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Relative controllability of impulsive multi-delay differential systems*

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Abstract. In this paper, relative controllability of impulsive multi-delay differential systems in finite dimensional space are studied. By introducing the impulsive multi-delay Gramian matrix, a necessary and sufficient condition, and the Gramian criteria, for the relative controllability of linear systems is given. Using Krasnoselskii's fixed point theorem, a sufficient condition for controllability of semilinear systems is obtained. Numerically examples are given to illustrate our theoretically results.

Keywords: impulsive multi-delay differential systems, impulsive multi-delay Gramian matrix, relative controllability.

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1 Introduction

In many motion processes of nature, science, and technology, the state of motion may be changed or interfered suddenly in a very short time, and then the system state will be changed. If the state change time of the disturbed system is very short, it can be regarded as instantaneous, and then this kind of instantaneous sudden change phenomenon is called pulse phenomenon. Time-delay systems are systems with aftereffect or dead time, genetic systems, equations with deviating arguments or differential difference equations. They are used to model various phenomena from population systems, viscoelasticity, biological sciences, chemistry, economics, mechanics, physics, physiology, and engineering sciences. In the real world, impulsive phenomena and time-delay effects are intertwined and interact with each other. Impulse technology is widely used in the state control of time-delay systems and has applications in military and civil fields.

The delayed exponential matrix functions approach was presented in [6, 10] for discrete and continuous delay systems with permutable matrices, respectively. This new approach has been used in the stability of solutions and control problems for linear and nonlinear delay systems (see [1–5, 7–9, 11, 13–20]).

Medved' and Pospíšil extended the idea of deriving the representation of delay differential equations in [6, 10] to multi-delay differential equations with linear parts defined by pairwise permutable matrices in [16] and obtained sufficient conditions for the asymptotic stability of solutions. You and Wang [22, 23] extended the multiple delayed exponential matrix function in [10] to the impulsive case and used it to discuss the representation and stability of solutions in [24]. However, there are still very few results for the relative controllability of impulsive multi-delay differential systems. In this paper, we study the following impulsive multi-delay differential systems:

$$\begin{aligned} &\nu'(t) = A\nu(t) + \sum_{m=1}^{n} B_m \nu(t-\vartheta_m) + f\big(t,\nu(t)\big) + Cu(t), \quad t \in J,\ t \notin \mathscr{T}, \\ &\Delta\nu(t_i) := \nu\big(t_i^+\big) - \nu\big(t_i^-\big) = D_i\nu(t_i), \quad t_i \in \mathscr{T}, \\ &\nu(t) = \psi(t), \quad -\vartheta \leqslant t \leqslant 0,\ \vartheta := \max\{\vartheta_1,\dots,\vartheta_n\}, \end{aligned} \tag{1}$$

where $\vartheta_m > 0$, A, B_m, C, D_i are constant $N \times N$ matrices, $AB_m = B_mA$, $B_jB_m = B_mB_j$, $AD_i = D_iA$, and $B_mD_i = D_iB_m$ for each $m, j = 1, 2, \dots, n$, $i = 1, 2, \dots$, $\psi \in C_\vartheta^1 := C^1([-\vartheta, 0], \mathbb{R}^N)$, and $\nu(t) \in \mathbb{R}^N$. Now $f \in C(J \times \mathbb{R}^N, \mathbb{R}^N)$, $J := [0, \tau_1]$, $\tau_1 > 0$, $0 < t_1 < t_2 < \dots < t_h < \tau_1$, and the control function $u(\cdot)$ takes values from $L^2(J, \mathbb{R}^N)$. Let $\nu(t_i^+) = \lim_{\epsilon\to 0^+} \nu(t_i + \epsilon)$ and $\nu(t_i^-) = \nu(t_i)$ represent respectively the right and left limits of $\nu(t)$ at $t = t_i$.

First, we investigate the relative controllability of the linear case of (1), i.e., $f = \mathbf{0} \in \mathbb{R}^N$ using the impulsive multi-delayed matrix exponential in (2). Next, we construct a suitable control function for (1), which means that we give a condition (necessary and sufficient) for $u \in L^2(J, \mathbb{R}^N)$ to lead the solution of (1) with $f = \mathbf{0}$ to ν_{τ_1} at the time τ_1. We apply Krasnoselskii's fixed point theorem to show that (1) is also relatively controllable under suitable conditions.

The rest of this paper is organized as follows. In Section 2, we give some notations, concepts, and important lemmas. In Section 3, we establish relative controllability results for linear and semilinear systems, respectively. Examples are given to illustrate our main results in the final section.

