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Feedback exponential stabilization of the semilinear heat equation with nonlocal initial conditions*

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Abstract. The present paper is devoted to the problem of stabilization of the one-dimensional semilinear heat equation with nonlocal initial conditions. The control is with boundary actuation. It is linear, of finite-dimensional structure, given in an explicit form. It allows to write the corresponding solution of the closed-loop equation in a mild formulation via a kernel, then to apply a fixed point argument in a convenient space.

Keywords: exponential stabilization, parabolic equations, nonlocal initial conditions, eigenvalue, feedback control, contraction mapping theorem.

1 Introduction

Here we are interested in the following equation:

$$\begin{aligned} \partial_t w(t,x) &= \partial_{xx} w(t,x) + \big[a(x)+b(t)\big] w(t,x) \\ &\quad + c(t,x) w^n(t,x), \quad t\in(0,\infty),\ x\in(0,1), \\ w(t,0) &= u_w(t), \quad w(t,1)=0, \quad t>0, \\ w(0,x) &= \sum_{k=1}^{\infty} c_k w(t_k,x), \quad x\in(0,1). \end{aligned} \tag{1}$$

Above, a, b, c are continuous functions, where, for c, there exist $C_c, m>0$ such that

$$\big|c(t,x)\big| \leqslant C_c t^m \quad \forall t>0,\ x\in[0,1]. \tag{2}$$

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The increasing set $0 < t_1 < t_2 < \cdots < t_k < \cdots$ satisfies $t_k \to \infty$ when $k \to \infty$; c_k are real numbers, $c_k \neq 0$, $k = 1, 2, \ldots$, such that there exists $q > 0$ for which

$$\sum_{k=1}^{\infty} \mathrm{e}^{-qt_k} |c_k| < \infty. \tag{3}$$

Here $n > 1$. We assume that the initial data $w(0, \cdot)$ is square integrable on $(0, 1)$. Finally, u_w is a boundary actuator.

Parabolic problems with nonlocal initial conditions, as (1), appear in the modelling of concrete problems such as heat conduction and in thermoelasticity. For example, if there is too little gas at the initial time, then the measurement $w(0, x)$ of the amount of the gas in this instant may be less precise than the measurement $w(0, x) + \sum_{k=1}^{\infty} c_k w(t_k, x)$ of the sum of the amounts of this gas (for details, see [4]). They can be as well used for modelling certain physical measurements performed repeatedly by the devices having relaxation time comparable to the delay between the measurements. Particular cases of the set $\{c_k\}_{k=1}^{\infty}$ cover many well-known physical phenomena such as: problems with periodic conditions $u(0) = u(t_1)$, problems with Bitsadze–Samarskii conditions $u(0) + c_1 u(t_1) = c_2 u(t_2)$, regularized backward problems, etc. (for more details, see [11] and the references therein).

In this work, we address the problem of asymptotic exponential stabilization of (1). More precisely, we look for a control u_w given in a feedback form, i.e., $u_w(t) = u_w(w(t))$, such that once inserted into equation (1) it yields that the corresponding solution of the closed-loop equation (1) satisfies the exponential decay

$$\int_0^1 w^2(t, x)\, \mathrm{d}x \leqslant C\mathrm{e}^{-\rho t} \quad \forall t \geqslant 0$$

for a constant $C > 0$ and arbitrarily large ρ. The problem of exponential stability associated to (1), i.e., whether the solution of the uncontrolled equation (1) ($u_w \equiv 0$) satisfies an exponential decay as above, has been addressed in many works, see, for example, [2,4,9]. There it is shown that, under some appropriate conditions on the coefficients, the solution decays exponentially fast at infinity. In our case, since we let free the coefficients a, b, c in (1), it is clear that one cannot expect such an exponential decay to hold true. In fact, we do not even know whether equation (1) has solutions at all. But if it has, the blow-up phenomenon may occur (see [6]). Equation (1) is in connection with the stabilization to states for the semilinear heat equation with nonlocal initial conditions. We stress that the time dependency of the linear governing operator is related to nonstationary states. In other words, we include in our study the case of stabilization to trajectories for the semilinear heat equation. The ideas in this paper rely on the controller design technique developed in [7]. There is an explicit feedback form control designed for stabilizing parabolic-type equations. Its simple form allows us to write the corresponding solution of the closed-loop equation in a mild formulation via a kernel similar to the heat kernel. In this way the solution becomes a fixed point of a nonlinear map. Then, applying a fixed point argument, in a convenient space, we prove simultaneously the well-posedness of the equation and the exponential decay of the solution not only in the L^2-norm, but also in the H^1-norm.

Let $\delta > \max_{x\in[0,1]} |b(x)|$, then we have

$$\left|\frac{1}{t}\int_0^t b(s)\,\mathrm{d}s\right| < \delta \quad \forall t > 0. \tag{4}$$

In (1), let us perform the transformation

$$y(t,x) := \mathrm{e}^{-\int_0^t b(s)\,\mathrm{d}s+\alpha t} w(t,x), \quad \alpha := \delta + \frac{m+q+1/4}{n-1}.$$

Recall that $n > 1$. We equivalently express (1) in terms of y as

$$\begin{aligned} \partial_t y(t,x) &= \partial_{xx} y(t,x) + a(x)y(t,x) + \alpha y(t,x) \\ &\quad + \tilde{c}(t,x)y^n(t,x), \quad t \in (0,\infty),\ x \in (0,1), \\ y(t,0) &= u_y(t), \qquad y(t,1) = 0, \\ y(0,x) &= \sum_{k=1}^{\infty} \tilde{c}_k y(t_k,x), \quad x \in (0,1). \end{aligned} \tag{5}$$

Here

$$\tilde{c}(t,x) := \mathrm{e}^{[\int_0^t b(s)\,\mathrm{d}s-\alpha t](n-1)} c(t,x), \qquad u_y(t) := \mathrm{e}^{-\int_0^t b(s)\,\mathrm{d}s+\alpha t} u_w(t)$$

and

$$\tilde{c}_k := \mathrm{e}^{\int_0^{t_k} b(s)\,\mathrm{d}s-\alpha t_k} c_k, \quad k \in \mathbb{N}^*.$$

