Nonlinear Analysis: Modelling and Control, Vol. 27, No. 2, 385–411
https://doi.org/10.15388/namc.2022.27.26314

Optimal harvesting in a unidirectional consumer–resource mutualisms system with size structure in the consumer*

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Received: January 21, 2021 / **Revised:** October 6, 2021 / **Published online:** February 15, 2022

Abstract. This paper considers the optimal harvesting problem for a size-structured model of unidirectional consumer–resource mutualisms in which the consumer species has both positive and negative effects on the resource species, while the resource has only a positive effect on the consumer. First, we show the existence of a unique nonnegative solution of the system and give the continuous dependence of solutions on the control variable. Next, the adjoint system is derived, which is necessary for optimality and the existence of a unique optimal policy. Then necessary conditions for optimality are established via the normal cone and adjoint system. Moreover, the existence of a unique optimal strategy is proved via Ekeland's variational principle and fixed-point reasoning in convex analysis. Finally, we use numerical simulations to verify the main results and find other dynamic properties of the system.

Keywords: consumer–resource interaction, size structure, optimal harvesting, Ekeland's variational principle.

1 Introduction

Ecological researches show that the outcomes resulting from interactions between species can vary with context-dependent factors (see [6]). The outcome of the interaction is that

*This work is supported by the National Natural Science Foundation of China (Nos. 12001341, 11971279), the Natural Science Foundation of Shanxi (No. 201901D211410) and Scientific and Technologial Innovation Programs of Higher Education Institutions in Shanxi (No. 2020L0258). All authors contributed equally to this manuscript.

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the density of each species may end up above, equal to, or below its carrying capacity in isolation from the other species (see [29]). Thus, the interaction outcomes of a two species system are classified into positive $(+)$, neutral (0) or negative $(-)$ effects (see [11]). Wang and Wu [30] pointed that the interaction outcomes between two species are not fixed but vary with biotic and/or abiotic factors. Moreover, Wang and DeAngelis [29] divided the outcomes between two species into six forms: mutualism $(+\,+)$, commensalism $(+\,0)$, predation/parasitism $(+\,-)$, amensalism $(-\,0)$, competition $(-\,-)$ and neutralism $(0\,0)$.

Holland and DeAngelis [11] established the consumer–resource theory by using differential equations, which provides a way to deal quantitatively with the problem. The consumer–resource system is a system that discusses the process of energy or nutrient transfer between a consumer organism and a resource in which a resource is defined as a biotic or abiotic factor, which could increase the consumers' growth, while a consumer will utilize the supply of resource and then reduces its growth rate (see [26]). The one-way and two-way flow of energy or matter between species makes that the consumer–resource system can be divided into unidirectional and bidirectional system (see [29]). In nature, there are many bidirectional consumer–resource interactions (such as lichens and plant mycorrhizal fungi) and unidirectional consumer–resource interactions (such as insect pollinator and host plant).

The traditional consumer–resource interaction is simulated by the $(+\,-)$ type relationship in which the consumer obtains some material benefits at the cost of the resource, such as the classic predator–prey models or parasite-host models. The unidirectional consumer–resource mutualisms are consistent with the traditional consumer–resource interaction (see [10, 27, 28]). Wang and DeAngelis [29] considered the following unidirectional consumer–resource system:

$$\begin{aligned}\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} &= N_1(t)\left[r_1 + \frac{\alpha_{12}N_2(t)}{\gamma_2 + N_2(t)} - \beta_1 N_2(t) - d_1 N_1(t)\right],\\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} &= N_2(t)\left[r_2 + \frac{\alpha_{21}N_1(t)}{\gamma_1 + N_1(t)} - d_2 N_2(t)\right].\end{aligned} \tag{1}$$

Here $N_1(t)$ and $N_2(t)$, respectively, represent the densities of the resource species and the consumer species at time t.

However, populations consist of individuals with many structural differences, such as age, size, location, status, movement, etc. According to these structural characteristics, structured population models distinguish individuals from one another to determine the birth, growth and death rates, interaction with each other and with environment, etc. (see [22]). In the last century, structured population models have played a significant role in the mathematical analysis and control of populations in biology and demography. Especially, age-structured first-order partial differential equations (PDEs) provide a main tool for modeling population systems and are recently employed in economics (see [2, 14, 20, 21, 23, 24, 31]). In [20], the authors investigated the oscillation theory of the following unidirectional consumer–resource model with age-structure in the consumer:

$$\begin{aligned}
&\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = N_1(t)\left[r - d_1N_1(t) + \frac{\alpha_{12}A(t)}{\gamma_2 + A(t)} - \beta_1 A(t)\right],\\
&\frac{\partial N_2(t,a)}{\partial t} + \frac{\partial N_2(t,a)}{\partial a} = -d_2N_2(t,a), \quad a \geqslant 0,\\
&N_2(t,0) = \frac{\alpha_{21}N_1(t)A(t)}{\gamma_1 + N_1(t)},\\
&N_1(0) = N_{10} \geqslant 0, \quad N_2(0,\cdot) = N_{20}(\cdot) \in L^1_+\big((0,+\infty), R\big),
\end{aligned} \tag{2}$$

where $N_1(t)$ is the density of the resource species at time t, and $N_2(t,a)$ is the density of the consumer species at time t with age a. $A(t) \doteq \int_0^{+\infty} \beta(a)N_2(t,a)\,\mathrm{d}a$ is the number of matured (reproducing) consumers with the age-dependent maturation function $\beta(a)$. r and d_1 are the intrinsic growth rate and logistic type limitation of resource species, respectively. d_2 is the death rate of the consumer species. $\alpha_{12}N_1(t)A(t)/(\gamma_2 + A(t))$ describes the positive effect on the growth of the resource species due to mutualistic interactions with the consumer species, where α_{12} denotes the saturation level of the functional response of the consumer species, and γ_2 denotes the half-saturation density of resource species. $\beta_1N_1(t)A(t)$ represents the consumption level of resource species by the matured consumer. $\alpha_{21}N_1(t)A(t)/(\gamma_1 + N_1(t))$ in the boundary condition denotes the new born individual of the consumer species N_2 depending on resource supplied by N_1, where α_{21} is the interaction strength, and γ_1 is the half-saturation constant.

Note that age is only a special kind of size and size of an individual has a strong influence upon dynamical processes like its feeding, growth and reproduction, which in turn affect the dynamics of the population as a whole [5]. Here sizes can be mass, length, diameter, surface area, volume, maturity, and so on. For some animals, the amount of food obtained by individuals is proportional to their surface area, and the cost of the metabolism is proportional to their volume (see [25]). As a result, modeling population dynamics, it is natural to assume that the vital rates, such as fertility, mortality, and growth rates of individuals, depend on their body size and time (see [3, 7–9, 15, 17–19, 32, 33]).

To the best of our knowledge, so far there is no investigation on the optimal control of size-structured population models of consumer–resource mutualisms. The purpose of this paper is to make some contribution in this direction. To build the model, we assume that $N_1(t)$ is the density of the resource at time t and $N_2(x,t)$ represents the density of the consumer at time t with size x. Let $Q = [0,l) \times [0,T]$. Here $l \in (0,+\infty)$ is the maximum size of any individual in the consumer species, and $T \in (0,+\infty)$ is a given time. Similar to [20], let

$$A(t) \doteq \int_0^l \beta(x)N_2(x,t)\,\mathrm{d}x \tag{3}$$

be the number of matured consumers with the size-dependent maturation function $\beta(x)$. In a similar way as to develop (2), we propose the following unidirectional consumer–

resource mutualisms system with size structure in the consumer to study the optimal harvest problem

$$
\begin{aligned}
&\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = N_1(t)\left[r - d_1N_1(t) + \frac{\alpha_{12}A(t)}{\gamma_2 + A(t)} - \beta_1 A(t) - u_1(t)\right], \quad t \in [0,T],\\
&\frac{\partial N_2(x,t)}{\partial t} + \frac{\partial(V(x,t)N_2(x,t))}{\partial x} = -d_2N_2(x,t) - u_2(x,t)N_2(x,t),\\
&\quad (t,x) \in Q,\\
&V(0,t)N_2(0,t) = \frac{\alpha_{21}N_1(t)A(t)}{\gamma_1 + N_1(t)}, \quad t \in [0,T],\\
&N_1(0) = N_{10} \geqslant 0, \quad N_2(\cdot,0) = N_{20}(\cdot) \in L^1_+\big((0,l),R_+\big).
\end{aligned}
\tag{4}
$$

All meanings of the parameters are exact to or similar as those for system (2) except the following. Here $V(x,t)$ is the growth rate of individual's size, that is, $\mathrm{d}x/\mathrm{d}t = V(x,t)$. The control variables $u_1(t)$ and $u_2(x,t)$ are the harvesting efforts for the resource species and the consumer species, respectively, which belong to

$$
\begin{aligned}
\mathcal{U} = \big\{&(u_1,u_2) \in L^\infty\big([0,T]\big) \times L^\infty(Q) \bigm| 0 \leqslant u_1(t) \leqslant H_1 \text{ a.e. } t \in [0,T],\\
&0 \leqslant u_2(x,t) \leqslant H_2 \text{ a.e. } (x,t) \in Q\big\}.
\end{aligned}
$$

Here H_1 and H_2 are positive constants. Let $(N_1(t), N_2(x,t))$ be solution of (4) corresponding to $(u_1,u_2) \in \mathcal{U}$. As done in [16], in this paper, we discuss the optimization problem as follows:

$$
\max_{(u_1,u_2)\in\mathcal{U}} J(u_1,u_2), \tag{5}
$$

where

$$
\begin{aligned}
J(u_1,u_2) = &\int_0^T \omega_1(t)u_1(t)N_1(t)\,\mathrm{d}t + \int_0^T\int_0^l \omega_2(x,t)u_2(x,t)N_2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&- \frac{1}{2}\int_0^T c_1\big[u_1(t)\big]^2\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_0^l c_2\big[u_2(x,t)\big]^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Here $\omega_1(t)$ and $\omega_2(x,t)$ are, respectively, the economic values of the individual of the resource and the consumer at time t; $c_i > 0$ $(i = 1,2)$ is the weight factor of the costs for implementing the controls. Thus, the optimization problem represents the total net economic benefit yielded from harvesting the resource and the consumer during a time of T.

Denote $R_+ \doteq [0,\infty)$, $L^1_+ \doteq L^1(0,l;R_+)$ and $L^\infty_+ \doteq L^\infty(0,l;R_+)$. We make the following assumptions throughout this paper.

(A1) $V : [0,l) \times [0,T] \to R_+$ is a bounded continuous function; V is of C^1-class with respect to $x \in [0,l)$ for each $t \in [0,T]$; $\lim_{x\uparrow l} V(x,t) = 0$ uniformly for $t \in [0, T]$. Further, there is a Lipschitz constant L_V such that $|V(x_1, t) - V(x_2, t)| \leqslant L_V|x_1 - x_2|$ for all $x_1, x_2 \in [0, l)$ and $t \in [0, T]$.

