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Solitons and other solutions of perturbed nonlinear Biswas–Milovic equation with Kudryashov's law of refractive index

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Abstract. We analytically study the exact solitary wave solutions of the perturbed nonlinear Biswas–Milovic equation with Kudryashov's law of refractive index, which describes the propagation of pulses of various types in optical fiber. We apply three efficient and reliable schemes, specifically, the simple equation method, the (G'/G)-expansion method, and the new Kudryashov method. These approaches lead to a range of solitons and other solutions comprising of the bright solitons, dark solitons, singular solitons, periodic, rational, and exponential solutions. These solutions are also presented graphically. Furthermore, all obtained solutions are verified by symbolic computations.

Keywords: perturbed Biswas–Milovic equation, simple equation method, (G'/G)-expansion method, new Kudryashov method, Kudryashov's law.

1 Introduction

The nonlinear Schrödinger (NLS) equation, which is a primary complete integrable nonlinear dispersive partial differential equation (PDE), has been crucial towards establishing a better understanding of a wide variety of systems from atomic physics and nonlinear optics to rogue waves, deep water waves, plasmas, among others [1, 11, 13, 15, 27, 30]. For several years, one of the most interesting and stimulating fields of research in the field of engineering and science has been the search for exact soliton solutions to nonlinear models [7, 8, 10, 12, 16, 24, 26, 32]. The study of solitons has a critical role to play in the creation of new theories in the field mathematical physics. The most significant inventions application of the soliton is that it is utilized in optical fibers to transmit digital information. The development of mathematical methods further provide us with more detailed findings for the extraction of these solitons. An effective approach to examine the exact solitons and other solutions of nonlinear models is to propose a transformation in a way to formulate at nonlinear ordinary differential equations (NODEs) that can be solved using computational techniques like modified simple equation [9], modified tanh-function method [31], sine-Gordon expansion method [2], subequation method [4], homogeneous balance method [18], new extended direct algebraic method [25], Jacobi elliptic function method [5], Riccati–Bernoulli's sub-ODE method [23], extended rational sine-cosine method [29], generalized exponential rational function method [14], functional variable method [17], and so on.

The nonlinear Biswas–Milovic (NLBM) equation in polarization preserving fibers without nonlinear perturbation terms is defined as [28, 33]

$$\mathrm{i}\big(Q^m\big)_t + \delta\big(Q^m\big)_{xx} + \lambda\mathcal{F}\big(|Q|^2\big)Q^m = 0, \quad m \geqslant 1,$$

where $Q(x,t)$ is a complex valued function, δ and λ specify respectively the coefficients of group velocity dispersion and nonlinearity. The spatial and temporal variables generally represent the independent variables x and t. The function $\mathcal{F}(|Q|^2)Q^m$ is believed to be r-times continuously differentiable, so

$$\mathcal{F}\big(|Q|^2\big)Q^m \in \bigcup_{k,l=1}^{\infty} \mathcal{C}^r\big[(-k,k)\times(-l,l);\mathcal{R}^2\big],$$

where $\mathcal{C}$ is a complex plane, while $\mathcal{R}^2$ is a two-dimensional linear space. The critical concept of this paper is to establish the soliton solutions of NLBM equation incorporated with Kudryashov's in polarization preserving fibers and nonlinear perturbation terms given as [34]

$$\begin{aligned}\mathrm{i}(Q^m)_t + \delta(Q^m)_{xx} + \left(\frac{\lambda_1}{|Q|^{2n}} + \frac{\lambda_2}{|Q|^n} + \lambda_3|Q|^n + \lambda_4|Q|^{2n}\right)Q^m \\ = \mathrm{i}\big(s\big(|Q|^{2n}Q^m\big)_x + \theta_1\big(|Q|^{2n}\big)_x Q^m + \theta_2|Q|^{2n}\big(Q^m\big)_x\big),\end{aligned} \tag{1}$$

where m and n are the maximum intensity and power nonlinearity respectively, λ_k, $k = 1,2,3,4$, indicate the coefficients of nonlinearity effects, while s is the coefficient of self-steepening term. To achieve this aim, we employed three efficient and reliable schemes, explicitly, the simple equation method, the (G'/G)-expansion method, and the new Kudryashov method. Recently, Zayed et al. in [34] studied this model with unified auxiliary equation method. Zayed et al. in [35] applied the modified Kudryashov's approach and the addendum to Kudryashov's approach to obtained optical soliton solutions to the cubic–quartic perturbation with Biswas–Milovic equation including Kudryashov's law of refractive index. This present study further complements and presents new solutions to this nonlinear problem, some of which do not exist in [34, 35] before.

The structure of this article is as follows: the mathematical formulation of soliton solutions to NLBM equation incorporated with Kudryashov's in polarization preserving fibers and nonlinear perturbation terms along with the application of three novel techniques, the simple equation method, the (G'/G)-expansion method, and the new Kudryashov method are detailed in Section 2. The graphic interpretations of some solutions are provided in Section 3. Finally, in Section 4 we give some conclusions.

2 Mathematical examination and solutions of the model

Consider the transformation

$$Q(x,t) = Q(\zeta)\mathrm{e}^{\mathrm{i}\Lambda}, \quad \zeta = b_1 x + c_1 t, \quad \Lambda = b_2 x + c_2 t, \tag{2}$$

where b_j and c_j, $j = 1, 2$, are arbitrary constants. We use the above transformation to reduce Eq. (1) to the below nonlinear ordinary differential equation (ODE)

$$\begin{aligned}
&\mathrm{i}\big(mc_1 Q^{m-1}Q' + \mathrm{i}mc_2 Q^m\big) + \delta\big(b_1^2 m(m-1)Q^{m-2}(Q')^2 \\
&\quad + b_1^2 m Q^{m-1}Q'' + 2\mathrm{i}b_1 b_2 m^2 Q^{m-1}Q' - m^2 b_2^2 Q^m\big) \\
&\quad + \big(\lambda_1 Q^{-2n+m} + \lambda_2 Q^{-n+m} + \lambda_3 Q^{n+m} + \lambda_4 Q^{2n+m}\big) \\
&\quad - \mathrm{i}\big(sb_1(2n+m)Q^{2n+m-1}Q' + \mathrm{i}sb_2 m Q^{2n+m} + 2\theta_1 b_1 n Q^{2n+m-1}Q' \\
&\quad + \theta_2 b_1 m Q^{2n+m-1}Q' + \mathrm{i}\theta_2 b_2 m Q^{2n+m}\big) = 0.
\end{aligned} \tag{3}$$