Let us notice that, in virtue of relations (4) and (2), we have

$$\begin{aligned} \left|\tilde{c}(t,x)\right| &\leqslant C_c \mathrm{e}^{(n-1)\int_0^t b(s)\,\mathrm{d}s} \mathrm{e}^{(-\delta(n-1)-m-q-1/4)t} t^m \\ &\leqslant C_c \mathrm{e}^{(n-1)[\int_0^t b(s)\,\mathrm{d}s-\delta t]} t^{-m} \mathrm{e}^{-mt} \mathrm{e}^{-qt} \mathrm{e}^{-t/4} \\ &\leqslant C_c t^{-1/4} \quad \forall t > 0,\ x \in (0,1), \end{aligned} \tag{6}$$

by using the obvious inequality (which will be frequently used below as well)

$$\mathrm{e}^{-\mu t} \leqslant t^{-\mu} \quad \forall t, \mu > 0.$$

Besides this, by (3) it is clearly seen that the series

$$\sum_{k=1}^{\infty} |\tilde{c}_k| \leqslant \sum_{k=1}^{\infty} \mathrm{e}^{\int_0^{t_k} b(s)\,\mathrm{d}s-\delta t_k} \mathrm{e}^{-qt_k} |c_k| < \infty.$$

We set

$$L^p(0,1) = \left\{ f : (0,1) \to \mathbb{R}\colon \int_0^1 \left|f(x)\right|^p \mathrm{d}x < \infty \right\}, \quad p > 1.$$

For the particular case $p = 2$, we denote by $\|\cdot\|$ and by $\langle \cdot, \cdot \rangle$ the standard norm and scalar product in $L^2(0,1)$, respectively. While for $p \neq 2$, we denote

$$\|f\|_p := \left(\int_0^1 \left|f(x)\right|^p \mathrm{d}x \right)^{1/p},$$

which is a norm in L^p.

Below, we denote by y' the derivative with respect to x, i.e., $y'(x) = (\mathrm{d}/\mathrm{d}x)y(x)$. Let us denote

$$\mathcal{A}y = -y'' - a(x)y - \alpha y \quad \forall y \in H^2(0,1) \cap H_0^1(0,1).$$

Here $H^r(0,1)$, $r \in \mathbb{N}^*$, stand for the standard Sobolev spaces on $(0,1)$, while $H_0^1(0,1)$ restricts to null trace functions. We recall that, via the Poincaré inequality, we have that the norm $\|\mathrm{d}/\mathrm{d}x \cdot \|$ is an equivalent norm in $H_0^1(0,1)$.

In this work, our results rely on the special properties of the spectrum of the operator $\mathcal{A}$. This is related to the Sturm–Liouville theory. In virtue of the results in [1, Sect. 2.4.1], considered for the particular case: definition interval $(0,1)$, function $p = 1$, $r(x) = a(x) + \alpha$, constants $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$, we have that $\mathcal{A}$ is self-adjoint and has a countable set of simple real eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}^*}$ with the corresponding eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}^*}$. Moreover, the eigenvalues can be arranged as an increasing sequence with $\lambda_k \to \infty$, and the eigenfunctions set forms an orthonormal basis in $L^2(0,1)$. Since $\lambda_k \to \infty$ when $k \to \infty$, we see that for $N \in \mathbb{N}$ large enough, $\lambda_k > 0$ for all $k \geqslant N$. Besides this, by [12, Thm. 4.3.1(7)] considered for the particular case: definition interval $(0,1)$, $p = 1$, $q(x) = -a(x) - \alpha$, $w = 1$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, we have that the eigenvalues of $\mathcal{A}$ satisfy $\lim_{j\to\infty} \lambda_j / j^2 = \mathrm{const}$. Hence, we have

$$\sum_{j=N}^{\infty} \frac{1}{\lambda_j} < \infty. \tag{7}$$

The energy space H_A of a positive definite self-adjoint operator $A : \mathcal{D}(A) \subset L^2(0,1) \to L^2(0,1)$ is a Hilbert space defined by introducing the inner product $\langle f, g \rangle_A := \langle Af, g \rangle$ and the energy norm $\|f\|_A := \sqrt{\langle f, f \rangle_A}$. The resulting space is then completed by including all limit elements. Moreover, the eigenfunctions system of A form a complete system in $L^2(0,1)$ and H_A. For details, see, e.g., [3]. It is clear that for $\zeta > 0$ sufficiently large, the operator $\mathcal{A} + \zeta I$ (I being the identity operator) is self-adjoint and positive definite. Thus, operator $A = \mathcal{A} + \zeta I$ introduces an energy space, H_A, which is in fact the space $H_0^1(0,1)$. The eigenvalues set of $\mathcal{A} + \zeta I$ is $\{\lambda_k + \zeta\}_{k=1}^{\infty}$, while the corresponding eigenfunctions are the same as of $\mathcal{A}$, namely $\{\varphi_k\}_{k\in\mathbb{N}^*}$. Moreover, in virtue of the results in [3], the set $\{\varphi_k\}_{k\in\mathbb{N}^*}$ is complete in $H_0^1(0,1)$, and there exists $\tilde{C} > 0$ such that

$$\|f\|_{H_0^1(0,1)} \leqslant \tilde{C}\left[\sum_{k=1}^{\infty}(\lambda_k+\zeta)\langle f,\varphi_k\rangle^2\right]^{1/2} \quad \forall f \in \mathcal{D}(\mathcal{A}). \tag{8}$$

Recall that $\lambda_j \to \infty$ when $j \to \infty$. So, we assume that N is large enough such that

$$\lambda_j \geqslant \zeta \quad \forall j \geqslant N+1. \tag{9}$$

It is easy to see that the fundamental solution associated to $-\mathcal{A}$, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = -\mathcal{A}z, \qquad z(0) = z_0,$$

can be written as

$$z(t,x) = \int_0^1 \mathrm{e}^{-\lambda_j t}\varphi_j(x)\varphi_j(\xi)z_0(\xi)\,\mathrm{d}\xi,$$

and, by [10], we know that

$$\sum_{j=1}^{\infty}\mathrm{e}^{-\lambda_j t}\varphi_j^2(x) < C\frac{1}{\sqrt{t}} \quad \forall t>0,\ x\in(0,1), \tag{10}$$

for some positive constant C.