(A2) $\beta \in L^\infty_+$ with $\|\beta\|_\infty \leqslant \bar{\beta}$, and $\bar{\beta}$ is a positive constant. Moreover, for any $(x, t) \in [0, l) \times [0, T]$, we assume that $V_x(x, t) \geqslant -d_2$.

(A3) $N_{20} \in L^1_+$, and there is a positive constant $\bar{N}_2$ such that $\int_0^l N_{20}(x)\,\mathrm{d}x \leqslant \bar{N}_2$.

2 Well-posedness of the state system

This section is devoted to the well-posedness of system (4). As in [16], we first introduce the definition of characteristic curve.

Definition 1. (See [16, Def. 1].) The unique solution $x = \varphi(t; t_0, x_0)$ of the initial-valued problem $\dot{x}(t) = V(x, t)$ with $x(t_0) = x_0$ is said to be a characteristic curve. Let $z(t) = \varphi(t; 0, 0)$ be the characteristic curve through $(0, 0)$ in x, t-plane.

For any point (x, t) in the first quadrant of x, t-plane such that $x \leqslant z(t)$, that is, $\varphi(t; t, x) \leqslant z(t)$, define initial time $\tau \doteq \tau(x, t)$. It is clear that $\varphi(t; \tau, 0) = x$ if and only if $\varphi(\tau; t, x) = 0$. Obviously, $\tau = \varphi^{-1}(0; t, x)$. Using characteristic curve technique as in [1], the solution of system (4) can be defined as follows.

Definition 2. A pair of functions $(N_1(t), N_2(x, t))$ is said to be a solution of system (4) if it satisfies

$$N_1(t) = N_{10} \exp\left\{ \int_0^t \left[r - d_1 N_1(s) + \frac{\alpha_{12} A(s)}{\gamma_2 + A(s)} - \beta_1 A(s) - u_1(s) \right] \mathrm{d}s \right\}, \tag{6}$$

$$N_2(x, t) = \begin{cases} \frac{F_{N_1}(\tau, N_2(\cdot, \tau))}{V(0, \tau)} + \int_\tau^t G_V(s, N_2(\cdot, s))(\varphi(s; t, x))\,\mathrm{d}s, & x \leqslant z(t), \\ N_{20}(\varphi(0; t, x)) + \int_0^t G_V(s, N_2(\cdot, s))(\varphi(s; t, x))\,\mathrm{d}s, & x > z(t), \end{cases} \tag{7}$$

where $A(s) = \int_0^l \beta(x) N_2(x, s)\,\mathrm{d}x$, F_{N_1} and G_V are given by

$$F_{N_1}(t, \phi) = \frac{\alpha_{21} N_1(t) \int_0^l \beta(x)\phi(x)\,\mathrm{d}x}{\gamma_1 + N_1(t)},$$

$$G_V(t, \phi)(x) = -d_2\phi(x) - V_x(x, t)\phi(x) - u_2(x, t)\phi(x)$$

for $t \in [0, T]$ and $\phi \in L^1$.

To discuss the well-posedness of (4), let $\mathbf{X} = L^\infty(0, T) \times L^\infty(0, T; L^1(0, l))$ and define a new norm in $\mathbf{X}$ by

$$\left\|(N_1, N_2)\right\|_* = \operatorname*{ess\,sup}_{t \in (0,T)} \left\{ \mathrm{e}^{-\lambda t} \left[\left|N_1(t)\right| + \int_0^l \left|N_2(x, t)\right| \mathrm{d}x \right] \right\}$$

for some $\lambda > 0$. Obviously, $\|\cdot\|_*$ is equivalent to the usual norm on $\mathbf{X}$. Thus, $\mathbf{X}$ is a Banach space with the norm $\|\cdot\|_*$. Further, denote

$$M = \max\bigl\{N_{10}\exp\bigl\{(r+\alpha_{12})T\bigr\},\ \bar{N}_{20}\bigl[1+\exp\bigl\{(L_V+\alpha_{21}\bar{\beta})T\bigr\}\bigr]\bigr\}$$

and define the space

$$\mathcal{X} = \Biggl\{(N_1,N_2)\in\mathbf{X}\ \Bigg|\ 0\leqslant N_1(t)\leqslant M \text{ a.e. } t\in(0,T),\ N_2(x,t)\geqslant 0 \\ \text{and } \int_0^l N_2(x,t)\,\mathrm{d}x \leqslant M \text{ a.e. } (x,t)\in Q\Biggr\}.$$

It is clear that $\mathcal{X}$ is a nonempty closed subset in $\mathbf{X}$. Define $\mathcal{A}:\mathcal{X}\to\mathbf{X}$ by

$$\mathcal{A}(N_1,N_2) = \bigl(\mathcal{A}_1(N_1,N_2),\mathcal{A}_2(N_1,N_2)\bigr),$$

where $\mathcal{A}_1(N_1,N_2)$ and $\mathcal{A}_2(N_1,N_2)$ are defined by the right-hand sides of (6) and (7), respectively. Clearly, if $(N_1(t),N_2(x,t))$ is a fixed point of $\mathcal{A}$, then it must be a solution of (4) and vice versa.

Next, we show that $\mathcal{A}$ is a contraction mapping on $\mathcal{X}$. To do this, we first introduce the following lemmas.

Lemma 1. (*See* [13, *Lemma* 3.3].) *For any $t\in[0,T]$, let $\tau_t(x)\doteq\tau(x,t)$. Then $\tau_t:[0,z(t)]\to[0,t]$ is continuous, decreasing and onto, and hence τ_t has the inverse $\tau_t^{-1}(\cdot)$, which is continuous from $[0,t]$ onto $[0,z(t)]$.*

Lemma 2. (*See* [13, *Lemma* 3.4].) *Let $x=\varphi(t;\tau,\eta)$. Then x is differentiable with respect to τ, and*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -V(\eta,\tau)\exp\Biggl(\int_\tau^t V_x(\varphi(\sigma;\tau,\eta),\sigma)\,\mathrm{d}\sigma\Biggr); \tag{8}$$

and x is differentiable with respect to η, and

$$\frac{\mathrm{d}x}{\mathrm{d}\eta} = \exp\Biggl(\int_\tau^t V_x(\varphi(\sigma;\tau,\eta),\sigma)\,\mathrm{d}\sigma\Biggr). \tag{9}$$

Theorem 1. *Assume that* (A1)–(A3) *hold. Then, for any $(u_1,u_2)\in\mathcal{U}$, system* (4) *has a unique solution $(N_1,N_2)\in\mathcal{X}$.*

Proof. First, we show that the mapping $\mathcal{A}$ maps $\mathcal{X}$ into $\mathcal{X}$. For any $(N_1,N_2)\in\mathcal{X}$, let $b(t)=V(0,t)N_2(0,t)$. It is clear that $b(t)=\alpha_{21}N_1(t)A(t)/(\gamma_1+N_1(t)) = F_{N_1}(t,N_2(\cdot,t))$. From (A2) it follows that $G_V(s,N_2(\cdot,s))\leqslant 0$. Thus, for $0<t<z^{-1}(l)$,

we have

$$
\begin{aligned}
b(t) &= \frac{\alpha_{21} N_1(t) A(t)}{\gamma_1 + N_1(t)} \leqslant \alpha_{21} A(t) = \alpha_{21} \int_0^l \beta(x) N_2(x,t)\,\mathrm{d}x \\
&\leqslant \alpha_{21}\bar{\beta} \int_0^{z(t)} \frac{F_{N_1}(\tau, N_2(\cdot,\tau))}{V(0,\tau)}\,\mathrm{d}x + \alpha_{21}\bar{\beta} \int_{z(t)}^{l} N_{20}\bigl(\varphi(0;t,x)\bigr)\,\mathrm{d}x \\
&\leqslant \alpha_{21}\bar{\beta}\bar{N}_{20} + \alpha_{21}\bar{\beta} \int_0^{z(t)} \frac{b(\tau)}{V(0,\tau)}\,\mathrm{d}x \doteq \alpha_{21}\bar{\beta}\bar{N}_{20} + \alpha_{21}\bar{\beta} I.
\end{aligned} \tag{10}
$$

Let $s = \tau = \varphi^{-1}(0;t,x)$. By Definition 1 $s = t$ when $x = 0$, while $s = 0$ when $x = z(t)$. Moreover, from $s = \varphi^{-1}(0;t,x)$ it follows that $x = \varphi(t;s,0)$. Then from Lemma 2, $\mathrm{d}x/\mathrm{d}s = -V(0,\tau)\exp(\int_\tau^t V_x(\varphi(\sigma;s,0),\sigma)\,\mathrm{d}\sigma)$. Thus, we have

$$
I = -\int_t^0 b(s) \exp\Biggl(\int_s^t V_x\bigl(\varphi(\sigma;s,0),\sigma\bigr)\,\mathrm{d}\sigma\Biggr)\,\mathrm{d}s \leqslant \int_0^t \mathrm{e}^{L_V(t-s)} b(s)\,\mathrm{d}s. \tag{11}
$$

This, together with (10), yields $\mathrm{e}^{-L_V t} b(t) \leqslant \alpha_{21}\bar{\beta}\bar{N}_{20} + \alpha_{21}\bar{\beta}\int_0^t \mathrm{e}^{-L_V s} b(s)\,\mathrm{d}s$. From Gronwall's inequality it follows that

$$
b(t) \leqslant \alpha_{21}\bar{\beta}\bar{N}_{20} \exp\bigl\{(\alpha_{21}\bar{\beta} + L_V)t\bigr\}. \tag{12}
$$

For $z^{-1}(l) < t < T$, inequality (12) still holds, and the proof is more simple.

For any $(N_1, N_2) \in \mathcal{X}$, we consider $\mathcal{A}(N_1,N_2) = (\mathcal{A}_1(N_1,N_2), \mathcal{A}_2(N_1,N_2))$. From (6) it is easy to see that

$$
\bigl|\mathcal{A}_1(N_1,N_2)\bigr|(t) \leqslant N_{10} \exp\bigl\{(r + \alpha_{12})T\bigr\}.
$$

Further, from (7), (11) and (12), for $0 < t < z^{-1}(l)$, we can see that

$$
\begin{aligned}
\int_0^l \bigl|\mathcal{A}_2(N_1,N_2)\bigr|(x,t)\,\mathrm{d}x &\leqslant \int_0^{z(t)} \frac{F_{N_1}(\tau, N_2(\cdot,\tau))}{V(0,\tau)}\,\mathrm{d}x + \int_{z(t)}^{l} N_{20}\bigl(\varphi\bigl(0;t,x\bigr)\bigr)\,\mathrm{d}x \\
&\leqslant \bar{N}_{20} + \int_0^t \mathrm{e}^{L_V(t-s)} b(s)\,\mathrm{d}s \\
&\leqslant \bar{N}_{20} + \alpha_{21}\bar{\beta}\bar{N}_{20}\mathrm{e}^{L_V t} \int_0^t \mathrm{e}^{\alpha_{21}\bar{\beta} s}\,\mathrm{d}s \\
&\leqslant \bar{N}_{20}\bigl[1 + \exp\bigl\{(L_V + \alpha_{21}\bar{\beta})T\bigr\}\bigr].
\end{aligned}
$$

For $z^{-1}(l) < t < T$, the above inequality still holds, and the proof is more simple. Thus, $\mathcal{A}$ maps $\mathcal{X}$ into itself.