The real and imaginary parts of Eq. (3) are attained as follows:

$$\begin{aligned}
&-m\big(c_2 + \delta m b_2^2\big)Q^m + \delta b_1^2 m Q^{m-1}Q'' \\
&\quad + \delta b_1^2 m(m-1)Q^{m-2}(Q')^2 + \lambda_1 Q^{-2n+m} + \lambda_2 Q^{-n+m} \\
&\quad + \lambda_3 Q^{n+m} + \big(\lambda_4 + b_2 m(s+\theta_2)\big)Q^{2n+m} = 0,
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\big(c_1 m + 2\delta m^2 b_1 b_2\big)Q^{m-1}Q' \\
&\quad - \big(sb_1(2n+m) + 2\theta_1 b_1 n + \theta_2 b_1 m\big)Q^{2n+m-1}Q' = 0.
\end{aligned} \tag{5}$$

Solving Eq. (5) yields

$$c_1 = -2\delta m b_1 b_2, \qquad s = -\frac{\theta_2 m + 2\theta_1 n}{m + 2n}. \tag{6}$$

Balance Q^{2n+m} with $Q^{m-1}Q''$ in Eq. (4). In accordance to the balancing procedure [21], we have $N = 1/n$. Therefore, we proposed another transformation of the form

$$Q = P^{1/n}, \qquad Q' = \frac{1}{n}P^{1/n-1}P',$$
$$Q'' = \frac{1}{n}\left(\frac{1}{n} - 1\right)P^{1/n-2}(P')^2 + \frac{1}{n}P^{1/n-1}P''$$

to reduce Eq. (4) to

$$\begin{aligned}&-mn^2\bigl(c_2+\delta b_2^2m\bigr)P^2+\delta b_1^2m\bigl(nPP''+(1-n)(P')^2\bigr)\\&\quad+\delta b_1^2m(m-1)(P')^2+\lambda_1n^2+\lambda_2n2^P+\lambda_3n^2P^3\\&\quad+n^2\bigl(\lambda_4+b_2m(s+\theta_2)\bigr)P^4=0.\end{aligned} \tag{7}$$

Balancing P^4 with PP'' in Eq. (4) yields $N=1$. Promptly, we now focus on the method of solutions to solve Eq. (7). We applied three procedures, which are the simple equation method, (G'/G)-expansion method, and new Kudryashov method.

2.1 The simple equation method (SEM)

The solution of Eq. (7) utilizing the SEM [19] can be described as

$$P(\zeta)=g_0+\sum_{j=1}^{N}g_j\Phi^j(\zeta),\quad g_N\neq 0,$$

where constants g_j, $j=0,1,2,\dots,N$, to be determined later. This function $\Phi(\zeta)$ satisfies the Bernoulli and Riccati equations, respectively, as follows:

$$\Phi'(\zeta)=\mu_1\Phi(\zeta)+\mu_2\Phi^2(\zeta) \tag{8}$$

and

$$\Phi'(\zeta)=\mu_1\Phi^2(\zeta)+\mu_2. \tag{9}$$

For Eq. (8), the solutions are given as

(i) the rational form

$$\Phi(\zeta)=\frac{1}{\mu_2(\zeta_0-\zeta)}\quad\text{when }\mu_1=0; \tag{10}$$

(ii) the exponential form

$$\begin{aligned}\Phi(\zeta)&=\frac{\mu_1\mathrm{e}^{(\mu_1(\zeta+\zeta_0))}}{1-\mu_2\,\mathrm{e}^{(\mu_1(\zeta+\zeta_0))}}\quad\text{when }\mu_1>0\text{ and }\mu_2<0,\\ \Phi(\zeta)&=-\frac{\mu_1\mathrm{e}^{(\mu_1(\zeta+\zeta_0))}}{1+\mu_2\mathrm{e}^{(\mu_1(\zeta+\zeta_0))}}\quad\text{when }\mu_1<0\text{ and }\mu_2>0.\end{aligned} \tag{11}$$

For Eq. (9), the solutions are as follows.

If $\mu_1\mu_2<0$, the hyperbolic form

$$\Phi(\zeta)=-\frac{\sqrt{-\mu_1\mu_2}}{\mu_1}\tanh\bigl(\sqrt{-\mu_1\mu_2}\,\zeta+\zeta_0\bigr),$$

$$\Phi(\zeta)=-\frac{\sqrt{-\mu_1\mu_2}}{\mu_1}\coth\bigl(\sqrt{-\mu_1\mu_2}\,\zeta+\zeta_0\bigr).$$

If $\mu_1\mu_2 > 0$, the periodic form

$$\begin{aligned}\Phi(\zeta) &= \frac{\sqrt{\mu_1\mu_2}}{\mu_1}\tan\left(\sqrt{\mu_1\mu_2}\,\zeta+\zeta_0\right),\\ \Phi(\zeta) &= -\frac{\sqrt{\mu_1\mu_2}}{\mu_1}\cot\left(\sqrt{\mu_1\mu_2}\,\zeta+\zeta_0\right),\end{aligned}\tag{12}$$

where ζ_0 is the constant of integration. With $N = 1$, the solution of Eq. (7) is

$$P(\zeta) = g_0 + g_1\Phi(\zeta), \quad g_1 \neq 0. \tag{13}$$

Substituting Eqs. (8) and (13) into Eq. (7), then collecting all the coefficient of $\Phi^j(\zeta)$, $j = 1, 2, 3, 4$, to zero, we achieve some equations involving g_0, g_1, and other constants. Now with the use of Mathematica assistance and in addition to Eq. (6), the following solutions are possible.