2 Construction of the stabilizer and the main result

We will apply the control design technique described in [7, Chap. 2]. The first idea is to lift the boundary control u into the equations via the Dirichlet map defined as (see [7, Eq. (2.16)]): for $\beta \in \mathbb{R}$ and $\gamma > 0$ large enough, we denote by $D_\gamma\beta := z$ the solution to the equation

$$\begin{aligned} &\mathcal{A}z - 2\sum_{k=1}^{N}\lambda_k\langle z,\varphi_k\rangle\varphi_k + \gamma z = 0 \quad \text{in } (0,1),\\ &z(0) = \beta \quad \text{and} \quad z(1) = 0. \end{aligned} \tag{11}$$

Notice that $\gamma > 0$ sufficiently large guarantees the unique existence of such solution z. The dual of D_γ, see [7, Eq. (2.17)] and [7, Ex. 2.4], depends on $D_\gamma^\diamond$, which, in our case, is given by $D_\gamma^\diamond\varphi_k = \varphi_k'(0)$, $k = 1,2,\dots$.

Let

$$\lambda_N < \gamma_1 < \gamma_2 < \cdots < \gamma_N,$$

N are the constants sufficiently large such that for each of them, the corresponding equation (11) has a unique solution D_{γ_k}, $k = 1, 2, \dots, N$. Then, following the notations in [7, Eqs. (2.20)–(2.26)], we denote by $\mathbf{B}$ the Gram matrix

$$\mathbf{B} := \bigl(\varphi_i'(0)\varphi_j'(0)\bigr)_{1\leqslant i,j\leqslant N} \tag{12}$$

and multiply it on both sides by

$$\Lambda_{\gamma_k} := \operatorname{diag}\left(\frac{1}{\gamma_k - \lambda_1}, \frac{1}{\gamma_k - \lambda_2}, \dots, \frac{1}{\gamma_k - \lambda_N}\right), \quad k = 1, 2, \dots, N,$$

to define

$$B_k := \Lambda_{\gamma_k}\mathbf{B}\Lambda_{\gamma_k}, \quad k = 1, \dots, N. \tag{13}$$

Then we introduce the matrix

$$A := (B_1 + \dots + B_N)^{-1} \tag{14}$$

and set the following feedback forms:

$$u_k(y) := \bigl\langle \Lambda_{\gamma_k} A\bigl(\langle y, \varphi_1\rangle, \langle y, \varphi_2\rangle, \dots, \langle y, \varphi_N\rangle\bigr)^{\mathrm{T}}, \bigl(\varphi_1'(0), \varphi_2'(0), \dots, \varphi_N'(0)\bigr)^{\mathrm{T}}\bigr\rangle_N.$$

Finally, we introduce u_y as

$$u_y(y) := -\bigl[u_1(y) + u_2(y) + \dots + u_N(y)\bigr].$$

Here $\langle\cdot,\cdot\rangle_N$ stands for the Euclidean scalar product in $\mathbb{R}^N$. For more details on the construction of u, one can see [7, Ex. 2.5].

Next, we plug this feedback into equation (5) and argue similarly as in [7, Eqs. (2.27)–(2.29)] in order to equivalently rewrite (5) as an internal-type control problem as follows:

$$\begin{aligned}
\partial_t y(t) &= -\mathcal{A}y(t) + \sum_{i=1}^{N}(\mathcal{A} + \gamma_i)D_{\gamma_i}u_i\bigl(y(t)\bigr) \\
&\quad - 2\sum_{i,j=1}^{N}\lambda_j\bigl\langle D_{\gamma_i}u_i\bigl(y(t)\bigr), \varphi_j\bigr\rangle\varphi_j + \tilde{c}(t)y^n(t), \quad t > 0; \\
y(0) &= \sum_{k=1}^{\infty}\tilde{c}_k y(t_k).
\end{aligned}$$

Following the ideas in [7, Lemma 7.1], we may arrive to the following result related to the linear operator that governs equation (15).

Lemma 1. *The solution z of*

$$\begin{aligned}
\partial_t z(t) &= -\mathcal{A}z(t) + \sum_{i=1}^{N}(\mathcal{A} + \gamma_i)D_{\gamma_i}u_i\bigl(z(t)\bigr) \\
&\quad - 2\sum_{i,j=1}^{N}\lambda_j\bigl\langle\bigl(z(t)\bigr)D_{\gamma_i}u_i\bigl(z(t)\bigr), \varphi_j\bigr\rangle\varphi_j, \quad t > 0; \\
z(0) &= z_0
\end{aligned} \tag{15}$$

can be written in a mild formulation as

$$z(t,x) = \int_0^1 p(t,x,\xi) z_0(\xi)\,\mathrm{d}\xi,$$

where the kernel

$$p(t,x,\xi) := p_1(t,x,\xi) + p_2(t,x,\xi) + p_3(t,x,\xi) \tag{16}$$

for $t \geqslant 0$*,* $x, \xi \in (0,1)$*. Here*

$$p_1(t,x,\xi) := \sum_{i=1}^{N} \left(\sum_{j=1}^{N} q_{ji}(t) \varphi_j(x) \right) \varphi_i(\xi),$$
$$p_2(t,x,\xi) := \sum_{i=N+1}^{\infty} \mathrm{e}^{-\lambda_i t} \varphi_i(x) \varphi_i(\xi),$$

and

$$p_3(t,x,\xi) := \sum_{i=1}^{N} \left(\sum_{j=N+1}^{\infty} w_i^j(t) \varphi_j(x) \right) \varphi_i(\xi).$$

The quantities $q_{ji}(t)$ *and* $w_i^j(t)$*, involved in the definition of* p*, obey the estimates*: *for some* $C_q > 0$ *depending on* N*,*

$$\left| q_{ji}(t) \right| \leqslant C_q \mathrm{e}^{-\gamma_1 t} \quad \forall t \geqslant 0 \tag{17}$$

for all $i, j = 1, 2, \dots, N$; *and for some* $C_w > 0$ *depending on* N*,*

$$\left| w_i^j(t) \right| \leqslant C_w \frac{1}{|\lambda_j - \gamma_1|} \mathrm{e}^{-\gamma_1 t} \quad \forall t \geqslant 0 \tag{18}$$

for all $i = 1, 2, \dots, N$ *and* $j = N+1,\ N+2, \dots$ *.*

Moreover, for all $z_0 \in L^2(0,1)$*, we have that*

$$\left\{ \sum_{j=N+1}^{\infty} \lambda_j \left[\sum_{i=1}^{N} w_i^j(t) \langle z_0, \varphi_i \rangle \right]^2 \right\}^{1/2} \leqslant C \mathrm{e}^{-\gamma_1 t} \sup_{l \in \{1,2,\dots,N\}} \left| \langle z_0, \varphi_l \rangle \right| \quad \forall t \geqslant 0. \tag{19}$$