Next, we discuss the compressibility of $\mathcal{A}$. For any (N_1, N_2) and $(N_1', N_2') \in \mathcal{X}$, from (6) it follows that

$$\begin{aligned}
&\big|\mathcal{A}_1(N_1, N_2) - \mathcal{A}_1(N_1', N_2')\big|(t) \\
&\quad = N_{10}\left|\mathrm{e}^{(r+\alpha_{12})t} \exp\left\{\int_0^t \left[-d_1 N_1(s) + \frac{\alpha_{12} A(s)}{\gamma_2 + A(s)} - \alpha_{12} - \beta_1 A(s) - u_1(s)\right] \mathrm{d}s\right\}\right. \\
&\qquad \left. - \mathrm{e}^{(r+\alpha_{12})t} \exp\left\{\int_0^t \left[-d_1 N_1'(s) + \frac{\alpha_{12} A'(s)}{\gamma_2 + A'(s)} - \alpha_{12} - \beta_1 A'(s) - u_1(s)\right] \mathrm{d}s\right\}\right| \\
&\quad \leqslant N_{10}\mathrm{e}^{(r+\alpha_{12})T} \\
&\qquad \times \int_0^t \left[d_1\big|N_1(s) - N_1'(s)\big| + \left(\frac{\alpha_{12}}{\gamma_2} + \beta_1\right)\int_0^l \beta(x)|N_2 - N_2'|(x, s)\,\mathrm{d}x\right] \mathrm{d}s \\
&\quad \leqslant M_1 \int_0^t \left[\big|N_1(s) - N_1'(s)\big| + \int_0^l \big|N_2(x, s) - N_2'(x, s)\big|\,\mathrm{d}x\right] \mathrm{d}s,
\end{aligned} \tag{13}$$

where $M_1 = N_{10}\mathrm{e}^{(r+\alpha_{12})T} \max\{d_1, (\alpha_{12}/\gamma_2 + \beta_1)\bar{\beta}\}$. For $0 < t < z^{-1}(l)$, we have

$$\begin{aligned}
&\int_0^l \big|\mathcal{A}_2(N_1, N_2) - \mathcal{A}_2(N_1', N_2')\big|(x, t)\,\mathrm{d}x \\
&\quad \leqslant \int_0^{z(t)} \frac{|F_{N_1}(\tau, N_2(\cdot, \tau)) - F_{N_1'}(\tau, N_2'(\cdot, \tau))|}{V(0, \tau)}\,\mathrm{d}x \\
&\qquad + \int_0^{z(t)} \int_\tau^t \big|G_V\big(s, N_2(\cdot, s)\big)\big(\varphi(s; t, x)\big) - G_V\big(s, N_2'(\cdot, s)\big)\big(\varphi(s; t, x)\big)\big|\,\mathrm{d}s\,\mathrm{d}x \\
&\qquad + \int_{z(t)}^l \int_0^t \big|G_V\big(s, N_2(\cdot, s)\big)\big(\varphi(s; t, x)\big) - G_V\big(s, N_2'(\cdot, s)\big)\big(\varphi(s; t, x)\big)\big|\,\mathrm{d}s\,\mathrm{d}x \\
&\quad \doteq I_1 + I_2 + I_3.
\end{aligned} \tag{14}$$

For I_1, let $s = \tau = \varphi^{-1}(0; t, x)$. By Definition 1 $s = t$ when $x = 0$, while $s = 0$ when $x = z(t)$. From $s = \varphi^{-1}(0; t, x)$ it follows that $x = \varphi(t; s, 0)$. Then from Lemma 2 it follows that $\mathrm{d}x/\mathrm{d}s = -V(0,\tau)\exp(\int_\tau^t V_x(\varphi(\sigma;s,0),\sigma)\,\mathrm{d}\sigma)$. Thus,

$$I_1 \leqslant \mathrm{e}^{L_V T}\int_0^t \bigl|F_{N_1}\bigl(\tau, N_2(\cdot,s)\bigr) - F_{N_1'}\bigl(\tau, N_2'(\cdot,s)\bigr)\bigr|\,\mathrm{d}s.$$

Further, we can obtain

$$\begin{aligned}
I_1 &\leqslant \mathrm{e}^{L_V T}\int_0^t \left|\frac{\alpha_{21}N_1(s)A(s)}{\gamma_1+N_1(s)} - \frac{\alpha_{21}N_1'(s)A'(s)}{\gamma_1+N_1'(s)}\right|\mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T}\int_0^t \frac{\alpha_{21}\gamma_1|N_1(s)A(s)-N_1'(s)A'(s)|}{(\gamma_1+N_1(s))(\gamma_1+N_1'(s))}\,\mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T}\int_0^t \left[\alpha_{21}\bigl|A(s)-A'(s)\bigr| + \frac{\alpha_{21}}{\gamma_1}A'(s)\bigl|N_1(s)-N_1'(s)\bigr|\right]\mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T}\alpha_{21}\bar{\beta}\left[\int_0^t\int_0^l |N_2-N_2'|(x,s)\,\mathrm{d}x\,\mathrm{d}s + \frac{M}{\gamma_1}\int_0^t |N_1-N_1'|(s)\,\mathrm{d}s\right]. \qquad (15)
\end{aligned}$$

For I_2+I_3, as in [12], using Fubini's theorem and Lemma 1, we have

$$\begin{aligned}
I_2+I_3 = {}&\int_0^t\int_{\tau_t^{-1}(s)}^{z(t)} \bigl|G_V\bigl(s,N_1(\cdot,s)\bigr)\bigl(\varphi(s;t,x)\bigr) - G_V\bigl(s,N_2'(\cdot,s)\bigr)\bigl(\varphi(s;t,x)\bigr)\bigr|\,\mathrm{d}x\,\mathrm{d}s \\
&+\int_0^t\int_{z(t)}^{l} \bigl|G_V\bigl(s,N_1(\cdot,s)\bigr)\bigl(\varphi(s;t,x)\bigr) - G_V\bigl(s,N_2'(\cdot,s)\bigr)\bigl(\varphi(s;t,x)\bigr)\bigr|\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

From Lemma 1 it follows that if $x=\tau_t^{-1}(s)$, then $s=\tau_t(x)=\tau(x,t)$. Further, let $\eta=\varphi(s;t,x)$. By Definition 1 $\eta=\varphi(\tau;t,x)=0$ when $x=\tau_t^{-1}(s)$ (i.e. $s=\tau(x,t)$), while $\eta=\varphi(s;t,l)<l$ when $x=l$. Moreover, it follows from $\eta=\varphi(s;t,x)$ that $x=\varphi(t;s,\eta)$. Thus, from Lemma 2 we have $\mathrm{d}x=\exp(\int_s^t V_x(\varphi(\sigma;s,\eta),\sigma)\,\mathrm{d}\sigma)\mathrm{d}\eta$. Thus, we have

$$\begin{aligned}
I_2+I_3 &\leqslant \mathrm{e}^{L_V T}\int_0^t\int_0^l \bigl|-\bigl(d_2+V_x(\eta,s)+u_2(\eta,s)\bigr)\bigl(N_2(\eta,s)-N_2'(\eta,s)\bigr)\bigr|\,\mathrm{d}\eta\,\mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T}(d_2+L_V+H_2)\int_0^t\int_0^l \bigl|N_2(\eta,s)-N_2'(\eta,s)\bigr|\,\mathrm{d}\eta\,\mathrm{d}s. \qquad (16)
\end{aligned}$$

Hence, from (14)–(16) we obtain

$$
\begin{aligned}
&\int_0^l \bigl|\mathcal{A}_2(N_1,N_2)-\mathcal{A}_2(N_1',N_2')\bigr|(x,t)\,\mathrm{d}x\\
&\quad\leqslant \mathrm{e}^{L_V T}(\alpha_{21}\bar\beta+d_2+L_V+H_2)\int_0^t\int_0^l \bigl|N_2(x,s)-N_2'(x,s)\bigr|\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+\mathrm{e}^{L_V T}\frac{\alpha_{21}\bar\beta M}{\gamma_1}\int_0^t \bigl|N_1(s)-N_1'(s)\bigr|\,\mathrm{d}s\\
&\quad\leqslant M_2\int_0^t\left[\bigl|N_1(s)-N_1'(s)\bigr|+\int_0^l \bigl|N_2(x,s)-N_2'(x,s)\bigr|\,\mathrm{d}x\right]\mathrm{d}s,
\end{aligned}
\tag{17}
$$

where $M_2=\mathrm{e}^{L_V T}\max\{\alpha_{21}\bar\beta+d_2+L_V+H_2,\ \alpha_{21}\bar\beta M/\gamma_1\}$. For $z^{-1}(l)<t<T$, the above inequality still holds, and the proof is more simple.

It follows from (13) and (17) that

$$
\begin{aligned}
&\bigl\|\mathcal{A}(N_1,N_2)-\mathcal{A}(N_1',N_2')\bigr\|_*\\
&\quad=\bigl\|\bigl(\mathcal{A}_1(N_1,N_2)-\mathcal{A}_1(N_1',N_2'),\mathcal{A}_2(N_1,\ N_2)-\mathcal{A}_2(N_1',N_2')\bigr)\bigr\|_*\\
&\quad\leqslant M_3\operatorname*{ess\,sup}_{t\in(0,T)}\left\{\mathrm{e}^{-\lambda t}\left[\int_0^t\left(|N_1-N_1'|(s)+\int_0^l|N_2-N_2'|(x,s)\,\mathrm{d}x\right)\mathrm{d}s\right]\right\}\\
&\quad\leqslant \frac{M_3}{\lambda}\bigl\|(N_1-N_1',N_2-N_2')\bigr\|_*.
\end{aligned}
$$

Choose λ such that $\lambda>M_3=M_1+M_2$. Thus, $\mathcal{A}$ is a contraction mapping on the Banach space $(\mathcal{X},\|\cdot\|_*)$. Hence, $\mathcal{A}$ owns a unique fixed point, which is the solution of (4). $\square$

To conclude this section, we will discuss the continuous dependence of solutions on the control variable. Let $L_1^\infty=L^\infty(0,T;L^1(0,l))$.

