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A new class of fractional impulsive differential hemivariational inequalities with an application*

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Abstract. We consider a new fractional impulsive differential hemivariational inequality, which captures the required characteristics of both the hemivariational inequality and the fractional impulsive differential equation within the same framework. By utilizing a surjectivity theorem and a fixed point theorem we establish an existence and uniqueness theorem for such a problem. Moreover, we investigate the perturbation problem of the fractional impulsive differential hemivariational inequality to prove a convergence result, which describes the stability of the solution in relation to perturbation data. Finally, our main results are applied to obtain some new results for a frictional contact problem with the surface traction driven by the fractional impulsive differential equation.

Keywords: fractional differential variational inequality, fractional impulsive equation, hemivariational inequality, frictional contact.

1 Introduction

Let Y, Z_1, Z_2 be three reflexive and separable Banach spaces, and let Z_2^* be the dual space of Z_2. For a prefixed $T > 0$, let $Q = [0, T]$, $f : Q \times Z_1 \times Z_2 \to Z_1$, $A : Q \times Z_2 \to Z_2^*$, $N : Z_2 \to Y$, $J : Q \times Y \to \mathbb{R}$, and $g : Q \times Z_1 \to Z_2^*$. This paper focuses on the following fractional impulsive differential hemivariational inequality (FIDHVI): find $z : Q \to Z_1$ and $y : Q \to Z_2$ such that

$$\begin{aligned}
&{}^C D_0^\kappa z(t) = f\big(t, z(t), y(t)\big), \quad t \in Q,\ t \neq \tau_j,\ j = 1, 2, \dots, m,\\
&z(0) = z_0, \qquad \Lambda z(\tau_j) = \Theta_j\big(z(\tau_j^-)\big), \quad j = 1, 2, \dots, m,\\
&\big\langle A\big(t, y(t)\big), x\big\rangle + J^\circ\big(t, Ny(t); Nx\big) \geqslant \big\langle g\big(t, z(t)\big), x\big\rangle \quad \forall (t, x) \in Q \times Z_2,
\end{aligned}$$

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where ${}^{C}D_0^{\kappa}$ ($0 < \kappa \leqslant 1$) stands for the Caputo derivative of fractional order κ, $\Theta_j : Z_1 \to Z_1$ is an impulsive function with $j = 1, 2, \dots, m$, $\Lambda z(\tau_j)$ is given by $\Lambda z(\tau_j) = z(\tau_j^+) - z(\tau_j^-)$ with $z(\tau_j^+)$ and $z(\tau_j^-)$ being the left and right limit of z at $t = \tau_j$, respectively, and $0 = \tau_0 < \tau_1 < \dots < \tau_m < \tau_{m+1} = T$.

We remark that for appropriate and suitable choices of the spaces and the above defined maps, FIDHVI includes a number of differential variational inequalities as special cases [5, 12, 13, 26, 29].

It is worth mentioning that FIDHVI is a new model, which captures the required characteristics of both the hemivariational inequality and the fractional impulsive differential equation within the same framework. In addition, FIDHVI can be used to describe the frictional contact problem with the surface traction driven by the fractional impulsive differential equation (see Section 5).

The study of differential variational inequality (DVI) can ascend to the work of Aubin and Cellina [1]. DVI described by the following generalized abstract system

$$\dot{y}(t) = f\big(t, y(t), x(t)\big), \quad G\big(y(0), y(T)\big) = 0 \quad \forall t \in Q,$$
$$\int_0^t \big(v - x(t)\big)^{\mathrm{T}} F\big(t, y(t), x(t)\big)\,\mathrm{d}t \geqslant 0 \quad \forall v \in K.$$

was then examined by Pang and Stewart [20] in finite dimensional Euclidean spaces. Here K is a nonempty, closed, and convex subset of $\mathbb{R}^m$, $f : Q \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $F : Q \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, and $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are three given functions. As pointed out by Pang and Stewart [20], DVI provides a powerful tool of describing many practical problems such as fluid mechanical problems, engineering operation research, dynamic traffic networks, economical dynamics, and frictional contact problems [6, 23, 25]. In 2010, Li et al. [11] discussed the solvability for a class of DVI in finite dimensional spaces. Later, Chen and Wang [4] employed the regularized time-stepping method to consider a class of parametric DVI and provided convergence analysis for this method in finite dimensional Euclidean spaces. Liu et al. [14] studied a class of nonlocal semilinear evolution DVI in Banach spaces. By using the theory of topological degree they obtained some existence results for their model under some suitable assumptions. Recently, in order to describe a free boundary problem raising from contact mechanics, Sofonea et al. [21] studied a differential quasivariational inequality and proved the stability of the solutions for such a problem. For more works related to DVIs, we refer the reader to [10, 13, 15, 28] and the the references therein.

As is well known, fractional calculus, that is, the noninteger calculus, allows us to define derivatives of arbitrary order and has many applications in practical problems [9]. Recently, by applying the fixed point approach, Ke et al. [8] discussed the solvability of a class of fractional DVI in finite dimensional spaces. Using the Rothe method, Zeng et al. [30] studied a class of parabolic fractional differential hemivariational inequalities in Banach spaces. Xue et al. [29] discussed the existence of the mild solutions of a class of fractional DVIs in Banach spaces under some appropriate hypotheses. Very recently, Weng et al. [27] considered a fractional nonlinear evolutionary delay system driven by a hemivariational inequality in Banach spaces and established an existence theorem for such a system by employing the KKM theorem, fixed point theorem for condensing set-valued operators, and the theory of fractional calculus.

It is worth noting that, in the real world, many systems are often disturbed suddenly, and systems changes suddenly in a short time. These phenomena are called impulsive effects. We note that diverse numerical methods and theoretical results have been widely studied for differential equations with impulsive effects using different assumptions in the literature; for instance, we refer the reader to [2] and the references therein. In [17] and [16], Migórski and Ochal studied the existence of the solutions for two class of nonlinear second-order impulsive evolution inclusions problems. Recently, Li et al. [12] introduced a class of impulsive DVI in finite dimensional spaces and presented some existence and stability results of the solutions under some suitable assumptions. However, in some practical situations applications, it is necessary to consider FIDHVI. To illustrate this point, a fractional contact problem with the surface traction driven by the fractional impulsive differential equation will be considered as an application of FIDHVI in Section 5. The discipline of FIDHVI is still not explored, and very little is known. To fill this gap, in this paper, we seek to make a contribution in this new direction.

The outline of this work is as follows. In the next section, we present some necessary preliminaries and notations. After that, Section 3 establishes an existence and uniqueness result concerning FIDHVI under some mild conditions. In Section 4, we provide a stability result of the solution of FIDHVI with respect to the perturbation of data. Finally, we apply our main results for FIDHVI to the frictional contact problem with the surface traction driven by the fractional impulsive differential equation in Section 5.

2 Preliminaries

For a Banach space X, we denote $C(Q;X)$ the space of all functions $x : Q \to X$ that is continuous, $L^p(Q;X)$ the space of all pth power Bochner integrable functions on Q taking values in X, $\mathcal{I}C(Q;X)$ the space of all functions $x : Q \to X$ such that $x : Q \setminus \cup_{j=1,\dots,m}\{\tau_j\} \to X$ is continuous, and $z(\tau_j^+)$ and $z(\tau_j^-)$ exist with $z(\tau_j) = z(\tau_j^-)$, $P(X)$ the set of all nonempty subsets of X, $P_c(X)$ the set of all closed subsets of X, $P_{k(cb)v}(X)$ the set of all compact (closed and bounded) convex subsets of X. For a set $U \subset X$, we define $\|U\|_X = \sup\{\|u\|_X | u \in U\}$. The norms in spaces $C(Q;X)$, $L^p(Q;X)$, and $\mathcal{I}C(Q;X)$ are respectively defined by $\|z\|_{C(Q;X)} = \max_{t\in Q}\|z(t)\|_X$, $\|z\|_{L^p(Q;X)} = (\int_Q \|z(t)\|_X^p \,\mathrm{d}t)^{1/p}$, and $\|z\|_{\mathcal{I}C(Q;X)} = \sup_{t\in Q}\|z(t)\|$.

In the sequel, let $\Gamma(\cdot)$ denote the gamma function.

Definition 1. (See [9].) The qth fractional integral of $z(s)$ with $q > 0$ is defined by

$$D_0^{-q} z(s) := \frac{1}{\Gamma(q)} \int_0^s (s-t)^{q-1} z(t)\,\mathrm{d}t, \quad s > 0.$$

Definition 2. (See [9].) For $\alpha \in (n-1, n)$, the Caputo fractional-order derivative of α of $z(s)$, denoted by ${}^C D_0^\alpha z(s)$, can be defined by setting

$$ {}^C D_0^\alpha z(s) := \frac{1}{\Gamma(n-\alpha)} \int_0^s (s-t)^{n-\alpha-1} z^{(n)}(t)\,\mathrm{d}t, \quad s > 0. $$

Definition 3. (See [3].) The generalized directional derivative of a locally Lipschitz functional $F : Z_2 \to \mathbb{R}$ at $x \in Z_2$ in the direction $z \in Z_2$ and the generalized gradient of function F at v, denoted respectively by $F^\circ(x; z)$ and $\partial F(v)$, are respectively defined by

$$ F^\circ(x; z) = \limsup_{y \to x,\ \mu \to 0^+} \frac{F(y + \mu z) - F(y)}{\mu} \quad \forall x, z \in Z_2 $$

and

$$ \partial F(v) = \bigl\{ \eta \in Z_2^* \bigm| F^\circ(v; z) \geqslant \langle \eta, z \rangle\ \forall z, v \in Z_2 \bigr\}. $$

Definition 4. (See [22].) An operator $B : Z_2 \to Z_2^*$ is said to be

(i) monotone if $\langle Bx_1 - Bx_2,\, x_1 - x_2 \rangle \geqslant 0$ for all $x_1, x_2 \in Z_2$;
(ii) strongly monotone if there exists $m_B > 0$ satisfying $\langle Bx_1 - Bx_2,\, x_1 - x_2 \rangle \geqslant m_B \|x_1 - x_2\|_{Z_2}^2$ for all $x_1, x_2 \in Z_2$;
(iii) pseudomomotone if B is bounded and $x_n \to x$ weakly in Z_2 with $\limsup \langle Bx_n,\, x_n - y \rangle \leqslant 0$ yields that $\liminf \langle Bx_n,\, x_n - y \rangle \geqslant \langle Bx,\, x - y \rangle$ for all $y \in Z_2$;
(iv) demicontinuous if $z_n \to z$ in Z_2 implies that $Bz_n \to Bz$ weakly in Z_2^*;
(v) bounded if $\Omega \subset Z_2$ is bounded implies $B(\Omega) \subset Z_2^*$ is bounded.