$$\begin{aligned}
c_1 &= -2\delta b_1 b_2 m, \quad s = -\frac{\theta_2 m+2\theta_1 n}{m+2n},\\
g_0 &= -\frac{\lambda_3(m+n)}{2(2m+n)(b_2m(s+\theta_2)+\lambda_4)} \mp \frac{b_1\mu_1\sqrt{-\delta m(m+n)(\lambda_4+b_2m(s+\theta_2))}}{2n(\lambda_4+b_2m(\theta_2+s))},\\
g_1 &= \mp\frac{b_1\mu_2\sqrt{-\delta m(m+n)(\lambda_4+b_2m(s+\theta_2))}}{n(\lambda_4+b_2m(s+\theta_2))},\\
\lambda_1 &= \frac{(m-n)(m+n)(\delta b_1^2\mu_1^2 m(2m+n)^2(\lambda_4+b_2m(s+\theta_2))+\lambda_3^2n^2(m+n))^2}{16n^4(2m+n)^4(\lambda_4+b_2m(s+\theta_2))^3},\\
\lambda_2 &= \frac{\lambda_3(2m-n)(m+n)(\delta b_1^2\mu_1^2 m(2m+n)^2(\lambda_4+b_2m(s+\theta_2))+\lambda_3^2n^2(m+n))}{4n^2(2m+n)^3(\lambda_4+b_2m(s+\theta_2))^2},\\
c_2 &= -\frac{1}{2n^2(2m+n)^2(\lambda_4+b_2m(s+\theta_2))}\\
&\quad\times\big(2\delta b_2^3m^2n^2(2m+n)^2(s+\theta_2)+\delta b_1^2b_2\mu_1^2m^2(2m+n)^2(s+\theta_2)\\
&\quad+2\delta b_2^2\lambda_4mn^2(2m+n)^2+\delta b_1^2\lambda_4\mu_1^2m(2m+n)^2+3\lambda_3^2n^2(m+n)\big).
\end{aligned}\tag{14}$$

Use Eq. (13) assisted with Eqs. (10), (11), and (14). The rational and exponential form solutions of Eq. (1) are provided below:

$$\begin{aligned}Q_1(x,t) &= \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4+b_2m(s+\theta_2))}\right.\\ &\qquad\left.\pm\sqrt{-\frac{\delta b_1^2m(m+n)}{n^2(\lambda_4+b_2m(s+\theta_2))}}\frac{1}{(\zeta_0-\zeta)}\right)^{1/n}\mathrm{e}^{\mathrm{i}(b_2x+c_2t)}\end{aligned}$$

in conjunction with $\mu_1 = 0$, $\lambda_3(\lambda_4 + b_2m(s + \theta_2)) < 0$, and $\delta(\lambda_4 + b_2m(s + \theta_2)) < 0$;

$$Q_2(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(b_2m(s+\theta_2)+\lambda_4)} \pm \sqrt{-\frac{\delta b_1^2\mu_1^2 m(m+n)}{4n^2(\lambda_4+b_2m(s+\theta_2))}}\left(\frac{1+\mu_2 e^{(\mu_1(\zeta+\zeta_0))}}{1-\mu_2 e^{(\mu_1(\zeta+\zeta_0))}}\right)\right)^{1/n} e^{i(b_2x+c_2t)}$$

along with $\mu_1 > 0$, $\mu_2 < 0$, $\lambda_3(\lambda_4 + b_2m(s+\theta_2)) < 0$, and $\delta(\lambda_4 + b_2m(s+\theta_2)) < 0$;

$$Q_3(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(b_2m(s+\theta_2)+\lambda_4)} \pm \sqrt{-\frac{\delta b_1^2\mu_1^2 m(m+n)}{4n^2(\lambda_4+b_2m(s+\theta_2))}}\left(\frac{1-\mu_2 e^{(\mu_1(\zeta+\zeta_0))}}{1+\mu_2 e^{(\mu_1(\zeta+\zeta_0))}}\right)\right)^{1/n} e^{i(b_2x+c_2t)} \tag{15}$$

along with $\mu_1 < 0$, $\mu_2 > 0$, $\lambda_3(\lambda_4 + b_2m(s+\theta_2)) < 0$, and $\delta(\lambda_4 + b_2m(s+\theta_2)) < 0$.

Again, inserting Eqs. (9) and (13) into (7), later collecting all the coefficient of $\Phi^j(\zeta)$, $j = 1,2,3,4$, to zero, we get certain equations in g_0, g_1, and other constants. With the use of Mathematica assistance and in addition to Eq. (6), the following solutions are achievable.

$$\begin{aligned}
&c_1 = -2\delta b_1 b_2 m, \qquad s = -\frac{\theta_2 m + 2\theta_1 n}{m+2n},\\
&g_0 = -\frac{\lambda_3(m+n)}{2(2m+n)(b_2m(s+\theta_2)+\lambda_4)}, \qquad g_1 = \pm\frac{ib_1\mu_1}{n}\sqrt{\frac{\delta m(m+n)}{\lambda_4+b_2m(s+\theta_2)}},\\
&\lambda_1 = \frac{(m-n)(m+n)(\lambda_3^2n^2(m+n) - 4b_1^2\delta\mu_1\mu_2 m(2m+n)^2(\lambda_4+b_2m(s+\theta_2)))^2}{16n^4(2m+n)^4(\lambda_4+b_2m(s+\theta_2))^3},\\
&\lambda_2 = -\frac{\lambda_3(2m-n)(m+n)(4\delta b_1^2\mu_1\mu_2 m(2m+n)^2(\lambda_4+b_2m(s+\theta_2)) - \lambda_3^2n^2(m+n))}{4n^2(2m+n)^3(\lambda_4+b_2m(s+\theta_2))^2},\\
&c_2 = \frac{1}{2n^2(2m+n)^2(\lambda_4+b_2m(s+\theta_2))}\\
&\quad\times\big(-2\delta b_2^3m^2n^2(2m+n)^2(s+\theta_2) + 4\delta b_1^2b_2\mu_1\mu_2 m^2(2m+n)^2(s+\theta_2)\\
&\quad - 2\delta b_2^2\lambda_4 mn^2(2m+n)^2 + 4\delta b_1^2\lambda_4\mu_1\mu_2 m(2m+n)^2 - 3\lambda_3^2n^2(m+n)\big).
\end{aligned} \tag{16}$$