2 Preliminaries

Let $\mathbb{R}^N$ be the N-dimensional Euclid space with the vector norm $\|\cdot\|$, and $\mathbb{R}^{N\times N}$ be the $N \times N$ matrix space with real value elements. For $\nu \in \mathbb{R}^N$ and $A \in \mathbb{R}^{N\times N}$, we introduce the vector infinite-norm $\|\nu\| = \max_{1\leqslant i\leqslant N} |\nu_i|$ and the matrix infinite-norm $\|A\| = \max_{1\leqslant i\leqslant N} \sum_{j=1}^{N} |a_{ij}|$, respectively, where ν_i and a_{ij} are the elements of the vector ν and matrix A. Let $L(\mathbb{R}^N)$ be the space of bounded linear operators in $\mathbb{R}^N$. Denote by $C(J, \mathbb{R}^N)$ the Banach space of vector-value bounded continuous functions from $J \to \mathbb{R}^N$ endowed with the norm $\|\nu\|_C = \sup_{t\in J} \|\nu(t)\|$. In addition, $\|\psi\|_C = \sup_{t\in[-\vartheta,0]} \|\psi(t)\|$. We introduce a space $C^1(\mathbb{R}^+, \mathbb{R}^N) = \{\nu \in C(\mathbb{R}^+, \mathbb{R}^N)\colon \nu' \in C(\mathbb{R}^+, \mathbb{R}^N)\}$. Denote $PC(J, \mathbb{R}^N) := \{\nu : J \to \mathbb{R}^N\colon \nu \in C((t_i, t_{i+1}], \mathbb{R}^N)$, there exist $\nu(t_i^-)$ and $\nu(t_i^+)$ with $\nu(t_i^-) = \nu(t_i)$ for any $i = 1, 2, \dots\}$ and $PC^1(J, \mathbb{R}^N) := \{\nu : J \to \mathbb{R}^N\colon \nu' \in PC(J, \mathbb{R}^N)\}$. Let X_1, X_2 be two Banach spaces, and $L_b(X_1, X_2)$ denotes the space of all bounded linear operators from X_1 to X_2. Next, $L^p(J, X_2)$ denotes the Banach space of functions $y : J \to X_2$, which are Bochner integrable normed by $\|y\|_{L^p(J,X_2)}$ for some $1 < p < \infty$.

We recall the notation of the multi-delayed matrix exponential given by [16]:

$$\mathcal{E}_{\vartheta_1,\cdots,\vartheta_j}^{B_1,\cdots,B_j t} = \begin{cases} \Theta, \quad t < -\vartheta_j, \\ \mathcal{X}_{j-1}(t+\vartheta_j), \quad -\vartheta_j \leqslant t < 0, \\ \mathcal{X}_{j-1}(t+\vartheta_j) + B_j \int_0^t \mathcal{X}_{j-1}(t-s_1)\mathcal{X}_{j-1}(s_1)\,\mathrm{d}s_1 + \cdots \\ \quad + B_j^z \int_{(z-1)\vartheta_j}^{t} \int_{(z-1)\vartheta_j}^{s_1} \cdots \int_{(z-1)\vartheta_j}^{s_{z-1}} \mathcal{X}_{j-1}(t-s_1) \\ \quad \times \prod_{i=1}^{z-1} \mathcal{X}_{j-1}(s_i - s_{i+1})\mathcal{X}_{j-1}(s_z-(z-1)\vartheta_j)\,\mathrm{d}s_z \cdots \mathrm{d}s_1, \\ (z-1)\vartheta_j \leqslant t < z\vartheta_j, \quad z = 1, 2, \dots, \end{cases} \tag{2}$$

where $\mathcal{X}_{j-1}(t) = \mathcal{E}_{\vartheta_1,\dots,\vartheta_{j-1}}^{B_1,\cdots,B_{j-1}(t-\vartheta_{j-1})}$, $j = 2, \dots, n$, and Θ is the zero matrix.

From [24] we know $\mathcal{Y}(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{N\times N}$ and

$$\mathcal{Y}(t,s) = \mathrm{e}^{A(t-s)}\mathcal{X}(t,\, s+\vartheta), \quad t > s, \tag{3}$$

where

$$\mathcal{X}(t,s) = \mathcal{E}_{\vartheta_1,\dots,\vartheta_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n(t-s)} + \sum_{s-\vartheta<t_j\leqslant t} D_j \mathcal{E}_{\vartheta_1,\dots,\vartheta_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n(t-\vartheta-t_j)} \mathcal{X}(t_j, s),$$

$\widetilde{B}_m = \mathrm{e}^{-A\vartheta_m} B_m$, $m = 1, \dots, n$.

Next, the solution of (1) has the form

$$\begin{aligned}
\nu(t) &= \mathcal{Y}(t,-\vartheta)\psi(-\vartheta) \\
&\quad + \int_{-\vartheta}^{0} \mathcal{Y}(t,s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}\big(t,\, s+\vartheta_m\big)\psi(s)\,\mathrm{d}s \\
&\quad + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \mathcal{Y}(t,s)\big[f\big(s,\nu(s)\big) + Cu(s)\big]\,\mathrm{d}s + \int_{t_i}^{t} \mathcal{Y}(t,s)\big[f\big(s,\nu(s)\big) + Cu(s)\big]\,\mathrm{d}s \\
&= \mathcal{Y}(t,-\vartheta)\psi(-\vartheta) \\
&\quad + \int_{-\vartheta}^{0} \mathcal{Y}(t,s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(t,\, s+\vartheta_m)\psi(s)\,\mathrm{d}s \\
&\quad + \int_{0}^{t} \mathcal{Y}(t,s)\big[f\big(s,\nu(s)\big) + Cu(s)\big]\,\mathrm{d}s. \qquad (4)
\end{aligned}$$