Definition 5. (See [18].) A set-valued operator $B : Z_2 \to P(Z_2^*)$ is said to be pseudomomotone if

(i) for every $x \in Z_2$, $Bx \in P_{cb}(Z_2^*)$;
(ii) for any subspace H of Z_2, B is upper semicontinuous from H to Z_2^* endowed with the weak topology;
(iii) if $z_n \to z$ weakly in Z_2 and $z_n^* \in Bz_n$ such that $\limsup \langle z_n^*,\, z_n - z \rangle \leqslant 0$, then for every $x \in Z_2$, there exists $z^* \in Bz$ such that $\liminf \langle z_n^*,\, z_n - x \rangle \geqslant \langle z^*,\, z - x \rangle$.

Lemma 1. (*See* [7, *Prop*. 5.6].) *Assume that U_1 and U_2 are two reflexive Banach spaces, $\psi : U_1 \to U_2$ is a linear, continuous, and compact operator, and $\psi^* : U_2^* \to U_1^*$ is the adjoint operator of ψ. If $\varphi : U \to \mathbb{R}$ is a locally Lipschitz functional satisfying $\|\partial\varphi(u)\|_{U_1^*} \leqslant c_\varphi(1 + \|u\|_{U_1})$ for all $u \in U_1$, where $c_\varphi > 0$ is a constant, then the set-valued operator $W : U_1 \to P(U_1^*)$, defined by $W(u) = \psi^* \partial\varphi(\psi(u))$ for all $u \in U_1$, is pseudomonotone.*

Lemma 2. (*See* [30, *Cor*. 7].) *Assume that U_0 is a reflexive Banach spaces, and let the following conditions hold*:

(i) *$T : U_0 \to U_0^*$ is pseudomomotone and strong monotone with constant $c_T > 0$;*

(ii) $G : U_0 \to P(U_0^*)$ *is pseudomomotone, and there exist two constants* $c_G, c^* > 0$ *satisfying* $\|G(u)\|_{U_0^*} \leqslant c_G \|u\|_{U_0} + c^*$ *for all* $u \in U_0$;

(iii) $c_G < c_T$.

Then $T + G$ *is surjective in* U_0^*.

According to [18, Prop. 3.37], we can rewrite FIDHVI as follows.

Problem 1. Find $z : Q \to Z_1$ and $y : Q \to Z_2$ such that

$$\begin{aligned}
&{}^C D_0^\kappa z(t) = f\bigl(t, z(t), y(t)\bigr), \quad t \in Q,\ t \neq \tau_j,\ j = 1, 2, \dots, m, \\
&z(0) = z_0, \qquad \Lambda z(\tau_j) = \Theta_j\bigl(z(\tau_j^-)\bigr), \quad j = 1, 2, \dots, m, \\
&A\bigl(t, y(t)\bigr) + N^* \partial J\bigl(t, Ny(t)\bigr) \ni g\bigl(t, z(t)\bigr) \quad \forall t \in Q.
\end{aligned}$$

To study Problem 1, we consider the following fractional impulsive Cauchy problem

$$\begin{aligned}
&{}^C D_0^\kappa z(t) = u(t), \quad t \in Q,\ t \neq \tau_j,\ j = 1, 2, \dots, m, \\
&z(0) = z_0, \quad \Lambda z(\tau_j) = \Theta_j\bigl(z(\tau_j^-)\bigr), \quad j = 1, 2, \dots, m.
\end{aligned}$$

Noting the fact that

$$z(t) = z_0 - \frac{1}{\Gamma(\kappa)} \int_0^a (a - s)^{\kappa - 1} u(s)\,\mathrm{d}s + \frac{1}{\Gamma(\kappa)} \int_0^t (t - s)^{\kappa - 1} u(s)\,\mathrm{d}s, \quad a > 0$$

solves the Cauchy problem

$${}^C D_0^\kappa z(t) = u(t), \qquad z(0) = z_0 - \frac{1}{\Gamma(\kappa)} \int_0^a (a - s)^{\kappa - 1} u(s)\,\mathrm{d}s, \quad t \in Q,$$

we have the following result immediately.

Lemma 3. *Let* $\kappa \in (0, 1)$ *and* $u \in C(Q; Z_1)$. *Then the Cauchy problem*

$${}^C D_0^\kappa z(t) = u(t), \quad t \in Q, \qquad z(a) = z_0, \quad a > 0,$$

is equivalent to the integral equation

$$z(t) = z_0 - \frac{1}{\Gamma(\kappa)} \int_0^a (a - s)^{\kappa - 1} u(s)\,\mathrm{d}s + \frac{1}{\Gamma(\kappa)} \int_0^t (t - s)^{\kappa - 1} u(s)\,\mathrm{d}s.$$

Lemma 4. *For* $\kappa \in (0, 1)$ *and* $u \in C(Q; Z_1)$, *the Cauchy problem*

$$\begin{aligned}
&{}^C D_0^\kappa z(t) = u(t), \quad t \in Q,\ t \neq \tau_j,\ j = 1, 2, \dots, m, \\
&z(0) = z_0, \qquad \Lambda z(\tau_j) = \Theta_j\bigl(z(\tau_j^-)\bigr), \quad j = 1, 2, \dots, m,
\end{aligned} \tag{1}$$

is equivalent to the integral equation

$$z(t) = z_0 + \sum_{i=1}^{j} \Theta_i\bigl(z(\tau_i^-)\bigr) + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} u(s)\,\mathrm{d}s \quad \forall t \in (t_j, t_{j+1}]. \tag{2}$$

Proof. Assume that (1) holds. If $t \in [0, \tau_1]$, then ${}^C D_0^\kappa z(t) = u(t)$ for all $t \in [0, \tau_1]$ with $z(0) = z_0$. Clearly,

$$z(t) = z_0 + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} u(s)\,\mathrm{d}s.$$

If $t \in (\tau_1, \tau_2]$, then

$${}^C D_0^\kappa z(t) = u(t), \quad t \in (\tau_1, \tau_2], \quad \text{with} \quad z(\tau_1^+) = z(\tau_1^-) + \Theta_1\bigl(z(\tau_j^-)\bigr),$$

and so Lemma 3 implies that

$$\begin{aligned} z(t) &= z(\tau_1^+) - \frac{1}{\Gamma(\kappa)} \int_0^{\tau_1} (\tau_1 - s)^{\kappa-1} u(s)\,\mathrm{d}s + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} u(s)\,\mathrm{d}s \\ &= z(\tau_1^-) + \Theta_1\bigl(z(\tau_1^-)\bigr) - \frac{1}{\Gamma(\kappa)} \int_0^{\tau_1} (\tau_1 - s)^{\kappa-1} u(s)\,\mathrm{d}s + \frac{1}{\Gamma(\kappa)} \int_0^t t - s^{\kappa-1} u(s)\,\mathrm{d}s \\ &= z_0 + \Theta_1(z(\tau_1^-)) + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} u(s)\,\mathrm{d}s. \end{aligned}$$

If $t \in (\tau_2, \tau_3]$, then using Lemma 3 again, we have

$$\begin{aligned} z(t) &= z\bigl(\tau_2^+\bigr) - \frac{1}{\Gamma(\kappa)} \int_0^{\tau_2} (\tau_1 - s)^{\kappa-1} u(s)\,\mathrm{d}s + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} u(s)\,\mathrm{d}s \\ &= z(\tau_2^-) + \Theta_2\bigl(z(\tau_2^-)\bigr) - \frac{1}{\Gamma(\kappa)} \int_0^{\tau_2} (\tau_1 - s)^{\kappa-1} u(s)\,\mathrm{d}s + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} u(s)\,\mathrm{d}s \\ &= z_0 + \Theta_1\bigl(z(\tau_1^-)\bigr) + \Theta_2\bigl(z(\tau_2^-)\bigr) + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} u(s)\,\mathrm{d}s. \end{aligned}$$

Similarly, if $t \in (\tau_j, \tau_{j+1}]$, then we can show that

$$z(t) = z_0 + \sum_{i=1}^{j} \Theta_i\bigl(z(\tau_i^-)\bigr) + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} u(s)\,\mathrm{d}s \quad \forall t \in (t_j, t_{j+1}].$$

Conversely, suppose that (2) holds. If $t \in (0, \tau_1]$, then we know that (1) holds by the fact that ${}^C D_0^\kappa$ is the inverse of $D_0^{-\kappa}$. If $t \in (\tau_j, \tau_{j+1}]$, $j = 1, 2, \dots, m$, since the Caputo fractional derivative for a constant is zero, one has ${}^C D_0^\kappa z(t) = u(t)$, $t \in (\tau_j, \tau_{j+1}]$ and $\Lambda z(\tau_j) = \Theta_j(z(\tau_j^-))$. □

From Lemma 4 we have the following definition.