Use Eq. (16) assisted with Eqs. (10), (11), and (14). The dark and singular solutions of Eq. (1) are listed below:

$$Q_4(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4+b_2m(s+\theta_2))} \pm \frac{ib_1}{n}\sqrt{-\frac{\delta m(m+n)\mu_1\mu_2}{\lambda_4+b_2m(s+\theta_2)}}\tanh(\sqrt{-\mu_1\mu_2}\,\zeta+\zeta_0)\right)^{1/n} e^{i(b_2x+c_2t)}, \tag{16}$$

$$Q_5(x,t) = \biggl(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4+b_2m(s+\theta_2))} \pm \frac{\mathrm{i}b_1}{n}\sqrt{-\frac{\delta m(m+n)\mu_1\mu_2}{\lambda_4+b_2m(s+\theta_2)}}\coth\bigl(\sqrt{-\mu_1\mu_2}\,\zeta+\zeta_0\bigr)\biggr)^{1/n}\mathrm{e}^{\mathrm{i}(b_2x+c_2t)}, \quad (17)$$

provided $\mu_1\mu_2 < 0$, $\lambda_3(\lambda_4 + b_2m(s+\theta_2)) < 0$, and $\delta(\lambda_4 + b_2m(s+\theta_2)) < 0$. The periodic solutions of Eq. (1) are listed below:

$$Q_6(x,t) = \biggl(-\frac{\lambda_3(m+n)}{2(2m+n)\bigl(\lambda_4+b_2m(s+\theta_2)\bigr)} \mp \frac{\mathrm{i}b_1}{n}\sqrt{\frac{\delta m(m+n)\mu_1\mu_2}{\lambda_4+b_2m(s+\theta_2)}}\tan\bigl(\sqrt{\mu_1\mu_2}\,\zeta+\zeta_0\bigr)\biggr)^{1/n}\mathrm{e}^{\mathrm{i}(b_2x+c_2t)}, \quad (18)$$

$$Q_7(x,t) = \biggl(-\frac{\lambda_3(m+n)}{2(2m+n)\bigl(\lambda_4+b_2m(s+\theta_2)\bigr)} \pm \frac{\mathrm{i}b_1}{n}\sqrt{\frac{\delta m(m+n)\mu_1\mu_2}{\lambda_4+b_2m(s+\theta_2)}}\cot\bigl(\sqrt{\mu_1\mu_2}\,\zeta+\zeta_0\bigr)\biggr)^{1/n}\mathrm{e}^{\mathrm{i}(b_2x+c_2t)}, \quad (19)$$

provided $\mu_1\mu_2 > 0$, $\lambda_3(\lambda_4 + b_2m(s+\theta_2)) < 0$, and $\delta(\lambda_4 + b_2m(s+\theta_2)) > 0$.

2.2 The (G'/G)-expansion method

Consider the (G'/G)-expansion method [6, 22], the solution to Eq. (7) is given as

$$P(\zeta) = g_0 + \sum_{j=1}^{N} g_j\left(\frac{G'(\zeta)}{G(\zeta)}\right)^j, \quad g_N \neq 0, \quad (20)$$

where $G(\xi)$ fulfills the below ODE

$$G''(\zeta) = -\mu_1 G'(\zeta) - \mu_2 G(\zeta). \quad (21)$$

Here the unknowns g_j, $j = 0, 1, 2, \dots, N$, and μ_j, $j = 1, 2$, can be determined subsequently. The solutions of Eq. (21) are given as

$$\frac{G'(\zeta)}{G(\zeta)} = \begin{cases} -\frac{\mu_1}{2} + \frac{\sqrt{\rho}}{2}\left(\frac{\Omega_1\sinh(\frac{1}{2}\sqrt{\rho}\,\zeta)+\Omega_2\cosh(\frac{1}{2}\sqrt{\rho}\,\zeta)}{\Omega_1\cosh(\frac{1}{2}\sqrt{\rho}\,\zeta)+\Omega_2\sinh(\frac{1}{2}\sqrt{\rho}\,\zeta)}\right), & \rho > 0, \\ -\frac{\mu_1}{2} + \frac{\sqrt{-\rho}}{2}\left(\frac{\Omega_1\sin(\frac{1}{2}\sqrt{-\rho}\,\zeta)+\Omega_2\cos(\frac{1}{2}\sqrt{-\rho}\,\zeta)}{\Omega_1\cos(\frac{1}{2}\sqrt{-\rho}\,\zeta)+\Omega_2\sin(\frac{1}{2}\sqrt{-\rho}\,\zeta)}\right), & \rho < 0, \\ -\frac{\mu_1}{2} + \frac{\Omega_2}{\Omega_1+\Omega_2\zeta}, & \rho = 0, \end{cases} \quad (22)$$

where Ω_1 and Ω_2 are arbitrary constants, and $\rho = \mu_1^2 - 4\mu_2$. We already obtained $N = 1$, thus Eq. (20) reads

$$P(\zeta) = g_0 + g_1\frac{G'(\zeta)}{G(\zeta)}, \quad g_1 \neq 0. \quad (23)$$

Putting Eqs. (21) and (23) into Eq. (7) and equating the $(G'/G)^i$, $i = 0, 1, 2, 3, 4$, coefficients to zero leads to some solvable algebraic equations. After solving the algebraic equations through Mathematica software with Eq. (6), we have

$$\begin{aligned}
c_1 &= -2\delta b_1 b_2 m, \qquad s = -\frac{\theta_2 m+2\theta_1 n}{m+2n},\\
g_0 &= -\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4+b_2m(s+\theta_2))} \mp \frac{b_1\mu_1\sqrt{-\delta m(m+n)(\lambda_4+b_2m(s+\theta_2))}}{2n(\lambda_4+b_2m(s+\theta_2))},\\
g_1 &= \mp\frac{b_1\sqrt{-\delta m(m+n)(\lambda_4+b_2m(s+\theta_2))}}{n(\lambda_4+b_2m(s+\theta_2))},\\
\lambda_1 &= \frac{(m-n)(m+n)(\delta b_1^2 m\rho(2m+n)^2(\lambda_4+b_2m(s+\theta_2))+\lambda_3^2n^2(m+n))^2}{16n^4(2m+n)^4(\lambda_4+b_2m(s+\theta_2))^3},\\
\lambda_2 &= \frac{\lambda_3(2m-n)(m+n)(\delta b_1^2 m\rho(2m+n)^2(\lambda_4+b_2m(s+\theta_2))+\lambda_3^2n^2(m+n))}{4n^2(2m+n)^3(\lambda_4+b_2m(s+\theta_2))^2},\\
c_2 &= -\frac{1}{2n^2(2m+n)^2(\lambda_4+b_2m(s+\theta_2))}\\
&\quad\times\big(2\delta b_2^3m^2n^2(2m+n)^2(s+\theta_2)+\delta b_1^2b_2m^2\rho(2m+n)^2(s+\theta_2)\\
&\quad+2\delta b_2^2\lambda_4mn^2(2m+n)^2+\delta b_1^2\lambda_4m\rho(2m+n)^2+3\lambda_3^2n^2(m+n)\big).
\end{aligned} \tag{24}$$

