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Sign-changing solutions for Kirchhoff-type problems involving variable-order fractional Laplacian and critical exponents

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Abstract. In this paper, we are concerned with the Kirchhoff-type variable-order fractional Laplacian problem with critical variable exponent. By using constraint variational method and quantitative deformation lemma we show the existence of one least energy solution, which is strictly larger than twice of that of any ground state solution.

Keywords: Kirchhoff-type problem, variable-order fractional Laplacian, variational method, sign-changing solution.

1 Introduction and main results

In this paper, we are interested in the existence of least energy nodal solutions for the following Kirchhoff-type variable-order fractional Laplacian problems with critical variable:

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growth:

$$\begin{aligned}&\big(a+b[u]_{s(\cdot)}^2\big)(-\Delta)^{s(\cdot)}u=|u|^{q(x)-2}u+\lambda f(x,u) \quad \text{in } \Omega,\\&u=0 \quad \text{in } \mathbb{R}^N\setminus\Omega,\end{aligned} \tag{1}$$

where

$$[u]_{s(\cdot)}:=\bigg(\iint\limits_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}x\,\mathrm{d}y\bigg)^{1/2},$$

$a,b>0$, $s(\cdot):\mathbb{R}^N\times\mathbb{R}^N\to(0,1)$ is a continuous function, Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $\lambda>0$ is a parameter, $N>2s(x,y)$ for all $(x,y)\in\Omega\times\Omega$, $(-\Delta)^{s(\cdot)}$ is the variable-order fractional Laplace operator, $4<q(x)\leqslant 2^\star(x):=2N/(N-2s(x,x))$ for all $x\in\Omega$. The variable-order fractional Laplace operator $(-\Delta)^{s(\cdot)}$ is defined as follows:

$$(-\Delta)^{s(\cdot)}\varphi(x)=2PV\int\limits_{\mathbb{R}^N}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}y$$

along any $\varphi\in C_0^\infty(\Omega)$, where PV denotes the Cauchy principle value. As $s(\cdot)\equiv\text{const}$, the variable-order fractional Laplace operator $(-\Delta)^{s(\cdot)}$ reduces to the usual fractional Laplace operator; see [14, 15] for the concise introduction to the fractional Laplace operator and related variational results.

We now impose the assumptions on the functions $s(\cdot)$ and f that will in full force throughout the paper. Firstly, we suppose that $s:\mathbb{R}^N\times\mathbb{R}^N\to(0,1)$ is a continuous function satisfying the following assumptions:

(s1) $0<s_-:=\min_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N}s(x,y)\leqslant s_+:=\max_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N}s(x,y)<1$;
(s2) $s(\cdot)$ is symmetric, that is, $s(x,y)=s(y,x)$ for all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$.

From now on, for the variable exponents m, we set

$$\underline{m}=\operatorname*{ess\,inf}_{x\in\Omega}m(x),\qquad \overline{m}=\operatorname*{ess\,sup}_{x\in\Omega}m(x).$$

Moreover, we suppose that $f\in C^1(\mathbb{R},\mathbb{R})$ satisfies the following conditions:

(f1) $\lim_{t\to0}f(x,t)/|t|^3=0$;
(f2) there exist $\theta(x)\in(4,2^\star(x))$ and $C>0$ such that $|f(x,t)|\leqslant C(1+|t|^{\theta(x)-1})$ for all $t\in\mathbb{R}$ and all $x\in\Omega$;
(f3) $f(x,t)/|t|^3$ is a strictly increasing function of $t\in\mathbb{R}\setminus\{0\}$.

A typical example of function fulfilling hypotheses (f1)–(f3) is as follows: $f(x,t)=|t|^{\theta(x)-2}t$, where $t\in\mathbb{R}$ and $x\in\Omega$.

The main driving force for studying problem (1) includes two aspects. On the one hand, when $s(\cdot)\equiv1$, Eq. (1) reduces to the general Kirchhoff-type model. Recently, some researchers also explored such equations in the study of nonlinear vibrations theoretically or experimentally. For example, Carrier [1] used a more rigorous method to deduce a more general Kirchhoff model. Moreover, the nonlocal Kirchhoff problems of parabolic type can model several biological systems such as population density; see, for instance, [4]. In fact, the energy functionals of (1) have obviously different properties from the case $b = 0$, and thus, several mathematical difficulties arise naturally in the study of the case $b \neq 0$ by variational and topological methods. It is worth mentioning that Fiscella and Valdinoci [9] deduced a new Kirchhoff model involving the fractional Laplacian by considering the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string; see [9, App. A] for more details.

In recent years, finding sign-changing solutions to the Kirchhoff-type problems has been an attractive subject, and many interesting results have been obtained. In the following, let us sketch some advances related to the subject of our paper. Concerning the advances of Kirchhoff-type problems in the bounded domains, Zhang and Perera in [26] applied the method of invariant sets of descent flow to investigate the existence of sign-changing solutions for Kirchhoff-type problems; see also Mao and Zhang in [12] for more related results via similar approaches. Using the constraint variational methods, Shuai in [18] obtained that Kirchhoff-type problems has one least energy sign-changing solution u_b and the energy of u_b strictly larger than the ground state energy. After that, with the help of non-Nehari manifold method, Tang and Cheng in [19] generalized some results obtained in [18]; see also [2] for more general Kirchhoff-type function in this direction. In [20], Wang obtained the following results for Kirchhoff-type equation with critical growth by employing the constraint variational method and the quantitative deformation lemma: the existence of least energy sign-changing solutions u_b and the energy of u_b is strictly larger than twice that of the ground state solutions. Concerning the advances in the abstract Kirchhoff framework, here we just review two papers as follows: by using the minimization argument and a quantitative deformation lemma, Figueiredo et al. in [7] investigated the existence of a sign-changing solution for the following Kirchhoff-type equation:

$$-M\biggl(\int\limits_{\Omega} |\nabla u|^2 \,\mathrm{d}x\biggr)\Delta u = g(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^3$, $M : \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function with some appropriate assumptions, and g is a superlinear C^1 class function with subcritical growth. In unbounded domains, Figueiredo and Santos Júnior in [8] obtained a least energy sign-changing solution to a class of nonlocal Schrödinger–Kirchhof problems involving only continuous functions by using a minimization argument and a quantitative deformation lemma. Moreover, the authors also proved that the problem has infinitely many nontrivial solutions when it presents symmetry. In [3], Cheng and Gao studied the following Kirchhoff-type problem, which involves a fractional Laplacian operator:

$$\biggl(a + b\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y\biggr)(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $0 < s < 1$ is a constant, f satisfies subcritical growth. The authors proved the existence of least energy sign-changing solutions for this problem by using the constraint variation method and quantitative deformation lemma.

On the other hand, variable-order fractional Laplacian problems was introduced by Xiang et al. in [25]. They studied the following variable-order fractional Laplacian problems involving variable exponents:

$$\begin{aligned} &(-\Delta)^{s(\cdot)}u + \lambda V(x)u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u \quad \text{in } \Omega, \\ &u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{2}$$

Under some suitable assumptions, they showed that problem (2) admits at least two distinct solutions by applying the mountain pass theorem and Ekeland's variational principle. Subsequently, Wang and Zhang in [21] proved the existence of infinitely many solutions for possibly degenerate Kirchhoff-type variable-order fractional Laplacian problems by using the new version of Clark's theorem due to Liu and Wang in [11]. Very recently, Xiang et al. in [24] obtained the existence of two solutions for a class of degenerate Kirchhoff-type variable-order fractional Laplacian problems by employing the Nehari manifold approach.

However, regarding the existence of sign-changing solutions for Kirchhoff-type variable-order fractional Laplacian problems involving variable exponents, there has been no paper in the literature as far as we know. Hence, a natural question is whether or not there exists sign-changing solutions of problem (1)? Another interesting question is whether or not the same conclusion still holds for critical exponent $q(x) = 2^\star(x) := 2N/(N - 2s(x,x))$? The goal of the present paper is to give an affirmative answer.

The corresponding energy functional $\mathcal{I}_{b,\lambda} : H_0^{s(\cdot)}(\Omega) \to \mathbb{R}$ to problem (1) is defined by

$$\mathcal{I}_{b,\lambda}(u) = \frac{a}{2}[u]_{s(\cdot)}^2 + \frac{b}{4}[u]_{s(\cdot)}^4 - \lambda \int_\Omega F(x,u)\,\mathrm{d}x - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x. \tag{3}$$

It is standard to verify that $\mathcal{I}_{b,\lambda}$ belongs to $C^1(H_0^{s(\cdot)}(\Omega), \mathbb{R})$, and the critical points of $\mathcal{I}_{b,\lambda}$ are the solutions of problem (1). Furthermore, if we write $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$ for $u \in H_0^{s(\cdot)}(\Omega)$, then every solution $u \in H_0^{s(\cdot)}(\Omega)$ of problem (1) with the property that $u^\pm \neq 0$ is a sign-changing solution of problem (1).