Lemma 1. (*See* [16, *Lemma* 13].) *If* $\|B_i\| \leqslant b_i \mathrm{e}^{b_i\vartheta_i}$, $b_i \in \mathbb{R}^+$, $i = 1,\cdots,n$, *then*

$$\big\|e_{\vartheta_1,\ldots,\vartheta_n}^{B_1,\ldots,B_n(t-\vartheta_n)}\big\| \leqslant \mathrm{e}^{(b_1+\cdots+b_n)t}, \quad t \in \mathbb{R}.$$

Lemma 2. *Suppose that* $\sum_{j=1}^{\infty}\|D_j\|$ *is convergent,* $\|\widetilde{B}_m\| \leqslant \alpha_m \mathrm{e}^{\alpha_m\vartheta_m}$, $\alpha_m \in \mathbb{R}^+$, $m = 1,\ldots,n$. *For any* $t > s$, *we have*

$$\big\|\mathcal{X}(t,s)\big\| \leqslant \bigg(\prod_{s-\vartheta<t_j\leqslant t} \big(\|D_j\|+1\big)\bigg)\mathrm{e}^{\alpha(t+\vartheta-s)}, \qquad (5)$$

$$\big\|\mathcal{Y}(t,s)\big\| \leqslant \bigg(\prod_{s<t_j\leqslant t} \big(\|D_j\|+1\big)\bigg)\mathrm{e}^{(\|A\|+\alpha)(t-s)}, \qquad (6)$$

$\alpha = \alpha_1 + \cdots + \alpha_n$.

Proof. Without loss of generality, we suppose that $t_i \leqslant s-\vartheta < t_{i+1}$ and $t_{i+l} \leqslant t < t_{i+l+1}$, $i, l = 0,1,2,\ldots$. We use mathematical induction.

For $l = 0$, by Lemma 1,

$$\big\|\mathcal{X}(t,s)\big\| \leqslant \big\|\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-s)}\big\| \leqslant \mathrm{e}^{(\alpha_1+\cdots+\alpha_n)(t+\vartheta_n-s)} \leqslant \mathrm{e}^{\alpha(t+\vartheta-s)}.$$

For $l = 1$, using Lemma 1, we have

$$\begin{aligned}
\big\|\mathcal{X}(t,s)\big\| &\leqslant \big\|\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-s)}\big\| + \big\|D_{i+1}\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-\vartheta-t_{i+1})}\mathcal{X}(t_{i+1},s)\big\| \\
&\leqslant \mathrm{e}^{\alpha(t+\vartheta-s)} + \|D_{i+1}\|\mathrm{e}^{(\alpha_1+\cdots+\alpha_n)(t-\vartheta+\vartheta_n-t_{i+1})}\mathrm{e}^{\alpha(t_{i+1}+\vartheta-s)} \\
&= \mathrm{e}^{\alpha(t+\vartheta-s)} + \|D_{i+1}\|\mathrm{e}^{\alpha(t+\vartheta_n-s)} \leqslant \big(\|D_{i+1}\|+1\big)\mathrm{e}^{\alpha(t+\vartheta-s)}.
\end{aligned}$$

For $l = k$, we suppose that

$$\left\|\mathcal{X}(t,s)\right\| \leqslant \left(\prod_{s-\vartheta<t_j\leqslant t}\left(\|D_j\|+1\right)\right)\mathrm{e}^{\alpha(t+\vartheta-s)}.$$

For $l = k+1$, using Lemma 1, we have

$$\begin{aligned}\left\|\mathcal{X}(t,s)\right\| &\leqslant \left\|\mathcal{E}_{\vartheta_1,\dots,\vartheta_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n(t-s)}\right\| + \sum_{s-\vartheta<t_j\leqslant t}\left\|D_j\mathcal{E}_{\vartheta_1,\dots,\vartheta_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n(t-\vartheta-t_j)}\mathcal{X}(t_j,s)\right\| \\ &\leqslant \mathrm{e}^{\alpha(t+\vartheta-s)} + \sum_{j=i+1}^{i+k+1}\|D_j\|\mathrm{e}^{\alpha(t-\vartheta+\vartheta_n-t_j)}\left(\prod_{z=i+1}^{j-1}\left(\|D_z\|+1\right)\right)\mathrm{e}^{\alpha(t_j+\vartheta-s)} \\ &\leqslant \left(1+\sum_{j=i+1}^{i+k+1}\|D_j\|\prod_{z=i+1}^{j-1}\left(\|D_z\|+1\right)\right)\mathrm{e}^{\alpha(t+\vartheta-s)} \\ &= \left(\prod_{j=i+1}^{i+k+1}\left(\|D_j\|+1\right)\right)\mathrm{e}^{\alpha(t+\vartheta-s)} = \left(\prod_{s-\vartheta<t_j\leqslant t}\left(\|D_j\|+1\right)\right)\mathrm{e}^{\alpha(t+\vartheta-s)}.\end{aligned}$$

Thus, we obtain (5).