Relying on the key lemma above, we rewrite (5) in a mild formulation as

$$y(t,x) = \int_0^1 p(t,x,\xi) y(0,\xi)\,\mathrm{d}\xi + \int_0^t \int_0^1 p(t-s,x,\xi) \tilde{c}(s,\xi) y^n(s,\xi)\,\mathrm{d}\xi\,\mathrm{d}s, \tag{20}$$

where p is defined in (16). We aim to express, in (20), the nonlocal initial condition inherited from equation (5). To this end, we denote by $\tilde{K}$ the following integral operator

$\tilde{K} : L^2(0,1) \to L^2(0,1)$:

$$(\tilde{K}y_0)(x) := \sum_{k=1}^{\infty} \tilde{c}_k \int_0^1 p(t_k, x, \xi) y_0(\xi)\, \mathrm{d}\xi.$$

We claim that the operator $I - \tilde{K}$ is invertible. To show this, we apply the result in [5, Thm. 1.1]. The main ingredient we will use is the exponential decay of the kernel p. In the spirit of [5], let us denote

$$K(x, \xi) := \sum_{k=1}^{\infty} \tilde{c}_k p(t_k, x, \xi).$$

Then we show that

$$\int_0^1 \sup_{x \in [0,1]} \big| K(x, \xi) \big| \,\mathrm{d}\xi < \infty.$$

Indeed, taking into account the particular form of K, we have

$$\begin{aligned}
\int_0^1 \sup_{x\in[0,1]} \big|K(x,\xi)\big|\,\mathrm{d}\xi &= \int_0^1 \sup_{x\in[0,1]} \sum_{k=1}^{\infty} \tilde{c}_k \big| p_1(t_k,x,\xi) + p_2(t_k,x,\xi) + p_3(t_k,x,\xi) \big| \,\mathrm{d}\xi \\
&\leqslant \int_0^1 \sup_{x\in[0,1]} \sum_{k=1}^{\infty} |\tilde{c}_k| \left| \sum_{i=1}^{N} \left(\sum_{j=1}^{N} q_{ji}(t_k)\varphi_j(x) \right) \varphi_i(\xi) \right| \mathrm{d}\xi \\
&\quad + \int_0^1 \sup_{x\in[0,1]} \sum_{k=1}^{\infty} \tilde{c}_k \left| \sum_{i=N+1}^{\infty} \mathrm{e}^{-\lambda_i t_k} \varphi_i(x)\varphi_i(\xi) \right| \mathrm{d}\xi \\
&\quad + \int_0^1 \sup_{x\in[0,1]} \sum_{k=1}^{\infty} |\tilde{c}_k| \left| \sum_{i=1}^{N} \left(\sum_{j=N+1}^{\infty} w_i^j(t_k)\varphi_j(x) \right) \varphi_i(\xi) \right| \mathrm{d}\xi.
\end{aligned}$$

Making use of the uniform bound of the eigenfunction system, the fact that

$$\int_0^1 \big|\varphi_j(\xi)\big| \,\mathrm{d}\xi \leqslant 1 \quad \forall j \in \mathbb{N}^*,$$

together with relations (17)–(18), it yields from above that

$$\begin{aligned}
&\int_0^1 \sup_{x\in[0,1]} \big|K(x,\xi)\big|\,\mathrm{d}\xi \\
&\quad \leqslant \mathcal{C}_1 \sum_{k=1}^{\infty} |\tilde{c}_k| \left\{ \mathrm{e}^{-\gamma_1 t_k} + \sum_{i=N+1}^{\infty} \mathrm{e}^{-t_k\lambda_i} + \sum_{j=N+1}^{\infty} \frac{1}{|\lambda_j - \gamma_1|} \mathrm{e}^{-\gamma_1 t_k} \right\},
\end{aligned}$$

where $\mathcal{C}_1 > 0$ is some constant. We bound the RHS of the above relation by taking into account that

$$\sum_{i=N+1}^{\infty} \mathrm{e}^{-t_k \lambda_i} = \sum_{i=N+1}^{\infty} \left(\frac{1}{\mathrm{e}^{t_k}}\right)^{\lambda_i} \leqslant \sum_{i=N}^{\infty} \left(\frac{1}{\mathrm{e}^{t_k}}\right)^{i} = \frac{1}{\mathrm{e}^{N t_k}} \frac{\mathrm{e}^{t_k}}{\mathrm{e}^{t_k} - 1} < \mathrm{e}^{-N t_k}\left(1 + \frac{1}{\mathrm{e}^{t_1} - 1}\right)$$

because $\lambda_i \to \infty$ when $i \to \infty$ and N is large enough; and by taking into account that, due to (7), we have $\sum_{j=N+1}^{\infty} 1/|\lambda_j - \gamma_1|$ is a convergent series. It yields that

$$\int_0^1 \sup_{x\in[0,1]} \bigl|K(x,\xi)\bigr| \,\mathrm{d}\xi \leqslant \mathcal{C}_2 \sum_{k=1}^{\infty} |\tilde{c}_k| \bigl(\mathrm{e}^{-\gamma_1 t_k} + \mathrm{e}^{-N t_k}\bigr) < \mathcal{C}_2 \bigl(\mathrm{e}^{-\gamma_1 t_1} + \mathrm{e}^{-N t_1}\bigr) \sum_{k=1}^{\infty} |\tilde{c}_k| < \infty. \tag{21}$$

Clearly seen, for γ_1, N large enough, the above quantity can be made arbitrary small.

We go on following [5] and introduce the quantities

$$M_\infty(V_\pm) := \int_0^1 \omega_\pm(\xi) \,\mathrm{d}\xi,$$

where

$$\omega_-(\xi) := \sup_{0\leqslant\xi\leqslant x\leqslant 1} \bigl|K(x,\xi)\bigr|, \qquad \omega_+(\xi) = \sup_{0\leqslant x\leqslant\xi\leqslant 1} \bigl|K(x,\xi)\bigr|.$$

It is easy to see that with similar arguments we used to obtain (21), based on the uniform bounds of the eigenfunction system, by taking γ_1, N sufficiently large, we may assume that $M_\infty(V_\pm)$ are small enough such that

$$\bigl(\mathrm{e}^{M_\infty(V_+)} - 1\bigr)\bigl(\mathrm{e}^{M_\infty(V_-)} - 1\bigr) < 1.$$

Thus, relation [5, Eq. (1.7)] (or, equivalently, [5, Eq. (1.5)]) is satisfied. Consequently, we are in power to apply the result in [5, Thm. 1.1], and we deduce that the operator $I - \tilde{K}$ is invertible with bounded inverse.