Now, we give the following first main result.

Theorem 1. *Assume that* (s1)–(s2) *and* (f1)–(f3) *hold. Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \geqslant \lambda^*$, *problem* (1) *has a least energy sign-changing solution* u_b.

Another objective of this paper is to establish the so-called energy doubling property (cf. [22]), i.e., the energy of any sign-changing solution of problem (1) is strictly bigger than twice that of the ground state solution. We have the following result.

Theorem 2. *Assume that* (s1)–(s2) *and* (f1)–(f3) *hold. Then there exists* $\lambda^{**} > 0$ *such that for all* $\lambda \geqslant \lambda^{**}$, $c^* := \inf_{u \in \mathcal{N}_{b,\lambda}} \mathcal{I}_{b,\lambda}(u) > 0$ *is achieved and* $\mathcal{I}_{b,\lambda}(u) > 2c^*$, *where* $\mathcal{N}_{b,\lambda} = \{u \in H_0^{s(\cdot)}(\Omega) \setminus \{0\}\colon \langle(\mathcal{I}_{b,\lambda})'(u), u\rangle = 0\}$, *and* u *is the least energy sign-changing solution obtained in Theorem* 1. *In particular,* $c^* > 0$ *is achieved either by a positive or a negative function.*

Remark 1. As an application of Theorem 2, problem (1) with $s = 9/10$, $2^\star(x) = 5$, and $\theta(x) = 9/2$,

$$\begin{aligned}&\left(a + b[u]_{9/10}^2\right)(-\Delta)^{9/10}u = |u|^3u + \lambda|u|^{5/2}u \quad \text{in } \Omega,\\ &u = 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega\end{aligned}$$

has a least energy sign-changing solution u with energy doubling property.

2 Preliminaries

In this section, we first recall some definitions and results of variable exponent Lebesgue spaces (see [5, 6, 16]), which will be used later.

Let $N \geqslant 1$ and $\Omega \subset \mathbb{R}^N$ be a nonempty open set. A measurable function $p : \overline{\Omega} \to [1, \infty)$ is named a variable exponent.

The variable exponent Lebesgue space is

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is a measurable function: } \eta_{p(x)}(u) = \int\limits_{\Omega} \left|u(x)\right|^{p(x)} \mathrm{d}x < \infty\right\}$$

with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\varrho > 0\colon \eta_{p(x)}\left(\varrho^{-1}u\right) \leqslant 1\right\},$$

then $L^{p(x)}(\Omega)$ is a Banach space, and when p is bounded, we have the following relations:

$$\min\left\{\|u\|_{L^{p(\cdot)}(\Omega)}^{\underline{p}}, \|u\|_{L^{p(\cdot)}(\Omega)}^{\overline{p}}\right\} \leqslant \eta_{p(\cdot)}(u) \leqslant \max\left\{\|u\|_{L^{p(\cdot)}(\Omega)}^{\underline{p}}, \|u\|_{L^{p(\cdot)}(\Omega)}^{\overline{p}}\right\}.$$

That is, if p is bounded, then norm convergence is equivalent to convergence with respect to the modular $\eta_{p(x)}$. For the bounded exponent, the dual space $(L^{p(\cdot)}(\Omega))'$ can be identified with $L^{p'(x)}(\Omega)$, where the conjugate exponent p' is defined by $p' = p(x)/(p(x) - 1)$. If $1 < \underline{p} \leqslant \overline{p} < \infty$, then the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is separable and reflexive. So we can see that Hölder's inequality is still valid in the variable exponent Lebesgue space. For all $\varphi \in L^{p(\cdot)}(\Omega)$, $\phi \in L^{p'(\cdot)}(\Omega)$ with $p(x) \in (1, \infty)$, the following inequality holds:

$$\int\limits_{\Omega} |\varphi\phi| \,\mathrm{d}x \leqslant \left(\frac{1}{\underline{p}} + \frac{1}{\underline{p'}}\right)\|\varphi\|_{L^{p(x)}(\Omega)}\|\phi\|_{L^{p'(x)}(\Omega)} \leqslant 2\|\varphi\|_{L^{p(x)}(\Omega)}\|\phi\|_{L^{p'(x)}(\Omega)}.$$

Next, we give some definitions and results of variable-order fractional Sobolev spaces.

Define $H^{s(\cdot)}(\Omega)$ as the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that any function $u = 0$ in $\mathbb{R}^N \setminus \Omega$ belongs to $L^2(\Omega)$ and

$$[u]_{s(\cdot)} := \left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} \,\mathrm{d}x \,\mathrm{d}y\right)^{1/2} < \infty.$$

Equip $H_0^{s(\cdot)}(\Omega)$ with the norm

$$\|u\|_{H_0^{s(\cdot)}(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + [u]^2_{s(\cdot)}\right)^{1/2}.$$

Similar to the proof of Lemma 7 in [17], we can show that $(H_0^{s(\cdot)}(\Omega), [\cdot]_{s(\cdot)})$ is a Hilbert space. In this paper, we used norm $\|\cdot\| = [\cdot]_{s(\cdot)}$ to study problem (1).

Lemma 1. (*See* [25, *Lemma* 2.1].) *The embeddings $H_0^{s_2}(\Omega) \hookrightarrow H_0^{s(\cdot)}(\Omega) \hookrightarrow H_0^{s_1}(\Omega)$ are continuous. Moreover, if $N > 2s_1$, for any fixed constant exponent $t \in [1, 2N/(N-2s_1)]$, $H_0^{s(\cdot)}(\Omega)$ can be continuously embedded into $L^t(\Omega)$.*

The following embedding theorem shows that the variable-order fractional Sobolev space is related with the variable exponent Sobolev spaces.

Lemma 2. (*See* [25, *Thm.* 2.1].) *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Assume that $s : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ and $p : \overline{\Omega} \to (1,\infty)$ are two continuous functions satisfying* (s1)–(s2) *and $2 < p(x) < 2N/(N-2s(x,x))$, respectively. Then there exists $C_p = C(N, \overline{p}, \overline{s}, \underline{s}) > 0$ such that for any $u \in H_0^{s(\cdot)}(\Omega)$, it holds that*

$$\|u\|_{L^{p(x)}(\Omega)} \leqslant C_p \|u\|_{H_0^{s(\cdot)}(\Omega)}.$$

That is, the embedding $H_0^{s(\cdot)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is continuous. Furthermore, this embedding is compact. If $u \in H_0^{s(\cdot)}(\Omega)$, then there exists $C_p = C(N, \overline{p}, \overline{s}, \underline{s}) > 0$ such that

$$\|u\|_{L^{p(x)}(\Omega)} \leqslant C_p [u]_{s(\cdot)}.$$

3 Some technical lemmas

Now, for fixed $u \in H_0^{s(\cdot)}(\Omega)$ with $u^\pm \neq 0$, we define function $\sigma_u : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ and mapping $T_u : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^2$ by

$$\sigma_u(\alpha,\beta) = \mathcal{I}_{b,\lambda}\left(\alpha u^+ + \beta u^-\right)$$

and

$$T_u(\alpha,\beta) = \left(\left\langle (\mathcal{I}_{b,\lambda})'\left(\alpha u^+ + \beta u^-\right), \alpha u^+\right\rangle, \left\langle (\mathcal{I}_{b,\lambda})'\left(\alpha u^+ + \beta u^-\right), \beta u^-\right\rangle\right). \tag{4}$$

Lemma 3. *Assume that* (s1)–(s2) *and* (f1)–(f3) *hold. If $u \in H_0^{s(\cdot)}(\Omega)$ with $u^\pm \neq 0$, then σ_u has the following properties*:

- (i) *The pair (α,β) is a critical point of σ_u with $\alpha,\beta > 0$ if and only if $\alpha u^+ + \beta u^- \in \mathcal{M}_{b,\lambda}$;*
- (ii) *The function σ_u has a unique critical point (α_u, β_u) on $(0,\infty)\times(0,\infty)$, which is also the unique maximum point of σ_u on $[0,\infty)\times[0,\infty)$. Furthermore, if $\langle (I_{b,\lambda})'(u), u^\pm\rangle \leqslant 0$, then $0 < \alpha_u, \beta_u < 1$.*

Proof. (i) By definition of σ_u we have that

$$\nabla\sigma_u(\alpha,\beta) = \left(\left\langle(\mathcal{I}_{b,\lambda})'\left(\alpha u^+ + \beta u^-\right), u^+\right\rangle, \left\langle(\mathcal{I}_{b,\lambda})'\left(\alpha u^+ + \beta u^-\right), u^-\right\rangle\right)$$
$$= \left(\frac{1}{\alpha}\left\langle(\mathcal{I}_{b,\lambda})'\left(\alpha u^+ + \beta u^-\right), \alpha u^+\right\rangle, \frac{1}{\beta}\left\langle(\mathcal{I}_{b,\lambda})'\left(\alpha u^+ + \beta u^-\right), \beta u^-\right\rangle\right).$$

Thus, item (i) holds.