Finally, using (3) and (5) via $\|\mathrm{e}^{At}\| \leqslant \mathrm{e}^{\|A\|t}$, one derives (6) immediately. The proof is finished. □

Lemma 3 [Krasnoselskii's fixed point theorem]. (*See* [12].) *Let $\mathcal{B}$ be a bounded closed and convex subset of Banach space X, and let F_1, F_2 be maps of $\mathcal{B}$ into X such that $F_1x + F_2y \in \mathcal{B}$ for every pair $x, y \in \mathcal{B}$. If F_1 is a contraction and F_2 is compact and continuous, then the equation $F_1x + F_2x = x$ has a solution on $\mathcal{B}$.*

Theorem 1 [PC-type Ascoli–Arzela theorem]. (*See* [21, *Thm.* 2.1].) *Let $\mathcal{Q} \subset PC(J,X)$, where X is a Banach space. Then $\mathcal{Q}$ is a relatively compact subset of $PC(J,X)$ if*:

(i) *$\mathcal{Q}$ is a uniformly bounded subset of $PC(J,X)$*;
(ii) *$\mathcal{Q}$ is equicontinuous in (t_i, t_{i+1}), $i = 0, 1, 2, \dots, h$ (here $t_0 = 0$ and $t_{h+1} = \tau_1$)*;
(iii) *$\mathcal{Q}(t) = \{\nu(t)\colon \nu \in \mathcal{Q},\ t \in J \setminus \mathcal{T}\}$, $\mathcal{Q}(t_i^+) = \{\nu(t_i^+)\colon \nu \in \mathcal{Q}\}$ and $\mathcal{Q}(t_i^-) = \{\nu(t_i^-)\colon \nu \in \mathcal{Q}\}$ are relatively compact subsets of X.*

3 Relative controllability

Definition 1. (See [11, Def. 4].) System (1) is called relatively controllable if for an arbitrary initial vector function $\psi \in C^1([-\vartheta,0],\mathbb{R}^N)$, the final state of the vector $\nu_{\tau_1} \in \mathbb{R}^N$ and time τ_1, there exists a control $u \in L^2(J,\mathbb{R}^N)$ such that system (1) has a solution $\nu \in C^1([-\vartheta,0] \cup J,\mathbb{R}^N)$ that satisfies the boundary conditions ν and $\nu(\tau_1) = \nu_{\tau_1}$.

3.1 Linear systems

Let $f(t,\nu(t)) \equiv \mathbf{0}$, $t \in J$. System (1) reduces to the following linear impulsive multi-delay controlled system:

$$\begin{aligned} &\nu'(t) = A\nu(t) + \sum_{m=1}^{n} B_m \nu(t-\vartheta_m) + Cu(t), \quad t \in J,\ t \notin \mathscr{T}, \\ &\Delta\nu(t_i) = D_i \nu(t_i), \quad t_i \in \mathscr{T}, \\ &\nu(t) = \psi(t), \quad -\vartheta \leqslant t \leqslant 0. \end{aligned} \tag{7}$$

The solution has a form

$$\begin{aligned} \nu(t) = \mathcal{Y}(t,-\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{Y}(t,s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s \\ - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(t,\ s+\vartheta_m)\psi(s)\,\mathrm{d}s + \int_{0}^{t} \mathcal{Y}(t,s)Cu(s)\,\mathrm{d}s. \end{aligned}$$

Similar to the classical Gramian matrix, we consider the impulsive multi-delay Gramian matrix as follows:

$$W_{\vartheta_1,\dots,\vartheta_n}[0,\tau_1] = \int_{0}^{\tau_1} \mathcal{Y}(\tau_1,s)CC^{\mathrm{T}}\mathcal{Y}^{\mathrm{T}}(\tau_1,s)\,\mathrm{d}s.$$

Theorem 2. *System* (7) *is relatively controllable if and only if* $W_{\vartheta_1,\dots,\vartheta_n}[0,\tau_1]$ *is nonsingular.*

Proof. First, we verify the sufficiency. Since $W_{\vartheta_1,\dots,\vartheta_n}[0,\tau_1]$ is nonsingular, its inverse $W_{\vartheta_1,\dots,\vartheta_n}[0,\tau_1]$ is well defined. For any final state $\nu_{\tau_1} \in \mathbb{R}^N$, one can select a control function as follows:

$$u(t) = C^{\mathrm{T}}\mathcal{Y}^{\mathrm{T}}(\tau_1,t)W^{-1}_{\vartheta_1,\dots,\vartheta_n}[0,\tau_1]\eta,$$

where

$$\begin{aligned} \eta = \nu_{\tau_1} - \mathcal{Y}(\tau_1,-\vartheta)\psi(-\vartheta) - \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1,s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s \\ + \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(\tau_1,\ s+\vartheta_m)\psi(s)\,\mathrm{d}s. \end{aligned}$$

Then

$$\begin{aligned} \nu(\tau_1) = \mathcal{Y}(\tau_1,-\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1,s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s \\ - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(\tau_1,\ s+\vartheta_m)\psi(s)\,\mathrm{d}s + \int_{0}^{\tau_1} \mathcal{Y}(\tau_1,s)Cu(s)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{Y}(\tau_1, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s \\
&\quad - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(\tau_1,\, s+\vartheta_m)\psi(s)\,\mathrm{d}s \\
&\quad + \int_{0}^{\tau_1} \mathcal{Y}(\tau_1, s)CC^{\mathrm{T}}\mathcal{Y}^{\mathrm{T}}(\tau_1, s)W^{-1}_{\vartheta_1,\dots,\vartheta_n}[0, \tau_1]\eta\,\mathrm{d}s \\
&= \nu_{\tau_1}.
\end{aligned}
$$