Definition 6. A pair $(z, y) \in \mathcal{IC}(Q; Z_1) \times \mathcal{IC}(Q; Z_2)$ is said to be a solution of Problem 1 if it satisfies the following system:

$$z(t) = z_0 + \sum_{i=1}^{j} \Theta_i\bigl(z(\tau_i^-)\bigr) + \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} f\bigl(s, z(s), y(s)\bigr)\,\mathrm{d}s \quad \forall t \in (t_j, t_{j+1}], \tag{3}$$

$$A\bigl(t, y(t)\bigr) + \eta = g\bigl(t, z(t)\bigr), \quad \eta \in N^* \partial J\bigl(t, Ny(t)\bigr) \quad \forall t \in Q. \tag{4}$$

Finally, we recall the following nonlinear impulsive Gronwall inequality.

Lemma 5. (*See* [24, *Lemma* 3.4].) *Let* $z \in \mathcal{IC}(Q; Z_1)$ *satisfy the following inequality*:

$$\bigl\|z(t)\bigr\| \leqslant k_1 + k_2 \int_0^t (t-s)^{\kappa-1} \bigl\|z(s)\bigr\|\,\mathrm{d}s + \sum_{0<\tau_j<t} d_j \bigl\|z(\tau_j^-)\bigr\|,$$

where $k_1, k_2, d_j \geqslant 0$ *are constants. Then*

$$\bigl\|z(t)\bigr\| \leqslant k_1 \bigl[1 + D^* E_\kappa(k_2 \Gamma(\kappa) t^\kappa)\bigr]^j E_\kappa\bigl(k_2 \Gamma(\kappa) t^\kappa\bigr) \quad \forall t \in (t_j, t_{j+1}],$$

where $D^* = \max\{d_j,\ j = 1, \dots, m\}$, *and* E_γ *is the Mittag-Leffler function* [9] *defined by* $E_\gamma(h) = \sum_{j=0}^\infty h^j / \Gamma(\gamma h + 1)$ *for all* $h \in \mathbb{C}$ *with* $\mathrm{Re}(\gamma) > 0$.

3 Existence and uniqueness

To study the solvability of Problem 1, we need the following assumptions.

(H_f) $f : Q \times Z_1 \times Z_2 \to Z_1$ is a map such that

- (i) for any given $(z, y) \in Z_1 \times Z_2$, $f(\cdot, z, y)$ is continuous;
- (ii) for any $(t, z_i, y_i) \in Q \times Z_1 \times Z_2$, $i = 1, 2$, there exists $M_1 > 0$ satisfying $\|f(t, z_1, y_1) - f(t, z_2, y_2)\|_{Z_1} \leqslant M_1(\|z_1 - z_2\|_{Z_1} + \|y_1 - y_2\|_{Z_2})$;
- (iii) there exists $\phi \in L_+^{1/p}[0, T]$ $(0 < p < \kappa < 1)$ satisfying $\|f(t, z, y)\|_{Z_1} \leqslant \phi(t)$ for all $(t, z, y) \in Q \times Z_1 \times Z_2$.

(H_I) For each $j \in \{1, 2, \dots, m\}$, $\Theta_j : Z_1 \to Z_1$ is bounded, and there exists $d_j > 0$ satisfying $\|\Theta_j(z_1) - \Theta_j(z_1)\|_{Z_1} \leqslant d_j \|z_1 - z_2\|_{Z_1}$ for all $z_1, z_2 \in Z_1$.

(H_A) $A : Q \times Z_2 \to Z_2^*$ is a map such that

- (i) for any given $y \in Z_2$, $A(\cdot, y)$ is continuous;
- (ii) for any given $t \in Q$, $A(t, \cdot)$ is bounded, demicontinuous, and strongly monotone with the constant m_A.

(H_N) $N \in L(Z_2, Y)$ is a compact operator.

(H_J) $J : Q \times Y \to \mathbb{R}$ is a functional satisfying

- (i) for any given $x \in Y$, $J(\cdot, x)$ is continuous;
- (ii) for any given $t \in Q$, $J(t, \cdot)$ is locally Lipschitz;
- (iii) there exists $m_J > 0$ satisfying $\|\partial J(t, x)\|_{Z_2^*} \leqslant m_J(\|x\|_Y + 1)$ for all $(t, x) \in Q \times Y$;
- (iv) there exists $c_J > 0$ satisfying $\langle \theta_1 - \theta_2, y_1 - y_2 \rangle \geqslant -c_J \|y_1 - y_2\|_Y^2$ for all $\theta_i \in \partial J(t, y_i)$, $(t, y_i) \in Q \times Y$, $i = 1, 2$.

(H_g) $g : Q \times Z_1 \to Z_2^*$ is a map such that

- (i) for any given $z \in Z_1$, $g(\cdot, z)$ is continuous;
- (ii) there exists $m_g > 0$ satisfying $\|g(t, z_1) - g(t, z_2)\|_{Z_2^*} \leqslant m_g \|z_1 - z_2\|_{Z_1}$ for all $(t, z_i) \in Q \times Z_1$, $i = 1, 2$.

(H_0)
- (i) $m_A > c_J \|N\|^2$, where $\|N\| = \|N\|_{L(Z_2, Y)}$;
- (ii) $T^\kappa M_1 m_g / (\kappa(m_A - c_J \|N\|^2)\Gamma(\kappa)) < 1$.

We first consider nonlinear inclusion (4).

Lemma 6. *For any given $z \in \mathcal{I}C(Q; Z_1)$, nonlinear inclusion* (4) *has a unique solution $y \in \mathcal{I}C(Q; Z_2)$ providing that assumptions* (H_A), (H_N), (H_J), (H_g), *and* (H_0) *hold. Moreover, for any $z_1, z_2 \in \mathcal{I}C(Q; Z_1)$, one has*

$$\big\|y_1(t) - y_2(t)\big\|_{Z_2} \leqslant \frac{m_g}{m_A - c_J \|N\|^2} \big\|z_1(t) - z_2(t)\big\|_{Z_1} \quad \forall t \in Q, \tag{5}$$

where $y_1, y_2 \in \mathcal{I}C(Q; Z_2)$ are the solutions of (4) *with respect to z_1 and z_2, respectively.*

Proof. For given $z \in \mathcal{I}C(Q; Z_1)$ and $t \in Q$, define two operators $\widehat{A} : Z_2 \to Z_2^*$ and $\widehat{N} : Z_2 \to P(Z_2^*)$ as $\widehat{A}y = A(t, y)$, $\widehat{N}y = N^* \partial J(t, Ny)$ for all $(t, y) \in Q \times Z_2$. For simplicity, we do not indicate their dependence t. Using (H_A), (H_N), (H_J), (H_0), Lemma 1, and [22, Lemma 3], we deduce that the operators $\widehat{A}$ and $\widehat{N}$ are pseudomomotone and

$$\begin{aligned} \|\widehat{N}x\|_{E_2^*} &\leqslant \|N^*\| \big\|\partial J(t, Nx)\big\| \leqslant \|N^*\| \big(m_{J_1} \|Nx\|_X + m_{J_2}\big) \\ &\leqslant m_{J_1} \|N\|^2 \|x\|_{Z_2} + m_{J_2} \|N\| \quad \forall x \in Z_2. \end{aligned}$$

By applying Lemma 2 with $B = \widehat{A}$ and $A = \widehat{N}$ we know that inclusion (4) has a solution $y(t)$ for all $t \in Q$. Next, we show that the solution $y(t)$ is unique. Let $y_1, y_2 \in Z_2$ be solutions to (4). Then there exist $\eta_1, \eta_2 \in N^* \partial J(t, Ny_i(t))$ satisfying $A(t, y_i) + \eta_i =$

$g(t, z(t))$ for all $t \in Q$, $i = 1, 2$. Subtracting the two equations and taking the result in duality with $y_1 - y_2$, we have

$$\langle A(t, y_1) - A(t, y_1),\, y_1 - y_2 \rangle_{Z_2^* \times Z_2} = \langle \eta_2 - \eta_1,\, y_1 - y_2 \rangle_{Z_2^* \times Z_2}.$$

By assumptions (H_A) and (H_J) one has

$$\left(m_A - c_J \|N\|^2\right) \|y_1 - y_2\|_{Z_2}^2 \leqslant 0,$$

and so assumption (H_0) implies that $y_1 = y_2$, which is our claim.