Theorem 2. *For any (u_1,u_2), $(u_1',u_2')\in\mathcal{U}$, let (N_1,N_2) and (N_1',N_2') be solutions of (4) corresponding to (u_1,u_2) and (u_1',u_2'), respectively. If T is small enough, then there are positive constants K_1 and K_2 such that*

$$
\begin{aligned}
&\|N_1-N_1'\|_{L^\infty(0,T)}+\|N_2-N_2'\|_{L_1^\infty}\\
&\quad\leqslant K_1T\bigl[\|u_1-u_1'\|_{L^\infty(0,T)}+\|u_2-u_2'\|_{L_1^\infty}\bigr]
\end{aligned}
$$

and

$$
\begin{aligned}
&\|N_1-N_1'\|_{L^1(0,T)}+\|N_2-N_2'\|_{L^1(Q)}\\
&\quad\leqslant K_2T\bigl[\|\alpha_i-\alpha_i'\|_{L^1(0,T)}+\|\alpha_1-\alpha_1'\|_{L^1(Q)}\bigr].
\end{aligned}
$$

Proof. We only prove the first estimate as the proof for the second one is similar. From (6) it follows that

$$
\begin{aligned}
&\left|N_1(t) - N_1'(t)\right| \\
&\quad \leqslant N_{10}\mathrm{e}^{(r+\alpha_{12})T} \int_0^t \left[d_1\left|N_1(s) - N_1'(s)\right| + \frac{\alpha_{12}\gamma_2 \int_0^l \beta(x)|N_2(x,s) - N_2'(x,s)|\,\mathrm{d}x}{(\gamma_2 + A(s))(\gamma_2 + A'(s))} \right. \\
&\qquad \left. + \beta_1 \int_0^l \beta(x)\left|N_2(x,s) - N_2'(x,s)\right|\mathrm{d}x + \left|u_1(s) - u_1'(s)\right| \right] \mathrm{d}s \\
&\quad \leqslant N_{10}\mathrm{e}^{(r+\alpha_{12})T} \\
&\qquad \times \int_0^t \left[d_1|N_1 - N_1'|(s) + \left(\frac{\alpha_{12}}{\gamma_2} + \beta_1\right) \int_0^l \beta(x)|N_2 - N_2'|(x,s)\,\mathrm{d}x \right] \mathrm{d}s \\
&\qquad + N_{10}\mathrm{e}^{(r+\alpha_{12})T} \int_0^t \left|u_1(s) - u_1'(s)\right| \mathrm{d}s \\
&\quad \leqslant M_3 \left[\int_0^t \left|N_1(s) - N_1'(s)\right| \mathrm{d}s + \int_0^t \int_0^l |N_2 - N_2'|(x,s)\,\mathrm{d}x\,\mathrm{d}s \right. \\
&\qquad \left. + \int_0^t \left|u_1(s) - u_1'(s)\right| \mathrm{d}s \right],
\end{aligned}
$$

where $M_3 = \max\{M_1, N_{10}\mathrm{e}^{(r+\alpha_{12})T}\}$. Further, from (7) we have

$$
\begin{aligned}
&\int_0^l \left|N_2(x,t) - N_2'(x,t)\right| \mathrm{d}x \\
&\quad \leqslant \int_0^{z(t)} \frac{|F_{N_1}(\tau, N_2(\cdot,\tau)) - F_{N_1'}(\tau, N_2'(\cdot,\tau))|}{V(0,\tau)}\,\mathrm{d}x \\
&\qquad + \int_0^{z(t)} \int_\tau^t \left|G_V\big(s, N_2(\cdot,s)\big)\big(\varphi(s;t,x)\big) - G_V\big(s, N_2'(\cdot,s)\big)\big(\varphi(s;t,x)\big)\right| \mathrm{d}s\,\mathrm{d}x \\
&\qquad + \int_{z(t)}^l \int_0^t \left|G_V\big(s, N_2(\cdot,s)\big)\big(\varphi(s;t,x)\big) - G_V\big(s, N_2'(\cdot,s)\big)\big(\varphi(s;t,x)\big)\right| \mathrm{d}s\,\mathrm{d}x \\
&\quad \doteq J_1 + J_2 + J_3.
\end{aligned}
$$

By a similar discussion as that in I_1 (changing variable $s = \varphi^{-1}(0; t, x)$) we have

$$
\begin{aligned}
J_1 &\leqslant \mathrm{e}^{L_V T} \int_0^t \left| F_{N_1}\big(\tau, N_2(\cdot, s)\big) - F_{N_1'}\big(\tau, N_2'(\cdot, s)\big) \right| \mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T} \int_0^t \frac{\alpha_{21}\gamma_1 |N_1(s) \int_0^l \beta(x) N_2(x, s)\, \mathrm{d}x - N_1'(s) \int_0^l \beta(x) N_2'(x, s)\, \mathrm{d}x|}{(\gamma_1 + N_1(s))(\gamma_1 + N_1'(s))} \mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T} \alpha_{21} \bar{\beta} \int_0^t \left[\int_0^l |N_2 - N_2'|(x, s)\, \mathrm{d}x + \frac{1}{\gamma_1} \int_0^l N_2(x, s)\, \mathrm{d}x\, |N_1 - N_1'|(s) \right] \mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T} \alpha_{21} \bar{\beta} \int_0^t \int_0^l |N_2 - N_2'|(x, s)\, \mathrm{d}x\, \mathrm{d}s + \mathrm{e}^{L_V T} \frac{\alpha_{21} \bar{\beta} M}{\gamma_1} \int_0^t |N_1 - N_1'|(s)\, \mathrm{d}s.
\end{aligned}
$$

By a similar discussion as that in $I_2 + I_3$ (changing variable $\eta = \varphi(s; t, x)$) we also have

$$
\begin{aligned}
J_2 + J_3 &\leqslant \mathrm{e}^{L_V T} \int_0^t \int_0^l \left| G_V\big(s, N_2(\cdot, s)\big)(\eta) - G_V\big(s, N_2'(\cdot, s)\big)(\eta) \right| \mathrm{d}\eta\, \mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T} \int_0^t \int_0^l \left| -\big(d_2 + V_x(\eta, s)\big)\big(N_2(\eta, s) - N_2'(\eta, s)\big) \right| \mathrm{d}\eta\, \mathrm{d}s \\
&\quad + \mathrm{e}^{L_V T} \int_0^t \int_0^l \left| u_2(\eta, s) N_2(\eta, s) - u_2'(\eta, s) N_2'(\eta, s) \right| \mathrm{d}\eta\, \mathrm{d}s \\
&\leqslant \mathrm{e}^{L_V T} (d_2 + L_V + H_2) \int_0^t \int_0^l \left| N_2(\eta, s) - N_2'(\eta, s) \right| \mathrm{d}\eta\, \mathrm{d}s \\
&\quad + \mathrm{e}^{L_V T} M \int_0^t \int_0^l \left| u_2(\eta, s) - u_2'(\eta, s) \right| \mathrm{d}\eta\, \mathrm{d}s.
\end{aligned}
$$

Hence, we can obtain

$$
\begin{aligned}
&\int_0^l \left| N_2(x, t) - N_2'(x, t) \right| \mathrm{d}x \\
&\quad \leqslant \mathrm{e}^{L_V T} \frac{\alpha_{21} \bar{\beta} M}{\gamma_1} \int_0^t \left| N_1(s) - N_1'(s) \right| \mathrm{d}s + \mathrm{e}^{L_V T} M \int_0^t \int_0^l \left| u_2(x, s) - u_2'(x, s) \right| \mathrm{d}x\, \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathrm{e}^{L_V T}(\alpha_{21}\bar{\beta} d_2 + L_V + H_2) \int_0^t \int_0^l \big|N_2(x,s) - N_2'(x,s)\big| \,\mathrm{d}x\,\mathrm{d}s \\
\leqslant{}& M_4 \Bigg[\int_0^t \big|N_1(s) - N_1'(s)\big| \,\mathrm{d}s + \int_0^t \int_0^l |N_2 - N_2'|(x,s) \,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \big|u_1(s) - u_1'(s)\big| \,\mathrm{d}s \Bigg],
\end{aligned}
$$

where $M_4 = \max\{M_2, M\}$. The result follows immediately from above analysis. □

3 The adjoint system

In this section, we will derive adjoint system of (4). Here and below we denote by $\mathcal{T}_{\mathcal{U}}(\alpha)$ and $\mathcal{N}_{\mathcal{U}}(\alpha)$ the tangent cone and normal cone of $\mathcal{U}$ at α, respectively.

Lemma 3. (*See* [4, *Prop.* 5.3].) *Suppose that* $\vartheta(x,t) \in L^\infty(Q)$ *satisfies*

$$
\int_0^T \int_0^l \big[\vartheta(x,t) v(x,t) + \rho\big|v(x,t)\big|\big] \,\mathrm{d}x\,\mathrm{d}t \geqslant 0 \quad \textit{for any } v \in \mathcal{T}_{\mathcal{U}}(\alpha).
$$

Then there is $\theta \in L^\infty(Q)$ *such that* $\|\theta\|_\infty \leqslant 1$ *and* $\rho\theta - \vartheta \in \mathcal{N}_{\mathcal{U}}(\alpha)$.

Lemma 4. *Let* (N_1, N_2) *be solution of* (4) *corresponding to* $(u_1, u_2) \in \mathcal{U}$. *For each* $(v_1, v_2) \in T_{\mathcal{U}}(u_1, u_2)$ *such that* $(u_1 + \varepsilon v_1,\, u_2 + \varepsilon v_2) \in \mathcal{U}$ *for sufficiently small* $\varepsilon > 0$, *we have*

$$
\frac{1}{\varepsilon}[N_1^\varepsilon - N_1] \to z_1, \qquad \frac{1}{\varepsilon}[N_2^\varepsilon - N_2] \to z_2
$$

as $\varepsilon \to 0$, *where* $(N_1^\varepsilon, N_2^\varepsilon)$ *is the solution of* (4) *corresponding to* $(u_1 + \varepsilon v_1,\, u_2 + \varepsilon v_2)$, *and* (z_1, z_2) *is the solution of the following system:*

$$
\begin{aligned}
&\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = z_1(t)\bigg[r - 2d_1 N_1(t) + \frac{\alpha_{12}\gamma_2 A(t)}{\gamma_2 + A(t)} - \beta_1 A(t) - u_1(t)\bigg] \\
&\qquad\qquad + N_1(t)\bigg[\frac{\alpha_{12}\gamma_2 B(t)}{(\gamma_2 + A(t))^2} - \beta_1 B(t) - v_1(t)\bigg], \\
&\frac{\partial z_2}{\partial t} + \frac{\partial [V(x,t) z_2]}{\partial x} = -\big[d_2 + u_2(x,t)\big] z_2(x,t) - v_2(x,t) N_2(x,t), \\
&V(0,t) z_2(0,t) = \frac{\alpha_{21} N_1(t) B(t)}{\gamma_1 + N_1(t)} + \frac{\alpha_{21}\gamma_1 z_1(t) A(t)}{(\gamma_1 + N_1(t))^2}, \\
&z_1(0) = 0, \quad z_2(\cdot, 0) = 0, \qquad B(t) = \int_0^l \beta(x) z_2(x,t) \,\mathrm{d}x.
\end{aligned}
\tag{18}
$$