Using Eq. (24) supported with Eqs. (22) and (23), we obtain the following solutions of Eq. (1):

$$\begin{aligned}
Q_8(x,t) &= \Bigg(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4+b_2m(s+\theta_2))}\\
&\qquad\pm\sqrt{-\frac{\delta b_1^2m(m+n)\rho}{4n^2(\lambda_4+b_2m(s+\theta_2))}}\left(\frac{\Omega_1\sinh(\frac{\sqrt{\rho}}{2}\zeta)+\Omega_2\cosh(\frac{\sqrt{\rho}}{2}\zeta)}{\Omega_1\cosh(\frac{\sqrt{\rho}}{2}\zeta)+\Omega_2\sinh(\frac{\sqrt{\rho}}{2}\zeta)}\right)\Bigg)^{1/n}\\
&\qquad\times \mathrm{e}^{\mathrm{i}(b_2x+c_2t)},
\end{aligned} \tag{25}$$

provided $\rho>0$, $\lambda_3(\lambda_4+b_2m(s+\theta_2))<0$, and $\delta(\lambda_4+b_2m(s+\theta_2))<0$;

$$\begin{aligned}
Q_9(x,t) &= \Bigg(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4+b_2m(s+\theta_2))}\\
&\qquad\pm\sqrt{\frac{\delta b_1^2m(m+n)\rho}{4n^2(\lambda_4+b_2m(s+\theta_2))}}\left(\frac{-\Omega_1\sin(\frac{\sqrt{-\rho}}{2}\zeta)+\Omega_2\cos(\frac{\sqrt{-\rho}}{2}\zeta)}{\Omega_1\cos(\frac{\sqrt{-\rho}}{2}\zeta)+\Omega_2\sin(\frac{\sqrt{-\rho}}{2}\zeta)}\right)\Bigg)^{1/n}\\
&\qquad\times \mathrm{e}^{\mathrm{i}(b_2x+c_2t)},
\end{aligned} \tag{26}$$

given $\rho<0$, $\lambda_3(\lambda_4+b_2m(s+\theta_2))<0$, and $\delta(\lambda_4+b_2m(s+\theta_2))<0$;

$$\begin{aligned}
Q_{10}(x,t) &= \Bigg(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4+b_2m(s+\theta_2))}\\
&\qquad\pm\sqrt{-\frac{\delta b_1^2m(m+n)\rho}{n^2(\lambda_4+b_2m(s+\theta_2))}}\frac{\Omega_2}{(\Omega_1+\Omega_2\zeta)}\Bigg)\Bigg)^{1/n}\mathrm{e}^{\mathrm{i}(b_2x+c_2t)},
\end{aligned}$$

provided $\rho=0$, $\lambda_3(\lambda_4+b_2m(s+\theta_2))<0$, and $\delta(\lambda_4+b_2m(s+\theta_2))<0$.

Remark 1. A specific example where $\Omega_1 \neq 0$ and $\Omega_2 = 0$ in Eq. (25) results to the dark soliton solution of Eq. (1) as

$$Q(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4 + b_2 m(s+\theta_2))} \pm \sqrt{-\frac{\delta b_1^2 m(m+n)\rho}{4n^2(\lambda_4 + b_2 m(s+\theta_2))}} \tanh\left(\frac{\sqrt{\rho}}{2}\zeta\right)\right)^{1/n} \mathrm{e}^{\mathrm{i}(b_2 x + c_2 t)}. \tag{27}$$

For $\Omega_1 = 0$ and $\Omega_2 \neq 0$ in Eq. (25), we get the singular soliton solution of Eq. (1) as

$$Q(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4 + b_2 m(s+\theta_2))} \pm \sqrt{-\frac{\delta b_1^2 m(m+n)\rho}{4n^2(\lambda_4 + b_2 m(s+\theta_2))}} \coth\left(\frac{\sqrt{\rho}}{2}\zeta\right)\right)^{1/n} \mathrm{e}^{\mathrm{i}(b_2 x + c_2 t)}. \tag{28}$$

Remark 2. A special example when $\Omega_1 \neq 0$ and $\Omega_2 = 0$ in Eq. (26) reveals the periodic solutions of Eq. (1) as

$$Q(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4 + b_2 m(s+\theta_2))} \mp \sqrt{\frac{\delta b_1^2 m(m+n)\rho}{4n^2(\lambda_4 + b_2 m(s+\theta_2))}} \tan\left(\frac{\sqrt{-\rho}}{2}\zeta\right)\right)^{1/n} \mathrm{e}^{\mathrm{i}(b_2 x + c_2 t)}. \tag{29}$$

For $\Omega_1 = 0$ and $\Omega_2 \neq 0$ in Eq. (26), we also get the periodic solutions of Eq. (1) as

$$Q(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4 + b_2 m(s+\theta_2))} \pm \sqrt{\frac{\delta b_1^2 m(m+n)\rho}{4n^2(\lambda_4 + b_2 m(s+\theta_2))}} \cot\left(\frac{\sqrt{-\rho}}{2}\zeta\right)\right)^{1/n} \mathrm{e}^{\mathrm{i}(b_2 x + c_2 t)}. \tag{30}$$

2.3 The new Kudryashov method

Based on the new Kudryashov method [3, 20], assume that the solution to Eq. (2) is

$$P(\zeta) = g_0 + \sum_{j=1}^{N} g_i \Phi^j(\zeta), \quad g_N \neq 0. \tag{31}$$

The function $\Phi(\zeta)$ satisfies an ODE expressed as

$$\left(\Phi'(\zeta)\right)^2 = \Phi^2(\zeta)\left(1 - \Omega\Phi^2(\zeta)\right). \tag{32}$$

The solution to the above-mentioned ODE is provided as

$$\Phi(\zeta) = \frac{4\Omega_1}{(4\Omega_1^2 - \Omega)\sinh(\zeta) + (4\Omega_1^2 + \Omega)\cosh(\zeta)}, \quad \Omega = 4\Omega_1\Omega_2, \tag{33}$$

for arbitrary constants Ω_1 and Ω_2. Again, $N = 1$, and from Eq. (31) we get

$$P(\zeta) = g_0 + g_1\Phi(\zeta), \quad g_1 \neq 0. \tag{34}$$

Inserting Eqs. (31) and (32) into Eq. (7), gathering all the coefficient of $\Phi^j(\zeta)$, $j = 0, 1, 2, 3, 4$, to zero, after solving the resulting equations with the unknown constants and taking int account Eq. (6), we obtain

