Nonlinear Analysis: Modelling and Control, Vol. 27, No. 3, 428–444
https://doi.org/10.15388/namc.2022.27.25473

Vilnius University Press

The uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model*

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Received: March 29, 2021 / **Revised:** November 17, 2021 / **Published online:** January 25, 2022

Abstract. In this paper, we consider the iterative properties of positive solutions for a general Hadamard-type singular fractional turbulent flow model involving a nonlinear operator. By developing a double monotone iterative technique we firstly establish the uniqueness of positive solutions for the corresponding model. Then we carry out the iterative analysis for the unique solution including the iterative schemes converging to the unique solution, error estimates, convergence rate and entire asymptotic behavior. In addition, we also give an example to illuminate our results.

Keywords: unique solutions, turbulent flow, nonlinear operator, iterative analysis.

1 Introduction

In recent years, many researchers were interested in the study of turbulent flow in which the fluid undergoes irregular fluctuations or mixing. For example, Leibenson [17] introduced a p-Laplacian differential equation $(\varphi_p(u'(t)))' = f(t, u(t))$, $t \in (0, 1)$, to model

*This research was supported by the National Natural Science Foundation of China (11871302).
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turbulent flow in a porous medium, where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. However, the transport of solute in highly heterogeneous porous media often exhibits anomalous diffusion phenomenon [14], and laboratory data and numerical experiments [2, 8] have indicated that solutes moving through a highly heterogeneous porous media violate the basic Fick's first law of Brownian motion. Thus the works [2, 8] indicated that the fractional differential equation is more suitable for describing the convection–dispersion process of solutes in porous media.

On the other hand, since fractional-order derivative possesses a nonlocal characteristics, so it can provide a possibility to represent the memory occurring in viscoelastic dynamical process [4, 10, 13], blood flow [3], quantum mechanics [7], advection–dispersion process in anomalous diffusion [22, 23, 25] and bioprocesses with genetic attribute [5, 20, 26]. As a powerful tool of modeling the above many abnormal phenomena, in the last few decades, the fractional calculus theory has been enriched, and several different derivatives and integrals such as Caputo, Atangana, Riemann–Liouville, Hadamard, Caputo–Fabrizio, Hilfer, Riesz derivative and so on have been developed. In comparison, the Hadamard derivative is a nonlocal fractional derivative with singular logarithmic kernel. So the study for Hadamard-type fractional differential equations is relatively difficult [9, 21, 27, 28].

Thus, in this paper, we choose a general Hadamard-type singular fractional differential equation involving a nonlinear operator from turbulent flow to study, more specifically, we consider the iterative properties of positive solutions for the equation

$$\begin{aligned}
&-{D_t}^{\alpha}\bigl(F\bigl(-{D_t}^{\beta}u(t)\bigr)\bigl(-{D_t}^{\beta}u(t)\bigr)\bigr) = f\bigl(t, u(t)\bigr), \quad 1 < t < e, \\
&u(1) = S(u(1)) = S(u(e)) = 0, \\
&R\bigl(-{D_t}^{\beta}u(1)\bigr) = S\bigl(R\bigl(-{D_t}^{\beta}u(1)\bigr)\bigr) = S\bigl(R\bigl(-{D_t}^{\beta}u(e)\bigr)\bigr) = 0,
\end{aligned} \tag{1}$$

where $\alpha, \beta \in (2, 3]$, ${D_t}^{\alpha}$, ${D_t}^{\beta}$ are α- and β-order Hadamard fractional derivatives, S is a differential operator denoted by $t(\mathrm{d}/\mathrm{d}t)$, that is, $Su(t) = t(\mathrm{d}/\mathrm{d}t)u(t)$, $f \in ((1, e) \times (0, +\infty), [0, +\infty))$ is a continuous function with singularities at $t = 0, 1$ and $u = 0$. $R(s) = sF(s)$, and F is a nonlinear operator satisfying

$$\begin{aligned}
\chi = \bigl\{F \in C^2\bigl([0, +\infty), [0, +\infty)\bigr)\colon &\text{ there exists a constant } p > 0 \text{ such that} \\
&\text{for any } 0 < k < 1,\ F(kx) \leqslant k^p F(x)\bigr\},
\end{aligned} \tag{2}$$

which possesses the following properties:

Proposition 1. (*See* [24].) *If $F \in \chi$, then*

(i) *$R(s)$ has a nonnegative increasing inverse mapping $R^{-1}(s)$;*
(ii) *For $0 < l < 1$, $R^{-1}(ls) \geqslant l^{1/(p+1)}R^{-1}(s)$;*
(iii) *For $l \geqslant 1$, $R^{-1}(ls) \leqslant l^{1/(p+1)}R^{-1}(s)$.*

Equation (1) involves a nonlinear operator F, which implies that Eq. (1) covers many interesting and important cases, in particular, if $F(x) \equiv 1$ and then $R(x) = xF(x) = x$, in this case, Eq. (1) reduces to the following form:

$$\begin{aligned}
&{D_t}^{\alpha}\big({D_t}^{\beta}u(t)\big) = f\big(t, u(t)\big), \quad 1 < t < e,\\
&u(1) = S\big(u(1)\big) = S\big(u(e)\big) = 0,\\
&\varphi_p\big({D_t}^{\beta}u(1)\big) = S\big(\varphi_p\big({D_t}^{\beta}u(1)\big)\big) = S\big(\varphi_p\big({D_t}^{\beta}u(e)\big)\big) = 0,
\end{aligned}$$

where $\alpha, \beta \in (2, 3]$, ${D_t}^{\alpha}$, ${D_t}^{\beta}$ are the α- and β-order Hadamard fractional derivatives, S is a differential operator denoted by $t(\mathrm{d}/\mathrm{d}t)$, that is, $Su(t) = t(\mathrm{d}/\mathrm{d}t)u(t)$. By using the fixed point index and the properties of nonnegative matrices Ding et al. [9] considered the existence of positive solutions for a system of the above Hadamard-type fractional differential equations with semipositone nonlinearities. If $F(x) = |x|^{p-2}$, $p > 1$, Eq. (1) takes the form

$$\begin{aligned}
&{D_t}^{\alpha}\big(\varphi_p\big({D_t}^{\beta}u(t)\big)\big) = f\big(t, u(t)\big), \quad 1 < t < e,\\
&u(1) = S\big(u(1)\big) = S\big(u(e)\big) = 0,\\
&\varphi_p\big({D_t}^{\beta}u(1)\big) = S\big(\varphi_p\big({D_t}^{\beta}u(1)\big)\big) = S\big(\varphi_p\big({D_t}^{\beta}u(e)\big)\big) = 0,
\end{aligned} \tag{3}$$

which is a p-Poisson turbulent flow equation in highly heterogeneous porous media. By using the fixed point theorem of the mixed monotone operator Zhang et al. [22] studied the uniqueness of positive solution for a fractional-order model of turbulent flow in a porous medium. In addition, on fractional differential equations, some significant work by using fixed theorems has been made by Karapinar and his collaborators [1, 11, 12]. Thus Eq. (1) is a more generalized p-Poisson turbulent flow equation in highly heterogeneous porous media (3). To the best of our knowledge, no results have been reported on the iterative analysis of positive solutions for Eq. (1) involving a nonlinear operator under singular case.