Proof. The existence and uniqueness of the solution to (18) can be established in a similar way as in Theorem 1. By [1, Lemma 3.1.3], $\lim_{\varepsilon\to 0}[N_1^\varepsilon - N_1]/\varepsilon$ and $\lim_{\varepsilon\to 0}[N_2^\varepsilon - N_2]/\varepsilon$ make sense. Note that $(N_1^\varepsilon, N_2^\varepsilon)$ and (N_1, N_2) are solutions of system (4) corresponding to $(u_1 + \varepsilon v_1,\, u_2 + \varepsilon v_2)$ and (u_1, u_2), respectively. For simplicity, we denote $A^\varepsilon(t) = \int_0^l \beta(x) N_2^\varepsilon(x,t)\,\mathrm{d}x$. It follows from Theorem 2 that

$$\begin{aligned}
&\frac{1}{\varepsilon}\bigl[A^\varepsilon(t) - A(t)\bigr]\\
&\quad= \int_0^l \beta(x)\frac{1}{\varepsilon}\bigl[N_2^\varepsilon(x,t) - N_2(x,t)\bigr]\,\mathrm{d}x \to \int_0^l \beta(x) z_2(x,t)\,\mathrm{d}x \doteq B(t)
\end{aligned}$$

as $\varepsilon \to 0$. Thus, $([N_1^\varepsilon - N_1]/\varepsilon,\, [N_1^\varepsilon - N_1]/\varepsilon)$ must be solution of

$$\begin{aligned}
&\frac{\mathrm{d}[\frac{1}{\varepsilon}(N_1^\varepsilon - N_1)]}{\mathrm{d}t}\\
&\quad= \frac{1}{\varepsilon}\bigl[rN_1^\varepsilon(t) - rN_1(t)\bigr] - \frac{1}{\varepsilon}\bigl[d_1\bigl(N_1^\varepsilon(t)\bigr)^2 - d_1\bigl(N_1(t)\bigr)^2\bigr]\\
&\qquad+ \frac{1}{\varepsilon}\biggl[\frac{\alpha_{12}N_1^\varepsilon(t)A^\varepsilon(t)}{\gamma_2 + A^\varepsilon(t)} - \frac{\alpha_{12}N_1(t)A(t)}{\gamma_2 + A(t)}\biggr] - \frac{1}{\varepsilon}\bigl[\beta_1 N_1^\varepsilon(t)A^\varepsilon(t) - \beta_1 N_1(t)A(t)\bigr]\\
&\qquad- \frac{1}{\varepsilon}\bigl[u_1(t)\bigl(N_1^\varepsilon(t) - N_1(t)\bigr)\bigr] - v_1(t)N_1^\varepsilon(t),\\
&\frac{\partial[\frac{1}{\varepsilon}(N_2^\varepsilon - N_2)]}{\partial t} + \frac{\partial[V\frac{1}{\varepsilon}(N_2^\varepsilon - N_2)]}{\partial x}\\
&\quad= -\frac{1}{\varepsilon}(d_2 + u_2)\bigl[N_2^\varepsilon(x,t) - N_2(x,t)\bigr] - v_2(x,t)N_2^\varepsilon(x,t),\\
&V(0,t)\frac{1}{\varepsilon}\bigl[N_2^\varepsilon(0,t) - N_2(0,t)\bigr] = \frac{1}{\varepsilon}\biggl[\frac{\alpha_{21}N_1^\varepsilon(t)A^\varepsilon(t)}{\gamma_1 + N_1^\varepsilon(t)} - \frac{\alpha_{21}N_1(t)A(t)}{\gamma_1 + N_1(t)}\biggr],\\
&\frac{1}{\varepsilon}\bigl[N_1^\varepsilon(0) - N_1(0)\bigr] = 0, \qquad \frac{1}{\varepsilon}\bigl[N_2^\varepsilon(\cdot,0) - N_2(\cdot,0)\bigr] = 0.
\end{aligned} \tag{19}$$

It follows from Theorem 2 that

$$\begin{gathered}
\frac{1}{\varepsilon}\bigl[rN_1^\varepsilon(t) - rN_1(t)\bigr] \to rz_1(t), \qquad \frac{1}{\varepsilon}\bigl[u_1(t)\bigl(N_1^\varepsilon(t) - N_1(t)\bigr)\bigr] \to u_1(t)z_1(t),\\
\frac{1}{\varepsilon}\bigl[d_1\bigl(N_1^\varepsilon(t)\bigr)^2 - d_1(N_1(t))^2\bigr] = d_1\bigl[N_1^\varepsilon + N_1\bigr]\frac{1}{\varepsilon}\bigl[N_1^\varepsilon(t) - N_1(t)\bigr] \to 2d_1N_1(t)z_1(t),\\
\frac{1}{\varepsilon}\biggl[\frac{\alpha_{12}N_1^\varepsilon(t)A^\varepsilon(t)}{\gamma_2 + A^\varepsilon(t)} - \frac{\alpha_{12}N_1(t)A(t)}{\gamma_2 + A(t)}\biggr] \to \frac{\alpha_{12}\gamma_2 N_1(t)B(t)}{(\gamma_2 + A(t))^2} + \frac{\alpha_{12}\gamma_2 z_1(t)A(t)}{\gamma_2 + A(t)},\\
\frac{1}{\varepsilon}\bigl[\beta_1 N_1^\varepsilon(t)A^\varepsilon(t) - \beta_1 N_1(t)A(t)\bigr] \to \beta_1 N_1(t)B(t) + \beta_1 A(t)z_1(t),\\
\frac{1}{\varepsilon}\bigl[d_2N_2^\varepsilon(x,t) - d_2N_2(x,t)\bigr] \to d_2z_2(x,t),
\end{gathered}$$

$$\frac{1}{\varepsilon}\big[u_2(x,t)\big(N_2^\varepsilon(x,t)-N_2(x,t)\big)\big] \to u_2(x,t)z_2(x,t),$$
$$\frac{1}{\varepsilon}\left[\frac{\alpha_{21}N_1^\varepsilon(t)A^\varepsilon(t)}{\gamma_1+N_1^\varepsilon(t)} - \frac{\alpha_{21}N_1(t)A(t)}{\gamma_1+N_1(t)}\right] \to \frac{\alpha_{21}N_1(t)\int_0^l B(t)}{\gamma_1+N_1(t)} + \frac{\alpha_{21}\gamma_1 z_1(t)A(t)}{(\gamma_1+N_1(t))^2}$$

as $\varepsilon \to 0$. Taking $\varepsilon \to 0$ in (19) and using the above results yield system (18). □

The adjoint system corresponding to control (u_1,u_2) and state (N_1,N_2) is

$$\begin{aligned}
\frac{\mathrm{d}\eta_1(t)}{\mathrm{d}t} &= -\eta_1(t)\left[r - 2d_1N_1(t) + \frac{\alpha_{12}\gamma_2A(t)}{\gamma_2+A(t)} - \beta_1A(t) - u_1(t)\right] \\
&\quad - \frac{\alpha_{21}\gamma_1\eta_2(0,t)A(t)}{(\gamma_1+N_1(t))^2} + \omega_1(t)u_1(t), \\
\frac{\partial\eta_2}{\partial t} + V(x,t)\frac{\partial\eta_2}{\partial x} &= \big[d_2+u_2(x,t)\big]\eta_2(x,t) - \frac{\alpha_{21}\eta_2(0,t)N_1(t)\beta(x)}{\gamma_1+N_1(t)} \\
&\quad + \beta_1\beta(x)N_1(t)\eta_1(t) - \frac{\alpha_{12}\gamma_2\beta(x)N_1(t)\eta_1(t)}{(\gamma_2+A(t))^2} \\
&\quad + \omega_2(x,t)u_2(x,t), \\
\eta_1(T) = 0, &\qquad \eta_2(x,T) = \eta_2(l,t) = 0.
\end{aligned} \tag{20}$$

Methods similar to Theorem 1 can be used to prove the existence of the solution to system (20). Moreover, for (20), by similar discussion as in Theorem 2 one has the following result.

Theorem 3. *For each $(u_1,u_2) \in \mathcal{U}$, the adjoint system* (20) *has a unique bounded solution $(\eta_1,\eta_2) \in L^\infty(0,T) \times L^\infty(Q)$. Moreover, for T sufficiently small, there is a positive constant K_3 such that*

$$\begin{aligned}
&\|\eta_1-\eta_1'\|_{L^\infty(0,T)} + \|\eta_2-\eta_2'\|_{L^\infty(Q)} \\
&\quad \leqslant K_3T\big[\|u_1-u_1'\|_{L^\infty(0,T)} + \|u_2-u_2'\|_{L^\infty(0,T)}\big],
\end{aligned}$$

where (η_1,η_2) and (η_1',η_2') are the solutions of system (20) *corresponding to (u_1,u_2) and $(u_1',u_2') \in \mathcal{U}$, respectively.*

4 Optimality conditions

In this section, we will give first-order necessary conditions of optimality in the form of an Euler–Lagrange system.

Theorem 4. *Let (u_1^*,u_2^*) be an optimal harvest policy for the optimization problem* (5)*, let and (N_1^*,N_2^*) be the corresponding optimal state of system* (4)*. Then*

$$\alpha_1^*(t) = \mathcal{F}_1\left[\frac{[(\omega_1+\eta_1)N_1^*](t)}{c_1}\right], \qquad \alpha_2^*(x,t) = \mathcal{F}_2\left[\frac{[(\omega_2+\eta_2)N_2^*](x,t)}{c_2}\right], \tag{21}$$

where the truncated mappings $\mathcal{F}_i$ are given by

$$\mathcal{F}_i(S) = \begin{cases} 0, & S < 0, \\ S, & 0 \leqslant S \leqslant H_i, \quad i = 1, 2, \\ H_i, & S > H_i, \end{cases} \tag{22}$$

where $(\eta_1(t), \eta_2(x,t))$ is the solution of the following system:

$$\begin{aligned} \frac{\mathrm{d}\eta_1(t)}{\mathrm{d}t} &= -\eta_1(t)\left[r - 2d_1 N_1^*(t) + \frac{\alpha_{12}\gamma_2 A^*(t)}{\gamma_2 + A^*(t)} - \beta_1 A^*(t) - u_1^*(t)\right] \\ &\quad - \frac{\alpha_{21}\gamma_1\eta_2(0,t)A^*(t)}{(\gamma_1 + N_1^*(t))^2} + \omega_1(t)u_1^*(t), \\ \frac{\partial\eta_2}{\partial t} + V(x,t)\frac{\partial\eta_2}{\partial x} &= \left[d_2 + u_2^*(x,t)\right]\eta_2(x,t) - \frac{\alpha_{21}\eta_2(0,t)N_1^*(t)\beta(x)}{\gamma_1 + N_1^*(t)} \\ &\quad + \beta_1\beta(x)N_1^*(t)\eta_1(t) - \frac{\alpha_{12}\gamma_2\beta(x)N_1^*(t)\eta_1(t)}{(\gamma_2 + A^*(t))^2} \\ &\quad + \omega_2(x,t)u_2^*(x,t), \\ \eta_1(T) = 0, \quad \eta_2(x,T) &= \eta_2(l,t) = 0, \qquad A^*(t) = \int_0^l \beta(x)N_2^*(x,t)\,\mathrm{d}x. \end{aligned} \tag{23}$$