(ii) For any $u \in H_0^{s(\cdot)}(\Omega)$ with $u^\pm \neq 0$, we prove the existence of α_u and β_u.

From (f1) and (f2), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\left|f(x,t)\right| \leqslant \varepsilon|t| + C_\varepsilon|t|^{\theta(x)-1} \quad \text{for all } t \in \mathbb{R}. \tag{5}$$

Then by the Sobolev embedding theorem we have

$$\begin{aligned}&\left\langle(\mathcal{I}_{b,\lambda})'\left(\alpha u^+ + \beta u^-\right), \alpha u^+\right\rangle \\ &\quad \geqslant (a-\lambda\varepsilon C_2)\alpha^2\left\|u^+\right\|^2 + b\alpha^4\left\|u^+\right\|^4 - \left(\alpha^{\underline{q}} + \alpha^{\overline{q}}\right)C_1\max\left\{\left\|u^+\right\|^{\overline{q}}, \left\|u^+\right\|^{\underline{q}}\right\} \\ &\qquad - \lambda C_\varepsilon C_3\left(\alpha^{\underline{\theta}} + \alpha^{\overline{\theta}}\right)\max\left\{\left\|u^+\right\|^{\overline{\theta}}, \left\|u^+\right\|^{\underline{\theta}}\right\}.\end{aligned}$$

Choose $\varepsilon > 0$ such that $(a - \lambda\varepsilon C_2) > 0$. Since $\underline{q}, \underline{\theta} > 4$, we have that $\langle(I_{b,\lambda})'(\alpha u^+ + \beta u^-), \alpha u^+\rangle > 0$ for α small enough and all $\beta \geqslant 0$. Similarly, we are also able to prove that $\langle\mathcal{I}'_{b,\lambda}(\alpha u^+ + \beta u^-), \beta u^-\rangle > 0$ for β small enough and all $\alpha \geqslant 0$. Therefore, there exists $\delta_1 > 0$ such that

$$\left\langle(\mathcal{I}_{b,\lambda})'\left(\delta_1 u^+ + \beta u^-\right), \delta_1 u^+\right\rangle > 0, \qquad \left\langle(\mathcal{I}_{b,\lambda})'\left(\alpha u^+ + \delta_1 u^-\right), \delta_1 u^-\right\rangle > 0. \tag{6}$$

On the other hand, by (f2) and (f3) we claim

$$f(x,t)t > 0, \quad t \neq 0; \qquad F(x,t) \geqslant 0, \quad t \in \mathbb{R}. \tag{7}$$

Therefore, choose $\alpha = \delta_2^* > \delta_1$. If $\beta \in [\delta_1, \delta_2^*]$ and δ_2^* is large enough, it follows that

$$\begin{aligned}&\left\langle(\mathcal{I}_{b,\lambda})'\left(\delta_2^* u^+ + \beta u^-\right), \delta_2^* u^+\right\rangle \\ &\quad \leqslant a(\delta_2^*)^2\left\|u^+\right\|^2 + b\left(\delta_2^*\right)^4\left\|u^+\right\|^4 + b(\delta_2^*)^4\left\|u^+\right\|^2\left\|u^-\right\|^2 \\ &\quad - (\delta_2^*)^{\underline{q}}\int_\Omega \left|u^+\right|^{q(x)}\,\mathrm{d}x \leqslant 0.\end{aligned}$$

Similarly, we have $\langle(\mathcal{I}_{b,\lambda})'(\alpha u^+ + \delta_2^* u^-), \delta_2^* u^-\rangle \leqslant 0$. Let $\delta_2 > \delta_2^*$ be large enough, we obtain

$$\left\langle(\mathcal{I}_{b,\lambda})'\left(\delta_2^* u^+ + \beta u^-\right), \delta_2^* u^+\right\rangle < 0 \quad \text{and} \quad \left\langle\left(\mathcal{I}_{b,\lambda}\right)'\left(\alpha u^+ + \delta_2^* u^-\right), \delta_2^* u^-\right\rangle < 0 \tag{8}$$

for all $\alpha, \beta \in [\delta_1, \delta_2]$. Combining (6) and (8) with Miranda's theorem [13], there exists $(\alpha_u, \beta_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $T_u(\alpha, \beta) = (0, 0)$, i.e., $\alpha u^+ + \beta u^- \in \mathcal{M}_\lambda$.

According to the proof in [20], we can prove the uniqueness of the pair (α_u, β_u).

Lastly, we prove that $0 < \alpha_u, \beta_u \leqslant 1$ if $\langle (\mathcal{I}_{b,\lambda})'(u), u^\pm \rangle \leqslant 0$.

Suppose $\alpha_u \geqslant \beta_u > 0$. By $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_{b,\lambda}$ we have

$$\begin{aligned} &a\alpha_u^2 \|u^+\|^2 + b\alpha_u^4 \|u^+\|^4 + b\alpha_u^4 \|u^+\|^2 \|u^-\|^2 \\ &\quad\geqslant a\alpha_u^2 \|u^+\|^2 + b\alpha_u^4 \|u^+\|^4 + b\alpha_u^2\beta_u^2 \|u^+\|^2 \|u^-\|^2 \\ &\quad= \lambda \int_\Omega f(x, \alpha_u u^+)\alpha_u u^+ \,\mathrm{d}x + \int_\Omega |\alpha_u u^+|^{q(x)} \,\mathrm{d}x. \end{aligned} \tag{9}$$

On the other hand, by $\langle (I_{b,\lambda})'(u), u^+ \rangle \leqslant 0$ we have

$$a\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 \leqslant \lambda \int_\Omega f(x, u^+)u^+ \mathrm{d}x + \int_\Omega |u^+|^{q(x)} \,\mathrm{d}x. \tag{10}$$

So, according to (9) and (10), we obtain

$$\begin{aligned} a\left(\frac{1}{\alpha_u^2} - 1\right)\|u^+\|^2 &\geqslant \lambda \int_\Omega \left[\frac{f(x, \alpha_u u^+)}{(\alpha_u u^+)^3} - \frac{f(x, u^+)}{(u^+)^3}\right](u^+)^4 \,\mathrm{d}x \\ &\quad+ \int_\Omega \left(\alpha_u^{(q(x)-2)} - 1\right)|u^+|^{q(x)} \,\mathrm{d}x. \end{aligned}$$

In view of (f3), we conclude that $\alpha_u \leqslant 1$. Thus, we have that $0 < \alpha_u, \beta_u \leqslant 1$. □

Lemma 4. *Let* $c_{b,\lambda} = \inf_{u \in \mathcal{M}_{b,\lambda}} \mathcal{I}_{b,\lambda}(u)$. *Then we have that* $\lim_{\lambda\to\infty} c_{b,\lambda} = 0$.

Proof. For any $u \in \mathcal{M}_{b,\lambda}$, we have

$$a\|u^\pm\|^2 + b\|u^\pm\|^4 + b\|u^+\|^2\|u^-\|^2 = \lambda \int_\Omega f(x, u^\pm)u^\pm \,\mathrm{d}x + \int_\Omega |u^\pm|^{q(x)} \,\mathrm{d}x.$$

Then by (5) and Sobolev inequalities we get

$$\begin{aligned} a\|u^\pm\|^2 &\leqslant \lambda \int_\Omega f(x, u^\pm)u^\pm \,\mathrm{d}x + \int_\Omega |u^\pm|^{q(x)} \,\mathrm{d}x \\ &\leqslant \lambda\varepsilon C_1\|u^\pm\|^2 + \lambda C_\varepsilon \min\{\|u^\pm\|^{\underline{\theta}}, \|u^\pm\|^{\overline{\theta}}\} \\ &\quad+ C\min\{\|u^\pm\|^{\underline{q}}, \|u^\pm\|^{\overline{q}}\}. \end{aligned}$$

Thus, we obtain

$$(a - \lambda\varepsilon C_1)\|u^\pm\|^2 \leqslant \lambda C_\varepsilon \min\{\|u^\pm\|^{\underline{\theta}}, \|u^\pm\|^{\overline{\theta}}\} + C\min\{\|u^\pm\|^{\underline{q}}, \|u^\pm\|^{\overline{q}}\}.$$