Now, returning to (20), we express the nonlocal initial condition as

$$\begin{aligned} y(t,x) &= \sum_{k=1}^{\infty} c_k \int_0^1 p(t,x,\xi)(I-\tilde{K})^{-1}\left[\int_0^{t_k}\int_0^1 p(t_k - s,\, \theta,\eta)\tilde{c}(s,\eta) y^n(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\right](\xi)\,\mathrm{d}\xi \\ &\quad + \int_0^t\int_0^1 p(t-s,\, x,\xi)\tilde{c}(s,\xi) y^n(s,\xi)\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned}$$

Thus, existence of a solution y is equivalent with the fact that the map $\mathcal{G}$, defined as

$$(\mathcal{G}y)(t) := \sum_{k=1}^{\infty} \tilde{c}_k \int_0^1 p(t,x,\xi)(I-\tilde{K})^{-1}(\mathcal{F}y)(t_k)\,\mathrm{d}\xi + (\mathcal{F}y)(t),$$

has a fixed point. Here

$$(\mathcal{F}y)(t) := \int_0^t \int_0^1 p(t-s,\, x,\xi)\tilde{c}(s,\xi)y^n(s,\xi)\,\mathrm{d}\xi\,\mathrm{d}s.$$

In the next section, we aim to prove the main result of the present work stated below.

Theorem 1. *Let $r > 0$ sufficiently small. Then there exists a unique fixed point $y \in B_r(0)$ of the map $\mathcal{G} : \mathcal{Y} \to \mathcal{Y}$, where $B_r(0)$ is the ball centered at the origin of radius r of the space*

$$\begin{aligned}\mathcal{Y} := \Big\{ & y \in C_b\big([0,\infty), H_0^1(0,1)\big)\colon \\ & |y|_{\mathcal{Y}} := \sup_{t\geqslant 0}\big[\mathrm{e}^{Nt}\big\|y(t)\big\| + \mathrm{e}^{Nt}t^{1/(2(n-1))}\big\|y'(t)\big\|\big] < \infty\Big\}.\end{aligned}$$

In particular, once we plug the feedback controller

$$\begin{aligned}u_w(t) := -\sum_{j=1}^{N}\big\langle & \Lambda_{\gamma_k}A\big(\langle w(t),\varphi_1\rangle, \langle w(t),\varphi_2\rangle, \dots, \langle w(t),\varphi_N\rangle\big)^{\mathrm{T}}, \\ & \big(\varphi_1'(0), \varphi_2'(0), \dots, \varphi_N'(0)\big)^{\mathrm{T}}\big\rangle_N\end{aligned}$$

into equation

$$\begin{aligned}\partial_t w(t,x) &= \partial_{xx}w(t,x) + a(x)w(t,x) + b(t)w(t,x) \\ &\quad + c(t,x)w^n(t,x), \quad t\in(0,\infty),\ x\in(0,1), \\ w(t,0) &= u_w(t), \quad w(t,1) = 0, \quad t>0, \\ w(0,x) &= \sum_{k=1}^{\infty} c_k w(t_k,x), \quad x\in(0,1),\end{aligned}$$

it yields that its unique solution is exponentially decaying in the H^1-norm. Here A, Λ_{γ_k} are given in (12)–(14)*, while φ_k, $k = 1,2,\dots,N$, are the first N eigenfunctions of the operator $\mathcal{A}$.*

Note that, due to the linearity of the control u and the definition of the transformation $w \to y$, we have

$$y(t,0) = u_y(t) \quad \text{is equivalent with} \quad w(t,0) = u_w(t).$$

3 Proof of the Theorem 1

It is clear that for all $y \in \mathcal{Y}$, we have

$$\mathrm{e}^{Nt}\|y(t)\| \leqslant |y|_{\mathcal{Y}} \quad \text{and} \quad \mathrm{e}^{(n-1)Nt}\|y'(t)\|^{n-1} \leqslant t^{-1/2}|y|_{\mathcal{Y}}^{n-1} \quad \forall t > 0. \tag{22}$$

We need to estimate the $|\cdot|_{\mathcal{Y}}$-norm of $\mathcal{G}y$. So, in particular, we need to estimate the $|\cdot|_{\mathcal{Y}}$-norm of $\mathcal{F}y$ for $y \in \mathcal{Y}$. We begin with the L^2-norm of $\mathcal{F}y$. We aim to use the Parseval's identity. In order to do this, based on the kernel's form (16), we conveniently rewrite the term $\mathcal{F}y$ as

$$\mathcal{F}y(t) = \int_0^t \left[\mathcal{F}_1\big(y(s)\big) + \mathcal{F}_2\big(y(s)\big) + \mathcal{F}_3\big(y(s)\big)\right] \mathrm{d}s,$$

where

$$\mathcal{F}_1(y)(t,s,x) := \sum_{j=1}^{N}\left[\sum_{i=1}^{N} q_{ji}(t-s)\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_i(\xi)\,\mathrm{d}\xi\right]\varphi_j(x),$$

$$\mathcal{F}_2(y)(t,s,x) := \sum_{j=N+1}^{\infty}\left[\mathrm{e}^{-\lambda_j(t-s)}\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_j(\xi)\,\mathrm{d}\xi\right]\varphi_j(x),$$

$$\mathcal{F}_3(y)(t,s,x) := \sum_{j=N+1}^{\infty}\left[\sum_{i=1}^{N} w_i^j(t-s)\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_i(\xi)\,\mathrm{d}\xi\right]\varphi_j(x).$$

We will use the well-known Sobolev embedding

$$H^1(0,1) \subset L^p(0,1) \quad \forall p > 1$$

and the fact that $\|\mathrm{d}/\mathrm{d}x \cdot \|$ is an equivalent norm in $H_0^1(0,1)$.