$$c_1 = -2\delta b_1 b_2 m, \qquad s = -\frac{\theta_2 m + 2\theta_1 n}{m + 2n},$$
$$g_0 = -\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4 + b_2 m(s+\theta_2))}, \quad g_1 = \pm\frac{b_1}{n}\sqrt{\frac{\delta\Omega m(m+n)}{\lambda_4 + b_2 m(s+\theta_2)}},$$
$$c_2 = -\delta b_2^2 m + \frac{\delta b_1^2 m}{n^2} - \frac{3\lambda_3^2(m+n)}{2(2m+n)^2(b_2 m(s+\theta_2)+\lambda_4)},$$
$$\lambda_1 = -\frac{\lambda_3^2(m-n)(m+n)^2(4\delta b_1^2 m(2m+n)^2(\lambda_4 + b_2 m(s+\theta_2)) - \lambda_3^2 n^2(m+n))}{16n^2(2m+n)^4(\lambda_4 + b_2 m(s+\theta_2))^3},$$
$$\lambda_2 = -\frac{\lambda_3(2m-n)(m+n)(2\delta b_1^2 m(2m+n)^2(\lambda_4 + b_2 m(s+\theta_2)) - \lambda_3^2 n^2(m+n))}{4n^2(2m+n)^3(\lambda_4 + b_2 m(s+\theta_2))^2}.$$

Incorporating these parameters into Eq. (34) assisted with Eq. (33), we obtain solutions

$$Q_{11}(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4 + b_2 m(s+\theta_2))} \pm\sqrt{\frac{\delta\Omega\, b_1^2 m(m+n)}{n^2(\lambda_4 + b_2 m(s+\theta_2))}}\left(\frac{4\Omega_1}{(4\Omega_1^2 - \Omega)\sinh\zeta + (4\Omega_1^2 + \Omega)\cosh\zeta}\right)\right)^{1/n} \times \mathrm{e}^{\mathrm{i}(b_2 x + c_2 t)}, \tag{35}$$

given that $\lambda_3(\lambda_4 + b_2 m(s+\theta_2)) < 0$, $\delta\Omega(\lambda_4 + b_2 m(s+\theta_2)) > 0$, and $\Omega = 4\Omega_1\Omega_2$.

Remark 3. Setting $\Omega_1 = \Omega_2 = 1$ in Eq. (35) results the bright soliton solutions of Eq. (1) as follows:

$$Q(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4 + b_2 m(s+\theta_2))} \pm\sqrt{\frac{\delta b_1^2 m(m+n)}{n^2(\lambda_4 + b_2 m(s+\theta_2))}}\operatorname{sech}\zeta\right)^{1/n} \mathrm{e}^{\mathrm{i}(b_2 x + c_2 t)}, \tag{36}$$

provided that $\lambda_3(\lambda_4 + b_2 m(s+\theta_2)) < 0$ and $\delta(\lambda_4 + b_2 m(s+\theta_2)) > 0$.

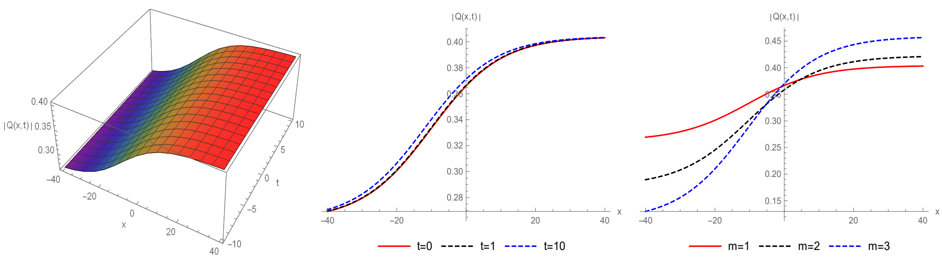
Remark 4. Setting $\Omega_1 = 1$ and $\Omega_2 = -1$ in Eq. (35) results the singular soliton solutions of Eq. (1) as

$$Q(x,t) = \left(-\frac{\lambda_3(m+n)}{2(2m+n)(\lambda_4 + b_2 m(s+\theta_2))} \mp\sqrt{-\frac{\delta b_1^2 m(m+n)}{n^2(\lambda_4 + b_2 m(s+\theta_2))}}\operatorname{csch}\zeta\right)^{1/n} \mathrm{e}^{\mathrm{i}(b_2 x + c_2 t)}, \tag{37}$$

provided that $\lambda_3(\lambda_4 + b_2 m(s+\theta_2)) < 0$ and $\delta(\lambda_4 + b_2 m(s+\theta_2)) < 0$.

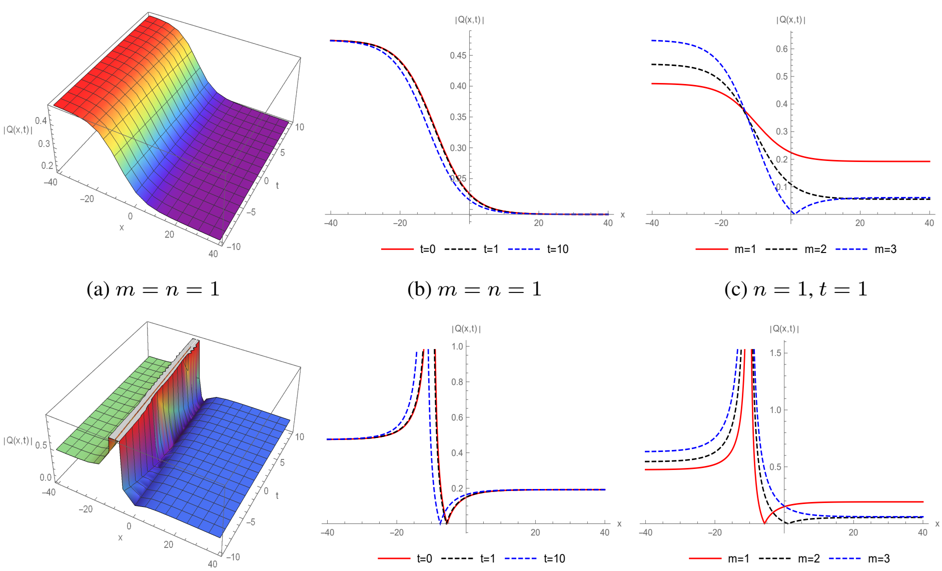
3 Graphical descriptions of some solutions