Next, by contradiction we prove the necessity. We assume that $W_{\vartheta_1,\dots,\vartheta_n}[0, \tau_1]$ is singular matrix, i.e., there exists at least one nonzero state $\tilde{\nu} \in \mathbb{R}^N$ such that

$$\tilde{\nu}^{\mathrm{T}} W_{\vartheta_1,\dots,\vartheta_n}[0, \tau_1]\tilde{\nu} = 0.$$

Then one obtains

$$
\begin{aligned}
0 &= \tilde{\nu}^{\mathrm{T}} W_{\vartheta_1,\dots,\vartheta_n}[0, \tau_1]\tilde{\nu} \\
&= \int_{0}^{\tau_1} \tilde{\nu}^{\mathrm{T}}\mathcal{Y}(\tau_1, s)CC^{\mathrm{T}}\mathcal{Y}^{\mathrm{T}}(\tau_1, s)\tilde{\nu}\,\mathrm{d}s = \int_{0}^{\tau_1} \big\|\tilde{\nu}^{\mathrm{T}}\mathcal{Y}(\tau_1, s)C\big\|^2\,\mathrm{d}s,
\end{aligned}
$$

which implies $\tilde{\nu}^{\mathrm{T}}\mathcal{Y}(\tau_1, s)C = \mathbf{0}^{\mathrm{T}}$ for all $s \in J$.

Since system (7) is relatively controllable, according to Definition 1, there exists a control $u_1(t)$ that drives the initial state to zero at τ_1, i.e.,

$$
\begin{aligned}
\nu(\tau_1) &= \mathcal{Y}(\tau_1, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s \\
&\quad - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(\tau_1,\, s+\vartheta_m)\psi(s)\,\mathrm{d}s + \int_{0}^{\tau_1} \mathcal{Y}(\tau_1, s)Cu_1(s)\,\mathrm{d}s \\
&= \mathbf{0}.
\end{aligned}
\tag{8}
$$

Similarly, there also exists a control $u_2(t)$ that drives the initial state to $\tilde{\nu}$ (nonzero) at τ_1, i.e.,

$$
\begin{aligned}
\nu(\tau_1) &= \mathcal{Y}(\tau_1, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s \\
&\quad - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(\tau_1,\, s+\vartheta_m)\psi(s)\,\mathrm{d}s + \int_{0}^{\tau_1} \mathcal{Y}(\tau_1, s)Cu_2(s)\,\mathrm{d}s \\
&= \tilde{\nu}.
\end{aligned}
\tag{9}
$$

Then from (8) and (9) we have

$$\tilde{\nu} = \int_0^{\tau_1} \mathcal{Y}(\tau_1, s)C\big[u_2(s) - u_1(s)\big]\,\mathrm{d}s. \tag{10}$$

Multiplying both sides of (10) by $\tilde{\nu}^{\mathrm{T}}$, we obtain

$$\tilde{\nu}^{\mathrm{T}}\tilde{\nu} = \int_0^{\tau_1} \tilde{\nu}^{\mathrm{T}}\mathcal{Y}(\tau_1, s)C\big[u_2(s) - u_1(s)\big]\,\mathrm{d}s = 0.$$

Thus, $\tilde{\nu} = \mathbf{0}$, which conflicts with $\tilde{\nu} \neq \mathbf{0}$. Thus, the impulsive multi-delay Gramian matrix $W_{\vartheta_1,\dots,\vartheta_n}[0, \tau_1]$ is nonsingular. The proof is complete. □

3.2 Semilinear systems

We assume the following:

- (H1) The operator $W : L^2(J, \mathbb{R}^N) \to \mathbb{R}^N$ defined by $Wu = \int_0^{\tau_1} \mathcal{Y}(\tau_1, s)Cu(s)\,\mathrm{d}s$ has an inverse operator W^{-1}, which takes values in $L^2(J, \mathbb{R}^N)/\ker W$. Then we set $M = \|W^{-1}\|_{L_b(\mathbb{R}^N, L^2(J,\mathbb{R}^N)/\ker W)}$. From [20, Remark 3.3] we know $M = (\|W^{-1}_{\vartheta_1,\dots,\vartheta_n}[0, \tau_1]\|)^{1/2}$.
- (H2) The function $f : J \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, and there exists a constant $q > 1$ and $L_f(\cdot) \in L^q(J, \mathbb{R}^+)$ such that

$$\big\|f\big(\cdot, \nu(\cdot)\big) - f\big(\cdot, \upsilon(\cdot)\big)\big\| \leqslant L_f(\cdot)\big\|\nu(\cdot) - \upsilon(\cdot)\big\|, \quad \nu, \upsilon \in \mathbb{R}^N.$$

Theorem 3. *Suppose that* (H1) *and* (H2) *are satisfied. Then system* (1) *is relatively controllable, provided that*

$$M_2\left[1 + \frac{a\|C\|M}{\|A\| + \alpha}\big(\mathrm{e}^{(\|A\|+\alpha)\tau_1} - 1\big)\right] < 1, \tag{11}$$

where $a = \prod_{j=1}^{h}(\|D_j\| + 1)$, *and* $M_2 = a[(1/(\|A\| + \alpha)p)(\mathrm{e}^{(\|A\|+\alpha)p\tau_1} - 1)]^{1/p}$, $\|L_f\|_{L^q(J,\mathbb{R}^+)}$, $1/p + 1/q = 1$, $p, q > 1$.

