ -1 & 0 & -\hat{c} \end{bmatrix}, \qquad g(x(t), t) = \begin{bmatrix} x_2(t)x_1(t \\ -x_1(t)x_1(t) \\ 0 \end{bmatrix}$$

such that $g(x(t),t)$ satisfies the Lipschitz condition with $\gamma = 3$. In order to stabilize the financial fractional-order differential system (32) by using a asymmetrical saturated controller with polytopic uncertainties $\hat{a}_\eta$ and $\hat{c}_\eta$, we consider the system in the following form:

$$^{C}D^{\varnothing}x(t) = (A_\eta + \Delta A)x(t) + g\big(x(t),t\big) + B\,\mathrm{sat}_{\underline{u},\overline{u}}\big(u(t)\big),$$

where

$$A_\eta = \begin{bmatrix} -\hat{a} & 0 & 1+\hat{a}_\eta \\ 0 & -\hat{b} & 0 \\ -1+\hat{c}_\eta & 0 & -\hat{c} \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0.02 & 0 & 0.01 \\ 0 & 0.01 & 0 \\ 0.03 & 0 & 0.05 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $\|\hat{a}_\eta\| \leqslant 0.5$ and $\|\hat{c}_\eta\| \leqslant 1$. Obviously, the above system belongs to the four vertex polytopic convex polyhedron in the form of (2) with the following parameters:

$$A_1 = \begin{bmatrix} -1 & 0 & 1.5 \\ 0 & -0.1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 1.5 \\ 0 & -0.1 & 0 \\ 2 & 0 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 0.5 \\ 0 & -0.1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} -1 & 0 & 0.5 \\ 0 & -0.1 & 0 \\ 2 & 0 & -1 \end{bmatrix},$$

and according to case (C1), the uncertainty ΔA gives that $\epsilon_A = 0.05$. Let us assume $a = 0.5$, $\alpha = 0.02$, $f_1 = 0.0001$, $\mathcal{T} = 0.04$ and the parameters of the controller gain fluctuation as $\epsilon_K = 0.2$. With these parameters, by solving the sufficient conditions of Theorem 1, we arrive at a feasible solution with $f_2 = 0.0703$ as the optimum finite-time bound value and

$$K = \begin{bmatrix} -28.1603 & 0 & -0.6209 \\ 0 & -91.3284 & 0 \\ -0.6107 & 0 & -28.6257 \end{bmatrix}, \qquad H = \begin{bmatrix} -0.0097 & 0 & -0.0001 \\ 0 & -0.1622 & 0 \\ -0.0001 & 0 & -0.0108 \end{bmatrix}$$

as the associated controller gain matrices. The corresponding simulation results for state responses with the asymmetrically saturated control is presented in Fig. 1.

Figure 3 depicts the evolution of $x^{\mathrm{T}}(t)Vx(t)$ with asymmetrically saturated control for various initial conditions within the interval $[0, 0.04]$. Hence, it is concluded that the financial fractional-order differential system (31) is bounded within a finite interval of time by means of the proposed asymmetrically saturated controller.

Next, for a known actuator fault $\tilde{G} = 0.5$ and $\mathcal{T} = 0.06$, we solve the LMIs of Theorem 2 and obtain $f_2 = 0.1467$ as the optimum finite-time bound value with

$$K = \begin{bmatrix} -43.9841 & 0 & -0.7610 \\ 0 & -139.7311 & 0 \\ -0.7559 & 0 & -44.4807 \end{bmatrix} \qquad H = \begin{bmatrix} -0.0557 & 0 & 0.0004 \\ 0 & -0.8428 & 0 \\ 0.0004 & 0 & -0.0566 \end{bmatrix}$$

as the associated controller gain matrices. The corresponding state responses of system (14) are shown in Fig. 2.

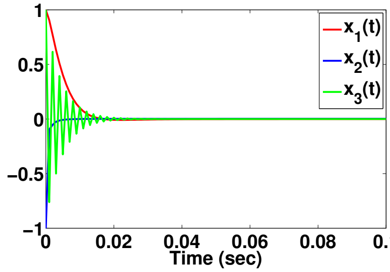

Figure 1. State responses of fractional-order financial system (31) under asymmetrically saturated control.

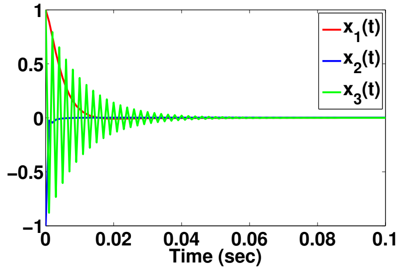

Figure 2. State responses of fractional-order financial system (31) under asymmetrically saturated reliable control.