Proof. For any $(v_1, v_2) \in \mathcal{T_U}(u_1^*, u_2^*)$, one has $(u_1^\varepsilon, u_2^\varepsilon) \doteq (u_1^* + \varepsilon v_1,\, u_2^* + \varepsilon v_2) \in \mathcal{U}$ for sufficiently small $\varepsilon > 0$. Let $(N_1^\varepsilon, N_2^\varepsilon)$ be the solution of system (4) corresponding to $(u_1^\varepsilon, u_2^\varepsilon)$. From the optimality of (u_1^*, u_2^*) it follows that $J(u_1^\varepsilon, u_2^\varepsilon) \leqslant J(u_1^*, u_2^*)$. Thus, from Lemma 4 it follows that

$$\begin{aligned} 0 &\geqslant \lim_{\varepsilon \to 0^+} \frac{J(u_1^\varepsilon, u_2^\varepsilon) - J(u_1^*, u_2^*)}{\varepsilon} \\ &= \int_0^T \omega_1(t)u_1^*(t)z_1(t)\,\mathrm{d}t + \int_0^T \left[\omega_1(t)N_1^*(t) - c_1 u_1^*(t)\right]v_1(t)\,\mathrm{d}t \\ &\quad + \int_0^T\int_0^l \omega_2(x,t)u_2^*(x,t)z_2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\int_0^l \left[\omega_2(x,t)N_2^*(x,t) - c_2 u_2^*(x,t)\right]v_2(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{24}$$

Here $(z_1(t), z_2(x,t))$ is the solution of (18) with (u_1, u_2) and (N_1, N_2) replaced by (u_1^*, u_2^*) and (N_1^*, N_2^*), respectively. Next, we show that

$$\int_0^T [\omega_1 u_i^* z_1]\,\mathrm{d}t + \int_0^T\int_0^l [\omega_2 u_2^* z_2]\,\mathrm{d}x\,\mathrm{d}t = \int_0^T [\eta_1 N_1^* v_1]\,\mathrm{d}t + \int_0^T\int_0^l [\eta_2 N_2^* v_2]\,\mathrm{d}x\,\mathrm{d}t. \tag{25}$$

In fact, multiplying the first equation in (23) by $z_1(t)$ and integrating on $[0, T]$, we obtain

$$\begin{aligned}\int_0^T \eta_1 \frac{\mathrm{d}z_1}{\mathrm{d}t}\,\mathrm{d}t &= \int_0^T \eta_1(t) z_1(t)\left[r - 2d_1 N_1^*(t) + \frac{\alpha_{12}\gamma_2 A^*(t)}{\gamma_2 + A^*(t)} - \beta_1 A^*(t) - u_1^*(t)\right]\mathrm{d}t \\ &\quad + \int_0^T \eta_2(0,t)\frac{\alpha_{21}\gamma_1 z_1(t) A^*(t)}{(\gamma_1 + N_1^*(t))^2}\,\mathrm{d}t - \int_0^T \omega_1(t) u_1^*(t) z_1(t)\,\mathrm{d}t. \qquad (26)\end{aligned}$$

Multiplying the second equation in (23) by $z_2(x,t)$ and integrating on Q, we obtain

$$\begin{aligned}&\int_0^T\int_0^l \eta_2\left[\frac{\partial z_2}{\partial t} + \frac{\partial (V z_2)}{\partial x}\right]\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\int_0^T\int_0^l \eta_2(x,t)\big[d_2 + u_2^*(x,t)\big] z_2(x,t)\,\mathrm{d}x\,\mathrm{d}t - \beta_1\int_0^T \eta_1(t) N_1^*(t) B(t)\,\mathrm{d}t \\ &\qquad + \int_0^T \eta_1(t) N_1^*(t)\frac{\alpha_{12}\gamma_2 B(t)}{(\gamma_2 + A^*(t))^2}\,\mathrm{d}t - \int_0^T \eta_2(0,t)\frac{\alpha_{21}\gamma_1 z_1(t) A^*(t)}{(\gamma_1 + N_1^*(t))^2}\,\mathrm{d}t \\ &\qquad - \int_0^T\int_0^l \omega_2(x,t) u_2^*(x,t) z_2(x,t)\,\mathrm{d}x\,\mathrm{d}t. \qquad (27)\end{aligned}$$

It follows from (26) and (27) that

$$\begin{aligned}&\int_0^T \eta_1 \frac{\mathrm{d}z_1}{\mathrm{d}t}\,\mathrm{d}t + \int_0^T\int_0^l \eta_2\left[\frac{\partial z_2}{\partial t} + \frac{\partial (V z_2)}{\partial x}\right]\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T \eta_1 z_1\left[r - 2d_1 N_1^*(t) + \frac{\alpha_{12}\gamma_2 A^*(t)}{\gamma_2 + A^*(t)} - \beta_1 A^*(t) - u_1^*\right]\mathrm{d}t - \beta_1\int_0^T \big[\eta_1 N_1^* B\big](t)\,\mathrm{d}t \\ &\qquad - \int_0^T\int_0^l \eta_2(x,t)\big[d_2 + u_2^*(x,t)\big] z_2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \eta_1(t) N_1^*(t)\frac{\alpha_{12}\gamma_2 B(t)}{(\gamma_2 + A^*(t))^2}\,\mathrm{d}t \\ &\qquad - \int_0^T \omega_1(t) u_1^*(t) z_1(t)\,\mathrm{d}t - \int_0^T\int_0^l \omega_2(x,t) u_2^*(x,t) z_2(x,t)\,\mathrm{d}x\,\mathrm{d}t. \qquad (28)\end{aligned}$$

Similarly, multiplying the first equation of (18) by $\eta_1(t)$ and multiplying the second equation of (18) by $\eta_2(x,t)$, with (u_1, u_2) and (N_1, N_2) replaced by (u_1^*, u_2^*) and (N_1^*, N_2^*)

in (18), respectively, we also have

$$\int_0^T \eta_1 \frac{\mathrm{d}z_1}{\mathrm{d}t}\,\mathrm{d}t + \int_0^T\int_0^l \eta_2\left[\frac{\partial z_2}{\partial t} + \frac{\partial(Vz_2)}{\partial x}\right]\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \eta_1 z_1\left[r - 2d_1N_1^*(t) + \frac{\alpha_{12}\gamma_2 A^*(t)}{\gamma_2+A^*(t)} - \beta_1 A^*(t) - u_1^*\right]\mathrm{d}t - \beta_1\int_0^T [\eta_1 N_1^* B](t)\,\mathrm{d}t$$
$$- \int_0^T\int_0^l \eta_2(x,t)\bigl[d_2 + u_2^*(x,t)\bigr]z_2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \eta_1(t)N_1^*(t)\frac{\alpha_{12}\gamma_2 B(t)}{(\gamma_2 + A^*(t))^2}\,\mathrm{d}t$$
$$- \int_0^T \eta_1(t)N_1^*(t)v_1(t)\,\mathrm{d}t - \int_0^T\int_0^l \eta_2(x,t)N_2^*(x,t)v_2(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{29}$$

By (28) and (29) we obtain that equality (26) is true. Substituting (26) into (24), for each $(v_1, v_2) \in \mathcal{T}_{\mathcal{U}}(u_1^*, u_2^*)$, we have

$$0 \geqslant \int_0^T \bigl\{\bigl[\omega_1(t) + \eta_1(t)\bigr]N_1^*(t) - c_1u_1^*(t)\bigr\}v_1(t)\,\mathrm{d}t$$
$$+ \int_0^T\int_0^l \bigl\{\bigl[\omega_2(x,t) + \eta_2(x,t)\bigr]N_2^*(x,t) - c_2u_2^*(x,t)\bigr\}v_2(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Hence, we have $([(\omega_1 + \eta_1)N_1^* - c_1u_1^*](t),\ [(\omega_2 + \eta_2)N_2^* - c_2u_2^*](x,t)) \in \mathcal{N}_{\mathcal{U}}(u_1^*, u_2^*)$. This implies the conclusion of this theorem. □

5 Existence of a unique optimal harvesting

The purpose of this section is to show that the optimization problem (5) has a unique solution by means of Ekeland's variational principle. First, we embed the functional $J(\cdot,\cdot)$ in the space $L^1(0,T) \times L^1(Q)$ by defining

$$\tilde{J}(u_1, u_2) = \begin{cases} J(u_1, u_2), & (u_1, u_2) \in \mathcal{U}, \\ -\infty, & (u_1, u_2) \notin \mathcal{U}. \end{cases}$$

Lemma 5. *The functional $\tilde{J}(u_1, u_2)$ is upper semicontinuous with respect to (u_1, u_2) in $L^1(0,T) \times L^1(Q)$.*

Proof. Assume that $(u_1^n, u_2^n) \to (u_1, u_2)$ as $n \to +\infty$. From Riesz theorem there is a subsequence of $\{(u_1^n, u_2^n)\}$ still denoted by $\{(u_1^n, u_2^n)\}$ such that

$$\bigl(u_1^n(t)\bigr)^2 \to \bigl(u_1(t)\bigr)^2 \quad \text{a.s. in } [0,T], \qquad \bigl(u_2^n(x,t)\bigr)^2 \to \bigl(u_2(x,t)\bigr)^2 \quad \text{a.s. in } Q$$

as $n \to \infty$. Then, using the Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_0^T \big(u_1^n(t)\big)^2\,\mathrm{d}t = \int_0^T \big(u_1(t)\big)^2\,\mathrm{d}t,$$

$$\lim_{n\to+\infty}\int_0^T\int_0^l \big(u_2^n(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_0^l \big(u_2(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t.$$

Let (N_{1n}, N_{2n}) and (N_1, N_2) be the solutions of system (4) corresponding to (u_1^n, u_2^n) and (u_1, u_2), respectively. From Theorem 2 it follows that

$$\begin{aligned}
&\left|\int_0^T \omega_1(t)u_1^n(t)N_{1n}(t)\,\mathrm{d}t - \int_0^T \omega_1(t)u_1(t)N_1(t)\,\mathrm{d}t\right| \\
&\quad\leqslant \int_0^T \omega_1(t)N_{1n}(t)\big|u_1^n(t)-u_1(t)\big|\,\mathrm{d}t + \int_0^T \omega_1(t)u_1(t)\big|N_{1n}(t)-N_1(t)\big|\,\mathrm{d}t \\
&\quad\leqslant M\|\omega_1\|_{L^\infty(0,T)}\big\|u_1^n-u_1\big\|_{L^1(0,T)} \\
&\qquad + \|\omega_1\|_{L^\infty(0,T)}H_1K_2T\big[\big\|u_1^n-u_1\big\|_{L^1(0,T)} + \big\|u_2^n-u_2\big\|_{L^1(Q)}\big].
\end{aligned}$$

Thus, we obtain

$$\lim_{n\to+\infty}\int_0^T \omega_1(t)u_1^n(t)N_{1n}(t)\,\mathrm{d}t = \int_0^T \omega_1(t)u_1(t)N_1(t)\,\mathrm{d}t.$$

Similarly, we also have

$$\lim_{n\to+\infty}\int_0^T\int_0^l \omega_2(x,t)u_2^n(x,t)N_{2n}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_0^l \omega_2(x,t)u_2(x,t)N_2(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

From Fatou's lemma it follows that $\limsup_{n\to+\infty}\tilde{J}(u_1^n,u_2^n) \leqslant \tilde{J}(u_1,u_2)$. This means that $\tilde{J}(u_1,u_2)$ is upper semicontinuous. □

Theorem 5. *If $T(c_1^{-1}+c_2^{-1})$ is small enough, there is a unique optimal harvesting policy $(u_1^*, u_2^*) \in \mathcal{U}$, which is in feedback form and determined by* (21)–(22).