Proof. Using hypothesis (H1), for arbitrary $\nu(\cdot) \in PC$ and $t \in J$, we define the control function $u_\nu(t)$ by

$$\begin{aligned} u_\nu(t) = W^{-1}\Bigg(&\nu_{\tau_1} - \mathcal{Y}(\tau_1, -\vartheta)\psi(-\vartheta) - \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s \\ &+ \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(\tau_1,\, s + \vartheta_m)\psi(s)\,\mathrm{d}s - \int_0^{\tau_1} \mathcal{Y}(\tau_1, s)f\big(s, \nu(s)\big)\,\mathrm{d}s\Bigg)(t). \end{aligned} \tag{12}$$

We show that, using this control, the operator $\mathcal{F} : PC \to PC$, defined by

$$\begin{aligned}(\mathcal{F}\nu)(t) &= \mathcal{Y}(t,-\vartheta)\psi(-\vartheta) \\ &\quad + \int_{-\vartheta}^{0} \mathcal{Y}(t,s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(t,\, s+\vartheta_m)\psi(s)\,\mathrm{d}s \\ &\quad + \int_{0}^{t} \mathcal{Y}(t,s) f\big(s,\nu(s)\big)\,\mathrm{d}s + \int_{0}^{t} \mathcal{Y}(t,s) C u_\nu(s)\,\mathrm{d}s,\end{aligned}$$

has a fixed point ν, which is a mild solution of (1).

We check that $(\mathcal{F}\nu)(\tau_1) = \nu_{\tau_1}$, which means that u_ν steers system (1) from $(\mathcal{F}\nu)(0)$ to ν_{τ_1} in finite time τ_1. This implies that system (1) is relatively controllable on J.

For each positive number r, let $\mathcal{B}_r = \{\nu \in PC\colon \|\nu\|_{PC} \leqslant r\}$ (a bounded, closed, and convex set of PC). Set $R_f = \sup_{t\in J} \|f(t,0)\|$.

We divide the proof into three steps.

Step 1. We claim that there exists a positive number r such that $\mathcal{F}(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

From (H2) and Hölder's inequality we obtain that

$$\begin{aligned}\int_{0}^{t} e^{(\|A\|+\alpha)(t-s)} L_f(s)\,\mathrm{d}s &\leqslant \left(\int_{0}^{t} \mathrm{e}^{p(\|A\|+\alpha)(t-s)}\,\mathrm{d}s\right)^{1/p} \left(\int_{0}^{t} L_f^q(s)\,\mathrm{d}s\right)^{1/q} \\ &\leqslant \left[\frac{1}{(\|A\|+\alpha)p}\big(\mathrm{e}^{(\|A\|+\alpha)pt} - 1\big)\right]^{1/p} \|L_f\|_{L^q(J,\mathbb{R}^+)},\end{aligned}$$

and

$$\int_{0}^{t} e^{(\|A\|+\alpha)(t-s)} \big\|f(s,0)\big\|\,\mathrm{d}s \leqslant R_f \int_{0}^{t} \mathrm{e}^{(\|A\|+\alpha)(t-s)}\,\mathrm{d}s = \frac{R_f}{\|A\|+\alpha}\big(\mathrm{e}^{(\|A\|+\alpha)t} - 1\big).$$

From (12), (H1) and (H2) we have

$$\begin{aligned}\big\|u_\nu(t)\big\| &\leqslant \|W^{-1}\|_{L_b(\mathbb{R}^N, L^2(J,\mathbb{R}^N)/\ker W)} \Bigg(\|\nu_{\tau_1}\| + \big\|\mathcal{Y}(\tau_1,-\vartheta)\big\|\big\|\psi(-\vartheta)\big\| \\ &\quad + \int_{-\vartheta}^{0} \big\|\mathcal{Y}(\tau_1,s)\big\|\big\|\psi'(s) - A\psi(s)\big\|\,\mathrm{d}s \\ &\quad + \sum_{m=1}^{n} \|B_m\| \int_{-\vartheta}^{-\vartheta_m} \big\|\mathcal{Y}(\tau_1,\, s+\vartheta_m)\big\|\big\|\psi(s)\big\|\,\mathrm{d}s \\ &\quad + \int_{0}^{\tau_1} \big\|\mathcal{Y}(\tau_1,s)\big\|\big\|f\big(s,\nu(s)\big)\big\|\,\mathrm{d}s \Bigg)\end{aligned}$$