In this section, our principal objective is to demonstrate that the newly obtain solutions are more general and useful when examining the graphical representations of these solutions and can undoubtedly play a prominent role in this virtue. In Figs. 1–7, the graphics of the soliton wave solutions of Eq. (1) are displayed in 2D and 3D. It can be found from the cited figures that the acquired soliton solutions consist of bright, dark, singular, periodic, exponential, and solitary waves. With some parameter values, Figs. 2(a)-(c) and Figs. 4(a)-(c) show respectively the structure of the dark soliton solutions. Figures 2(d)–(f), 4(d)–(f),



(a) $m = n = 1$ (b) $m = n = 1$ (c) $n = 1, t = 1$

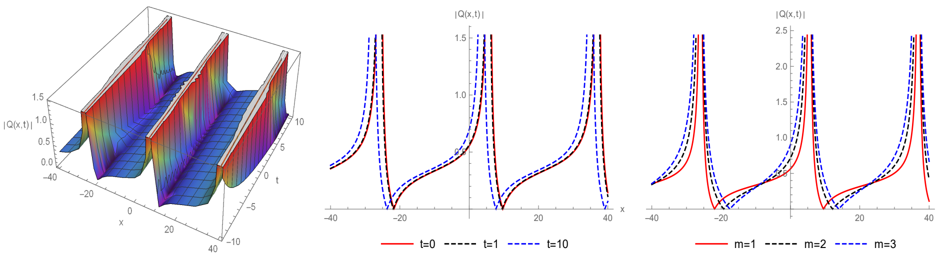
Figure 1. The plots of exponential solution for Eq. (15) with additional parameters $\delta = -1$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\theta_1 = \theta_2 = 1$, $\zeta_0 = 1$, $b_1 = b_2 = 0.1$, $\mu_1 = -1$, and $\mu_2 = 1$.



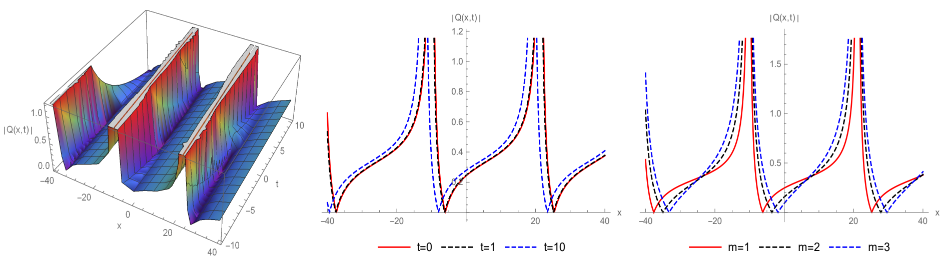
(a) $m = n = 1$ (b) $m = n = 1$ (c) $n = 1, t = 1$

(d) $m = n = 1$ (e) $m = n = 1$ (f) $n = 1, t = 1$

Figure 2. The plots of dark soliton for Eq. (16) in (a)–(c) and singular soliton for Eq. (17) in (d)–(f) with additional parameters $\delta = -1$, $\lambda_3 = -1$, $\lambda_4 = \theta_1 = \theta_2 = \zeta_0 = 1$, $b_1 = b_2 = 0.1$, $\mu_1 = -1$, and $\mu_2 = 1$.

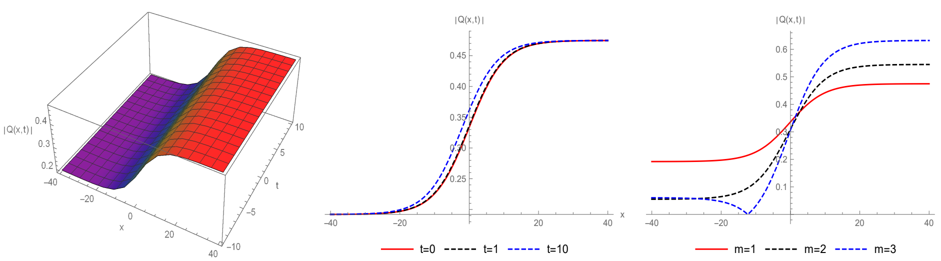


(a) $m = n = 1$ (b) $m = n = 1$ (c) $n = 1, t = 1$

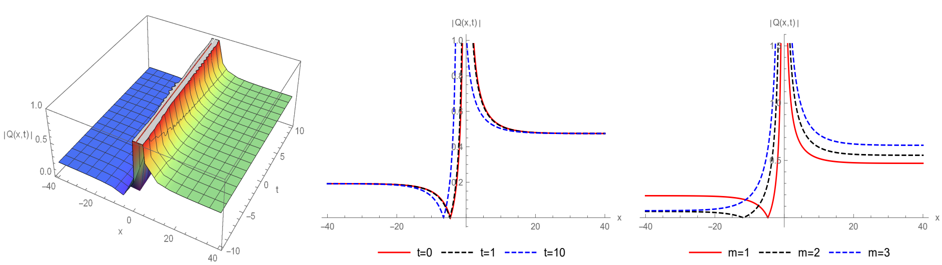


(d) $m = n = 1$ (e) $m = n = 1$ (f) $n = 1, t = 1$

Figure 3. The plots of periodic solutions for Eq. (18) in (a)–(c) and for Eq. (19) in (d)–(f) with additional parameters $\delta = -1$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\theta_1 = \theta_2 = 1$, $b_1 = b_2 = 0.1$, and $\mu_1 = \mu_2 = 1$.