Below, C will stand for different constants that may change from line to line, however, we keep denote them by C for the ease of notations. It follows via the Parseval's identity and the fact that the eigenfunctions are uniformly bounded and relation (6) that

$$\begin{aligned}\big\|\mathcal{F}_1(y)\big\| &= \left\{\sum_{j=1}^{N}\left[\sum_{i=1}^{N} q_{ji}(t-s)\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_i(\xi)\,\mathrm{d}\xi\right]^2\right\}^{1/2} \\ &\leqslant C\sum_{i,j=1}^{N}\big|q_{ji}(t-s)\big|s^{-1/4}\int_0^1 \big|y(s,\xi)\big|\big|y(s,\xi)\big|^{n-1}\,\mathrm{d}\xi.\end{aligned}$$

Involving relation (17) and the Schwarz's inequality, it yields

$$\begin{aligned}\big\|\mathcal{F}_1(y)\big\| &\leqslant C\mathrm{e}^{-\gamma_1(t-s)}s^{-1/4}\big\|y(s)\big\|\big\|y(s)\big\|_{2(n-1)}^{n-1} \\ &= C\mathrm{e}^{-nNt}\mathrm{e}^{(-\gamma_1+nN+1/4)(t-s)}\mathrm{e}^{-(t-s)/4}s^{-1/4} \\ &\quad\times \mathrm{e}^{Ns}\big\|y(s)\big\|\mathrm{e}^{(n-1)Ns}\big\|y(s)\big\|_{2(n-1)}^{n-1},\end{aligned}$$

where using the Sobolev embedding, we deduce that

$$\begin{aligned}\|\mathcal{F}_1(y)\| &\leqslant C\mathrm{e}^{-nNt}\mathrm{e}^{(-\gamma_1+nN+1/4)(t-s)}\mathrm{e}^{-(t-s)/4}s^{-1/4}\mathrm{e}^{Ns}\|y(s)\| \\ &\quad\times \mathrm{e}^{(n-1)Ns}\|y'(s)\|^{n-1}.\end{aligned}$$

It follows by (22) and the fact that $-\gamma_1 + nN + 1/4 < 0$ for γ_1 large enough that

$$\begin{aligned}\|\mathcal{F}_1(y)\| &\leqslant C\mathrm{e}^{-Nt}(t-s)^{-1/4}s^{-1/4}s^{-1/2}|y|_{\mathcal{Y}}^n \\ &= C\mathrm{e}^{-Nt}(t-s)^{-1/4}s^{-3/4}|y|_{\mathcal{Y}}^n\end{aligned} \tag{23}$$

for all $t \geqslant s \geqslant 0$.

Again involving Parseval's identity, we get

$$\begin{aligned}\|\mathcal{F}_2(y)\| &= \left\{\sum_{j=N+1}^{\infty}\left[\mathrm{e}^{-\lambda_j(t-s)}\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_j(\xi)\,\mathrm{d}\xi\right]^2\right\}^{1/2} \\ &= \mathrm{e}^{-nNt}\left\{\sum_{j=N+1}^{\infty}\left[\mathrm{e}^{-(\lambda_j-nN)(t-s)}\mathrm{e}^{nNs}\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_j(\xi)\,\mathrm{d}\xi\right]^2\right\}^{1/2} \\ &\leqslant \mathrm{e}^{-Nt}\left\{\sum_{j=N+1}^{\infty}\left[\int_0^1 \left(\mathrm{e}^{-(\lambda_j-nN)(t-s)}\mathrm{e}^{Ns}y(s,\xi)\varphi_j(\xi)\right)\right.\right. \\ &\qquad \left.\left.\times\left(\tilde{c}(s,\xi)\mathrm{e}^{(n-1)Ns}y^{n-1}(s,\xi)\right)\mathrm{d}\xi\right]^2\right\}^{1/2}.\end{aligned} \tag{24}$$

In virtue of Schwarz's inequality, we obtain

$$\begin{aligned}\|\mathcal{F}_2(y)\| &\leqslant C\mathrm{e}^{-Nt}\left\{\sum_{j=N+1}^{\infty}\int_0^1 \mathrm{e}^{-2(\lambda_j-nN)(t-s)}\varphi_j^2(\xi)\mathrm{e}^{2Ns}y^2(s,\xi)\,\mathrm{d}\xi\right. \\ &\qquad\left.\times\int_0^1 \tilde{c}^2(s,\xi)\mathrm{e}^{2(n-1)Ns}y^{2(n-1)}(s,\xi)\,\mathrm{d}\xi\right\}^{1/2} \\ &= C\mathrm{e}^{-Nt}\left\{\int_0^1\left[\sum_{j=N+1}^{\infty}\mathrm{e}^{-2(\lambda_j-2N)(t-s)}\varphi_j^2(\xi)\right]\mathrm{e}^{2Ns}y^2(s,\xi)\,\mathrm{d}\xi\right. \\ &\qquad\left.\times\int_0^1 \tilde{c}^2(s,\xi)\mathrm{e}^{2(n-1)Ns}y^{2(n-1)}(s,\xi)\,\mathrm{d}\xi\right\}^{1/2}.\end{aligned} \tag{25}$$

Then, making use of (10), we get

$$\left\|\mathcal{F}_2(y)\right\| \leqslant C\mathrm{e}^{-Nt}\left\{\int_0^1 (t-s)^{-1/2}\mathrm{e}^{2Ns}y^2(s,\xi)\,\mathrm{d}\xi \times \int_0^1 \tilde{c}^2(s,\xi)\mathrm{e}^{2(n-1)Ns}y^{2(n-1)}(s,\xi)\,\mathrm{d}\xi\right\}^{1/2}. \tag{26}$$

Taking advantage of (6) and the Sobolev embedding, we arrive at

$$\begin{aligned}\left\|\mathcal{F}_2(y)\right\| &\leqslant C\mathrm{e}^{-Nt}(t-s)^{-1/4}s^{-1/4}\mathrm{e}^{Ns}\left\|y(s)\right\|\mathrm{e}^{(n-1)Ns}\left\|y'(s)\right\|^{n-1}\\ &\leqslant C\mathrm{e}^{-Nt}(t-s)^{-1/4}s^{-3/4}|y|_{\mathcal{Y}}^n \quad \forall\, 0<s<t\end{aligned} \tag{27}$$

in virtue of relation (22).

Next,

$$\begin{aligned}\left\|\mathcal{F}_3(y)\right\| &= \left\{\sum_{j=N+1}^{\infty}\left[\sum_{i=1}^{N} w_i^j(t-s)\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_i(\xi)\,\mathrm{d}\xi\right]^2\right\}^{1/2}\\ &\leqslant C\left(\sum_{j=N+1}^{\infty}\frac{1}{|\lambda_j-\gamma_1|}\right)\mathrm{e}^{-\gamma_1(t-s)}s^{-1/4}\left\|y(s)\right\|\left\|y'(s)\right\|^{n-1},\end{aligned} \tag{28}$$

by taking into account relations (6), (18) and the uniform boundedness of the eigenfunctions. Recalling relation (7), we see that the above series converge. Thus,

$$\begin{aligned}\left\|\mathcal{F}_3(y)\right\| &\leqslant C\mathrm{e}^{-nNt}\mathrm{e}^{(-\gamma_1+nN+1/4)(t-s)}\\ &\quad\times \mathrm{e}^{-(t-s)/4}s^{-1/4}\mathrm{e}^{Ns}\left\|y(s)\right\|\mathrm{e}^{(n-1)Ns}\left\|y'(s)\right\|^{n-1}\\ &\leqslant C\mathrm{e}^{-Nt}(t-s)^{-1/4}s^{-3/4}|y|_{\mathcal{Y}}^n \quad \forall 0<s<t,\end{aligned} \tag{29}$$

where we used relation (22) and the fact that $-\gamma_1 + nN + 1/4 < 0$ for γ_1 large enough.