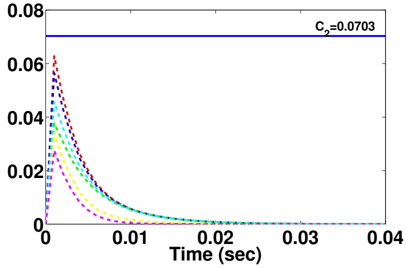

Figure 3. Evolution of the trajectories of $x^{\mathrm{T}}(t)Vx(t)$.

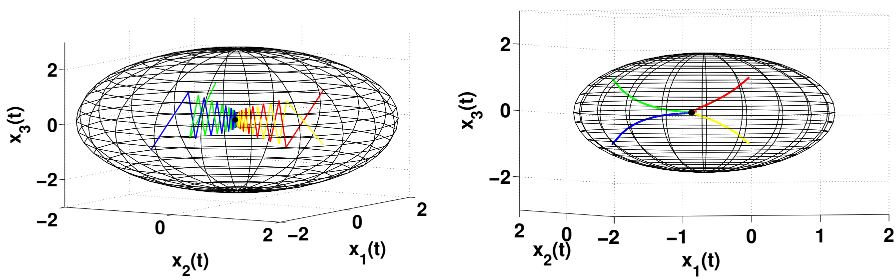

Figure 4. State responses under asymmetrical saturated control and symmetrical saturated control inside the polyhedral set of saturation $Ł(K)$.

From Figs. 1 and 2 we can observe that the financial fractional-order differential system achieves stability, although it consumes more time in the presence of actuator faults. Figure 4 shows the evolution of trajectories of the state vector $x(t)$ under various initial states x_0 inside $Ł(K)$, which is already defined as the polyhedral set of saturation in the Section 2. In the case of asymmetrically saturated control, if $x_0 \in \varepsilon_s(P_\eta, \rho)$, then the trajectories of the state vector converges to the point of equilibrium $x_e = -(A_\eta + BK)^{-1} \times E\tau$, which lies close to the origin due to the presence of $\tau(t)$, which is nothing but the pseudo permanent perturbation. In the case of symmetrically saturated control, if $x_0 \in \varepsilon_s(P_\eta, \rho)$, then the state trajectories converges to x_0.

Example 2. The permanent magnet synchronous motor fractional-order differential chaotic system is described in [20] as

$$\begin{aligned} {}^{C}D^{\o_1}x_1(t) &= -x_1(t) + x_2(t)x_3(t), \\ {}^{C}D^{\o_2}x_2(t) &= x_2(t) - x_1(t)x_3(t) + \tilde{a}x_3(t), \\ {}^{C}D^{\o_3}x_3(t) &= \tilde{b}\big(x_2(t) - x_3(t)\big), \end{aligned} \tag{33}$$

where the fractional orders are $\o_1 = 0.98$, $\o_2 = 0.95$ and $\o_3 = 0.99$. With the initial values $(x_1(0), x_2(0), x_3(0))^{\mathrm{T}} = (35, 0.02, 0.01)^{\mathrm{T}}$, when $\tilde{a} = 50$ and $\tilde{b} = 4$, system (33) exhibits a chaotic behavior. Now, the state representation of permanent magnet synchronous motor fractional-order differential chaotic system (33) can be written as

$${}^{C}D^{\o}x(t) = Ax(t) + g\big(x(t), t\big), \tag{34}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is the system state. Further, from (33) and (34) we have

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & \tilde{a} \\ 0 & \tilde{b} & -\tilde{b} \end{bmatrix}, \qquad g(x(t), t) = \begin{bmatrix} x_2(t)x_3(t \\ -x_1(t)x_3(t) \\ 0 \end{bmatrix}$$

such that $g(x(t), t)$ satisfies the Lipschitz condition. Let the permanent magnet synchronous motor fractional-order differential chaotic system (33) with uncertainty $\tilde{b}_\eta$ takes the parameters $\Delta A = D(I - \Delta_1 F)^{-1}\Delta_1 E$ with $F = 0.1I$, $E = I$, $\Delta_1 = \mathrm{diag}\{\Delta_{11}, \Delta_{12}, \Delta_{13}\}$, $\|\Delta_{11}\| \leqslant 1$, $\|\Delta_{12}\| \leqslant 1$, $\|\Delta_{13}\| \leqslant 1$,

$$D = \begin{bmatrix} 0.05 & 0.1 & 0 \\ 0.1 & -0.1 & 0 \\ 0 & 0 & 0.02 \end{bmatrix} \quad \text{and} \quad A_\eta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & \tilde{a} \\ 0 & \tilde{b} + \tilde{b}_\eta & -\tilde{b} \end{bmatrix}$$

with $\|\tilde{b}_\eta\| \leqslant 0.05$. Now, this is a two vertex polytopic convex polyhedron in the form of (2) with

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 50 \\ 0 & 4.05 & -4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 50 \\ 0 & 3.95 & -4 \end{bmatrix}.$$