In addition, a fluid in highly heterogeneous porous media may push the transmission process from a phase into another different phase or state. At absolute zero, this change always leads to transformation process losing continuity and further forms some singular points or singular domains. Normally, near singular points and domains, the "bad" properties such as blow-up properties [24], impulse interference [19], chaotic influence [6] obstruct people's apperceiving for the essence of related natural phenomena. Thus it is important and interesting to explore the properties of dynamic process governed by singular differential equations, which can deepen people's comprehension for the natural law of dynamic system.

Thus, to overcome the difficult associated with singular logarithmic kernel and singular nonlinearity of Eq. (1) and follow the work Nieto [18] and Ladde, Lakshmikantham and Vatsala [16], a new double monotone iterative technique will be developed, and more new estimates are given. This paper is organized as follows. In Section 2, we firstly give the definition of Hadamard fractional integral and differential operators and then claim the properties of Green function. The main results are summarized in Section 3.

2 Preliminaries and lemmas

In this section, we firstly review the definitions of the Hadamard-type fractional integrals and derivatives; for details, see [15].

Let $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $n = [\operatorname{Re}\alpha]$, and let (a, b) be a finite or infinite interval of $\mathbb{R}^+$. The α-order left Hadamard fractional integral is defined by

$$\big(I_a^\alpha x\big)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{x(s)}{s}\,\mathrm{d}s, \quad t \in (a, b),$$

and the α-order left Hadamard fractional derivative is defined by

$$\big({D_t}^\alpha\big)x(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} \frac{x(s)}{s}\,\mathrm{d}s, \quad t \in (a, b).$$

In the following, we firstly consider the linear Hadamard fractional equation

$$\begin{aligned} &-{D_t}^\alpha v(t) = h(t), \quad t \in (1, e), \\ &v(1) = Sv(1) = Sv(e) = 0. \end{aligned} \tag{4}$$

Equation (4) is equivalent to the following Hammerstein-type integral equation [9]:

$$v(t) = \int_1^{\mathrm{e}} G(t, s)h(s)\frac{\mathrm{d}s}{s}, \quad t \in [1, \mathrm{e}], \tag{5}$$

and

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\ln t)^{\alpha-1}(1-\ln s)^{\alpha-2} - (\ln t - \ln s)^{\alpha-1}, & 1 \leqslant s \leqslant t \leqslant e, \\ (\ln t)^{\alpha-1}(1-\ln s)^{\alpha-2}, & 1 \leqslant t \leqslant s \leqslant e. \end{cases}$$

is the Green's function of Eq. (4).

Let

$$H(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} (\ln t)^{\beta-1}(1-\ln s)^{\beta-2} - (\ln t - \ln s)^{\beta-1}, & 1 \leqslant s \leqslant t \leqslant e, \\ (\ln t)^{\beta-1}(1-\ln s)^{\beta-2}, & 1 \leqslant t \leqslant s \leqslant e, \end{cases}$$

for $h \in L^1[0, 1]$, and then we consider the associated linear boundary value problem

$$\begin{aligned} &-{D_t}^\alpha\big(F\big(-{D_t}^\beta u(t)\big)\big(-{D_t}^\beta u(t)\big)\big) = h(t), \quad 1 < t < e, \\ &u(1) = S\big(u(1)\big) = S\big(u(e)\big) = 0, \\ &R\big(-{D_t}^\beta u(1)\big) = S\big(R\big(-{D_t}^\beta u(1)\big)\big) = S\big(R\big(-{D_t}^\beta u(e)\big)\big) = 0. \end{aligned} \tag{6}$$

We have the following lemma.

Lemma 1. *The associated linear boundary value problem* (6) *has a unique solution if and only if it solves the Hammerstein-type integral equation*

$$u(t) = \int_1^{\mathrm{e}} H(t,s) R^{-1}\left(\int_1^{\mathrm{e}} G(s,\tau) h(\tau) \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{d}s}{s}.$$

Proof. Let $y(t) = -{D_t}^{\beta} u(t)$, $v(t) = F(y(t))y(t)$, then the associated linear boundary value problem (6) reduces to

$$\begin{aligned} &-{D_t}^{\alpha} v(t) = h(t), \quad t \in (1, e), \\ &v(1) = Sv(1) = Sv(e) = 0. \end{aligned}$$

By (4) it is equivalent to the Hammerstein-type integral equation

$$v(t) = \int_1^{\mathrm{e}} G(t,s) h(s) \frac{\mathrm{d}s}{s}, \quad t \in [1, \mathrm{e}].$$

Noting that $-{D_t}^{\beta} u(t) = y$, $y = R^{-1}(v)$ as well as (5) and Proposition 1, the associated linear boundary value problem (6) is equivalent to the following equation:

$$\begin{aligned} &-{D_t}^{\beta} u(t) = R^{-1}\left(\int_1^{\mathrm{e}} G(t,s) h(s) \frac{\mathrm{d}s}{s} \right), \quad t \in (1, e), \\ &u(1) = S\bigl(u(1)\bigr) = S\bigl(u(e)\bigr) = 0, \end{aligned}$$

and it follows from [9] that if and only if it solves the following Hammerstein-type integral equation

$$u(t) = \int_1^{\mathrm{e}} H(t,s) R^{-1}\left(\int_1^{\mathrm{e}} G(s,\tau) h(\tau) \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{d}s}{s}, \quad t \in [1, \mathrm{e}],$$

the proof is completed. □

Lemma 2. (See [9].) *Let* $\psi_i(t) = \ln t (1 - \ln t)^{i-2}$, $i = \alpha, \beta$, $t \in [1, \mathrm{e}]$. *Then the Green's functions* H *and* G *has the following properties*:

(i) $H, G \in C([1, \mathrm{e}] \times [1, \mathrm{e}], \mathbb{R}^+)$.