Hence, it follows by (23)–(29) that

$$\begin{aligned}\left\|\mathcal{F}(y)(t)\right\| &\leqslant C\mathrm{e}^{-Nt}\int_0^t s^{-3/4}(t-s)^{-1/4}\,\mathrm{d}s\,|y|_{\mathcal{Y}}^n\\ &= \mathrm{e}^{-Nt}CB\left(\frac{1}{4},\frac{3}{4}\right)|y|_{\mathcal{Y}}^n \quad \forall t\geqslant 0,\end{aligned} \tag{30}$$

where $B(x,y)$ is the classical beta function.

By the exponential semigroup property we have that

$$\left\|\int_0^1 p(t,\cdot,\xi)y_0(\xi)\,\mathrm{d}\xi\right\| \leqslant C\mathrm{e}^{-\gamma_1 t}\|y_0\| \quad \forall y_0\in L^2(0,1).$$

Consequently, we have

$$\left\| \sum_{k=1}^{\infty} \tilde{c}_k \int\limits_0^1 p(t,\cdot,\xi)(I-\tilde{K})^{-1}(\mathcal{F}y)(t_k) \right\| \leqslant C \sum_{k=1}^{\infty} |\tilde{c}_k| \big\|(I-\tilde{K})^{-1}\big\| \big\|(\mathcal{F}y)(t_k)\big\|.$$

We know by [5] that $(I-\tilde{K})^{-1}$ is bounded, so, it follows from above together with (30) that

$$\left\| \sum_{k=1}^{\infty} \tilde{c}_k \int\limits_0^1 p(t,\cdot,\xi)(I-\tilde{K})^{-1}(\mathcal{F}y)(t_k) \right\| \leqslant C \sum_{k=1}^{\infty} |\tilde{c}_k| \mathrm{e}^{-Nt} B\left(\frac{1}{4},\frac{3}{4}\right) |y|_{\mathcal{Y}}^n. \tag{31}$$

We go on with the estimates in the H^1-norm. We have, in virtue of (8), that

$$\begin{aligned}
\big\|\big(\mathcal{F}_1(y)\big)'\big\| &\leqslant \tilde{C}\left\{ \sum_{j=1}^{N} (\lambda_j+\zeta) \left[\sum_{i=1}^{N} q_{ij}(t-s) \int\limits_0^1 \tilde{c}(s,\xi) y^n(s,\xi)\varphi_i(\xi)\,\mathrm{d}\xi \right]^2 \right\}^{1/2} \\
&\leqslant \max\big\{\sqrt{\lambda_j+\zeta}\colon\ j=1,2,\dots,N\big\} \\
&\quad \times \left\{ \sum_{j=1}^{N} \left[\sum_{i=1}^{N} q_{ij}(t-s) \int\limits_0^1 \tilde{c}(s,\xi) y^n(s,\xi)\varphi_i(\xi)\,\mathrm{d}\xi \right]^2 \right\}^{1/2}.
\end{aligned} \tag{32}$$

Involving relation (17), the Schwarz's inequality, and the Sobolev embedding, we obtain

$$\begin{aligned}
\big\|\big(\mathcal{F}_1(y)\big)'\big\| &\leqslant C\mathrm{e}^{-\gamma_1(t-s)} s^{-1/4} \big\|y(s)\big\| \big\|y'(s)\big\|^{n-1} \\
&= C\mathrm{e}^{-nNt} \mathrm{e}^{(-\gamma_1+nN+3/4)(t-s)} \mathrm{e}^{-(3/4)(t-s)} s^{-1/4} \\
&\quad \times \mathrm{e}^{Ns} \big\|y(s)\big\| \mathrm{e}^{(n-1)Ns} \big\|y'(s)\big\|^{n-1}.
\end{aligned}$$

Then by (22) and the fact that $-\gamma_1+nN+3/4<0$, for γ_1 large enough, we see that

$$\big\|\big(\mathcal{F}_1(y)\big)'\big\| \leqslant C\mathrm{e}^{-Nt}(t-s)^{-3/4} s^{-3/4} |y|_{\mathcal{Y}}^n \quad \forall 0<s<t. \tag{33}$$

Next,

$$\begin{aligned}
\big\|\big(\mathcal{F}_2(y)\big)'\big\| &\leqslant \tilde{C}\left\{ \sum_{j=N+1}^{\infty} (\lambda_j+\zeta) \left[\mathrm{e}^{-\lambda_j(t-s)} \int\limits_0^1 \tilde{c}(s,\xi) y^n(s,\xi)\varphi_j(\xi)\,\mathrm{d}\xi \right]^2 \right\}^{1/2} \\
&= (t-s)^{-1/2} \left\{ \sum_{j=N+1}^{\infty} \left[(t-s)^{1/2} (\lambda_j+\zeta)^{1/2} \mathrm{e}^{-\lambda_j(t-s)} \right.\right. \\
&\quad \left.\left. \times \int\limits_0^1 \tilde{c}(s,\xi) y^n(s,\xi)\varphi_j(\xi)\,\mathrm{d}\xi \right]^2 \right\}^{1/2}.
\end{aligned} \tag{34}$$

Then, using the obvious inequality

$$\big[(t-s)(\lambda_j+\zeta)\big]^{1/2} \leqslant \mathrm{e}^{(\lambda_j+\zeta)(t-s)/2}, \tag{35}$$

we arrive at

$$\begin{aligned}\big\|\big(\mathcal{F}_2(y)\big)'\big\| &\leqslant (t-s)^{-1/2} \\ &\quad\times \Bigg\{\sum_{j=N+1}^{\infty}\Bigg[\mathrm{e}^{-(\lambda_j-\zeta)(t-s)/2}\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_j(\xi)\,\mathrm{d}\xi\Bigg]^2\Bigg\}^{1/2}. \end{aligned}\tag{36}$$