Proof. Define the mapping $\mathcal{B} : \mathcal{U} \to L^\infty(0,T) \times L^\infty(Q)$ by

$$\mathcal{B}(u_1,u_2) = \left(\mathcal{F}_1\left[\frac{(\omega_1+\eta_1)N_1}{c_1}\right], \mathcal{F}_2\left[\frac{(\omega_2+\eta_2)N_2}{c_2}\right]\right),$$

where (N_1, N_2) and (η_1, η_2) are, respectively, the solutions of (4) and (20) corresponding to $(u_1, u_2) \in \mathcal{U}$. Now, we show $\mathcal{B}$ owns a unique fixed point, which maximizes J.

From Lemma 5 and Ekeland's variational principle it follows that for each $\varepsilon > 0$, there exists $(u_1^\varepsilon, u_2^\varepsilon) \in \mathcal{U}$ such that

$$\tilde{J}(u_1^\varepsilon, u_2^\varepsilon) \geqslant \sup_{(u_1,u_2)\in\mathcal{U}} \tilde{J}(u_1, u_2) - \varepsilon, \tag{30}$$

$$\tilde{J}(u_1^\varepsilon, u_2^\varepsilon) \geqslant \sup_{(u_1,u_2)\in\mathcal{U}} \tilde{J}_\varepsilon(u_1, u_2), \tag{31}$$

where

$$\tilde{J}_\varepsilon(u_1, u_2) = \tilde{J}(u_1, u_2) - \sqrt{\varepsilon}\big\|u_1^\varepsilon - u_1\big\|_{L^1(0,T)} - \sqrt{\varepsilon}\big\|u_2^\varepsilon - u_2\big\|_{L^1(Q)}.$$

It is clear that perturbed functional $\tilde{J}_\varepsilon(u_1, u_2)$ attains its supremum at $(u_1^\varepsilon, u_2^\varepsilon)$. In the same manner as that in the proof of Theorem 4, we obtain

$$u_1^\varepsilon(t) = \mathcal{F}_1\left[\frac{[\omega_1(t) + \eta_1^\varepsilon(t)]N_1^\varepsilon(t)}{c_1} + \frac{\sqrt{\varepsilon}\theta_1(t)}{c_1}\right],$$

$$u_2^\varepsilon(x,t) = \mathcal{F}_2\left[\frac{[\omega_2(x,t) + \eta_2^\varepsilon(x,t)]N_2^\varepsilon(x,t)}{c_2} + \frac{\sqrt{\varepsilon}\theta_2(x,t)}{c_2}\right],$$

where $(N_1^\varepsilon, N_2^\varepsilon)$ and $(\eta_1^\varepsilon, \eta_2^\varepsilon)$ are solutions of (4) and (20) corresponding to $(u_1^\varepsilon, u_2^\varepsilon)$, $\theta_1 \in L^\infty(0,T)$ with $|\theta_1(t)| \leqslant 1$ a.e. in $(0,T)$, and $\theta_2 \in L^\infty(Q)$ with $|\theta_2(x,t)| \leqslant 1$ a.e. in Q.

Step 1. We show that the mapping $\mathcal{B}$ has only one fixed point.

(i) For any $(u_1, u_2) \in \mathcal{U}$, from (22) it follows that $(0,0) \leqslant \mathcal{B}(u_1, u_2) \leqslant (H_1, H_2)$. Thus, $\mathcal{B}$ maps $\mathcal{U}$ into itself.

(ii) From Theorems 2 and 3 we know that (N_1, N_2) and (η_1, η_2) are continuous about the control variable (u_1, u_2). Thus, for any (u_1, u_2), $(u_1', u_2') \in \mathcal{U}$, we have

$$\begin{aligned}\big\|\mathcal{B}(u_1, u_2) - \mathcal{B}(u_1', u_2')\big\| &\leqslant \left\|\frac{(\omega_1+\eta_1)N_1}{c_1} - \frac{(\omega_1+\eta_1')N_1'}{c_1}\right\|_{L^\infty(0,T)} \\ &\quad + \left\|\frac{(\omega_2+\eta_2)N_2}{c_2} - \frac{(\omega_2+\eta_2')N_2'}{c_2}\right\|_{L^\infty(Q)} \\ &\leqslant K_4 T\big(c_1^{-1} + c_2^{-1}\big) \\ &\quad \times \big[\|u_1 - u_1'\|_{L^\infty(0,T)} + \|u_2 - u_2'\|_{L^\infty(Q)}\big],\end{aligned}$$

where $K_4 > 0$ is a constant. Obviously, $\mathcal{B}$ is a contraction if $T(c_1^{-1} + c_2^{-1})$ is small enough. Thus, $\mathcal{B}$ has a unique fixed point $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}$. In addition, Theorem 4 shows that if the optimal policy exists, it must be the fixed point of $\mathcal{B}$. Thus, the uniqueness holds.

Step 2. We show that $(\bar{u}_1, \bar{u}_2)$ is the optimal policy. That is, $(u_1^\varepsilon, u_2^\varepsilon) \to (\bar{u}_1, \bar{u}_2)$ as $\varepsilon \to 0^+$.

Note that

$$\big\|(\bar{u}_1, \bar{u}_2) - \big(u_1^\varepsilon, u_2^\varepsilon\big)\big\|_\infty = \big\|\bar{u}_1 - u_1^\varepsilon\big\|_{L^\infty(0,T)} + \big\|\bar{u}_2 - u_2^\varepsilon\big\|_{L^\infty(Q)}$$

and

$$\begin{aligned}\left\|\mathcal{B}(u_1^\varepsilon, u_2^\varepsilon) - (u_1^\varepsilon, u_2^\varepsilon)\right\|_\infty &\leqslant c_1^{-1}\sqrt{\varepsilon}\left\|\theta_1(t)\right\|_{L^\infty(0,T)} + c_2^{-1}\sqrt{\varepsilon}\left\|\theta_2(x,t)\right\|_{L^\infty(Q)} \\ &\leqslant \sqrt{\varepsilon}\left(c_1^{-1} + c_2^{-1}\right).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\left\|(\bar{u}_1, \bar{u}_2) - \left(u_1^\varepsilon, u_2^\varepsilon\right)\right\|_\infty \\ &\quad\leqslant \left\|\mathcal{B}(\bar{u}_1, \bar{u}_2) - \mathcal{B}(u_1^\varepsilon, u_2^\varepsilon)\right\|_\infty + \left\|\mathcal{B}\left(u_1^\varepsilon, u_2^\varepsilon\right) - \left(u_1^\varepsilon, u_2^\varepsilon\right)\right\|_\infty \\ &\quad\leqslant K_4 T\left(c_1^{-1} + c_2^{-1}\right)\left\|(\bar{u}_1, \bar{u}_2) - \left(u_1^\varepsilon, u_2^\varepsilon\right)\right\|_\infty + \sqrt{\varepsilon}\left(c_1^{-1} + c_2^{-1}\right).\end{aligned}$$

If $T(c_1^{-1} + c_2^{-1})$ is sufficiently small such that $K_4T(c_1^{-1} + c_2^{-1}) < 1$, then

$$\left\|(\bar{u}_1, \bar{u}_2) - \left(u_1^\varepsilon, u_2^\varepsilon\right)\right\|_\infty \leqslant \frac{\sqrt{\varepsilon}(c_1^{-1} + c_2^{-1})}{1 - K_4T(c_1^{-1} + c_2^{-1})}.$$

Thus, $(u_1^\varepsilon, u_2^\varepsilon) \to (\bar{u}_1, \bar{u}_2)$ as $\varepsilon \to 0^+$. From Lemma 5 it follows that

$$\tilde{J}(\bar{u}_1, \bar{u}_2) = \sup_{(u_1,u_2)\in\mathcal{U}} \tilde{J}(u_1, u_2).$$

This means that $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}$ is the optimal policy. □

6 Numerical tests

In this section, we provide some examples to illustrate the effectiveness of the obtained results. Note that our problem is highly nonlinear, and one cannot expect an explicit optimal controller. In the following examples, we do not consider the interaction between resource and consumer species and do not consider the costs of controls. We take $r = 2$, $d_1 = 0.8$, $d_2 = 0.1$, $\gamma_1 = \gamma_2 = \beta_1 = 0$, $\alpha_{12} = \alpha_{21} = 0.8$, $\beta(x) = 10x^2(1 + x)$, $V(x,t) = 1 - x$, $l = 1$, $T = 1$, $\omega_1(t) = 0.35(1 + \sin(4\pi t))$, $\omega_2(x,t) = 1/120 \times (5\pi x + \sin(8\pi t) + 1)$.

Example 1. Take $H_1 = 0.5$, $H_2 = 2$, $N_{10} = 1.5$ and $N_{20}(x) = 2(1 + x)^2(1 - x)^2$ (see Figs. 1–3).

Example 2. Take $H_1 = 0.5$, $H_2 = 2$, $N_{10} = 4$ and $N_{20}(x) = 8(1 + x)^2(1 - x)^2$ (see Fig. 4).

Example 3. Take $H_1 = H_1(t) = 0.5 + 0.2\sin(4\pi t)$, $H_2 = H_2(x,t) = 2 + 0.2x + 0.3\sin(8\pi t)$, $N_{10} = 1.5$ and $N_{20}(x) = 2(1 + x)^2(1 - x)^2$ (see Figs. 5–7).