(ii) *For all* $t, s \in [1, \mathrm{e}]$, *the following inequalities hold*:

$$\begin{aligned} \frac{1}{\Gamma(\beta)} (\ln t)^{\beta-1} \psi_\beta(s) \leqslant H(t,s) \leqslant \frac{1}{\Gamma(\beta)} (\ln t)^{\beta-1} (1 - \ln s)^{\beta-2}, \\ \frac{1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} \psi_\alpha(s) \leqslant G(t,s) \leqslant \frac{1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} (1 - \ln s)^{\alpha-2}. \end{aligned}$$

Our work space of this paper is Banach space $E = C[1, \mathrm{e}]$ equipped with the norm $\|u\| = \max_{t\in[1,\mathrm{e}]} |u(t)|$. Let $P = \{u \in E\colon u(t) \geqslant 0,\ t \in [1, \mathrm{e}]\}$, then P is a normal cone of E with normality constant 1. Now let us define a subset of P by

$$\begin{aligned}\varLambda = \bigl\{u(x) \in P\colon &\text{ there exists a positive number } 0 < l_u < 1 \text{ such that} \\ &l_u(\ln t)^{\beta-1} \leqslant u(x) \leqslant l_u^{-1}(\ln t)^{\beta-1},\ x \in [1, \mathrm{e}]\bigr\}\end{aligned}$$

and a nonlinear operator $T : E \to E$:

$$(Tu)(x) = \int\limits_1^{\mathrm{e}} H(t, s)R^{-1}\left(\int\limits_1^{\mathrm{e}} G(s, \tau)f\bigl(\tau, u(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}, \quad x \in [1, \mathrm{e}].$$

It follows from Lemma 1 that $u \in C[1, \mathrm{e}]$ is a solution of the Hadamard-type fractional turbulent flow model (1) if and only if $u \in C[1, \mathrm{e}]$ is a fixed point of the nonlinear operator T.

3 Main results

In order to carry out the iterative analysis of positive solution for the singular Hadamard-type fractional turbulent flow equation (1), we need the following assumption:

(H1) $f : (1, e) \times (0, +\infty) \to [0, +\infty)$ is continuous and decreasing with regard to second variable, and for any $c \in (0, 1)$, there exists a constant $0 < \mu < p + 1$, where p is defined by (2), such that for all $(t, u) \in (1, e) \times (0, +\infty)$,

$$f(t, cu) \leqslant c^{-\mu} f(t, u). \tag{7}$$

Remark 1. Assumption (7) allows f to have a singularity at $t = 1, \mathrm{e}$, and $u = 0$. For example, $f(t, u) = (t - 1)^{-1}(\mathrm{e} - t)^{-1/3}u^{-p}$ satisfies assumption (H1).

Remark 2. Suppose that condition (H1) holds. It is easy to prove that for any $c \geqslant 1$ and for all $(t, u) \in (1, e) \times (0, +\infty)$, the following formula is also valid:

$$f(t, cu) \geqslant c^{-\mu} f(t, u). \tag{8}$$

Now we give our main results as follows:

Theorem 1. *Suppose that* (H1) *holds, and the following condition is satisfied*:

$$0 < R^{-1}\left(\int\limits_1^{\mathrm{e}} (1 - \ln\tau)^{\alpha-2} f\bigl(\tau, (\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\right) < +\infty.$$

Then

(i) *The singular Hadamard-type fractional turbulent flow equation* (1) *has a unique positive solution* $u^* \in \varLambda$;

(ii) *For any initial value $v_0 \in \Lambda$, we construct the iterative sequence*

$$v_n = \int_1^{\mathrm{e}} H(t,s)R^{-1}\left(\int_1^{\mathrm{e}} G(s,\tau)f\bigl(\tau, v_{n-1}(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}, \quad n = 1, 2, \dots,$$

then $\{v_n\}_{n\geqslant 1}$ converges uniformly to the unique positive solution u^* of Eq. (1) *on* $[1, \mathrm{e}]$;

(iii) *The error of the iterative value v_n and the exact solution u^* can be estimated by the following formula*:

$$\|v_n - u^*\| \leqslant \bigl(1 - \zeta^{2(\mu/(p+1))^{2n}}\bigr)\zeta^{-1}$$

with an exact convergence rate

$$\|v_n - u^*\| = o\bigl(1 - \zeta^{2(\mu/(p+1))^{2n}}\bigr),$$

where $0 < \zeta < 1$ is a positive constant, which is determined by $u_0 = \rho^\kappa \ln^{\beta-1}(t)$ and some $\rho \in (0,1)$;

(iv) *The exact solution u^* of Eq.* (1) *has an entire asymptotic estimate*

$$l(\ln t)^{\beta-1} \leqslant u^*(t) \leqslant l^{-1}(\ln t)^{\beta-1}, \quad t \in [1,\mathrm{e}],\ 0 < l < 1.$$

Proof. Firstly, we show that $T\colon \Lambda \to \Lambda$ is a compact operator.

To do this, for any $u \in \Lambda$, by the definition of the set Λ there exists a constant $0 < l_u < 1$ such that

$$l_u(\ln t)^{\beta-1} \leqslant u(t) \leqslant l_u^{-1}(\ln t)^{\beta-1}, \quad t \in [1,\mathrm{e}]. \tag{9}$$

Notice that $f(t,x)$ is decreasing in x, it follows from Lemma 2, (7), (9) and Proposition 1 that

$$\begin{aligned}
T(u)(t) &= \int_1^{\mathrm{e}} H(t,s)R^{-1}\left(\int_1^{\mathrm{e}} G(s,\tau)f\bigl(\tau,u(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\
&\leqslant \int_1^{\mathrm{e}} \frac{(\ln t)^{\beta-1}}{\Gamma(\beta)}R^{-1}\left(\frac{1}{\Gamma(\alpha)}\int_1^{\mathrm{e}}(1-\ln\tau)^{\alpha-2}f\bigl(\tau, l_u(\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\
&\leqslant \frac{l_u^{-\mu/(p+1)}}{\Gamma(\beta)}\left(\frac{1}{\Gamma(\alpha)}+1\right)^{1/(p+1)}R^{-1}\left(\int_1^{\mathrm{e}}(1-\ln\tau)^{\alpha-2}f\bigl(\tau,(\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\right) \\
&< +\infty.
\end{aligned} \tag{10}$$

So T is well defined and uniformly bounded.

On the other hand, notice that $H(s,t)$ is uniformly continuous on $[1,\mathrm{e}]\times[1,\mathrm{e}]$, and let $1\leqslant t_1<t_2\leqslant e$ for all $u\in\Lambda$. Then we have

$$\begin{aligned}
&\bigl|(Tu)(t_1)-(Tu)(t_2)\bigr| \\
&\quad\leqslant \int_0^1 \bigl|H(t_1,s)-H(t_2,s)\bigr| R^{-1}\Biggl(\int_1^{\mathrm{e}} G(s,\tau) f\bigl(\tau,u(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \int_0^1 \bigl|H(t_1,s)-H(t_2,s)\bigr| R^{-1}\Biggl(\frac{1}{\Gamma(\alpha)}\int_1^{\mathrm{e}} (1-\ln\tau)^{\alpha-2} f\bigl(\tau,l_u(\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant l_u^{-\mu/(p+1)}\biggl(\frac{1}{\Gamma(\alpha)}+1\biggr)^{1/(p+1)} R^{-1}\Biggl(\int_1^{\mathrm{e}} (1-\ln\tau)^{\alpha-2} f\bigl(\tau,(\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr) \\
&\qquad\times\int_0^1 \bigl|H(t_1,s)-H(t_2,s)\bigr|\frac{\mathrm{d}s}{s},
\end{aligned}$$

that is, $T(\Lambda)$ is equicontinuous, and then T is a compact operator in Λ.