Choosing ε small enough such that $a - \lambda\varepsilon C_1 > 0$, since $\underline{\theta}, \overline{\theta}, \underline{q}, \overline{p} > 4$, there exists $\rho > 0$ such that

$$\|u^{\pm}\| \geqslant \rho \quad \text{for all } u \in \mathcal{M}_{b,\lambda}. \tag{11}$$

On the other hand, for any $u \in \mathcal{M}_{b,\lambda}$, it is obvious that $\langle (\mathcal{I}_{b,\lambda})'(u), u\rangle = 0$. Thanks to (f2) and (f3), we obtain that

$$f'(x,t)t - 3f(x,t) > (<)\, 0 \quad \text{for all } t > 0,$$

This fact implies that

$$\Theta(x,t) := f(x,t)t - 4F(x,t) \geqslant 0 \tag{12}$$

is increasing when $t > 0$ and decreasing when $t < 0$ for almost every $x \in \Omega$. Then we have

$$\mathcal{I}_{b,\lambda}(u) = \mathcal{I}_{b,\lambda}(u) - \frac{1}{4}\bigl\langle (\mathcal{I}_{b,\lambda})'(u), u\bigr\rangle \geqslant \frac{a}{4}\|u\|^2.$$

From above discussions we have that $\mathcal{I}_{b,\lambda}(u) > 0$ for all $u \in \mathcal{M}_{b,\lambda}$. Therefore, $\mathcal{I}_{b,\lambda}$ is bounded below on $\mathcal{M}_{b,\lambda}$, that is, $c_{b,\lambda} = \inf_{u\in\mathcal{M}_{b,\lambda}} \mathcal{I}_{b,\lambda}(u)$ is well defined.

Let $u \in H_0^{s(\cdot)}(\Omega)$ with $u^{\pm} \neq 0$ be fixed. By Lemma 3, for each $\lambda > 0$, there exist $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda u^+ + \beta_\lambda u^- \in \mathcal{M}_{b,\lambda}$. By Lemma 3 we have

$$\begin{aligned}
0 \leqslant c_{b,\lambda} &= \inf_{u\in\mathcal{M}_{b,\lambda}} \mathcal{I}_{b,\lambda}(u) \leqslant \mathcal{I}_{b,\lambda}(\alpha_\lambda u^+ + \beta_\lambda u^-) \\
&\leqslant \frac{a}{2}\bigl\|\alpha_\lambda u^+ + \beta_\lambda u^-\bigr\|^2 + \frac{b}{4}\bigl\|\alpha_\lambda u^+ + \beta_\lambda u^-\bigr\|^4 \\
&\leqslant a\alpha_\lambda^2\bigl\|u^+\bigr\|^2 + a\beta_\lambda^2\bigl\|u^-\bigr\|^2 + 2b\alpha_\lambda^4\bigl\|u^+\bigr\|^4 + 2b\beta_\lambda^4\bigl\|u^-\bigr\|^4.
\end{aligned}$$

For our purpose, we just prove that $\alpha_\lambda \to 0$ and $\beta_\lambda \to 0$ as $\lambda \to \infty$.

Let

$$\mathcal{T}_u = \bigl\{(\alpha_\lambda, \beta_\lambda) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \colon T_u(\alpha_\lambda, \beta_\lambda) = (0,0),\ \lambda > 0\bigr\},$$

where T_u is defined as (4). By (5) there holds

$$\begin{aligned}
&\min\bigl\{\alpha_\lambda^{\overline{q}}, \alpha_\lambda^{\underline{q}}\bigr\} \int\limits_\Omega \bigl|u^+\bigr|^{q(x)}\,\mathrm{d}x + \min\bigl\{\beta_\lambda^{\overline{q}}, \beta_\lambda^{\underline{q}}\bigr\} \int\limits_\Omega \bigl|u^-\bigr|^{q(x)}\,\mathrm{d}x \\
&\leqslant \int\limits_\Omega \alpha_\lambda^{q(x)}\bigl|u^+\bigr|^{q(x)}\,\mathrm{d}x + \int\limits_\Omega \alpha_\beta^{q(x)}\bigl|u^-\bigr|^{q(x)}\,\mathrm{d}x \\
&\quad + \lambda\int\limits_\Omega f\bigl(x, \alpha_\lambda u^+\bigr)\alpha_\lambda u^+\,\mathrm{d}x + \lambda\int\limits_\Omega f\bigl(x, \beta_\lambda u^-\bigr)\beta_\lambda u^-\,\mathrm{d}x \\
&= a\bigl\|\alpha_\lambda u^+ + \beta_\lambda u^-\bigr\|^2 + b\bigl\|\alpha_\lambda u^+ + \beta_\lambda u^-\bigr\|^4 \\
&\leqslant 2a\alpha_\lambda^2\bigl\|u^+\bigr\|^2 + 2a\beta_\lambda^2\bigl\|u^-\bigr\|^2 + 4b\alpha_\lambda^4\bigl\|u^+\bigr\|^4 + 4b\beta_\lambda^4\bigl\|u^-\bigr\|^4.
\end{aligned}$$

Hence, $\mathcal{T}_u$ is bounded. Let $\{\lambda_n\} \subset \mathbb{R}^+$ be such that $\lambda_n \to \infty$ as $n \to \infty$. Then there exist α_0 and β_0 such that $(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$ as $n \to \infty$.

Now, we claim $\alpha_0 = \beta_0 = 0$. Suppose, by contradiction, that $\alpha_0 > 0$ or $\beta_0 > 0$. By $\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^- \in \mathcal{M}_{b,\lambda_n}$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} &a\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|^2 + b\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|^4 \\ &\quad = \lambda_n \int\limits_{\Omega} f(x, \alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)\,\mathrm{d}x \\ &\qquad + \int\limits_{\Omega} |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^{q(x)}\,\mathrm{d}x. \end{aligned} \tag{13}$$

Thanks to $\alpha_{\lambda_n} u^+ \to \alpha_0 u^+$ and $\beta_{\lambda_n} u^- \to \beta_0 u^-$ in $H_0^{s(\cdot)}(\Omega)$, (5) and (7), we have that

$$\begin{aligned} &\int\limits_{\Omega} f(x, \alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)\,\mathrm{d}x \\ &\quad \to \int\limits_{\Omega} f(x, \alpha_0 u^+ + \beta_0 u^-)(\alpha_0 u^+ + \beta_0 u^-)\,\mathrm{d}x > 0 \end{aligned}$$

as $n \to \infty$. It follows a contradiction with equality (13) from two facts: $\lambda_n \to \infty$ as $n \to \infty$, and $\{\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\}$ is bounded in $H_0^{s(\cdot)}(\Omega)$. Hence, $\alpha_0 = \beta_0 = 0$. Therefore, we conclude that $\lim_{\lambda\to\infty} c_{b,\lambda} = 0$. ☐

Lemma 5. *There exists $\lambda^* > 0$ such that for all $\lambda \geqslant \lambda^*$, the infimum $c_{b,\lambda}$ is achieved.*

Proof. By the definition of $c_{b,\lambda}$ there exists a sequence $\{u_n\} \subset \mathcal{M}_{b,\lambda}$ such that $\lim_{n\to\infty} \mathcal{I}_{b,\lambda}(u_n) = c_{b,\lambda}$. Obviously, $\{u_n\}$ is bounded in $H_0^{s(\cdot)}(\Omega)$. Then, up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in H_0^{s(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$. Since the embedding $H_0^{s(\cdot)}(\Omega) \hookrightarrow L^t(\Omega)$ is compact, for all $t \in (2, 2^\star(x))$, we have

$$u_n \to u \quad \text{in } L^t(\Omega), \qquad u_n \to u \quad \text{a.e. } x \in \Omega.$$

Hence, $u_n^\pm \to u^\pm$, a.e. $x \in \Omega$, and

$$u_n^\pm \rightharpoonup u^\pm \quad \text{in } H_0^{s(\cdot)}(\Omega), \qquad u_n^\pm \to u^\pm \quad \text{in } L^t(\Omega).$$

By Lemma 3 we have

$$I_{b,\lambda}(\alpha u_n^+ + \beta u_n^-) \leqslant I_{b,\lambda}(u_n) \quad \text{for all } \alpha, \beta \geqslant 0.$$