With remaining parameters as in Example 1 and the control fluctuation parameter as $\Delta K = X(I - \Delta_2 Y)^{-1}\Delta_2 Z$ with $X = 0.5I$, $Y = 0.2I$, $Z = I$, $\Delta_2 = \mathrm{diag}\{\Delta_{21}, \Delta_{22}, \Delta_{23}\}$, $\|\Delta_{21}\| \leqslant 1$, $\|\Delta_{22}\| \leqslant 1$, $\|\Delta_{23}\| \leqslant 1$, we solve the LMIs in Theorem 3 and arrive at a feasible solution with $f_2 = 0.1041$ as the optimum finite-time bound value and

$$K = \begin{bmatrix} -5.0960 & 0.2157 & 0.5237 \\ 0.0183 & -36.2074 & -60.2708 \\ -0.0680 & -10.0624 & -77.9895 \end{bmatrix}, \qquad H = \begin{bmatrix} 0.0040 & -0.0029 & 0.0038 \\ -0.0014 & -0.8940 & 0.8008 \\ -0.0002 & 0.1350 & -0.4396 \end{bmatrix}$$

as the associated controller gain matrices.

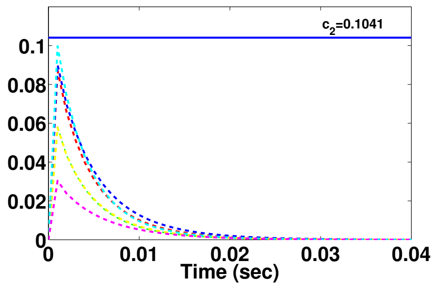

Figure 5. Evolution of the trajectories of $x^{\mathrm{T}}(t)Vx(t)$.

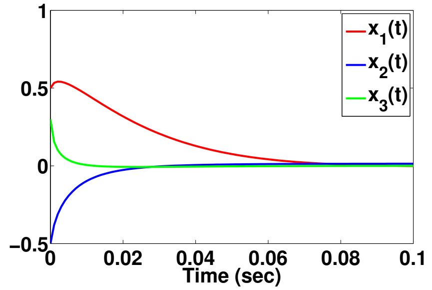

Figure 6. State responses of permanent magnet synchronous motor fractional-order differential chaotic system (33) under asymmetrically saturated reliable control.

Figure 5 depicts the evolution of the trajectories of $x^{\mathrm{T}}(t)Vx(t)$ with asymmetrically saturated control for various initial conditions within the interval $[0, 0.04]$. Hence, we arrive at a conclusion that the permanent magnet synchronous motor fractional-order differential chaotic system (33) is finite-time bounded by means of the proposed asymmetrically saturated controller.

Next, for a known actuator fault $\tilde{G} = 0.5$ and $T = 0.06$, we solve the LMIs of Theorem 4 and obtain $f_2 = 0.0899$ as the optimum finite-time bound value with

$$K = \begin{bmatrix} -11.2472 & -0.0233 & -0.2809 \\ -0.1264 & -35.6318 & -0.0316 \\ -0.2745 & -0.0101 & -11.3718 \end{bmatrix}, \qquad H = \begin{bmatrix} -0.0111 & -0.0007 & -0.0008 \\ -0.0015 & -0.2017 & -0.0003 \\ -0.0007 & -0.0001 & -0.0130 \end{bmatrix}$$

as the associated controller gain matrices. The corresponding state responses of (14) are depicted in Fig 6. The time taken for convergence in the presence of actuator failures with a saturated controller, which is asymmetrically constraint is more when compared to absence of actuator faults. However, proposed controller makes the system stable within a finite-time, even if there are actuator faults.

Remark 3. In [9] and [5], the authors have studied the stabilization problem of fractional-order systems without considering the effect of saturation and uncertainties. In this paper, we have extended the results to fractional-order systems in the presence of asymmetrical saturation and mixed uncertainties. Moreover, in [20], stabilization of fractional-order uncertain chaotic systems is studied with the conventional controller, whereas this paper considers a more generalized form with asymmetrical saturated controller and mixed uncertainties in the state and control parameters.

6 Conclusion

In this study, a finite-time controller has been designed for the stabilization of the fractional-order systems in the presence of structured uncertainties, actuator faults, asymmetric saturation and gain fluctuations. The primary concern of this study is to broaden the results of fractional-order system with symmetric saturation to asymmetric constrained saturation by using LMIs. By means of a suitable Lyapunov functional candidate, finite-time boundedness of closed-loop system has been guaranteed through a set of LMI-based sufficient conditions. Through these sufficient conditions, we have obtained the reliable resilient controller gain matrix for obtaining the required result. Finally, two examples are used by means of financial fractional-order differential system and permanent magnet synchronous motor chaotic fractional-order differential system to depict the effectiveness of the asymmetrically saturated controller design strategy. The nonfragile control problem for nonlinear fractional-order stochastic systems driven by G-Brownian motion with quantization effects will be our future research work in this direction.

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