Now we show that $T(\Lambda)\subset\Lambda$. In fact, similar to (10), for any $u\in\Lambda$, one has

$$\begin{aligned}
T(u)(t) &= \int_1^{\mathrm{e}} H(t,s)R^{-1}\Biggl(\int_1^{\mathrm{e}} G(s,\tau) f\bigl(\tau,u(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\leqslant \int_1^{\mathrm{e}} \frac{(\ln t)^{\beta-1}}{\Gamma(\beta)} R^{-1}\Biggl(\frac{1}{\Gamma(\alpha)}\int_1^{\mathrm{e}} (1-\ln\tau)^{\alpha-2} f\bigl(\tau,l_u(\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\leqslant \frac{l_u^{-\mu/(p+1)}}{\Gamma(\beta)}\biggl(\frac{1}{\Gamma(\alpha)}+1\biggr)^{1/(p+1)} \\
&\qquad\times R^{-1}\Biggl(\int_1^{e} (1-\ln\tau)^{\alpha-2} f\bigl(\tau,(\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)(\ln t)^{\beta-1} \\
&< \widetilde{l}_{T_u}^{-1}(\ln t)^{\beta-1}
\end{aligned}\tag{11}$$

and

$$\begin{aligned}
(Tu)(t) &= \int_1^{\mathrm{e}} H(t,s)R^{-1}\Biggl(\int_1^{\mathrm{e}} G(s,\tau) f\bigl(\tau,u(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\geqslant (\ln t)^{\beta-1}\int_1^{\mathrm{e}} \frac{1}{\Gamma(\beta)}\psi_\beta(s) R^{-1}\Biggl(\int_1^{\mathrm{e}} G(s,\tau) f\bigl(\tau,l_u^{-1}(\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{l_u^{-\mu/(p+1)}}{\Gamma(\beta)} \int_1^{\mathrm{e}} \psi_\beta(s) R^{-1}\left(\int_1^{\mathrm{e}} G(s,\tau) f\bigl(\tau, (\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}(\ln t)^{\beta-1} \\
&\geqslant (\ln t)^{\beta-1}\frac{l_u^{-\mu/(p+1)}}{\Gamma(\beta)}\left(\frac{1}{\Gamma(\alpha)}\right)^{1/(p+1)} \int_1^{\mathrm{e}} \psi_\beta(s)(\ln s)^{(\alpha-1)/(p+1)}\frac{\mathrm{d}s}{s} \\
&\quad \times R^{-1}\left(\int_1^{\mathrm{e}} \psi_\alpha(\tau) f\bigl(\tau, (\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\right) \\
&\geqslant (\ln t)^{\beta-1}\frac{l_u^{-\mu/(p+1)}\Gamma(\frac{\alpha-1}{p+1}+2)}{\Gamma(\beta+\frac{\alpha-1}{p+1}+2)}\left(\frac{1}{\Gamma(\alpha)}\right)^{1/(p+1)} \\
&\quad \times R^{-1}\left(\int_1^{\mathrm{e}} \psi_\alpha(\tau) f\bigl(\tau, (\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\right) \\
&\geqslant \widetilde{l}_{T_u}(\ln t)^{\beta-1},
\end{aligned}
\tag{12}
$$

where $\widetilde{l}_{T_u}$ satisfies

$$
\begin{aligned}
0 &< \widetilde{l}_{T_u} \\
&< \min\left\{\frac{1}{2}, \left\{\frac{l_u^{-\mu/(p+1)}}{\Gamma(\beta)}\left(\frac{1}{\Gamma(\alpha)}+1\right)^{1/(p+1)}\right.\right. \\
&\qquad \left.\times R^{-1}\left(\int_1^{\mathrm{e}} (1-\ln\tau)^{\alpha-2} f\bigl(\tau, (\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\right\}^{-1}, \\
&\qquad \frac{l_u^{-\mu/(p+1)}\Gamma(\frac{\alpha-1}{p+1}+2)}{\Gamma(\beta+\frac{\alpha-1}{p+1}+2)}\left(\frac{1}{\Gamma(\alpha)}\right)^{1/(p+1)} \\
&\qquad \left.\times R^{-1}\left(\int_1^{\mathrm{e}} \psi_\alpha(\tau) f\bigl(\tau, (\ln\tau)^{\beta-1}\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\right\}.
\end{aligned}
\tag{13}
$$

Hence we have $T(\Lambda) \subset \Lambda$ from (11) and (12).

Next, by developing the double monotone iterative technique we will show that Eq. (1) has a unique positive solution u^* in Λ.

Taking $\omega(t) = (\ln t)^{\beta-1}$, we have $\omega \in \Lambda$, and then $T\omega \in \Lambda$, which implies that there exists a constant $0 < l_{T_\omega} < 1$ such that

$$
l_{T_\omega}\omega(t) \leqslant T\omega(t) \leqslant l_{T_\omega}^{-1}\omega(t), \tag{14}
$$

where l_{T_ω} is chosen according to (13). Notice that $0 < \mu/(p+1) < 1$ for some $\rho \in (0,1)$ and take a sufficiently large positive constant κ such that

$$
\bigl[\rho^{1-\mu/(p+1)}\bigr]^\kappa \leqslant l_{T_\omega}. \tag{15}
$$

Now we construct an iterative sequence with initial value $u_0(t) = \rho^\kappa \omega(t)$

$$u_n(t) = Tu_{n-1}(t), \quad t \in [1, \mathrm{e}],\ n = 1, 2, \dots . \tag{16}$$

Then we assert

$$u_0 \leqslant u_2 \leqslant \cdots \leqslant u_{2n} \leqslant \cdots \leqslant u_{2n+1} \leqslant \cdots \leqslant u_3 \leqslant u_1. \tag{17}$$