In what follows, we start by showing that (5) holds. Let $z_i(t) \in Z_1$ ($i = 1, 2$) and denote $z_i(t) = z_i$, $y_i(t) = y_i$, $g(t, z_i(t)) = g_i$ with $i = 1, 2$. It follows from (4) that $A(t, y_i) + \varsigma_i = g_i$, $\varsigma_i \in N^* \partial J(t, N y_i)$ ($i = 1, 2$). Subtracting the two equations and taking the result in duality with $y_1 - y_2$, we have

$$\begin{aligned} &\langle A(t, y_1) - A(t, y_1),\, y_1 - y_2 \rangle_{Z_2^* \times Z_2} + \langle \varsigma_1 - \varsigma_2,\, y_1 - y_2 \rangle_{Z_2^* \times Z_2} \\ &\quad = \langle g_1 - g_2,\, y_1 - y_2 \rangle_{Z_2^* \times Z_2}. \end{aligned}$$

By assumptions (H_J), (H_A), and (H_g) one has

$$\begin{aligned} &\left(m_A - c_J \|N\|^2\right) \|y_1 - y_2\|_{Z_2}^2 \\ &\quad \leqslant \|g_1 - g_2\|_{Z_2^*} \|y_1 - y_2\|_{Z_2} \leqslant m_g \|z_1 - z_2\|_{Z_1} \|y_1 - y_2\|_{Z_2}. \end{aligned}$$

Thus, assumption (H_0) implies that

$$\|y_1 - y_2\|_{Z_2} \leqslant \frac{m_g}{m_A - c_J \|N\|^2} \|z_1 - z_2\|_{Z_1}. \tag{6}$$

It follows from (6) that the map $Z_1 \ni z(t) \mapsto y(t) \in Z_2$ is continuous for all $t \in Q$. Since $z \in \mathcal{IC}(Q; Z_1)$, we know that $y \in \mathcal{IC}(Q; Z_2)$. By (6) we conclude that, for any given $z \in \mathcal{IC}(Q; Z_1)$, nonlinear inclusion (4) has a unique solution $y \in \mathcal{IC}(Q; Z_2)$. Moreover, for any given $z_1, z_2 \in \mathcal{IC}(Q; Z_1)$, (5) holds due to (6). □

Theorem 1. *Problem* 1 *admits a unique solution* $(z, y) \in \mathcal{IC}(Q; Z_1) \times \mathcal{IC}(Q; Z_2)$ *providing that assumptions* (H_A), (H_f), (H_I), (H_N), (H_J), (H_g), *and* (H_0) *hold.*

Proof. For any given $z \in \mathcal{IC}(Q; Z_1)$, Lemma 6 shows that nonlinear inclusions (4) admits a unique solution y_z. Define an operator $\Sigma : \mathcal{IC}(Q; Z_1) \to \mathcal{IC}(Q; Z_1)$ by setting

$$\Sigma z(t) = z_0 + \sum_{i=1}^{j} \Theta_i\left(z(\tau_i^-)\right) + \frac{1}{\Gamma(\kappa)} \int_0^t (t - s)^{\kappa - 1} f\left(s, z(s), y_z(s)\right) \mathrm{d}s.$$

Then assumption (H_f) implies that Σ is well defined. To prove Theorem 1, we only need to show that Σ admits a unique fixed point in $\mathcal{IC}(Q; Z_1)$.

To this end, we first show that $\Sigma z \in \mathcal{I}C(Q; Z_1)$ for any $z \in \mathcal{I}C(Q; Z_1)$. In fact, let $z \in C([0, \tau_1], Z_1)$ and $\iota > 0$ be given. When $t \in [0, \tau_1]$, by the Hölder inequality and assumption (H_f) we have

$$\begin{aligned}
&\big\|(\Sigma z)(t+\iota) - (\Sigma z)(t)\big\|_{Z_1} \\
&\quad \leqslant \frac{1}{\Gamma(\kappa)} \int_0^t \big((t-s)^{\kappa-1} - (t+\iota-s)^{\kappa-1}\big) \big\|f\big(s, z(s), y_z(s)\big)\big\|_{Z_1} \,\mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\kappa)} \int_t^{t+\iota} (t+\iota-s)^{\kappa-1} \big\|f\big(s, z(s), y_z(s)\big)\big\|_{Z_1} \,\mathrm{d}s \\
&\quad \leqslant \frac{1}{\Gamma(\kappa)} \int_0^t \big((t-s)^{\kappa-1} - (t+\iota-s)^{\kappa-1}\big) \phi(s) \,\mathrm{d}s + \frac{1}{\Gamma(\kappa)} \int_t^{t+\iota} (t+\iota-s)^{\beta-1} \phi(s) \,\mathrm{d}s \\
&\quad \leqslant \frac{M}{\Gamma(\kappa)} \left(\int_0^t \big((t-s)^{\kappa} - (t+\iota-s)^{\alpha}\big) \,\mathrm{d}s \right)^{1-p} + \frac{M}{\Gamma(\kappa)} \left(\int_t^{t+\iota} (t+\iota-s)^{\alpha} \,\mathrm{d}s \right)^{1-p} \\
&\quad \leqslant \frac{M}{\Gamma(\kappa)(1+\alpha)^{1-p}} \big(\big|(t+\iota)^{1+\alpha} - t^{1+\alpha}\big| + \iota^{1+\alpha}\big)^{1-p} + \frac{M}{\Gamma(\kappa)(1+\alpha)^{1-p}} \iota^{(1+\alpha)(1-p)} \\
&\quad \leqslant \frac{2M}{\Gamma(\kappa)(1+\alpha)^{1-p}} \iota^{(1+\alpha)(1-p)} + \frac{M}{\Gamma(\kappa)(1+\alpha)^{1-p}} \iota^{(1+\alpha)(1-p)} \\
&\quad \leqslant \frac{3M}{\Gamma(\kappa)(1+\alpha)^{1-p}} \iota^{(1+\alpha)(1-p)} \to 0
\end{aligned}$$

as $\iota \to 0$, where $M = \|\phi\|_{L^{1/p}[0,T]}$ and $\alpha = (\kappa-1)/(1-p) \in (-1, 0)$. This shows that $\Sigma z \in C([0, \tau_1], Z_1)$. When $t \in (\tau_1, \tau_2]$, using the same argument, one has

$$\big\|(\Sigma z)(t+\iota) - (\Sigma z)(t)\big\|_{Z_1} \leqslant \frac{3M}{\Gamma(\kappa)(1+\alpha)^{1-p}} \iota^{(1+\alpha)(1-p)} \to 0 \quad \text{as } \iota \to 0,$$

which implies that $\Sigma z \in C((\tau_1, \tau_2], Z_1)$. Similarly, when $t \in (\tau_j, \tau_{j+1}]$, $j = 1, 2, \dots, m$, we can show that

$$\big\|(\Sigma z)(t+\iota) - (\Sigma z)(t)\big\|_{Z_1} \leqslant \frac{3M}{\Gamma(\kappa)(1+\alpha)^{1-p}} \iota^{(1+\alpha)(1-p)} \to 0 \quad \text{as } \iota \to 0$$

and so $\Sigma z \in C((\tau_j, \tau_{j+1}], Z_1)$.

Combining all the above, we see that $\Sigma z \in \mathcal{I}C(Q; Z_1)$ for any $z \in \mathcal{I}C(Q; Z_1)$.

Next, we prove that Σ is a contractive map. For given $z_1, z_2 \in \mathcal{I}C(Q; Z_1)$, by assumption ($\mathrm{H}_f$) it follows from (5) that

$$\begin{aligned}
&\big\|(\Sigma z_1)(t) - (\Sigma z_2)(t)\big\|_{Z_1} \\
&\quad \leqslant \frac{M_1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1}\big(\big\|z_1(s) - z_2(s)\big\|_{Z_1} + \big\|y_{z_1}(s) - y_{z_2}(s)\big\|_{Z_2}\big)\,\mathrm{d}s \\
&\quad \leqslant \frac{M_1 m_g}{(m_A - c_J\|N\|^2)\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1}\big\|z_1(s) - z_2(s)\big\|_{Z_1}\,\mathrm{d}s \\
&\quad \leqslant \frac{T^\beta M_1 m_g}{\beta(m_A - c_J\|N\|^2)\Gamma(\kappa)}\|z_1 - z_2\|_{\mathcal{I}C(Q;Z_1)},
\end{aligned}$$

and so

$$\|\Sigma z_1 - \Sigma z_2\|_{\mathcal{I}C(Q;Z_1)} \leqslant \frac{T^\beta M_1 m_g}{\beta(m_A - c_J\|N\|^2)\Gamma(\kappa)}\|z_1 - z_2\|_{\mathcal{I}C(Q;Z_1)}.$$

Now assumption (H_0) implies that Σ is a contractive map, and so Σ admits a unique solution $z \in \mathcal{I}C(Q; Z_1)$ by employing the Banach fixed point theorem. □

4 A convergence result

We investigate the perturbation problem of Problem 1 to prove a convergence result, which describes the stability of the solution in relation to perturbation data. To this end, let $\delta > 0$ and J_δ be the perturbed data of J such that J_δ satisfies assumptions (H_J) and (H_0). More precisely, we examine the following perturbation problem: find a pair of functions $(z_\delta, y_\delta) \in \mathcal{I}C(Q; Z_1) \times \mathcal{I}C(Q; Z_2)$ such that

$$\begin{aligned}
&{}^C D_0^\kappa z_\delta(t) = f\big(t, z_\delta(t), y_\delta(t)\big), \quad t \in Q,\ t \neq \tau_j,\ j = 1, 2, \dots, m, \\
&z(0) = z_0, \qquad \varLambda z_\delta(\tau_j) = \varTheta_j\big(z_\delta(\tau_j^-)\big), \quad j = 1, 2, \dots, m, \\
&\big\langle A\big(t, y_\delta(t)\big), x\big\rangle + J_\delta^\circ\big(t, y_\delta(t), N y_\delta(t); N x\big) \geqslant \big\langle g\big(t, z_\delta(t)\big), x\big\rangle \quad \forall (t, x) \in Q \times Z_2.
\end{aligned} \tag{7}$$

We denote the constants involved in assumption (H_J) by $m_{J\delta}$ and c_{J_δ}. Furthermore, we introduce the following assumptions.

(H_{J^*}) $J_\delta : Q \times Y \to \mathbb{R}$ is a functional satisfying

(i) there exists a function $V : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying, for any $(t, y) \in Q \times Z_2$ and $\delta > 0$, $\|\zeta - \zeta_\delta\|_{Z_2^*} \leqslant V(\delta)$ for all $(\zeta, \zeta_\delta) \in N^*\partial J(t, Ny(t)) \times N^*\partial J_\delta(t, Ny(t))$;

(ii) $\lim_{\delta \to 0} V(\delta) = 0$.