Then by Brézis–Lieb lemma and Fatou's lemma we get

$$\begin{aligned} &\liminf_{n\to\infty} \mathcal{I}_{b,\lambda}(\alpha u_n^+ + \beta u_n^-) \\ &\quad \geqslant I_{b,\lambda}(\alpha u^+ + \beta u^-) + \frac{a\alpha^2}{2} \lim_{n\to\infty} \|u_n^+ - u^+\|^2 + \frac{a\beta^2}{2} \lim_{n\to\infty} \|u_n^- - u^-\|^2 \\ &\qquad + \frac{b\alpha^4}{2} \lim_{n\to\infty} \|u_n^+ - u^+\|^2 \|u^+\|^2 + \frac{b\beta^4}{2} \lim_{n\to\infty} \|u_n^- - u^-\|^2 \|u^-\|^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{b\alpha^4}{4}\Bigl(\lim_{n\to\infty}\|u_n^+-u^+\|^2\Bigr)^2+\frac{b\beta^4}{4}\bigl(\lim_{n\to\infty}\|u_n^--u^-\|^2\bigr)^2\\
&-\int\limits_\Omega\frac{\alpha^{q(x)}}{q(x)}\bigl|u_n^+-u^+\bigr|^{q(x)}\,\mathrm{d}x-\int\limits_\Omega\frac{\beta^{q(x)}}{q(x)}\bigl|u_n^--u^-\bigr|^{q(x)}\,\mathrm{d}x\\
\geqslant{}& I_{b,\lambda}\bigl(\alpha u^++\beta u^-\bigr)+\frac{a\alpha^2}{2}A_1+\frac{b\alpha^4}{2}A_1\|u^+\|^2+\frac{b\alpha^4}{4}A_1^2-\frac{\max\{\alpha^{\underline{q}},\alpha^{\overline{q}}\}}{\underline{q}}B_1\\
&+\frac{a\beta^2}{2}A_2+\frac{b\beta^4}{2}A_2\|u^-\|^2+\frac{b\beta^4}{4}A_2^2-\frac{\max\{\alpha^{\underline{q}},\alpha^{\overline{q}}\}}{\underline{q}}B_2,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1&=\lim_{n\to\infty}\|u_n^+-u^+\|^2, & A_2&=\lim_{n\to\infty}\|u_n^--u^-\|^2,\\
B_1&=\lim_{n\to\infty}|u_n^+-u^+|_{q(x)}^{q(x)}, & B_2&=\lim_{n\to\infty}|u_n^--u^-|_{q(x)}^{q(x)}.
\end{aligned}
$$

That is, one has

$$
\begin{aligned}
&\mathcal{I}_{b,\lambda}\bigl(\alpha u^++\beta u^-\bigr)+\frac{a\alpha^2}{2}A_1+\frac{b\alpha^4}{2}A_1\|u^+\|^2+\frac{b\alpha^4}{4}A_1^2-\frac{\max\{\alpha^{\underline{q}},\alpha^{\overline{q}}\}}{\underline{q}}B_1\\
&\quad+\frac{a\beta^2}{2}A_2+\frac{b\beta^4}{2}A_2\|u^-\|^2+\frac{b\beta^4}{4}A_2^2-\frac{\max\{\beta^{\underline{q}},\beta^{\overline{q}}\}}{\underline{q}}B_2\leqslant c_{b,\lambda}
\end{aligned}\tag{14}
$$

for all $\alpha\geqslant 0$ and all $\beta\geqslant 0$.

Now, we claim that $u^\pm\neq 0$. In fact, since the situation $u^-\neq 0$ is analogous, we just prove $u^+\neq 0$. By contradiction we suppose $u^+=0$. Hence, let $\beta=0$ in (14), and we have

$$
\frac{a\alpha^2}{2}A_1+\frac{b\alpha^4}{4}A_1^2-\frac{\max\{\alpha^{\underline{q}},\alpha^{\overline{q}}\}}{\underline{q}}B_1\leqslant c_{b,\lambda}\quad\text{for all }\alpha\geqslant 0.\tag{15}
$$

Case 1: $B_1=0$.

If $A_1=0$, that is, $u_n^+\to u^+$ in $H_0^{s(\cdot)}(\Omega)$. From Lemma (11) we obtain $\|u^+\|>0$, which contradicts our supposition. If $A_1>0$, by (14) and Lemma 4 we have

$$
0<\frac{a\alpha^2}{2}A_1+\frac{b\alpha^4}{4}A_1^2\leqslant c_{b,\lambda}\to 0\quad\text{for all }\alpha\geqslant 0\text{ and }\lambda\to+\infty.
$$

The inequality is absurd. Anyway, we have a contradiction.

Case 2: $B_1>0$.

One the one hand, by Lemma 4 there exists $\lambda^*>0$ such that

$$
c_{b,\lambda}<\Lambda:=\min\bigl\{C_q^{-\underline{q}},C_q^{-\overline{q}}\bigr\}a^{\underline{q}/(\underline{q}-2)}\quad\text{for all }\lambda\geqslant\lambda^*.
$$

On the other hand, since $B_1 > 0$, we obtain $A_1 > 0$. Hence, by means of (15) we have

$$\Lambda \leqslant \left[\frac{(aA_1)^{\underline{q}/2}}{B_1}\right]^{2/(\underline{q}-2)} \leqslant \max_{0\leqslant\alpha\leqslant1}\left\{\frac{a\alpha^2}{2}A_1 - \frac{\alpha^{\underline{q}}}{\underline{q}}B_1\right\}$$
$$\leqslant \max_{0\leqslant\alpha}\left\{\frac{a\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{\underline{q}}}{\underline{q}}B_1\right\} \leqslant c_{b,\lambda},$$

which is a contradiction. Hence, we deduce that $u^{\pm} \neq 0$.

Second, we prove $B_1 = B_2 = 0$.

Since the situation $B_2 = 0$ is analogous, we only prove $B_1 = 0$. By contradiction we suppose that $B_1 > 0$.

Case 1: $B_2 > 0$.

According to $B_1, B_2 > 0$ and Sobolev embedding, we obtain that $A_1, A_2 > 0$. Let

$$\varphi(\alpha) = \frac{a\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\max\{\alpha^{\underline{q}}, \alpha^{\overline{q}}\}}{\underline{q}}B_1 \quad \text{for all } \alpha \geqslant 0.$$

It is easy to see that $\varphi(\alpha) > 0$ for $\alpha > 0$ small enough and $\varphi(\alpha) < 0$ for $\alpha < 0$ large enough. Hence, by the continuity of $\varphi(\alpha)$ there exists $\tilde{\alpha} > 0$ such that

$$\frac{a\tilde{\alpha}^2}{2}A_1 + \frac{b\tilde{\alpha}^4}{4}A_1^2 - \frac{\max\{\tilde{\alpha}^{\underline{q}}, \tilde{\alpha}^{\overline{q}}\}}{\underline{q}}B_1$$
$$= \max_{s\geqslant0}\left\{\frac{a\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\max\{\alpha^{\underline{q}}, \alpha^{\overline{q}}\}}{\underline{q}}B_1\right\}.$$

Similarly, there exists $\tilde{\beta} > 0$ such that

$$\frac{a\tilde{\beta}^2}{2}A_2 + \frac{b\tilde{\beta}^4}{4}A_2^2 - \frac{\max\{\tilde{\beta}^{\underline{q}}, \tilde{\beta}^{\overline{q}}\}}{\underline{q}}B_2$$
$$= \max_{s\geqslant0}\left\{\frac{a\beta^2}{2}A_2 + \frac{b\beta^4}{4}A_2^2 - \frac{\max\{\beta^{\underline{q}}, \beta^{\overline{q}}\}}{\underline{q}}B_2\right\}.$$

Since $[0, \tilde{\alpha}] \times [0, \tilde{\beta}]$ is compact and σ is continuous, there exists $(\alpha_u, \beta_u) \in [0, \tilde{\alpha}] \times [0, \tilde{\beta}]$ such that

$$\sigma(\alpha_u, \beta_u) = \max_{(\alpha,\beta)\in[0,\tilde{\alpha}]\times[0,\tilde{\beta}]} \sigma(\alpha, \beta).$$

Now, we prove that $(\alpha_u, \beta_u) \in (0, \tilde{\alpha}) \times (0, \tilde{\beta})$. Note that if β is small enough, we obtain

$$\sigma(\alpha, 0) = \mathcal{I}_{b,\lambda}(\alpha u^+) < \mathcal{I}_{b,\lambda}(\alpha u^+) + \mathcal{I}_{b,\lambda}(\beta u^-)$$
$$\leqslant I_{b,\lambda}(\alpha u^+ + \beta u^-) = \sigma(\alpha, \beta)$$

for all $\alpha \in [0, \tilde{\alpha}]$. Hence, there exists $\beta_0 \in [0, \tilde{\beta}]$ such that

$$\sigma(\alpha, 0) \leqslant \sigma(\alpha, \beta_0) \quad \text{for all } \alpha \in [0, \tilde{\alpha}].$$

That is, any point of $(\alpha, 0)$ with $0 \leqslant \alpha \leqslant \tilde{\alpha}$ is not the maximizer of σ. This yields that $(\alpha_u, \beta_u) \notin [0, \tilde{\alpha}] \times \{0\}$. Similarly, we obtain $(\alpha_u, \beta_u) \notin \{0\} \times [0, \tilde{\alpha}]$.