In fact, noticing that T is a decreasing operator in u, by (14)–(16) we have

$$\begin{aligned} u_0(t) &= \rho^\kappa \omega(t) \leqslant \omega(t), \\ u_1(t) &= Tu_0(t) \geqslant T\omega(t) \geqslant l_{T_\omega}\omega(t) \geqslant \bigl(\rho^{-\mu/(p+1)+1}\bigr)^\kappa \omega(t) \\ &= \bigl(\rho^{\mu/(p+1)}\bigr)^{-\kappa} \rho^\kappa \omega(t) = \bigl(\rho^{\mu/(p+1)}\bigr)^{-\kappa} u_0(t) \geqslant u_0(t), \end{aligned} \tag{18}$$

and then

$$u_2(t) = Tu_1(t) \leqslant Tu_0(t) = u_1(t). \tag{19}$$

On the other hand, from (7) and (14) we have the following estimates:

$$\begin{aligned} u_1(t) &= Tu_0(t) \\ &= \int_1^{\mathrm{e}} H(t,s) R^{-1}\Biggl(\int_1^{\mathrm{e}} G(s,\tau) f\bigl(\tau, \rho^\kappa \omega(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\ &\leqslant \rho^{-\mu\kappa/(p+1)} \int_1^{\mathrm{e}} H(t,s) R^{-1}\Biggl(\int_1^{\mathrm{e}} G(s,\tau) f\bigl(\tau, \omega(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\ &= \rho^{-\mu\kappa/(p+1)} T\omega \leqslant \rho^{-\mu\kappa/(p+1)} l_{T_\omega}^{-1}\omega(t) \leqslant \rho^{-\kappa}\omega(t). \end{aligned} \tag{20}$$

By using (8), (14), (20) and the monotonicity of T we get

$$\begin{aligned} u_2(t) &= Tu_1(t) \\ &\geqslant T\bigl(\rho^{-\kappa}\omega(t)\bigr) = \int_1^{\mathrm{e}} H(t,s) R^{-1}\Biggl(\int_1^{\mathrm{e}} G(s,\tau) f\bigl(\tau, \rho^{-\kappa}\omega(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\ &\geqslant \rho^{\kappa\mu/(p+1)} T\omega(t) \geqslant \rho^{\kappa\mu/(p+1)} l_{T_\omega}\omega(t) \geqslant \rho^\kappa \omega(t) = u_0(t). \end{aligned} \tag{21}$$

Thus it follows from (18), (19) and (21) that

$$u_0(t) \leqslant u_2(t) \leqslant u_1(t).$$

So by mathematical induction the conclusion (17) holds.

On the other hand, for any $c \in (0, 1)$, it follows from Proposition 1, (7) and (8) that

$$T^2(cu) \geqslant c^{(\mu/(p+1))^2} T^2 u. \tag{22}$$

Since T^2 is increasing operator with respect to u, thus let us apply (22) repeatedly, one gets

$$\begin{aligned}
u_{2n}(t) &= Tu_{2n-1}(t) = T^{2n}u_0(t) = T^{2n}\big(\rho^{\kappa}\omega(t)\big) = T^{2n}\big(\rho^{2\kappa}\rho^{-\kappa}\omega(t)\big) \\
&\geqslant T^{2n-2}\big(T^2\big(\rho^{2\kappa}u_1(t)\big)\big) \geqslant T^{2n-2}\big(\big(\rho^{2\kappa}\big)^{(\mu/(p+1))^2}T^2u_1(t)\big) \\
&= T^{2n-4}T^2\big(\big(\rho^{2\kappa}\big)^{(\mu/(p+1))^2}T^2u_1(u)\big) \geqslant T^{2n-4}\big(\big(\rho^{2\kappa}\big)^{(\mu/(p+1))^4}T^4u_1(u)\big) \\
&\geqslant \cdots \geqslant \big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}T^{2n}u_1(u) = \big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}T^{2n+1}u_0(t) \\
&= \big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}u_{2n+1}(t),
\end{aligned}$$

that is,

$$\big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}u_{2n+1}(t) \leqslant u_{2n}(t) \leqslant u_{2n+1}(t).$$

So for any natural numbers n and m, we have

$$\begin{aligned}
0 &\leqslant u_{2(n+m)}(t) - u_{2n}(t) \leqslant u_{2n+1}(t) - u_{2n}(t) \\
&\leqslant \big(1 - \big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}\big)u_{2n+1} \leqslant \big(1 - \big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}\big)u_1(t) \\
&\leqslant \big(1 - \big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}\big)\rho^{-\kappa}\omega(t)
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
0 &\leqslant u_{2n+1}(t) - u_{2(n+m)+1}(t) \leqslant u_{2n+1}(t) - u_{2n}(t) \\
&\leqslant \big(1 - \big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}\big)\rho^{-\kappa}\omega(t).
\end{aligned} \tag{24}$$

Notice that P is a normal cone with normality constant 1, thus it follows from (23), (24) that

$$\|u_{n+m} - u_n\| \leqslant \big(1 - \big(\rho^{2\kappa}\big)^{(\mu/(p+1))^{2n}}\big)\rho^{-\kappa} \to 0, \quad n \to +\infty,$$

which implies that $\{u_n\}$ is a Cauchy sequence of compact set. Since $\{u_n\} \in \Lambda$ and $T(\Lambda) \subset \Lambda$ is compact, consequently, $\{u_n\}$ converges to some $u^* \in \Lambda$ with

$$u_{2n} \leqslant u^* \leqslant u_{2n+1},$$

and then

$$u_{2n+2} = Tu_{2n+1} \leqslant Tu^* \leqslant Tu_{2n} = u_{2n+1}. \tag{25}$$

Let $n \to \infty$ in (25), one has $u^*(t) = Tu^*(t)$, that is, u^* is a positive solution of Eq. (1).

In the end, we show the uniqueness of u^* in Λ and the asymptotic properties of solution for Eq. (1). Suppose that $\tilde{u}$ is another positive solution u^*. Let $r_1 = \sup\{r > 0: \tilde{u} \geqslant ru^*\}$. Clearly, $0 < r_1 < +\infty$. We claim that $r_1 \geqslant 1$. Otherwise, we have $0 < r_1 < 1$, which as well as (22) implies

$$\tilde{u} = T\tilde{u} = T^2\tilde{u} \geqslant T^2(r_1u^*) \geqslant r_1^{(\mu/(p+1))^2}T^2u^* = r_1^{(\mu/(p+1))^2}u^*.$$

Since $r_1^{(\mu/(p+1))^2} > r_1$, this contradicts with the definition of r_1. So we have $r_1 \geqslant 1$, and then $\tilde{u} \geqslant u^*$. In the same way, we have $\tilde{u} \leqslant u^*$. Consequently, $\tilde{u} = u^*$, so the positive solution of Eq. (1) in Λ is unique.