(H_{0^*}) There exists $m_{A0} > 0$ such that

(i) $m_A > m_{A0} > c_{J\delta}\|N\|^2$, where $\|N\| = \|N\|_{L(Z_2,Y)}$;
(ii) $T^\kappa M_1 m_g/(\kappa(m_A - c_{J\delta}\|N\|^2)\Gamma(\kappa)) < 1$.

The following example indicates that assumption (H_{J^*}) can be satisfied for some functions.

Example 1. Let $0 < m < n$. Consider the functions $J : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $J_\delta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by

$$J(b,a) = \begin{cases} \frac{m-n}{m}a + n, & a \leqslant m,\\ ma + \frac{m(n-m)}{2}, & a > m, \end{cases}$$

$$J_\delta(b,a) = \begin{cases} \frac{m-n}{2m}(a+\delta)^2 + n(a+\delta), & a \leqslant m,\\ m(a+\delta) + \frac{m(n-m)}{2}, & a > m. \end{cases}$$

Then it is easy to check that $J(b,\cdot)$ and $J_\delta(b,\cdot)$ are locally Lipschitz and nonconvex for all $b \in \mathbb{R}^+$. Moreover, their Clarke subgradients are given by

$$\partial J(b,a) = \begin{cases} \frac{m-n}{m}a + n, & a \leqslant m,\\ a, & a > m, \end{cases}$$

$$\partial J_\delta(b,a) = \begin{cases} \frac{m-n}{m}(a+\delta) + n, & a \leqslant m,\\ a+\delta, & a > m. \end{cases}$$

Thus, we can see that condition (H_{J^*}) holds with $V(\delta) = \delta$.

Next, we show the stability result for FDQHVI as follows.

Theorem 2. *Suppose that assumptions* (H_A), (H_f), (H_I), (H_N), (H_J), (H_g), (H_0), (H_{0^*}), *and* (H_{J^*}) *hold. Then*

(i) *for each* $\delta > 0$*, the perturbation problem* (7) *has a unique solution* $(z_\delta, y_\delta) \in \mathcal{IC}(Q;Z_1)\times\mathcal{IC}(Q;Z_2)$*;*
(ii) (z_δ, y_δ) *converges to* $(z(t), y(t))$*, the solution of Problem* 1.

Proof. (i) In view of Theorem 1, the proof is obvious.

(ii) By Definition 6 we consider the problem

$$z_\delta(t) = z_0 + \sum_{i=1}^{j}\Theta_i\big(z_\delta(\tau_i^-)\big) + \frac{1}{\Gamma(\kappa)}\int_0^t (t-s)^{\kappa-1} f\big(s, z_\delta(s), y_\delta(s)\big)\,\mathrm{d}s$$
$$\forall t \in (t_j, t_{j+1}],\ j = 1,2,\dots,m, \tag{8}$$

$$A\big(t, y_\delta(t)\big) + \eta_\delta \ni g\big(t, z_\delta(t)\big), \quad \eta_\delta \in N^*\partial J_\delta\big(t, N y_\delta(t)\big) \quad \forall (t,x) \in Q\times Z_2. \tag{9}$$

Subtracting (9) from (4) and multiplying the result by $y(t) - y_\delta(t)$, we have

$$\begin{aligned}&\bigl\langle A\bigl(t, y(t)\bigr) - A\bigl(t, y_\delta(t)\bigr),\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2} + \bigl\langle \eta - \eta_\delta,\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2}\\ &\quad = \bigl\langle g\bigl(t, z(t)\bigr) - g\bigl(t, z_\delta(t)\bigr),\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2}\\ &\quad \forall (t, \eta, \eta_\delta) \in Q \times N^* \partial J\bigl(t, Ny(t)\bigr) \times N^* \partial J_\delta\bigl(t, Ny_\delta(t)\bigr).\end{aligned}$$

Since

$$\begin{aligned}&\bigl\langle \eta - \eta_\delta,\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2}\\ &\quad = \bigl\langle \eta - \xi_\delta,\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2} + \langle \xi_\delta - \eta_\delta,\, y(t) - y_\delta(t)\rangle_{Z_2^* \times Z_2}\\ &\quad \forall (t, \eta, \xi_\delta, \eta_\delta) \in Q \times N^* \partial J\bigl(t, Ny(t)\bigr) \times N^* \partial J\bigl(t, Ny_\delta(t)\bigr) \times N^* \partial J_\delta\bigl(t, Ny_\delta(t)\bigr),\end{aligned}$$

one has

$$\begin{aligned}&\bigl\langle A\bigl(t, y(t)\bigr) - A\bigl(t, y_\delta(t)\bigr),\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2} + \bigl\langle \eta - \xi_\delta,\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2}\\ &\quad = \bigl\langle \eta_\delta - \xi_\delta,\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2} + \bigl\langle g\bigl(t, z(t)\bigr) - g\bigl(t, z_\delta(t)\bigr),\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2}\\ &\quad \forall (t, \eta, \xi_\delta, \eta_\delta) \in Q \times N^* \partial J\bigl(t, Ny(t)\bigr) \times N^* \partial J\bigl(t, Ny_\delta(t)\bigr) \times N^* \partial J_\delta\bigl(t, Ny_\delta(t)\bigr).\end{aligned}$$

Note that assumption (H_A) implies

$$\begin{aligned}&\bigl\langle A\bigl(t, y(t)\bigr) - A\bigl(t, y_\delta(t)\bigr),\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2}\\ &\quad \geqslant m_A \bigl\|y(t) - y_\delta(t)\bigr\|_{Z_2}^2 \quad \forall t \in Q.\end{aligned} \tag{10}$$

Using assumptions (H_J) and (H_{J^*}), for any

$$(t, \eta, \xi_\delta) \in Q \times N^* \partial J\bigl(t, Ny(t)\bigr) \times N^* \partial J\bigl(t, Ny_\delta(t)\bigr)$$

and

$$(t, \xi_\delta, \eta_\delta) \in Q \times N^* \partial J\bigl(t, Ny_\delta(t)\bigr) \times N^* \partial J_\delta\bigl(t, Ny_\delta(t)\bigr),$$

we have

$$\bigl\langle \eta - \xi_\delta,\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2} \geqslant -c_J \|N\|^2 \bigl\|y(t) - y_\delta(t)\bigr\|_{Z_2}^2 \tag{11}$$

and

$$\bigl\langle \eta_\delta - \xi_\delta,\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2} \leqslant V(\delta) \bigl\|y(t) - y_\delta(t)\bigr\|_{Z_2}. \tag{12}$$

We conclude from assumption (H_g) that, for any $t \in Q$,

$$\begin{aligned}&\bigl\langle g\bigl(t, z(t)\bigr) - g\bigl(t, z_\delta(t)\bigr),\, y(t) - y_\delta(t)\bigr\rangle_{Z_2^* \times Z_2}\\ &\quad \leqslant m_g \bigl\|y(t) - y_\delta(t)\bigr\|_{Z_2} \bigl\|z(t) - z_\delta(t)\bigr\|_{Z_1}.\end{aligned}$$

Combining (10)–(12), one has

$$\begin{aligned}&\big(m_A - c_J\|N\|^2\big)\big\|y(t) - y_\delta(t)\big\|_{Z_2}^2\\&\quad\leqslant V(\delta)\big\|y(t) - y_\delta(t)\big\|_{Z_2} + m_g\big\|y(t) - y_\delta(t)\big\|_{Z_2}\big\|z(t) - z_\delta(t)\big\|_{Z_1}.\end{aligned}$$

Thus, assumption (H_0) yields that

$$\big\|y(t) - y_\delta(t)\big\|_{Z_2} \leqslant \frac{V(\delta)}{m_A - c_J\|N\|^2} + \frac{m_g}{m_A - c_J\|N\|^2}\big\|z(t) - z_\delta(t)\big\|_{Z_1}. \tag{13}$$

Subtracting (8) from (3), by assumptions (H_f), (H_I) and estimation (13) one has

$$\begin{aligned}&\big\|z_\delta(t) - z(t)\big\|_{Z_1}\\
&\quad\leqslant \frac{M_1}{\Gamma(\kappa)}\int_0^t (t-s)^{\kappa-1}\big(\big\|z(t) - z_\delta(t)\big\|_{Z_1} + \big\|y(t) - y_\delta(t)\big\|_{Z_2}\big)\,\mathrm{d}s\\
&\qquad + \sum_{i=1}^{j} d_j\big\|z_\delta(\tau_i^-) - z(\tau_i^-)\big\|_{Z_1}\\
&\quad\leqslant \frac{M_1}{\Gamma(\kappa)}\int_0^t (t-s)^{\kappa-1}\bigg[\frac{V(\delta)}{m_A - c_J\|N\|^2} + \bigg(\frac{m_g}{m_A - c_J\|N\|^2} + 1\bigg)\big\|z(t) - z_\delta(t)\big\|_{Z_1}\bigg]\,\mathrm{d}s\\
&\qquad + \sum_{i=1}^{j} d_j\big\|z_\delta(\tau_i^-) - z(\tau_i^-)\big\|_{Z_1}\\
&\quad\leqslant \frac{T^\kappa M_1}{\kappa\Gamma(\kappa)(m_A - c_J\|N\|^2)}V(\delta)\\
&\qquad + \frac{M_1}{\Gamma(\kappa)}\bigg(\frac{m_g}{m_A - c_J\|N\|^2} + 1\bigg)\int_0^t (t-s)^{\kappa-1}\big\|z(t) - z_\delta(t)\big\|_{Z_1}\,\mathrm{d}s\\
&\qquad + \sum_{i=1}^{j} d_j\big\|z_\delta(\tau_i^-) - z(\tau_i^-)\big\|_{Z_1}.\end{aligned}$$

By Lemma 5 with

$$k_1 = \frac{T^\kappa M_1}{\kappa\Gamma(\kappa)(m_A - c_J\|N\|^2)}V(\delta) \quad \text{and} \quad k_2 = \frac{M_1}{\Gamma(\kappa)}\bigg(\frac{m_g}{m_A - c_J\|N\|^2} + 1\bigg)$$

there exists $H^* > 0$ such that $\|z_\lambda(t) - z(t)\|_{Z_1} \leqslant H^* V(\delta)$, where H^* is independent of z, z_λ, y, y_λ and t. By assumption (H_{J^*}) we assert that $\|z_\lambda(t) - z(t)\|_{Z_1} \to 0$ as $\delta \to 0$. It follows from (13) and (H_{J^*}) that $\|y(t) - y_\delta(t)\|_{Z_2} \to 0$ as $\delta \to 0$. □

5 An application

In this section, we show that the results obtained in Sections 3 and 4 can be applied to study the frictional contact problem (Problem 2) between an elastic body and a foundation over time interval Q. We suppose that the surface traction may change suddenly in a short time, such as shocks, and consequently, which can be described by a fractional impulsive differential equations. We show that the weak form of Problem 2 leads to Problem 1 analyzed in Sections 3 and 4. Then Theorems 1 and 2 are applied to obtain the unique solvability of the frictional contact problem mentioned above as well as the convergence result of the perturbation problem.