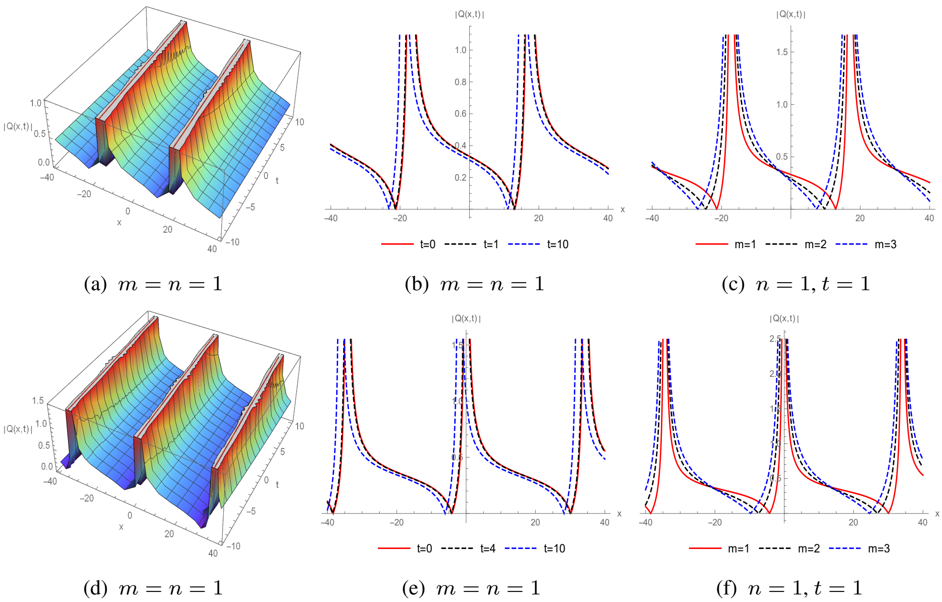


(a) $m = n = 1$ (b) $m = n = 1$ (c) $n = 1, t = 1$



(d) $m = n = 1$ (e) $m = n = 1$ (f) $n = 1, t = 1$

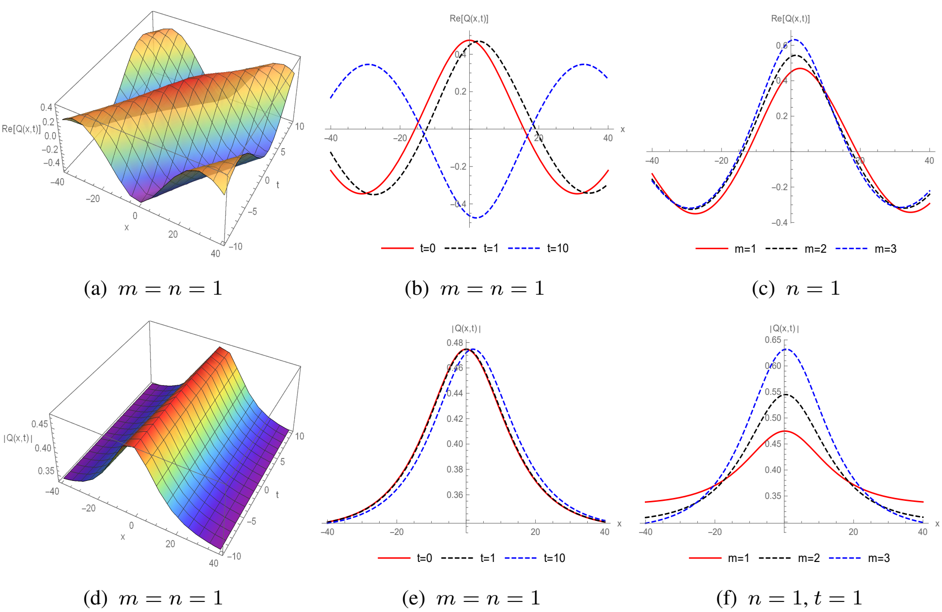
Figure 4. The plots of dark soliton for Eq. (27) in (a)–(c) and singular soliton for Eq. (28) in (d)–(f) with additional parameters $\delta = -1$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\theta_1 = \theta_2 = 1$, $b_1 = b_2 = 0.1$, $\mu_1 = \sqrt{8}$, and $\mu_2 = 1$.



(a) $m = n = 1$ (b) $m = n = 1$ (c) $n = 1, t = 1$

(d) $m = n = 1$ (e) $m = n = 1$ (f) $n = 1, t = 1$

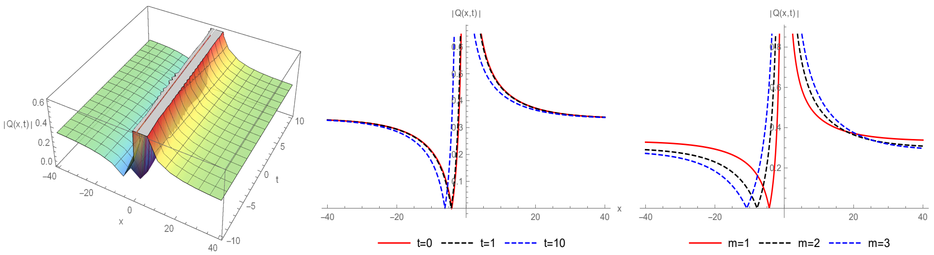
Figure 5. (a)–(c) The plots of periodic solutions of Eq. (29); (d)–(f) of Eq. (30) with additional parameters $\delta = -1$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\theta_1 = \theta_2 = 1$, $b_1 = b_2 = 0.1$, $\mu_1 = 0.8$, and $\mu_2 = 1$.



(a) $m = n = 1$ (b) $m = n = 1$ (c) $n = 1$

(d) $m = n = 1$ (e) $m = n = 1$ (f) $n = 1, t = 1$

Figure 6. The plots of bright soliton of Eq. (36) with additional parameters $\delta = -1$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\theta_1 = \theta_2 = 1$, and $b_1 = b_2 = 0.1$.



(a) $m = n = 1$ (b) $m = n = 1, t = 1$ (c) $n = 1, t = 1$

Figure 7. The plots of singular soliton for Eq. (37) with additional parameters $\delta = 1$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\theta_1 = \theta_2 = 1$, and $b_1 = b_2 = 0.1$.

and 7(a)–(b) reveal the singular soliton solutions. Figures 3, 5 represent the periodic wave solutions, while Figs. 6(a)–(f) exhibit the bright soliton solution. The plot in Fig. 1 exhibits the exponential solution of NLBM equation with Kudryashov's in polarization preserving fibers and nonlinear perturbation terms. Nevertheless, the remaining plots are not depicted since some of the aforementioned solutions display identical behavior. The obtained solutions and the cited figures presented in this work provide us with some physical explanation of the proposed problem.

4 Concluding remarks

We have successfully investigated the exact solitons and other solutions of the perturbed nonlinear Biswas–Milovic equation with Kudryashov's law of refractive index that describes the propagation of pulses of various types in optical fiber. Three different schemes, specifically, the simple equation method, the (G'/G)-expansion method, and the new Kudryashov method, have been implemented to construct several exact solutions, which include multiple soliton solutions, singular solutions, periodic solutions, solitary wave solutions, bright and dark soliton solutions of different structures. These solutions are presented under constraint conditions. The proposed methods obtained the results promptly and need simple algorithms in programming. Moreover, we presented the graphical representations of some solutions, which apparently reveal that the obtained solutions are more practical and clear to understand. Our results further strengthened the fact that the proposed methods are powerful, efficient, and easy mathematical tools for constructing solutions to numerous nonlinear problems in mathematical physics.

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