On the other hand, for any initial value $v_0 \in \Lambda$, there exists a constant $l_{v_0} \in (0,1)$ such that

$$l_{v_0}\omega(t) \leqslant v_0(t) \leqslant l_{v_0}^{-1}\omega(t), \quad t \in [1, \mathrm{e}].$$

Let

$$v_n = \int_1^{\mathrm{e}} H(t,s)R^{-1}\left(\int_1^{\mathrm{e}} G(s,\tau)f\bigl(\tau, v_{n-1}(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}, \quad n = 1, 2, \dots .$$

Since $T(\Lambda) \subset \Lambda$, there still exists a constant $l_{v_1} \in (0,1)$ such that

$$l_{v_1}\omega(t) \leqslant v_1 = Tv_0 \leqslant l_{v_1}^{-1}\omega(t), \quad t \in [1, \mathrm{e}].$$

As (15), let us choose a sufficiently large $\kappa > 0$ such that

$$\bigl[\rho^{(1-\mu/(p+1))}\bigr]^{\kappa} \leqslant \min\{l_{v_0}, l_{v_1}, l_{T_\omega}\},$$

where $\rho \in (0,1)$. Thus

$$u_0 = \rho^{\kappa}\omega(t) \leqslant \bigl[\rho^{1-\mu/(p+1)}\bigr]^{\kappa}\omega(t) \leqslant l_{v_0}\omega(t) \leqslant v_0,$$
$$u_0 = \rho^{\kappa}\omega(t) \leqslant \bigl[\rho^{1-\mu/(p+1)}\bigr]^{\kappa}\omega(t) \leqslant l_{v_1}\omega(t) \leqslant v_1,$$

which implies that $v_1 = Tv_0 \leqslant Tu_0 = u_1$, and then

$$u_0 \leqslant v_1 \leqslant u_1, \qquad u_2 \leqslant v_2 \leqslant u_1. \tag{26}$$

According to the fact that T^2 is increasing operator with respect to u and (26), we have

$$u_{2n}(t) \leqslant v_{2n+1}(t) \leqslant u_{2n+1}(t), \qquad u_{2n+2}(t) \leqslant v_{2n+2}(t) \leqslant u_{2n+1}(t). \tag{27}$$

Letting $n \to \infty$ in (27), we get that v_n uniformly converges to the unique positive solution u^* of Eq. (1).

Moreover it follows from (24) and (27) that the unique positive solution u^* satisfies the following estimate of error:

$$\|v_n - u^*\| \leqslant \bigl(1 - \bigl(\rho^{2\kappa}\bigr)^{(\mu/(p+1))^{2n}}\bigr)\rho^{-\kappa} = \bigl(1 - \zeta^{2(\mu/(p+1))^{2n}}\bigr)\zeta^{-1}$$

with a exact convergence rate

$$\|v_n - u^*\| = o\bigl(1 - \zeta^{2(\mu/(p+1))^{2n}}\bigr),$$

where $0 < \zeta = \rho^{\kappa} < 1$ is a positive constant, which is determined by $u_0 = \rho^{\kappa}\omega(t)$.

Finally, since $u^* \in \Lambda$, we have the following entire asymptotic analysis, that is, there exists a constant $0 < l < 1$ such that

$$l(\ln t)^{\beta-1} \leqslant u^*(t) \leqslant l^{-1}(\ln t)^{\beta-1}.$$

The proof is completed. □

4 Example

The following example shows that conditions of Theorem 1 are easily verified.

Consider the following singular Hadamard-type fractional turbulent flow model:

$$\begin{aligned}
&-D_t^{5/2}\big(F\big(-D_t^{9/4}u(t)\big)\big(-D_t^{9/4}u(t)\big)\big) = \frac{(1-\ln t)^{-3/4}}{u^{1/2}(t)} + \ln t, \\
&\quad 1 < t < e, \\
&u(1) = S\big(u(1)\big) = S\big(u(e)\big) = 0, \\
&R\big(-D_t{}^{9/4}u(1)\big) = S\big(R\big(-D_t{}^{9/4}u(1)\big)\big) = S\big(R\big(-D_t{}^{9/4}u(e)\big)\big) = 0
\end{aligned} \tag{28}$$

with a nonlinear operator $F(x) = x^{1/2}$, where $R(x) = x^{3/2}$.

Then we have the following conclusions:

(i) The singular Hadamard-type fractional turbulent flow equation (28) has a unique positive solution $u^* \in \Lambda$;

(ii) For any initial value $v_0 \in \Lambda$, we construct the iterative sequence

$$v_n(t) = \int_1^{\mathrm{e}} H(t,s)\left(\int_1^{\mathrm{e}} G(s,\tau)\big[(1-\ln\tau)^{-3/4}v_{n-1}^{-1/2}(\tau) + \ln\tau\big]\frac{\mathrm{d}\tau}{\tau}\right)^{3/2}\frac{\mathrm{d}s}{s},$$
$$t \in [1,\mathrm{e}],\ n = 1,2,\dots,$$

then $\{v_n\}_{n\geqslant 1}$ converges uniformly to the unique positive solution u^* of Eq. (28) on $[1,\mathrm{e}]$;

(iii) The error of the iterative value v_n and the exact solution u^* can be estimated by the following formula:

$$\|v_n - u^*\| \leqslant \big(1 - \zeta^{2(1/2)^{2n}}\big)\zeta^{-1},$$

and the convergence rate is

$$\|v_n - u^*\| = o\big(1 - \zeta^{2(1/2)^{2n}}\big),$$

where $0 < \zeta < 1$ is a positive constant;

(iv) The exact solution u^* of Eq. (28) has an entire asymptotic estimate

$$l(\ln t)^{5/4} \leqslant u^*(t) \leqslant l^{-1}(\ln t)^{5/4}, \quad t \in [1,\mathrm{e}],\ 0 < l < 1.$$

First, we define a set as

$$\begin{aligned}
\Lambda = \big\{u(x) \in P\colon &\text{ there exists a positive number } 0 < l_u < 1 \text{ such that} \\
&l_u(\ln t)^{5/4} \leqslant u(x) \leqslant l_u^{-1}(\ln t)^{5/4},\ x \in [1,\mathrm{e}]\big\}.
\end{aligned}$$

Let $\alpha = 5/2$, $\beta = 9/4$, $p = 1/2$, $\mu = 5/4$,

$$f(t,u) = (1-\ln t)^{-3/4}u^{-1/2},$$

then we have

$$G(t,s)=\frac{1}{\Gamma(\frac{5}{2})}\begin{cases}(\ln t)^{3/2}(1-\ln s)^{1/2}-(\ln t-\ln s)^{3/2}, & 1\leqslant s\leqslant t\leqslant e,\\ (\ln t)^{3/2}(1-\ln s)^{1/2}, & 1\leqslant t\leqslant s\leqslant e,\end{cases}$$

and

$$H(t,s)=\frac{1}{\Gamma(\frac{9}{4})}\begin{cases}(\ln t)^{5/4}(1-\ln s)^{1/4}-(\ln t-\ln s)^{5/4}, & 1\leqslant s\leqslant t\leqslant e,\\ (\ln t)^{5/4}(1-\ln s)^{1/4}, & 1\leqslant t\leqslant s\leqslant e.\end{cases}$$