On the other hand, it is easy to see that

$$\frac{a\alpha^2}{2} A_1 + \frac{b\alpha^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha^4}{4} A_1^2 - \frac{\max\{\alpha^{\underline{q}}, \alpha^{\overline{q}}\}}{\underline{q}} B_1 > 0 \tag{16}$$

and

$$\frac{a\beta^2}{2} A_2 + \frac{b\beta^4}{2} A_2 \|u^-\|^2 + \frac{b\beta^4}{4} A_2^2 - \frac{\max\{\beta^{\underline{q}}, \beta^{\overline{q}}\}}{\underline{q}} B_2 > 0 \tag{17}$$

for $\alpha \in (0, \tilde{\alpha}]$, $\beta \in (0, \tilde{\beta}]$.

Then we have

$$\begin{aligned} \varLambda \leqslant{} & \frac{a\tilde{\alpha}^2}{2} A_1 + \frac{b\tilde{\alpha}^4}{4} A_1^2 - \frac{\max\{\alpha^{\underline{q}}, \alpha^{\overline{q}}\}}{\underline{q}} B_1 + \frac{b\tilde{\alpha}^4}{2} A_1 \|u^+\|^2 \\ & + \frac{a\beta^2}{2} A_2 + \frac{b\beta^4}{2} A_2 \|u^-\|^2 + \frac{b\beta^4}{4} A_2^2 - \frac{\max\{\beta^{\underline{q}}, \beta^{\overline{q}}\}}{\underline{q}} B_2 \end{aligned}$$

and

$$\begin{aligned} \varLambda \leqslant{} & \frac{a\tilde{\beta}^2}{2} A_2 + \frac{b\tilde{\beta}^4}{4} A_2^2 - \frac{\max\{\beta^{\underline{q}}, \beta^{\overline{q}}\}}{\underline{q}} B_2 + \frac{b\tilde{\beta}^4}{2} A_2 \|u^-\|^2 \\ & + \frac{a\alpha^2}{2} A_1 + \frac{b\alpha^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha^4}{4} A_1^2 - \frac{\max\{\alpha^{\underline{q}}, \alpha^{\overline{q}}\}}{\underline{q}} B_1 \end{aligned}$$

for all $\alpha \in [0, \tilde{\alpha}]$ and all $\beta \in [0, \tilde{\beta}]$. Therefore, according to (14), we conclude $\sigma(\alpha, \tilde{\beta}) \leqslant 0$, $\sigma(\tilde{\alpha}, \beta) \leqslant 0$ for all $\alpha \in [0, \tilde{\alpha}]$ and all $\beta \in [0, \tilde{\beta}]$. Thus, $(\alpha_u, \beta_u) \notin \{\tilde{\alpha}\} \times [0, \tilde{\beta}]$ and $(\alpha_u, \beta_u) \notin [0, \tilde{\alpha}] \times \{\tilde{\beta}\}$. Finally, we get that $(\alpha_u, \beta_u) \in (0, \tilde{\alpha}) \times (0, \tilde{\beta})$. Hence, it follows that (α_u, β_u) is a critical point of σ. This implies that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_{b,\lambda}$. From (14), (16), and (17) we have

$$\begin{aligned} c_{b,\lambda} \geqslant{} & \mathcal{I}_{b,\lambda}\left(\alpha_u u^+ + \beta_u u^-\right) \\ & + \frac{a\alpha_u^2}{2} A_1 + \frac{b\alpha_u^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha_u^4}{4} A_1^2 - \frac{\max\{\alpha^{\underline{q}}, \alpha^{\overline{q}}\}}{\underline{q}} B_1 \\ & + \frac{a\beta_u^2}{2} A_2 + \frac{b\beta_u^4}{2} A_2 \|u^-\|^2 + \frac{b\beta_u^4}{4} A_2^2 - \frac{\max\{\beta^{\underline{q}}, \beta^{\overline{q}}\}}{\underline{q}} B_2 \\ > {} & \mathcal{I}_{b,\lambda}\left(\alpha_u u^+ + \beta_u u^-\right) \geqslant c_{b,\lambda}, \end{aligned}$$

which is a contradiction.

Case 2: $B_2 = 0$.

In this case, we can maximize in $[0, \tilde{\alpha}] \times \mathbb{R}_0^+$. Indeed, it is possible to show that there exist $\beta_0 \in \mathbb{R}_0^+$ such that

$$\mathcal{I}_{b,\lambda}\big(\alpha_u u^+ + \beta_u u^-\big) \leqslant 0 \quad \text{for all } (\alpha, \beta) \in [0, \tilde{\alpha}] \times [\beta_0, \infty).$$

Hence, there is $(\alpha_u, \beta_u) \in [0, \tilde{\alpha}] \times [0, \infty)$ such that

$$\sigma(\alpha_u, \beta_u) = \max_{(\alpha,\beta)\in[0,\tilde{\alpha}]\times[0,\infty)} \sigma(\alpha, \beta).$$

In the following, we prove that $(\alpha_u, \beta_u) \in (0, \tilde{\alpha}) \times \mathbb{R}^+$. It is noted that $\sigma(\alpha, 0) < \sigma(\alpha, \beta)$ for $\alpha \in [0, \tilde{\alpha}]$ and β small enough, so we have $(\alpha_u, \beta_u) \notin [0, \tilde{\alpha}] \times \{0\}$. Meanwhile, $\sigma(0, \beta) < \sigma(\alpha, \beta)$ for $\beta \in [0, \infty)$ and α small enough, then we have $(\alpha_u, \beta_u) \notin \{0\} \times \mathbb{R}_0^+$.

On the other hand, it is obvious that

$$\begin{aligned}
\Lambda \leqslant{}& \frac{a\tilde{\alpha}^2}{2} A_1 + \frac{b\tilde{\alpha}^4}{4} A_1^2 - \frac{\max\{\alpha^{\underline{q}}, \alpha^{\overline{q}}\}}{\underline{q}} B_1 + \frac{b\tilde{\alpha}^4}{2} A_2 \|u^+\|^2 \\
&+ \frac{a\beta^2}{2} A_2 + \frac{b\beta^4}{2} A_2 \|u^-\|^2 + \frac{b\beta^4}{4} A_2^2 \quad \text{for all } \beta \in [0, \infty).
\end{aligned}$$

Hence, we obtain that $\sigma(\tilde{\alpha}, \beta) \leqslant 0$ for all $\beta \in \mathbb{R}_0^+$. Thus, $(\alpha_u, \beta_u) \notin \{\tilde{\alpha}\} \times \mathbb{R}_0^+$. Hence, $(\alpha_u, \beta_u) \in (0, \tilde{\alpha}) \times \mathbb{R}^+$. That is, (α_u, β_u) is an inner maximizer of σ in $[0, \tilde{\alpha}) \times \mathbb{R}_0^+$. Hence, $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_{b,\lambda}$. Then it follows from (16) that

$$\begin{aligned}
c_{b,\lambda} \geqslant{}& \mathcal{I}_{b,\lambda}\big(\alpha_u u^+ + \beta_u u^-\big) \\
&+ \frac{a\alpha_u^2}{2} A_1 + \frac{b\alpha_u^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha_u^4}{4} A_1^2 - \frac{\max\{\alpha^{\underline{q}}, \alpha^{\overline{q}}\}}{\underline{q}} B_1 \\
&+ \frac{a\beta_u^2}{2} A_2 + \frac{b\beta_u^4}{2} A_2 \|u^-\|^2 + \frac{b\beta_u^4}{4} A_2^2 \\
>{}& \mathcal{I}_{b,\lambda}\big(\alpha_u u^+ + \beta_u u^-\big) \geqslant c_{b,\lambda},
\end{aligned}$$

which is absurd.

Therefore, from the above arguments we have that $B_1 = B_2 = 0$.