$$\begin{aligned}
&\leqslant M\Biggl[\|\nu_{\tau_1}\| + \Biggl(\prod_{j=1}^{h}\bigl(\|D_j\|+1\bigr)\Biggr)\mathrm{e}^{(\|A\|+\alpha)(\tau_1+\vartheta)}\bigl\|\psi(-\vartheta)\bigr\| \\
&\quad + \Biggl(\prod_{j=1}^{h}\bigl(\|D_j\|+1\bigr)\Biggr)\int_{-\vartheta}^{0}\mathrm{e}^{(\|A\|+\alpha)(\tau_1-s)}\bigl\|\psi'(s)-A\psi(s)\bigr\|\,\mathrm{d}s \\
&\quad + \Biggl(\prod_{j=1}^{h}\bigl(\|D_j\|+1\bigr)\Biggr)\sum_{m=1}^{n}\|B_m\|\int_{-\vartheta}^{-\vartheta_m}\mathrm{e}^{(\|A\|+\alpha)(\tau_1-\vartheta_m-s)}\bigl\|\psi(s)\bigr\|\,\mathrm{d}s \\
&\quad + \Biggl(\prod_{j=1}^{h}\bigl(\|D_j\|+1\bigr)\Biggr)\int_{0}^{\tau_1}\mathrm{e}^{(\|A\|+\alpha)(\tau_1-s)}\bigl(L_f(s)\bigl\|\nu(s)\bigr\|+\bigl\|f(s,0)\bigr\|\bigr)\,\mathrm{d}s\Biggr] \\
&\leqslant M\Biggl[\|\nu_{\tau_1}\| + a\mathrm{e}^{(\|A\|+\alpha)(\tau_1+\vartheta)}\bigl\|\psi(-\vartheta)\bigr\| + \int_{-\vartheta}^{0}a\mathrm{e}^{(\|A\|+\alpha)(\tau_1-s)}\bigl\|\psi'(s)-A\psi(s)\bigr\|\,\mathrm{d}s \\
&\quad + \sum_{m=1}^{n}a\|B_m\|\int_{-\vartheta}^{-\vartheta_m}\mathrm{e}^{(\|A\|+\alpha)(\tau_1-\vartheta_m-s)}\bigl\|\psi(s)\bigr\|\,\mathrm{d}s \\
&\quad + a\biggl[\frac{1}{(\|A\|+\alpha)p}\bigl(\mathrm{e}^{(\|A\|+\alpha)p\tau_1}-1\bigr)\biggr]^{1/p}\|L_f\|_{L^q(J,\mathbb{R}^+)}\|\nu\|_{PC} \\
&\quad + \frac{aR_f}{\|A\|+\alpha}\bigl(\mathrm{e}^{(\|A\|+\alpha)\tau_1}-1\bigr)\Biggr] \\
&\leqslant M\|\nu_{\tau_1}\| + MM_1 + MM_2\|\nu\|_{PC},
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= a\mathrm{e}^{(\|A\|+\alpha)(\tau_1+\vartheta)}\|\psi(-\vartheta)\| + \int_{-\vartheta}^{0}a\mathrm{e}^{(\|A\|+\alpha)(\tau_1-s)}\bigl\|\psi'(s)-A\psi(s)\bigr\|\,\mathrm{d}s \\
&\quad + \sum_{m=1}^{n}a\|B_m\|\int_{-\vartheta}^{-\vartheta_m}\mathrm{e}^{(\|A\|+\alpha)(\tau_1-\vartheta_m-s)}\bigl\|\psi(s)\bigr\|\,\mathrm{d}s + \frac{aR_f}{\|A\|+\alpha}\bigl(\mathrm{e}^{(\|A\|+\alpha)\tau_1}-1\bigr).
\end{aligned}$$