Clearly, $f:(1,e)\times(0,+\infty)\to[0,+\infty)$ is continuous and decreasing with respect to variable u, and for any $c\in(0,1)$, there exists a constant $0<\mu<p+1$ such that for all $(t,u)\in(1,e)\times(0,+\infty)$,

$$\begin{aligned}f(t,cu)&=(1-\ln t)^{-3/4}(cu)^{-1/2}+\ln t\\ &\leqslant c^{-1/2}\big[(1-\ln t)^{-3/4}u^{-1/2}+\ln t\big]\\ &\leqslant c^{-3/4}f(t,u).\end{aligned}$$

Moreover, we also have

$$\begin{aligned}0&<\int_1^{\mathrm{e}}(1-\ln\tau)^{\alpha-2}f\big(\tau,(\ln\tau)^{\beta-1}\big)\frac{\mathrm{d}\tau}{\tau}\\ &=\int_1^{\mathrm{e}}(1-\ln\tau)^{-1/4}\big[(\ln\tau)^{-5/8}+\ln\tau\big]\frac{\mathrm{d}\tau}{\tau}\\ &=\frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{8})}{\Gamma(\frac{9}{8})}+\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{11}{4})}=3.8469<+\infty,\end{aligned}$$

and then

$$\begin{aligned}0&<R^{-1}\Bigg(\int_1^{\mathrm{e}}(1-\ln\tau)^{\alpha-2}f\big(\tau,(\ln\tau)^{\beta-1}\big)\frac{\mathrm{d}\tau}{\tau}\Bigg)\\ &=\Bigg(\int_1^{\mathrm{e}}(1-\ln\tau)^{-1/4}(\ln\tau)^{-5/8}\frac{\mathrm{d}\tau}{\tau}\Bigg)^{2/3}<+\infty.\end{aligned}$$

Thus all conditions of Theorem 1 are satisfied. From Theorem 1 it follows that all of the conclusions hold.

Now we give the graphical illustration of the iterative process to show the effectiveness of approximate solution converging to the exact solution.

Figure 1 shows that the convergence speed of iterative sequence is fairly fast, especially when $n=4$, the iterative solution has almost approximated exact solution of Eq. (28). Tables 1 and 2 give a numerical approximation to exact solution, which also shows that iterative process is very effective and the iterative convergence speed is robust.

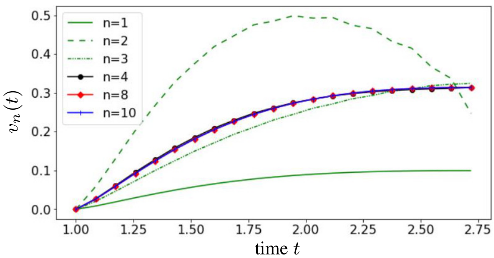

Figure 1. The sequence series $v_n(t)$ converges when $n \geqslant 4$.

Table 1. The numerical approximation of the solution for Eq. (28) in $[1.0859, 1.8591]$.

n	t									
	1.0859	1.1718	1.2577	1.3437	1.4296	1.5155	1.6014	1.6873	1.7732	1.8591
$n = 1$	0.0085	0.019	0.0297	0.0399	0.0495	0.058	0.0657	0.0723	0.0781	0.0829
$n = 2$	0.0585	0.13	0.2023	0.2669	0.3277	0.3748	0.4191	0.447	0.475	0.4848
$n = 3$	0.0205	0.0464	0.0731	0.0998	0.1249	0.1495	0.1714	0.1931	0.2112	0.2299
$n = 4$	0.0273	0.0611	0.0956	0.1279	0.1583	0.1849	0.2092	0.2293	0.2474	0.2617
$n = 5$	0.0263	0.0589	0.0924	0.1242	0.1542	0.1811	0.2054	0.2265	0.2451	0.2606
$n = 6$	0.0263	0.0589	0.0923	0.1241	0.1539	0.1807	0.2049	0.2259	0.2443	0.2598
$n = 7$	0.0263	0.059	0.0924	0.1242	0.1541	0.1809	0.2051	0.2261	0.2445	0.2599
$n = 8$	0.0263	0.0589	0.0924	0.1242	0.1541	0.1809	0.2051	0.2261	0.2445	0.2599
$n = 9$	0.0263	0.0589	0.0924	0.1242	0.1541	0.1809	0.2051	0.2261	0.2445	0.2599
$n = 10$	0.0263	0.0589	0.0924	0.1242	0.1541	0.1809	0.2051	0.2261	0.2445	0.2599

Table 2. The numerical approximation of the solution for Eq. (28) in $[1.9451, 2.7183]$.

m	t									
	1.9451	2.031	2.1169	2.2028	2.2887	2.3746	2.4605	2.5465	2.6324	2.7183
$m = 1$	0.087	0.0904	0.0931	0.0952	0.0968	0.098	0.0988	0.0993	0.0995	0.0996
$m = 2$	0.4984	0.4923	0.4937	0.4738	0.4652	0.4324	0.4147	0.3663	0.3367	0.2465
$n = 3$	0.2444	0.2603	0.2715	0.2848	0.2932	0.3041	0.3101	0.3187	0.3219	0.3247
$n = 4$	0.2744	0.2838	0.2921	0.2978	0.3027	0.3059	0.3084	0.3102	0.3115	0.3136
$n = 5$	0.2738	0.2845	0.2933	0.3	0.3052	0.3088	0.3114	0.3127	0.3134	0.3133
$n = 6$	0.273	0.2836	0.2924	0.2992	0.3045	0.3083	0.3109	0.3126	0.3134	0.3136
$n = 7$	0.2731	0.2838	0.2925	0.2993	0.3045	0.3083	0.3109	0.3125	0.3134	0.3136
$n = 8$	0.2731	0.2838	0.2925	0.2993	0.3045	0.3083	0.3109	0.3125	0.3134	0.3136
$n = 9$	0.2731	0.2838	0.2925	0.2993	0.3045	0.3083	0.3109	0.3125	0.3134	0.3136
$n = 10$	0.2731	0.2838	0.2925	0.2993	0.3045	0.3083	0.3109	0.3125	0.3134	0.3136

5 Conclusion

In this paper, we study the uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model involving a nonlinear operator. The singularities and nonlinear operator lead to some difficulties of study. To overcome these difficulties, we introduce a new double monotone iterative technique and give some useful estimations. Then we establish some new results of positive solutions including the uniqueness, the iterative sequence converging to the unique solution, error estimates, convergence rate and entire asymptotic behavior. These properties can describe nicely the dynamic behaviour of the turbulent flow. The example also indicates that the conditions of theorem are easy to be verified, the graphic and numerical approximation show that the convergence speed of iterative sequence is robust and effective.