Arguing as in (24)-(27), we get

$$\begin{aligned}\big\|\big(\mathcal{F}_2(y)\big)'\big\| &\leqslant C(t-s)^{-1/2}\mathrm{e}^{-Nt}(t-s)^{-1/4}s^{-3/4}|y|^n_{\mathcal{Y}} \\ &= C\mathrm{e}^{-Nt}(t-s)^{-3/4}s^{-3/4}|y|^n_{\mathcal{Y}} \quad \forall 0<s<t. \end{aligned}\tag{37}$$

Finally, recalling relation (9),

$$\begin{aligned}\big\|\big(\mathcal{F}_3(y)\big)'\big\| &\leqslant \tilde{C}\Bigg\{\sum_{j=N+1}^{\infty}(\lambda_j+\zeta)\Bigg[\sum_{i=1}^{N} w_i^j(t-s)\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_i(\xi)\,\mathrm{d}\xi\Bigg]^2\Bigg\}^{1/2} \\ &\leqslant \tilde{C}\Bigg\{\sum_{j=N+1}^{\infty}2\lambda_j\Bigg[\sum_{i=1}^{N} w_i^j(t-s)\int_0^1 \tilde{c}(s,\xi)y^n(s,\xi)\varphi_i(\xi)\,\mathrm{d}\xi\Bigg]^2\Bigg\}^{1/2}. \end{aligned}\tag{38}$$

Then, in virtue of (19) and arguing like before, we have

$$\begin{aligned}\big\|\big(\mathcal{F}_3(y)\big)'\big\| &\leqslant C\mathrm{e}^{-\gamma_1(t-s)}\sup_{l=1,2,\dots,N}\big|\big\langle\tilde{c}(s,\cdot)y^n(s,\cdot),\varphi_l(\cdot)\big\rangle\big| \\ &\leqslant C\mathrm{e}^{-Nt}(t-s)^{-3/4}s^{-3/4}|y|^n_{\mathcal{Y}} \quad \forall\, 0<s<t. \end{aligned}\tag{39}$$

Therefore, (32)–(39) imply that

$$\begin{aligned}\big\|\big(\mathcal{F}(y)(t)\big)'\big\| &\leqslant C\mathrm{e}^{-Nt}\int_0^t (t-s)^{-3/4}s^{-3/4}\,\mathrm{d}s\,|y|^n_{\mathcal{Y}} \\ &= \mathrm{e}^{-Nt}t^{-1/2}CB\bigg(\frac{1}{4},\frac{1}{4}\bigg)|y|^n_{\mathcal{Y}} \quad \forall t>0. \end{aligned}\tag{40}$$

Heading towards the end of the proof, we note that

$$\int_0^{\infty}\mathrm{e}^{Nt}t^{1/2}\Bigg[\int_0^1\bigg(\frac{\partial p}{\partial x}(t,x,\xi)\bigg)^2\mathrm{d}\xi\Bigg]^{1/2}\mathrm{d}t<\infty,$$

since the presence of the λ_j in the infinite summation is controlled as in (35) by the presence of $t^{1/2}$. Consequently, via the semigroup property, we deduce that

$$\left\| \int_0^1 \frac{\partial p}{\partial \cdot}(t,\cdot,\xi)\,\mathrm{d}\xi \right\| \leqslant C\mathrm{e}^{-Nt} t^{-1/2} \|y_0\|.$$

So, arguing as in (31) and using (40), we arrive at

$$\left\| \left[\sum_{k=1}^{\infty} \tilde{c}_k \int_0^1 p(t,\cdot,\xi)(I-\tilde{K})^{-1}(\mathcal{F}y)(t_k) \right]' \right\| \leqslant C\mathrm{e}^{-Nt} t^{-1/2} B\left(\frac{1}{4},\frac{1}{4}\right) |y|_{\mathcal{Y}}^n. \tag{41}$$

Now, gathering together relations (30), (31), (40), (41) and observing that the beta functions $B(1/4,3/4)$, $B(1/4,1/4)$ are finite, we obtain that

$$|\mathcal{G}y|_{\mathcal{Y}} \leqslant C|y|_{\mathcal{Y}}^n \quad \forall y \in \mathcal{Y} \tag{42}$$

for some positive constant C.

Similar computations lead to

$$|\mathcal{G}y_1 - \mathcal{G}y_2|_{\mathcal{Y}} \leqslant C\left|y_1^n - y_2^n\right|_{\mathcal{Y}} \quad \forall y_1, y_2 \in \mathcal{Y}. \tag{43}$$

Next, we set $B_r(0) := \{y \in \mathcal{Y}\colon |y|_{\mathcal{Y}} \leqslant r\}$, $r > 0$. In virtue of (42) and (43), we get that for $y \in B_r(0)$, $|\mathcal{G}y|_{\mathcal{Y}} \leqslant Cr^n \leqslant r$ for r sufficiently small. Hence, $\mathcal{G}$ maps the ball $B_r(0)$ into itself. Then for $y_1, y_2 \in B_r(0)$, we have $|\mathcal{G}y_1 - \mathcal{G}y_2|_{\mathcal{Y}} \leqslant nCr^{n-1}|y_1 - y_2|_{\mathcal{Y}}$ with $nCr^{n-1} < 1$ for r sufficiently small. Thus, $\mathcal{G}$ is a contraction on $B_r(0)$. We conclude by the contraction mapping theorem that $\mathcal{G}$ has a unique fixed point in $B_r(0)$ when r is sufficiently small. This implies that equation (15) has a unique solution, which satisfies

$$|y|_{\mathcal{Y}} \leqslant r < \infty.$$

Returning to the transformation $y \to w$, the conclusion of the theorem follows immediately.

4 Conclusions

Here we discussed about the semilinear heat equation on the rod with polynomial nonlinearity and with nonlocal initial conditions. We addressed the problem of boundary exponential stabilization, but in the same time, we showed the well-posedness of the model since there was no result guaranteeing the existence of solutions. The exponential decay is of order e^{-Nt}, where N is some natural number standing for the dimension of the controller. It is clear that taking N large enough, the exponential decay can be made arbitrarily fast, but with large dimension of the controller. We stress that, instead of the nonlinearity w^n, one can consider the term $w\partial_x w$, and with slight adjustments the proof is similar. It remains as an open problem the multidimensional space case. In this case the problem comes from the estimates of the eigenfunction system and for the fundamental solution (10), which are not as good as in the one-dimensional case. For the two-dimensional case, one can argue for cubic nonlinearities as in [8].

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