We shortly review the basic notations and its mechanical interpretations. A deformable elastic body occupies a regular Lipschitz domain $V \subseteq \mathbb{R}^n (n = 2, 3)$ with the boundary ∂V. The boundary ∂V consists of three measurable disjoint parts Σ_1, Σ_2, and Σ_3 with $\operatorname{meas}\Sigma_1 > 0$. The body is clamped on Σ_1 and subjected to the action of volume force with density $\boldsymbol{f}_0$. An unknown surface traction (for convenience, we denote by $\boldsymbol{f}_2$ its density) with impulsive effect is applied on Σ_2. On Σ_3, the body may contact with an obstacle. We do not show expressly the relation of various functions and $\boldsymbol{y}$.

Let $\boldsymbol{\nu}$ be unit outward normal vector, $\mathbb{S}^n$ be the space of symmetric matrix of order two on $\mathbb{R}^n$. $\mathbb{S}^n$ and $\mathbb{R}^n$ are equipped with, respectively, the following inner products and norms: $\boldsymbol{\xi} \cdot \boldsymbol{\zeta} = \xi_{ij}\zeta_{ij}$ with $\|\boldsymbol{\xi}\| = (\boldsymbol{\xi} \cdot \boldsymbol{\xi})^{1/2}$ for all $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{S}^n$ and $\boldsymbol{m} \cdot \boldsymbol{n} = m_i n_i$ with $\|\boldsymbol{m}\| = (\boldsymbol{m} \cdot \boldsymbol{m})^{1/2}$ for all $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{R}^n$. Here the summation convention is adopted. For any $\boldsymbol{\eta} \in \mathbb{R}^n$ and $\boldsymbol{\sigma} \in \mathbb{S}^n$, we denote by $\eta_\nu = \boldsymbol{\eta} \cdot \boldsymbol{\nu}$ the normal components of $\boldsymbol{\eta}$, $\boldsymbol{\eta}_\tau = \boldsymbol{\eta} - \eta_\nu \boldsymbol{\nu}$ the tangential components of $\boldsymbol{\eta}$, $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ the normal components of $\boldsymbol{\sigma}$, $\boldsymbol{\sigma_\tau} = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ the tangential components of $\boldsymbol{\sigma}$. We also denote by $\boldsymbol{u} = (u_i) \in \mathbb{R}^n$, $\boldsymbol{\sigma} \in \mathbb{S}^n$, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})) \in \mathbb{S}^n$, respectively, the displacement vector, the stress tensor, and the linearized (small) strain tensor, where

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad u_{i,j} = \frac{\partial u_i}{\partial y_i}, \quad \boldsymbol{y} = (y_i) \in V \cup \partial V, \quad i, j = 1, \dots, n.$$

For more details, we refer the reader to [17, 18]. We now turn to present a new contact problem with the surface traction governed by a fractional impulsive differential equation.

Problem 2. Find a stress $\boldsymbol{\sigma} : V \times Q \to \mathbb{S}^n$, a surface traction density $\boldsymbol{f}_2 : \Sigma_2 \times Q \to \mathbb{R}^n$, and a displacement field $\boldsymbol{u} : V \times Q \to \mathbb{R}^n$ such that

$$\boldsymbol{\sigma}(t) = \mathsf{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) \quad \text{in } V \times Q, \tag{14}$$

$$\operatorname{div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \quad \text{in } V \times Q, \tag{15}$$

$$\boldsymbol{u}(t) = \mathbf{0} \quad \text{on } \Sigma_1 \times Q, \tag{16}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t), \quad {}_0^C D_t^\kappa \boldsymbol{f}_2(t) = F\big(t, \boldsymbol{f}_2(t), \boldsymbol{u}(t)\big) \quad \text{on } \Sigma_2 \times Q, \tag{17}$$

$$\boldsymbol{f}_2(0) = \boldsymbol{f}_2^0, \qquad \Lambda \boldsymbol{f}_2(\tau_j) = \Theta_j\big(\boldsymbol{f}_2(\tau_j^-)\big) \quad \text{on } \Sigma_2 \times Q, \tag{18}$$

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau\big(\boldsymbol{u}_\tau(t)\big), \quad -\sigma_\nu(t) \in \partial j_\nu\big(u_\nu(t)\big) \quad \text{on } \Sigma_3 \times Q, \tag{19}$$

$t \in Q$, $0 < \kappa < 1$, $t \neq \tau_j$, $j = 1, 2, \dots, m$. Here relation (14) presents an elastic constitutive law with A being the elasticity operator. Equation (15) is the equilibrium equation, and equation (16) implies that the body is clamped on Σ_1. Equalities (17)–(18) show that the traction is acted on Σ_2, and the density of the surface traction is governed by a fractional impulsive differential equation, where F is a function to be specified later. The set-valued relations in (19) denote the friction and contact conditions, respectively, where j_τ and j_ν are locally Lipschitz functionals.

To deduce the weak formulation of Problem 2, we consider spaces $\mathcal{H} = L^2(V; \mathbb{S}^n)^{n\times n}$ and $\mathcal{V} = \{\boldsymbol{v} \in H^1(V; \mathbb{R}^n) \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Sigma_1\}$ equipped with the inner products

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int\limits_V \sigma_{ij}\tau_{ij}\,\mathrm{d}x, \qquad (\boldsymbol{u}, \boldsymbol{v})_{\mathcal{V}} = \big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}}$$

and corresponding norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{V}}$, respectively. We denote by $\mathcal{V}^*$ the dual space of $\mathcal{V}$, $\langle\cdot,\cdot\rangle_{\mathcal{V}^*\times\mathcal{V}}$ the duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$. The trace theorem states

$$\|\gamma \boldsymbol{v}\|_{L^2(\Sigma_3;\mathbb{R}^n)} \leqslant \|\gamma\|\|\boldsymbol{v}\|_{\mathcal{V}} \quad \forall \boldsymbol{v} \in \mathcal{V},$$

where γ is the trace operator defined by $\gamma : \mathcal{V} \to L^2(\Sigma_3; \mathbb{R}^n)$. In order to study Problem 2, we impose some hypotheses on the relevant data.

($\widetilde{\mathrm{H}}_{\mathsf{A}}$) The elasticity operator $\mathsf{A} = (\mathsf{A}_{ijkl}) : V \times \mathbb{S}^n \to \mathbb{S}^n$ satisfies the conditions:

- (i) $\mathsf{A}_{ijkl} = \mathsf{A}_{klij} = \mathsf{A}_{jikl} \in L^\infty(V)$, i.e., $\mathsf{A}(\boldsymbol{y}, \cdot)$ is symmetric and linear for a.e. $\boldsymbol{y} \in V$;
- (ii) there exists $L_{\mathsf{A}} > 0$ such that $\|\mathsf{A}(\boldsymbol{y}, \boldsymbol{\zeta}_1) - \mathsf{A}(\boldsymbol{y}, \boldsymbol{\zeta}_2)\| \leqslant L_{\mathsf{A}}\|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|$ for all $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in \mathbb{S}^n$, a.e. $\boldsymbol{y} \in V$;
- (iii) there exists $m_{\mathsf{A}} > 0$ such that $(\mathsf{A}(\boldsymbol{y}, \boldsymbol{\zeta}_1) - \mathsf{A}(\boldsymbol{y}, \boldsymbol{\zeta}_2))(\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) \geqslant m_{\mathsf{A}}\|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|^2$ for all $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in \mathbb{S}^n$.

($\widetilde{\mathrm{H}}_F$) The function $F : Q \times \Sigma_2 \times L^2(\Sigma_2; \mathbb{R}^n) \times \mathcal{V} \to L^2(\Sigma_2; \mathbb{R}^n)$ is such that

- (i) $F(\cdot, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is continuous for all $(\boldsymbol{y}, \boldsymbol{z}) \in L^2(\Sigma_2; \mathbb{R}^n) \times \mathcal{V}$, a.e. $\boldsymbol{x} \in \Sigma_2$;
- (ii) there exists $M_1 > 0$ such that $\|F(t, \boldsymbol{x}, \boldsymbol{z}_1, \boldsymbol{y}_1) - F(t, \boldsymbol{x}, \boldsymbol{z}_2, \boldsymbol{y}_2)\| \leqslant M_1(\|\boldsymbol{z}_1 - \boldsymbol{z}_2\| + \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|)$ for all $(t, \boldsymbol{z}_i, \boldsymbol{y}_i) \in Q \times L^2(\Sigma_2; \mathbb{R}^n) \times \mathcal{V}$ $(i = 1, 2)$, a.e. $\boldsymbol{x} \in \Sigma_2$;
- (iii) there exists $\phi \in L_+^{1/p}[0, T]$ $(0 < p < \kappa < 1)$ satisfying $\|F(t, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y})\| \leqslant \phi(t)$ for all $(t, \boldsymbol{z}, \boldsymbol{y}) \in Q \times L^2(\Sigma_2; \mathbb{R}^n) \times \mathcal{V}$, a.e. $\boldsymbol{x} \in \Sigma_2$.