References

1. T. Abdeljawad, R.P. Agarwal, E. Karapınar, P.S. Kumari, Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space, *Symmetry*, **11**(5):686, 2019, `https://doi.org/10.3390/sym11050686`.
2. E.E. Adams, L.W. Gelhar, Field study of dispersion in a heterogeneous aquifer: 2. Spatial moments analysis, *Water Resour. Res.*, **28**(12):3293–3307, 1992, `https://doi.org/10.1029/92WR01757`.
3. R.S. Adigüzel, Ü. Aksoy, E. Karapinar, İ.M. Erhan, On the solution of a boundary value problem associated with a fractional differential equation, *Math. Methods Appl. Sci.*, 2020, `https://doi.org/10.1002/mma.6652`.
4. H. Afshari, S. Kalantari, E. Karapinar, Solution of fractional differential equations via coupled fixed point, *Electron. J. Differ. Equ*, **2015**:286, 2015.
5. H. Afshari, E. Karapınar, A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces, *Adv. Difference Equ.*, **2020**:616, 2020, `https://doi.org/10.1186/s13662-020-03076-z`.
6. I. Ahmad, A Lyapunov-based direct adaptive controller for the suppression and synchronization of a perturbed nuclear spin generator chaotic system, *Appl. Math. Comput.*, **395**: 125858, 2021, `https://doi.org/10.1016/j.amc.2020.125858`.
7. B. Alqahtani, H. Aydi, E. Karapınar, V. Rakočević, A solution for volterra fractional integral equations by hybrid contractions, *Mathematics*, **7**(8):694, 2019, `https://doi.org/10.3390/math7080694`.
8. D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, The fractional-order governing equation of Lévy motion, *Water Resour. Res.*, **36**(6):1413–1423, 2000, `https://doi.org/10.1029/2000WR900032`.
9. Y. Ding, J. Jiang, D. O'Regan, J. Xu, Positive solutions for a system of Hadamard-type fractional differential equations with semipositone nonlinearities, *Complexity*, **2020**:9742418, 2020, `https://doi.org/10.1155/2020/9742418`.
10. J. He, X. Zhang, L. Liu, Y. Wu, Y. Cui, A singular fractional Kelvin–Voigt model involving a nonlinear operator and their convergence properties, *Boundary Value Probl.*, **2019**:112, 2019, `https://doi.org/10.1186/s13661-019-1228-7`.
11. E. Karapınar, T. Abdeljawad, F. Jarad, Applying new fixed point theorems on fractional and ordinary differential equations, *Adv. Difference Equ.*, **2019**:421, 2019, `https://doi.org/10.1186/s13662-019-2354-3`.
12. E. Karapinar, H.D. Binh, N.H. Luc, N.H. Can, On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems, *Adv. Difference Equ.*, **2021**:70, 2021, `https://doi.org/10.1186/s13662-021-03232-z`.
13. E. Karapınar, A. Fulga, M. Rashid, L. Shahid, H. Aydi, Large contractions on quasi-metric spaces with an application to nonlinear fractional differential equations, *Mathematics*, **7**(5): 444, 2019, `https://doi.org/10.3390/math7050444`.
14. M. Khan, A. Rasheed, M.S. Anwar, Z. Hussain, T. Shahzad, Modelling charge carrier transport with anomalous diffusion and heat conduction in amorphous semiconductors using fractional calculus, *Phys. Scr.*, **96**(4):045204, 2021, `https://doi.org/10.1088/1402-4896/abde0f`.

15. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

16. G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston, 1985.

17. L.S. Leibenson, General problem of the movement of a compressible fluid in a porous medium, *Izv. Akad. Nauk Kirg. SSSR*, **9**(1):7–10, 1983.

18. J.J. Nieto, An abstract monotone iterative technique, *Nonlinear Anal., Theory, Methods Appl.*, **28**(12):1923–1933, 1997, https://doi.org/10.1016/S0362-546X(97)89710-6.

19. W. Ren, J. Xiong, Stochastic stability of impulsive switched systems with constrained switching and impulses, *IEEE Control Syst. Lett.*, **5**(4):1441–1446, 2020, https://doi.org/10.1109/LCSYS.2020.3040234.

20. A. Salim, M. Benchohra, E. Karapınar, J.E. Lazreg, Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations, *Adv. Difference Equ.*, **2020**:601, 2020, https://doi.org/10.1186/s13662-020-03063-4.

21. X. Zhang, J. Jiang, L. Liu, Y. Wu, Extremal solutions for a class of tempered fractional turbulent flow equations in a porous medium, *Mathematical Problems in Engineering*, **2020**:2492193, 2020, https://doi.org/10.1155/2020/2492193.

22. X. Zhang, L. Liu, Y. Wu, The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium, *Appl. Math. Lett.*, **37**:26–33, 2014, https://doi.org/10.1016/j.aml.2014.05.002.

23. X. Zhang, L. Liu, Y. Wu, Variational structure and multiple solutions for a fractional advection–dispersion equation, *Comput. Math. Appl.*, **68**(12):1794–1805, 2014, https://doi.org/10.1016/j.camwa.2014.10.011.

24. X. Zhang, L. Liu, Y. Wu, Y. Cui, A sufficient and necessary condition of existence of blow-up radial solutions for a k-Hessian equation with a nonlinear operator, *Nonlinear Anal. Model. Control*, **25**(1):126–143, 2020, https://doi.org/10.15388/namc.2020.25.15736.

25. X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, Nontrivial solutions for a fractional advection dispersion equation in anomalous diffusion, *Appl. Math. Lett.*, **66**:1–8, 2017, https://doi.org/10.1016/j.aml.2016.10.015.

26. X. Zhang, C. Mao, L. Liu, Y. Wu, Exact iterative solution for an abstract fractional dynamic system model for bioprocess, *Qual. Theory Dyn. Syst.*, **16**(1):205–222, 2017, https://doi.org/10.1007/s12346-015-0162-z.

27. X. Zhang, L. Yu, J. Jiang, Y. Wu, Y. Cui, Positive solutions for a weakly singular Hadamard-type fractional differential equation with changing-sign nonlinearity, *J. Funct. Spaces*, **2020**:5623589, 2020, https://doi.org/10.1155/2020/5623589.

28. X. Zhang, L. Yu, J. Jiang, Y. Wu, Y. Cui, Solutions for a singular Hadamard-type fractional differential equation by the spectral construct analysis, *J. Funct. Spaces*, **2020**:8392397, 2020, https://doi.org/10.1155/2020/8392397.