From the numerical simulations given in Figs. 1 and 4 we can see that the optimal harvesting strategies for both the resource species and the consumer species basically have a bang-bang structure. Further, by comparing Fig. 1 and Fig. 4 it can be seen that given other parameters, the optimal harvesting strategy for the resource species has nothing

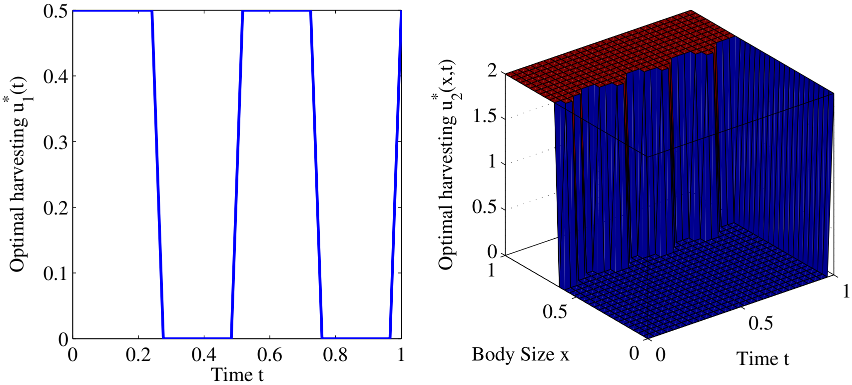

Figure 1. Optimal harvesting efforts $u_1^*(t)$ in Example 1 (left) and $u_2^*(x,t)$ in Example 1 (right).

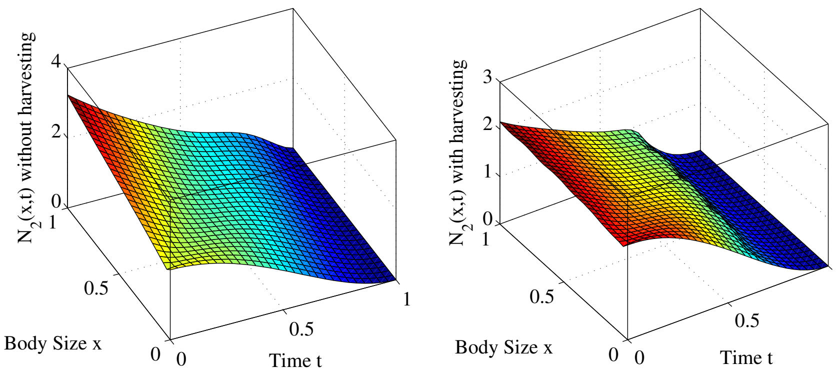

Figure 2. Population density $N_2(x,t)$ in Example 1 with $u_2 = 0$ (left); with $u_2 = u_2^*$ (right).

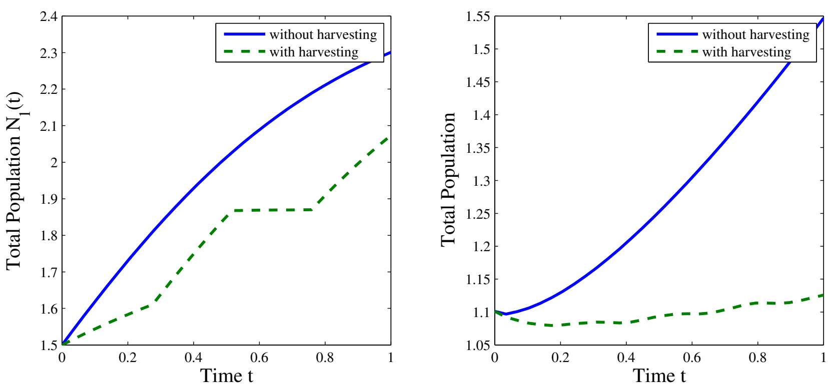

Figure 3. Trend of the total population in the case of harvest and no harvest in Example 1. Resource population (left); consumer population (right).

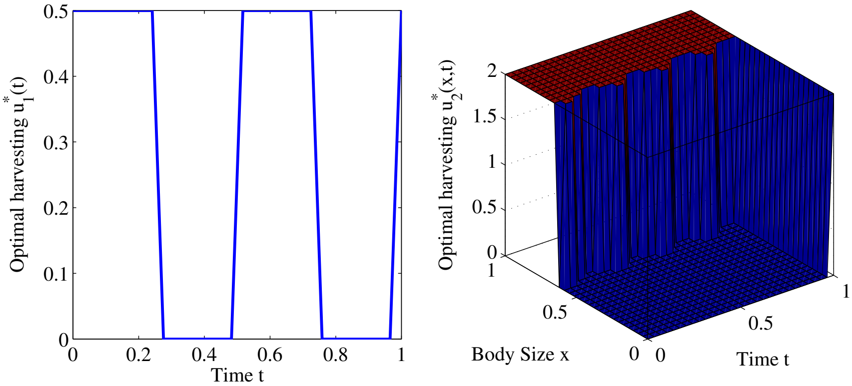

Figure 4. Optimal harvesting efforts $u_1^*(t)$ in Example 2 (left) and $u_2^*(x,t)$ in Example 2 (right).

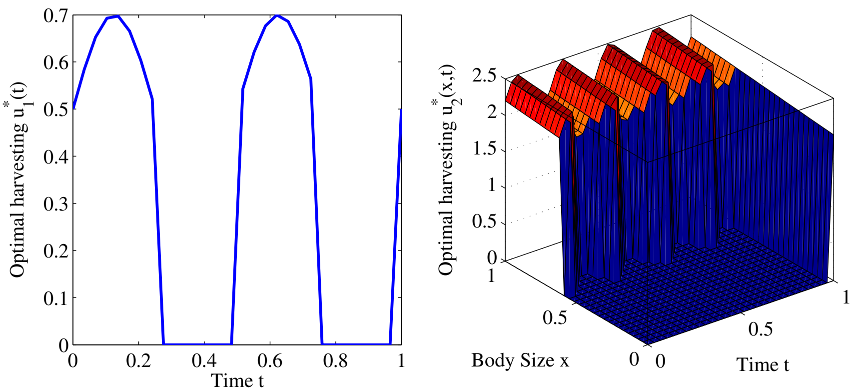

Figure 5. Optimal harvesting efforts $u_1^*(t)$ in Example 3 (left) and $u_2^*(x,t)$ in Example 3 (right).

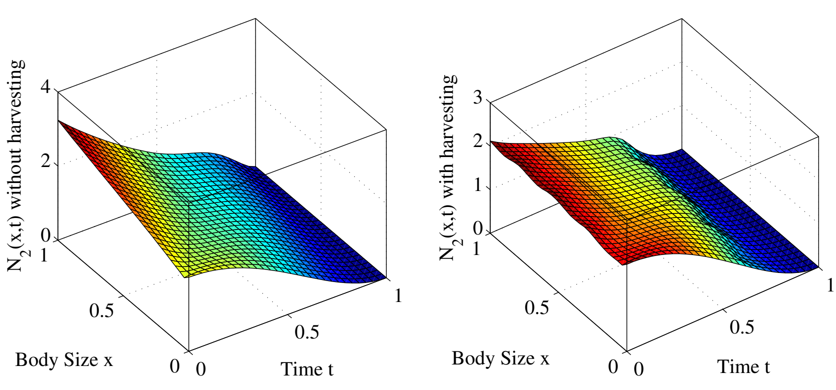

Figure 6. Population density $N_2(x,t)$ in Example 3 with $u_2 = 0$ (left); with $u_2 = u_2^*$ (right).

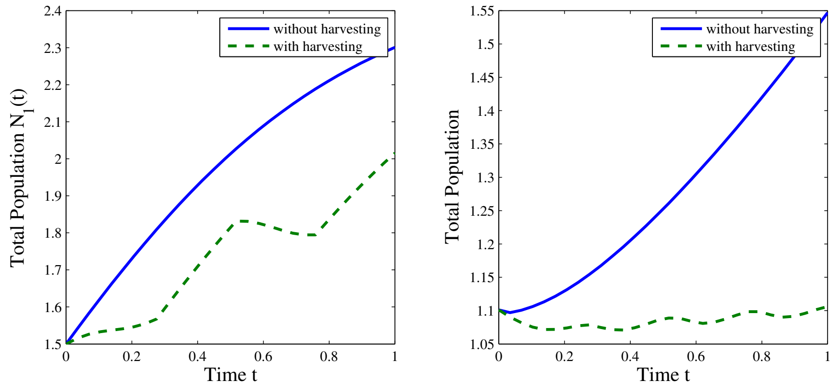

Figure 7. Trend of the total population in the case of harvest and no harvest in Example 3. Resource population (left); consumer population (right).

to do with its initial value, and the optimal harvesting strategy for the consumer species has nothing to do with its initial size distribution. Thus, it leads to the conclusion that the bang-bang structure of optimal policies is much more common in optimal population management. In this paper, we assume that the maximum harvesting efforts for the resource species and the consumer species are, respectively, positive constants. However, from the numerical simulations in Example 3 it can be seen that if the maximum harvesting effort for the resource species is a bounded function with respect to time t, and the maximum harvest effort for the consumer species is a bounded function with respect to time t and individual size x, the optimal harvesting strategies for both the resource species and the consumer species basically have a bang-bang structure. From the right part of Figs. 1, 4 and 5 it can be seen that for consumer species, harvesting individuals with larger sizes is conducive to obtaining more economic benefits. This has obvious biological significance because we assume that individuals with larger size have greater economic value.

7 Conclusion

This paper is concerned with the harvesting problem for a size-structured model of unidirectional consumer–resource mutualisms in which the consumer species has both positive and negative effects on the resource species, while the resource has only a positive effect on the consumer. In the previous sections, we have established the well-posedness of the system by constructing a suitable solution space and equivalent norm. Then the continuous dependence of solutions on the control variable and the adjoint system of the state system are investigated. More important result is the existence of a unique optimal harvesting policy, which provides a theoretical basis for practical application. As for the structure of the optimal policy, in Theorem 4, we have presented a feedback strategy.

Let us make some comments on the difference of our results and methods with those of closely related works. For the optimal control problems of size-structured population models, the authors in [15,17,18] proved that the optimization problems admit at least one solution but paid no attention to the uniqueness. In addition, the structure of the optimal strategy did not considered in [15].

In our paper, we show that there is a unique optimal harvesting policy, and the structure of the optimal policy is given in the form of feedback. As far as we know, most of optimal control problems for population systems are naturally formed in an infinite time horizon. However, in this paper, we consider the optimal harvest problem with a fixed horizon $[0, T]$, where $T < \infty$. To our knowledge, even for the population model of ordinary differential equations, the infinite-horizon optimal control problems are still challenging. For example, it is difficult to establish a suitable transversality condition so that one can choose the correct solution of adjoint system for which Pontryagin maximum principle is applicable. For more details of the infinite-horizon optimal control (including age-structured systems and size-structured systems), please refer to [24]. We leave these for our future work. Moreover, as done in [20], we can investigate the existence and stability of positive equilibrium and the existence of nontrivial periodic solution of the system.

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