($\widetilde{\mathrm{H}}_I$) $\Theta_j : L^2(\Sigma_2; \mathbb{R}^n) \to L^2(\Sigma_2; \mathbb{R}^n)$ $(j = 1, 2, \dots, m)$ is bounded, and there exist $d_j > 0$ satisfying $\|\Theta_j(\boldsymbol{z}_1) - \Theta_j(\boldsymbol{z}_1)\|_{L^2(\Sigma_2;\mathbb{R}^n)} \leqslant d_j\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{L^2(\Sigma_2;\mathbb{R}^n)}$ for all $\boldsymbol{z}_1, \boldsymbol{z}_2 \in L^2(\Sigma_2; \mathbb{R}^n)$.

($\widetilde{\mathrm{H}}_{j_\nu}$) The function $j_\nu : \Sigma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

- (i) For a.e. $\boldsymbol{y} \in \Sigma_3$, $j_\nu(\boldsymbol{y}, \cdot)$ is locally Lipschitz on $\mathbb{R}$;
- (ii) For all $r \in \mathbb{R}$, $j_\nu(\cdot, r)$ is measurable on Σ_3 ;

(iii) For all $r \in \mathbb{R}$ and a.e. $\boldsymbol{y} \in \Sigma_3$, there exist $\overline{c}_0 \geqslant 0$ such that $|\partial j_\nu(\boldsymbol{y}, r)| \leqslant \overline{c}_0(1 + |r|)$;

(iv) For all $s_i \in \mathbb{R}$ $(i = 1, 2)$ and a.e. $\boldsymbol{y} \in \Sigma_3$, there exist $\alpha_{\nu 1} > 0$ such that $j_\nu^\circ(\boldsymbol{y}, s_1; s_2 - s_1) + j_\nu^\circ(\boldsymbol{y}, s_2; s_1 - s_2) \leqslant \alpha_{\nu 1}|s_1 - s_2|^2$.

($\widetilde{\mathrm{H}}_{j_\tau}$) The function $j_\tau : \Sigma_3 \times \mathbb{R}^n \to \mathbb{R}$ is such that

(i) $j_\tau(\boldsymbol{y}, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $\boldsymbol{y} \in \Sigma_3$;

(ii) $j_\tau(\cdot, \boldsymbol{r})$ is measurable on Σ_3 for all $\boldsymbol{r} \in \mathbb{R}^n$;

(iii) there exist $\overline{c}_1 \geqslant 0$ such that $|\partial j_\tau(\boldsymbol{y}, \boldsymbol{r})| \leqslant \overline{c}_1(1 + \|\boldsymbol{r}\|)$ for all $\boldsymbol{r} \in \mathbb{R}^n$, a.e. $\boldsymbol{y} \in \Sigma_3$;

(iv) there exist $\alpha_{\nu 2} > 0$ such that $j_\tau^\circ(\boldsymbol{y}, \boldsymbol{s}_1; \boldsymbol{s}_2 - \boldsymbol{s}_1) + j_\tau^\circ(\boldsymbol{y}, \boldsymbol{s}_2; \boldsymbol{s}_1 - \boldsymbol{s}_2) \leqslant \alpha_{\nu 2}\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2$ for all $\boldsymbol{s}_i \in \mathbb{R}$ $(i = 1, 2)$, a.e. $\boldsymbol{y} \in \Sigma_3$.

($\widetilde{\mathrm{H}}_f$) The densities of body force satisfies $\boldsymbol{f}_0 \in \mathcal{IC}(Q; L^2(V; \mathbb{R}^n))$.

($\widetilde{\mathrm{H}}_0$) (i) $m_{\mathsf{A}} > (\alpha_{\nu 1} + \alpha_{\nu 2})c_0^2$;

(ii) $T^\kappa M_1 c_0 / (\kappa[m_{\mathsf{A}} - (\alpha_{\nu 1} + \alpha_{\nu 2})c_0^2]\Gamma(\kappa)) < 1$.

Utilizing the Green formula, we get the variational form of Problem 2.

Problem 3. Find a displacement field $\boldsymbol{u} : Q \to \mathcal{V}$ and a surface traction density $\boldsymbol{f}_2 : Q \to L^2(\Sigma_2; \mathbb{R}^n)$ such that

$$
\begin{aligned}
&{}_0^C D_t^\kappa \boldsymbol{f}_2(t) = F\big(t, \boldsymbol{f}_2(t), \boldsymbol{u}(t)\big), \quad t \in Q,\ 0 < \kappa < 1,\ t \neq \tau_j,\ j = 1, 2, \dots, m, \\
&\boldsymbol{f}_2(0) = \boldsymbol{f}_2^0, \qquad \varLambda \boldsymbol{f}_2(\tau_j) = \varTheta_j\big(\boldsymbol{f}_2(\tau_j^-)\big), \quad j = 1, 2, \dots, m, \\
&\big(\mathsf{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}} + \int_{L_3} \big(j_\nu^\circ\big(u_\nu(t); v_\nu\big) + j_\tau^\circ\big(\boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau\big)\big)\,\mathrm{d}a \\
&\quad \geqslant \int_{L_2} \boldsymbol{f}_2(t)\boldsymbol{v}\,\mathrm{d}a + \int_V \boldsymbol{f}_0(t)\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \quad \forall (t, \boldsymbol{v}) \in Q \times \mathcal{V}.
\end{aligned}
$$

5.1 Existence and uniqueness for the contact problem

We define the maps $A : \mathcal{V} \to \mathcal{V}^*$, $f : Q \times L^2(\Sigma_2; \mathbb{R}^n) \times \mathcal{V} \to L^2(\Sigma_2; \mathbb{R}^n)$, $J : \mathcal{V} \to \mathbb{R}$, and $\boldsymbol{g} : L^2(\Sigma_2; \mathbb{R}^n) \to \mathcal{V}^*$ by setting

$$
\big\langle A\boldsymbol{u}, \boldsymbol{v}\big\rangle_{\mathcal{V}^* \times \mathcal{V}} = \big(\mathsf{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}}, \quad f(t, \boldsymbol{f}_2, \boldsymbol{v}) = F\big(t, \boldsymbol{f}_2(t), \boldsymbol{v}(t)\big),
$$

$$
J(\boldsymbol{u}) = \int_{\Sigma_3} \big(j_\nu\big(u_\nu(t)\big) + j_\tau\big(\boldsymbol{u}_\tau(t)\big)\big)\,\mathrm{d}a, \tag{20}
$$

$$
\big\langle \boldsymbol{g}(t), \boldsymbol{v}\big\rangle_{\mathcal{V}^* \times \mathcal{V}} = \int_V \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_{\Sigma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\,\mathrm{d}a \tag{21}
$$

for all $(t, \boldsymbol{f}_2, \boldsymbol{u}, \boldsymbol{v}) \in Q \times \mathbb{R}^n \times \mathcal{V} \times \mathcal{V}$.

Then Problem 3 is equivalent to the problem:

Problem 4. Find a displacement vector $\boldsymbol{u} : Q \to \mathcal{V}$ and a surface traction density $\boldsymbol{f}_2 : Q \to L^2(\Sigma_2; \mathbb{R}^n)$ such that

$$\begin{aligned}
&{}^C_0D^\kappa_t \boldsymbol{f}_2(t) = f\big(t, \boldsymbol{f}_2(t), \boldsymbol{u}(t)\big), \quad t \in Q,\ 0 < \kappa < 1,\ t \neq \tau_j,\ j = 1, 2, \dots, m, \\
&\boldsymbol{f}_2(0) = \boldsymbol{f}_2^0, \qquad \Lambda \boldsymbol{f}_2(\tau_j) = \Theta_j\big(\boldsymbol{f}_2(\tau_j^-)\big), \quad j = 1, 2, \dots, m, \\
&\big\langle A\boldsymbol{u}(t), \boldsymbol{v}\big\rangle_{\mathcal{V}^* \times \mathcal{V}} + J^\circ(\boldsymbol{u}; \boldsymbol{v}) \geqslant \big\langle \boldsymbol{g}(t), \boldsymbol{v}\big\rangle_{\mathcal{V}^* \times \mathcal{V}} \quad \forall (t, \boldsymbol{v}) \in t \times \mathcal{V}.
\end{aligned}$$

Clearly, Problem 4 is the form of Problem 1 with $Z_1 = L^2(\Sigma_2; \mathbb{R}^n)$, $Z_2 = \mathcal{V}$, $Y = L^2(\Sigma_3; \mathbb{R}^n)$.

Theorem 3. *Problem* 4 *has a unique solution* $(\boldsymbol{f}_2, \boldsymbol{u}(t)) \in \mathcal{IC}(Q; L^2(\Sigma_2; \mathbb{R}^n)) \times \mathcal{IC}(Q; \mathcal{V})$ *providing that hypotheses* ($\widetilde{\mathrm{H}}_{\mathsf{A}}$), ($\widetilde{\mathrm{H}}_F$), ($\widetilde{\mathrm{H}}_I$), ($\widetilde{\mathrm{H}}_{j_\nu}$), ($\widetilde{\mathrm{H}}_{j_\tau}$), ($\widetilde{\mathrm{H}}_f$), *and* ($\widetilde{\mathrm{H}}_0$) *hold.*

Proof. To prove Theorem 3, we only need to check the validity of assumptions (H_A), (H_f), (H_I), (H_N), (H_J), (H_g), and (H_0).