Finally, we prove that $c_{b,\lambda}$ is achieved. Since $u^\pm \neq 0$, by Lemma 1, there exist $\alpha_u, \beta_u > 0$ such that

$$\widehat{u} := \alpha_u u^+ + \beta_u u^- \in \mathcal{M}_{b,\lambda}.$$

Furthermore, it is easy to see that $\langle (\mathcal{I}_{b,\lambda})'(u), u^\pm \rangle \leqslant 0$. By Lemma 3 we obtain $0 < \alpha_u, \beta_u < 1$. Since $u_n \in \mathcal{M}_{b,\lambda}$, according to Lemma 4, we get

$$\mathcal{I}_{b,\lambda}\big(\alpha_u u_n^+ + \beta_u u_n^-\big) \leqslant \mathcal{I}_{b,\lambda}\big(u_n^+ + u_n^-\big) = I_{b,\lambda}(u_n).$$

Thanks to (f3), $B_1 = B_2 = 0$ and the norm in $H_0^{s(\cdot)}(\Omega)$ is lower semicontinuous, we have

$$\begin{aligned}
c_{b,\lambda} &\leqslant \mathcal{I}_{b,\lambda}(\widehat{u}) - \frac{1}{4}\bigl\langle (\mathcal{I}_{b,\lambda})'(\widehat{u}), \widehat{u}\bigr\rangle \\
&\leqslant \frac{a}{4}\|\widehat{u}\|^2 + \int\limits_{\Omega} \left(\frac{1}{4} - \frac{1}{q(x)}\right) |\widehat{u}|^{q(x)}\,\mathrm{d}x + \frac{\lambda}{4}\int\limits_{\Omega} \bigl[f(x,\widehat{u})\bar{u} - 4F(x,\widehat{u})\bigr]\,\mathrm{d}x \\
&\leqslant \frac{a}{4}\|u\|^2 + \int\limits_{\Omega} \left(\frac{1}{4} - \frac{1}{q(x)}\right) |u|^{q(x)}\,\mathrm{d}x + \frac{\lambda}{4}\int\limits_{\Omega} \bigl[f(x,u)u - 4F(x,u)\bigr]\,\mathrm{d}x \\
&\leqslant \liminf_{n\to\infty} \left[\mathcal{I}_{b,\lambda}(u_n) - \frac{1}{4}\bigl\langle (I_{b,\lambda})'(u_n), u_n\bigr\rangle\right] \leqslant c_{b,\lambda}.
\end{aligned}$$

Therefore, $\alpha_u = \beta_u = 1$, and $c_{b,\lambda}$ is achieved by $u_b := u^+ + u^- \in \mathcal{M}_{b,\lambda}$. □

4 Proof of main results

In this section, we prove our main results. First, we prove Theorem 1. In fact, thanks to Lemma 5, we just prove that the minimizer u_b for $c_{b,\lambda}$ is indeed a sign-changing solution of problem (1).

Proof of Theorem 1. Since $u_b \in \mathcal{M}_{b,\lambda}$, we have

$$\bigl\langle (\mathcal{I}_{b,\lambda})'(u_b), u_b^+\bigr\rangle = \bigl\langle (\mathcal{I}_{b,\lambda})'(u_b), u_b^-\bigr\rangle = 0.$$

By Lemma 3 and Lemma 4, for $(\alpha,\beta) \in (\mathbb{R}_+ \times \mathbb{R}_+) \setminus (1,1)$, we have

$$\mathcal{I}_{b,\lambda}\bigl(\alpha u_b^+ + \beta u_b^-\bigr) < \mathcal{I}_{b,\lambda}\bigl(u_b^+ + u_b^-\bigr) = c_{b,\lambda}. \tag{18}$$

If $(\mathcal{I}_{b,\lambda})'(u_b) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$\bigl\|(\mathcal{I}_{b,\lambda})'(v)\bigr\| \geqslant \theta \quad \text{for all } \|v - u_b\| \geqslant 3\delta.$$

Choose $\tau \in (0, \min\{1/2, \delta/(\sqrt{2}\|u_b\|)\})$. Let

$$D := (1-\tau,\, 1+\tau) \times (1-\tau,\, 1+\tau)$$

and

$$g(\alpha,\beta) = \alpha u_b^+ + \beta u_b^- \quad \text{for all } (\alpha,\beta) \in D.$$

In view of (18), it is easy to see that

$$\bar{c}_\lambda := \max_{\partial\Omega} I_{b,\lambda} \circ g < c_{b,\lambda}.$$

Let $\varepsilon := \min\{(c_{b,\lambda} - \bar{c}_\lambda)/2, \theta\delta/8\}$ and $S_\delta := B(u_b,\delta)$, by Lemma 2.3 in [23] there exists a deformation $\eta \in C([0,1] \times D, D)$ such that

(a) $\eta(1, v) = v$ if $v \notin (\mathcal{I}_{b,\lambda})^{-1}([c_{b,\lambda} - 2\varepsilon, c_{b,\lambda} + 2\varepsilon] \cap S_{2\delta})$;
(b) $\eta(1, (\mathcal{I}_{b,\lambda})^{c_{b,\lambda}+\varepsilon} \cap S_\delta) \subset (\mathcal{I}_{b,\lambda})^{c_{b,\lambda}-\varepsilon}$;
(c) $\mathcal{I}_{b,\lambda}(\eta(1, v)) \leqslant \mathcal{I}_{b,\lambda}(v)$ for all $v \in H_0^{s(\cdot)}(\Omega)$.

First, we need to prove that

$$\max_{(\alpha,\beta)\in\bar{D}} \mathcal{I}_{b,\lambda}\bigl(\eta\bigl(1, g(\alpha, \beta)\bigr)\bigr) < c_{b,\lambda}. \tag{19}$$

In fact, it follows from Lemma 1 that $\mathcal{I}_{b,\lambda}(g(\alpha, \beta)) \leqslant c_{b,\lambda} < c_{b,\lambda} + \varepsilon$. That is,

$$g(\alpha, \beta) \in (\mathcal{I}_{b,\lambda})^{c_{b,\lambda}+\varepsilon}.$$

On the other hand, we have

$$\begin{aligned}\bigl\|g(\alpha, \beta) - u_b\bigr\|^2 &= \bigl\|(\alpha - 1)u_b^+ + (\beta - 1)u_b^-\bigr\|^2\\ &\leqslant 2\bigl((\alpha - 1)^2\bigl\|u_b^+\bigr\|^2 + (\beta - 1)^2\bigl\|u_b^-\bigr\|^2\bigr)\\ &\leqslant 2\tau\|u_b\|^2 < \delta^2,\end{aligned}$$

which shows that $g(\alpha, \beta) \in S_\delta$ for all $(\alpha, \beta) \in \bar{D}$.

Therefore, by (b) we have $\mathcal{I}_{b,\lambda}(\eta(1, g(s, t))) < c_{b,\lambda} - \varepsilon$. Hence, (19) holds.

In the following, we prove that $\eta(1, g(D)) \cap \mathcal{M}_{b,\lambda} \neq \emptyset$, which contradicts the definition of $c_{b,\lambda}$.

Let $h(\alpha, \beta) := \eta(1, g(\alpha, \beta))$,

$$\begin{aligned}\Psi_0(\alpha, \beta) &:= \bigl(\bigl\langle(\mathcal{I}_{b,\lambda})'\bigl(g(\alpha, \beta)\bigr), u_b^+\bigr\rangle, \bigl\langle(\mathcal{I}_{b,\lambda})'\bigl(g(\alpha, \beta)\bigr), u_b^-\bigr\rangle\bigr)\\ &= \bigl(\bigl\langle(\mathcal{I}_{b,\lambda})'\bigl(\alpha u_b^+ + \beta u_b^-\bigr), u_b^+\bigr\rangle, \bigl\langle(\mathcal{I}_{b,\lambda})'\bigl(\alpha u_b^+ + \beta u_b^-\bigr), u_b^-\bigr\rangle\bigr)\\ &:= \bigl(\varphi_u^1(\alpha, \beta), \varphi_u^2(\alpha, \beta)\bigr),\end{aligned}$$

and

$$\Psi_1(\alpha, \beta) := \biggl(\frac{1}{\alpha}\bigl\langle(\mathcal{I}_{b,\lambda})'\bigl(h(\alpha, \beta)\bigr), \bigl(h(\alpha, \beta)\bigr)^+\bigr\rangle, \frac{1}{\beta}\bigl\langle(\mathcal{I}_{b,\lambda})'\bigl(h(\alpha, \beta)\bigr), \bigl(h(\alpha, \beta)\bigr)^-\bigr\rangle\biggr).$$

By the direct calculation we have

$$\begin{aligned}\frac{\varphi_u^1(\alpha, \beta)}{\partial\alpha}\bigg|_{(1,1)} &= a\bigl\|u_b^+\bigr\|^2 + 3b\bigl\|u_b^+\bigr\|^4 + b\bigl\|u_b^+\bigr\|^2\bigl\|u_b^-\bigr\|^2\\ &\quad - \int_\Omega \bigl(q(x) - 1\bigr)\bigl|u_b^+\bigr|^{q(x)}\,\mathrm{d}x - \lambda\int_\Omega \partial_\alpha f\bigl(x, u_b^+\bigr)\bigl(u_b^+\bigr)^2\,\mathrm{d}x,\\ \frac{\varphi_u^1(\alpha, \beta)}{\partial\beta}\bigg|_{(1,1)} &= 2b\bigl\|u_b^+\bigr\|^2\bigl\|u_b^-\bigr\|^2, \qquad \frac{\varphi_u^2(\alpha, \beta)}{\partial\alpha}\bigg|_{(1,1)} = 2b\bigl\|u_b^+\bigr\|^2\bigl\|u_b^-\bigr\|^2,\\ \frac{\varphi_u^2(\alpha, \beta)}{\partial\beta}\bigg|_{(1,1)} &= a\bigl\|u_b^-\bigr\|^2 + 3b\bigl\|u_b^-\bigr\|^4 + b\bigl\|u_b^+\bigr\|^2\bigl\|u_b^-\bigr\|^2\\ &\quad - \int_\Omega \bigl(q(x) - 1\bigr)\bigl|u_b^-\bigr|^{q(x)}\,\mathrm{d}x - \lambda\int_\Omega \partial_\beta f\bigl(x, u_b^-\bigr)\bigl(u_b^-\bigr)^2\,\mathrm{d}x.\end{aligned}$$

Let

$$M = \begin{bmatrix} \frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\big|_{(1,1)} & \frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\big|_{(1,1)} \\ \frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\big|_{(1,1)} & \frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\big|_{(1,1)} \end{bmatrix}.$$

By (f3), for $t \neq 0$, we have

$$\partial_t f(x,t)t^2 - 3f(x,t)t > 0 \quad \text{for a.e. } x \in \Omega.$$

Then, since $u_b \in \mathcal{M}_{b,\lambda}$, we have

$$\det M = \frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\bigg|_{(1,1)} \cdot \frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\bigg|_{(1,1)} - \frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\bigg|_{(1,1)} \cdot \frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\bigg|_{(1,1)} > 0.$$

Since $\Psi_0(\alpha,\beta)$ is a C^1 function and $(1,1)$ is the unique isolated zero point of Ψ_0, by using the degree theory we deduce that $\deg(\Psi_0, D, 0) = 1$.

Hence, combining (19) with (a), we obtain

$$g(\alpha,\beta) = h(\alpha,\beta) \quad \text{on } \partial D.$$

Consequently, we obtain $\deg(\Psi_1, D, 0) = 1$. Therefore, $\Psi_1(\alpha_0,\beta_0) = 0$ for some $(\alpha_0,\beta_0) \in D$ so that

$$\eta\bigl(1, g(\alpha_0,\beta_0)\bigr) = h(\alpha_0,\beta_0) \in \mathcal{M}_{b,\lambda},$$

which contradicts (19).

From the above discussions we deduce that u_b is a sign-changing solution for problem (1).

Finally, we prove that u has exactly two nodal domains. To this end, we assume by contradiction that

$$u_b = u_1 + u_2 + u_3 \quad \text{with } u_i \neq 0,\ u_1 \geqslant 0,\ u_2 \leqslant 0$$

and

$$\operatorname{suppt}(u_i) \cap \operatorname{suppt}(u_j) = \emptyset \quad \text{for } i \neq j,\ i,j = 1,2,3,$$

and

$$\bigl\langle (\mathcal{I}_{b,\lambda})'(u), u_i \bigr\rangle = 0 \quad \text{for } i = 1,2,3.$$

Setting $v := u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \neq 0$. Then there exist a unique pair (α_v, β_v) of positive numbers such that $\alpha_v u_1 + \beta_v u_2 \in \mathcal{M}_{b,\lambda}$. Hence,

$$\mathcal{I}_{b,\lambda}(\alpha_v u_1 + \beta_v u_2) \geqslant c_b^\lambda.$$

Moreover, using the fact that $\langle (I_{b,\lambda})'(u), u_i \rangle = 0$, we obtain $\langle (I_{b,\lambda})'(v), v^\pm \rangle < 0$.

From Lemma 3(ii) we have that

$$(\alpha_v, \beta_v) \in (0,1] \times (0,1].$$

On the other hand, we have

$$0 = \frac{1}{4}\bigl\langle (\mathcal{I}_{b,\lambda})'(u), u_3 \bigr\rangle < \mathcal{I}_{b,\lambda}(u_3) + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2.$$

Hence, by (12) we obtain

$$\begin{aligned}
c_{b,\lambda} &\leqslant \mathcal{I}_{b,\lambda}(\alpha_v u_1 + \beta_v u_2)\\
&= \mathcal{I}_{b,\lambda}(\alpha_v u_1 + \beta_v u_2) - \frac{1}{4}\big\langle (\mathcal{I}_{b,\lambda})'(\alpha_v u_1 + \beta_v u_2), (\alpha_v u_1 + \beta_v u_2)\big\rangle\\
&= \frac{a}{4}\big(\|\alpha_v u_1\|^2 + \|\beta_v u_2\|^2\big) + \frac{\lambda}{4}\int\limits_{\Omega} \big[f(x,\alpha_v u_1)(\alpha_v u_1) - 4F(x,\alpha_v u_1)\big]\,\mathrm{d}x\\
&\quad + \frac{\lambda}{4}\int\limits_{\Omega} \big[f(x,\beta_v u_2)(\beta_v u_2) - 4F(x,\beta_v u_2)\big]\,\mathrm{d}x\\
&\quad + \int\limits_{\Omega}\left(\frac{1}{4} - \frac{1}{q(x)}\right)\alpha_v^{q(x)}|u_1|^{q(x)}\,\mathrm{d}x + \int\limits_{\Omega}\left(\frac{1}{4} - \frac{1}{q(x)}\right)\beta_v^{q(x)}|u_2|^{q(x)}\,\mathrm{d}x\\
&\leqslant \mathcal{I}_{b,\lambda}(u_1 + u_2) - \frac{1}{4}\big\langle (I_{b,\lambda})'(u_1 + u_2), (u_1 + u_2)\big\rangle\\
&= \mathcal{I}_{b,\lambda}(u_1 + u_2) + \frac{1}{4}\big\langle (I_{b,\lambda})'(u), u_3\big\rangle + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2\\
&< \mathcal{I}_{b,\lambda}(u_1) + I_{b,\lambda}(u_2) + I_{b,\lambda}(u_3) + \frac{b}{4}\big(\|u_2\|^2 + \|u_3\|^2\big)\|u_1\|^2\\
&\quad + \frac{b}{4}\big(\|u_1\|^2 + \|u_3\|^2\big)\|u_2\|^2 + \frac{b}{4}\big(\|u_1\|^2 + \|u_2\|^2\big)\|u_3\|^2\\
&= \mathcal{I}_{b,\lambda}(u) = c_{b,\lambda},
\end{aligned}$$

which is a contradiction, that is, $u_3 = 0$ and u_b has exactly two nodal domains. □

By Theorem 1 we obtain a least energy sign-changing solution u_b of problem (1). Next, we prove that the energy of u_b is strictly larger than two times the ground state energy.

Proof of Theorem 2. Similar to the proof of Lemma 5, there exists $\lambda_1^* > 0$ such that for all $\lambda \geqslant \lambda_1^*$ and for each $b > 0$, there exists $v_b \in \mathcal{N}_{b,\lambda}$ such that $\mathcal{I}_{b,\lambda}(v_b) = c^* > 0$. By standard arguments (see [10, Cor. 2.13]) the critical points of the functional $\mathcal{I}_{b,\lambda}$ on $\mathcal{N}_{b,\lambda}$ are critical points of $\mathcal{I}_{b,\lambda}$ in $H_0^{s(\cdot)}(\Omega)$, and we obtain $(\mathcal{I}_{b,\lambda})'(v_b) = 0$. That is, v_b is a ground state solution of (1).

According to Theorem 1, we know that problem (1) has a least energy sign-changing solution u_b, which changes sign only once when $\lambda \geqslant \lambda^*$.

Let $\lambda^{**} = \max\{\lambda^*, \lambda_1^*\}$. Suppose that $u_b = u_b^+ + u_b^-$. As in the proof of Lemma 3, there exist $\alpha_{u_b^+} > 0$ and $\beta_{u_b^-} > 0$ such that $\alpha_{u_b^+}u_b^+ \in \mathcal{N}_{b,\lambda}$, $\beta_{u_b^-}u_b^- \in \mathcal{N}_{b,\lambda}$. Furthermore, Lemma 3 implies that $\alpha_{u_b^+}, \beta_{u_b^-} \in (0,1)$. Therefore, in view of Lemma 3, we have

$$\begin{aligned}
2c^* &\leqslant \mathcal{I}_{b,\lambda}\big(\alpha_{u_b^+}u_b^+\big) + \mathcal{I}_{b,\lambda}\big(\beta_{u_b^-}u_b^-\big) \leqslant \mathcal{I}_{b,\lambda}\big(\alpha_{u_b^+}u_b^+ + \beta_{u_b^-}u_b^-\big)\\
&< \mathcal{I}_{b,\lambda}\big(u_b^+ + u_b^-\big) = c_{b,\lambda}.
\end{aligned}$$

Hence, it follows that $c^* > 0$ cannot be achieved by a sign-changing function. □

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