Firstly, conditions ($\widetilde{\mathrm{H}}_{\mathsf{A}}$), ($\widetilde{\mathrm{H}}_F$), and ($\widetilde{\mathrm{H}}_I$) indicate that assumptions (H_A), (H_f), and (H_I) are fulfilled with $m_A = m_{\mathsf{A}}$. Since the trace operator is compact and surjective, we see that assumption (H_N) holds. Clearly, (21) implies that assumption (H_g) holds with $m_g = \|\gamma\|$. By hypotheses ($\widetilde{\mathrm{H}}_{j_\nu}$), ($\widetilde{\mathrm{H}}_{j_\tau}$) and Lemma 14 in [19] it follows from Lemma 14 in [19] that the functional J in (20) is locally Lipschitz on $\mathcal{V}$, and

$$J^\circ(\boldsymbol{u}; \boldsymbol{w}) = \int\limits_{L_3} \big(j_\nu^\circ\big(u_\nu(t); w_\nu\big) + j_\tau^\circ\big(\boldsymbol{u}_\tau(t); \boldsymbol{w}_\tau\big)\big)\,\mathrm{d}a \quad \forall \boldsymbol{u}, \boldsymbol{w} \in \mathcal{V}$$

is the generalized directional derivative of J at $\boldsymbol{u}$ in the directional $\boldsymbol{w}$. Moreover, assumption (H_J) holds with $c_J = \alpha_{\nu 1} + \alpha_{\nu 2}$ and $m_J = \max\{\bar{c}_0, \bar{c}_1\}$. Combining Theorem 1 with hypothesis ($\widetilde{\mathrm{H}}_0$), we see that Theorem 3 holds. □

5.2 A convergence result for the contact problem

The above analysis reveals that the solution of Problem 4 relies on the data j_ν and j_τ. In what follows, we present a continuous dependence result of the solution in relation to these data. We consider the perturbation data $j_{\nu\delta}$ and $j_{\tau\delta}$ of j_ν and j_τ, respectively, which satisfy hypotheses ($\widetilde{\mathrm{H}}_{j_\nu}$) and ($\widetilde{\mathrm{H}}_{j_\tau}$). For each $\delta > 0$, define a function $J_\delta : \mathcal{V} \to \mathbb{R}$ by setting

$$J_\delta(\boldsymbol{u}) = \int\limits_{\Sigma_3} \big(j_{\nu\delta}\big(u_\nu(t)\big) + j_{\tau\delta}\big(\boldsymbol{u}_\tau(t)\big)\big)\,\mathrm{d}a \quad \forall \boldsymbol{u} \in \mathcal{V}.$$

The perturbation problem of Problem 4 can be formulated as follows.

Problem 5. Find a displacement vector $\boldsymbol{u}_\delta : Q \to \mathcal{V}$ and a surface traction density $\boldsymbol{f}_{2\delta} : Q \to L^2(\Sigma_2; \mathbb{R}^n)$ such that

$$\begin{aligned}
&{}_0^C D_t^\kappa \boldsymbol{f}_{2\delta}(t) = f\big(t, \boldsymbol{f}_{2\delta}(t), \boldsymbol{u}_\delta(t)\big), \quad t \in Q,\ 0 < \kappa < 1,\ t \neq \tau_j,\ j = 1, 2, \dots, m, \\
&\boldsymbol{f}_{2\delta}(0) = \boldsymbol{f}_2^0, \qquad \varLambda \boldsymbol{f}_{2\delta}(\tau_j) = \varTheta_j\big(\boldsymbol{f}_{2\delta}(\tau_j^-)\big), \quad j = 1, 2, \dots, m, \\
&\big\langle A\boldsymbol{u}_\delta(t), \boldsymbol{v}\big\rangle + J_\delta^\circ(\boldsymbol{u}_\delta; \boldsymbol{v}) \geqslant \big\langle \boldsymbol{g}(t), \boldsymbol{v}\big\rangle \quad \forall (t, \boldsymbol{v}) \in t \times \mathcal{V}.
\end{aligned}$$

Denote the constants involved in hypotheses $(\widetilde{\mathrm{H}}_{j_\nu\delta})$(iv) and $(\widetilde{\mathrm{H}}_{j_\tau\delta})$(iv) by $\alpha_{\nu 1\delta}$ and $\alpha_{\nu 2\delta}$, respectively. In addition, we impose the following hypotheses on the data.

$(\widetilde{\mathrm{H}}_{j^*})$ There exists a function $\overline{V} : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

- (i) $|\partial j_\nu(\boldsymbol{x}, r) - \partial j_{\nu\delta}(\boldsymbol{x}, r)| \leqslant \overline{V}(\delta)|r|$ for all $(\delta, r) \in \mathbb{R}^+ \times \mathbb{R}$, a.e. $\boldsymbol{x} \in \Sigma_3$;
- (ii) $\|\partial j_\tau(\boldsymbol{x}, \boldsymbol{b}) - \partial j_{\tau\delta}(\boldsymbol{x}, \boldsymbol{b})\| \leqslant \overline{V}(\delta)\|\boldsymbol{b}\|$ for all $(\boldsymbol{x}, \boldsymbol{b}) \in \Sigma_3 \times \mathbb{R}^n$;
- (iii) $\lim_{\delta \to 0} \overline{V}(\delta) = 0$.

$(\widetilde{\mathrm{H}}_{0^*})$ There exists $m_{\mathsf{A}_0} > 0$ such that

- (i) $m_{\mathsf{A}} > m_{\mathsf{A}_0} > (\alpha_{\nu 1\delta} + \alpha_{\nu 2\delta})c_0^2$;
- (ii) $T^\kappa M_1 c_0 / (\kappa[m_{\mathsf{A}} - (\alpha_{\nu 1\delta} + \alpha_{\nu 2\delta})c_0^2]\Gamma(\kappa)) < 1$.

Remark 1. Assumption $(\widetilde{\mathrm{H}}_{j^*})$ means that the perturbations of j_ν and j_τ must satisfy the locally Lipschitz conditions. Moreover, it is easy to see that the functions given in Example 1 satisfy condition $(\widetilde{\mathrm{H}}_{j^*})$.

Theorem 4. *Assume that hypotheses* $(\widetilde{\mathrm{H}}_{\mathsf{A}})$, $(\widetilde{\mathrm{H}}_F)$, $(\widetilde{\mathrm{H}}_I)$, $(\widetilde{\mathrm{H}}_{j_\nu})$, $(\widetilde{\mathrm{H}}_{j_\tau})$, $(\widetilde{\mathrm{H}}_f)$, $(\widetilde{\mathrm{H}}_{j^*})$, $(\widetilde{\mathrm{H}}_0)$, *and* $(\widetilde{\mathrm{H}}_{0^*})$ *hold. Then*

- (i) *Problem* 5 *has a unique solution* $(\boldsymbol{f}_{2\delta}, \boldsymbol{u}_\delta(t)) \in \mathcal{IC}(Q; L^2(\Sigma_2; \mathbb{R}^n)) \times \mathcal{IC}(Q; \mathcal{V})$ *for each* $\delta > 0$;
- (ii) $(\boldsymbol{f}_{2\delta}, \boldsymbol{u}_\delta(t))$ *converges to* $(\boldsymbol{f}_2, \boldsymbol{u}(t))$*, the solution of Problem* 4.

Proof. (i) In view of Theorem 3, the proof is obvious.

(ii) We employ Theorem 2 to prove the conclusion. To this end, we only need to check the validity of assumptions (H_{0^*}) and (H_{J^*}). Clearly, hypothesis (H_0^*) implies that assumption (H_{0^*}) holds. By Proposition 3.35 of [18], Corollary 4.15 in [18], and hypothesis $(\widetilde{\mathrm{H}}_{j^*})$, for any $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\xi}_\delta) \in \mathcal{V} \times \gamma^*\partial J(\gamma\boldsymbol{u}) \times \gamma^*\partial J_\delta(\gamma\boldsymbol{u})$ and $(\xi_\nu, \xi_{\nu\delta}, \boldsymbol{\xi}_\tau, \boldsymbol{\xi}_{\tau\delta}) \in \partial j_\nu(u_\nu(t)) \times \partial j_{\nu\delta}(u_\nu(t)) \times \partial j_\tau(\boldsymbol{u}_\tau(t)) \times \partial j_{\tau\delta}(\boldsymbol{u}_\tau(t))$, we have

$$\begin{aligned}
\|\boldsymbol{\xi} - \boldsymbol{\xi}_\delta\| &\leqslant \|\gamma^*\| \int\limits_{L_3} \big(|\xi_\nu - \xi_{\nu\delta}| + \|\boldsymbol{\xi}_\tau - \boldsymbol{\xi}_{\tau\delta}\|\big)\,\mathrm{d}a \\
&\leqslant \|\gamma^*\|\overline{V}(\delta) \int\limits_{L_3} \big(|u_\nu| + \|\boldsymbol{u}_\tau\|\big)\,\mathrm{d}a \leqslant \big(\|\gamma\|^2 \operatorname{meas}\Sigma_3 \|\boldsymbol{u}\|\big)\overline{V}(\delta),
\end{aligned}$$

which shows that assumption (H_{J^*}) holds with $V(\delta) = (\|\gamma\|^2 \operatorname{meas}\Sigma_3\|\boldsymbol{u}\|)\overline{V}(\delta)$. The convergence result now follows from Theorem 2. □

Remark 2. It is worth noting that the results in this paper are local in time thanks to (H_0)(ii), and $(\widetilde{\mathrm{H}}_0)$(ii) provide some constraints on the length T.

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