From (H1) and (H2) we have

$$\begin{aligned}
\bigl\|(\mathcal{F}\nu)(t)\bigr\| &\leqslant \bigl\|\mathcal{Y}(t,-\vartheta)\bigr\|\bigl\|\psi(-\vartheta)\bigr\| + \int_{-\vartheta}^{0}\bigl\|\mathcal{Y}(t,s)\bigr\|\bigl\|\psi'(s)-A\psi(s)\bigr\|\,\mathrm{d}s \\
&\quad + \sum_{m=1}^{n}\|B_m\|\int_{-\vartheta}^{-\vartheta_m}\bigl\|\mathcal{Y}(t,\, s+\vartheta_m)\bigr\|\bigl\|\psi(s)\bigr\|\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \big\|\mathcal{Y}(t,s)\big\|\big\|f\big(s,\nu(s)\big)\big\|\,\mathrm{d}s + \int_0^t \big\|\mathcal{Y}(t,s)\big\|\|C\|\big\|u_\nu(s)\big\|\,\mathrm{d}s \\
&\leqslant \left(\prod_{j=1}^{h}\big(\|D_j\|+1\big)\right)\mathrm{e}^{(\|A\|+\alpha)(t+\vartheta)}\big\|\psi(-\vartheta)\big\| \\
&\quad + \left(\prod_{j=1}^{h}\big(\|D_j\|+1\big)\right)\int_{-\vartheta}^{0}\mathrm{e}^{(\|A\|+\alpha)(t-s)}\big\|\psi'(s)-A\psi(s)\big\|\,\mathrm{d}s \\
&\quad + \left(\prod_{j=1}^{h}\big(\|D_j\|+1\big)\right)\sum_{m=1}^{n}\|B_m\|\int_{-\vartheta}^{-\vartheta_m}\mathrm{e}^{(\|A\|+\alpha)(t-\vartheta_m-s)}\big\|\psi(s)\big\|\,\mathrm{d}s \\
&\quad + \int_0^t\left(\prod_{s<t_j\leqslant t}\big(\|D_j\|+1\big)\right)\mathrm{e}^{(\|A\|+\alpha)(t-s)}\big(L_f(s)\big\|\nu(s)\big\| + \big\|f(s,0)\big\|\big)\,\mathrm{d}s \\
&\quad + \int_0^t\left(\prod_{s<t_j\leqslant t}\big(\|D_j\|+1\big)\right)\mathrm{e}^{(\|A\|+\alpha)(t-s)}\|C\| \\
&\qquad \times \big(M\|\nu_{\tau_1}\| + MM_1 + MM_2\|\nu\|_{PC}\big)\,\mathrm{d}s \\
&\leqslant M_1 + M_2\|\nu\|_{PC} + \left(\prod_{j=1}^{h}\big(\|D_j\|+1\big)\right)\|C\| \\
&\quad \times \big(M\|\nu_{\tau_1}\| + MM_1 + MM_2\|\nu\|_{PC}\big)\int_0^t \mathrm{e}^{(\|A\|+\alpha)(t-s)}\,\mathrm{d}s \\
&\leqslant M_1\left[1 + \left(\prod_{j=1}^{h}\big(\|D_j\|+1\big)\right)\frac{\|C\|M}{\|A\|+\alpha}\big(\mathrm{e}^{(\|A\|+\alpha)t}-1\big)\right] \\
&\quad + \left(\prod_{j=1}^{h}\big(\|D_j\|+1\big)\right)\frac{\|C\|M}{\|A\|+\alpha}\big(\mathrm{e}^{(\|A\|+\alpha)t}-1\big)\|\nu_{\tau_1}\| \\
&\quad + M_2\left(\prod_{j=1}^{h}\big(\|D_j\|+1\big)\right)\frac{\|C\|M}{\|A\|+\alpha}\big(\mathrm{e}^{(\|A\|+\alpha)t}-1\big)\|\nu\|_{PC} \\
&\leqslant M_1\left[1 + \frac{a\|C\|M}{\|A\|+\alpha}\big(\mathrm{e}^{(\|A\|+\alpha)\tau_1}-1\big)\right] + \frac{a\|C\|M}{\|A\|+\alpha}\big(\mathrm{e}^{(\|A\|+\alpha)\tau_1}-1\big)\|\nu_{\tau_1}\| \\
&\quad + M_2\left[1 + \frac{a\|C\|M}{\|A\|+\alpha}\big(\mathrm{e}^{(\|A\|+\alpha)\tau_1}-1\big)\right]r \\
&= r,
\end{aligned}
$$

where

$$r = \frac{M_1[1 + \frac{a\|C\|M}{\|A\|+\alpha}(\mathrm{e}^{(\|A\|+\alpha)\tau_1} - 1)] + \frac{a\|C\|M}{\|A\|+\alpha}(\mathrm{e}^{(\|A\|+\alpha)\tau_1} - 1)\|\nu_{\tau_1}\|}{1 - M_2[1 + \frac{a\|C\|M}{\|A\|+\alpha}(\mathrm{e}^{(\|A\|+\alpha)\tau_1} - 1)]}.$$

Hence, we obtain $\mathcal{F}(\mathcal{B}_r) \subseteq \mathcal{B}_r$ for such an r.

Now, we define operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{B}_r$ as

$$\begin{aligned}(\mathcal{F}_1\nu)(t) &= \mathcal{Y}(t, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{Y}(t, s)\big[\psi'(s) - A\psi(s)\big]\,\mathrm{d}s \\ &\quad - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(t, s + \vartheta_m)\psi(s)\,\mathrm{d}s + \int_{0}^{t} \mathcal{Y}(t, s)Cu_\nu(s)\,\mathrm{d}s, \quad t \in J,\end{aligned}$$

and

$$(\mathcal{F}_2\nu)(t) = \int_{0}^{t} \mathcal{Y}(t, s) f\big(s, \nu(s)\big)\,\mathrm{d}s, \quad t \in J.$$

Step 2. We claim that $\mathcal{F}_1$ is a contraction mapping.

Let $\nu, \gamma \in \mathcal{B}_r$. From (H1) and (H2), for each $t \in J$, we have

$$\begin{aligned}&\big\|u_\nu(t) - u_\gamma(t)\big\| \\ &\quad\leqslant M \int_{0}^{\tau_1} \big\|\mathcal{Y}(t, s)\big\|\big\|f\big(s, \nu(s)\big) - f\big(s, \gamma(s)\big)\big\|\,\mathrm{d}s \\ &\quad\leqslant M \int_{0}^{\tau_1} \bigg(\prod_{s<t_j\leqslant\tau_1} \big(\|D_j\| + 1\big)\bigg)\mathrm{e}^{(\|A\|+\alpha)(\tau_1-s)} L_f(s)\big(\|\nu(s) - \gamma(s)\|\big)\,\mathrm{d}s \\ &\quad\leqslant M\bigg(\prod_{j=1}^{h} \big(\|D_j\| + 1\big)\bigg) \int_{0}^{\tau_1} \mathrm{e}^{(\|A\|+\alpha)(\tau_1-s)} L_f(s)\,\mathrm{d}s\,\|\nu - \gamma\|_{PC} \\ &\quad\leqslant Ma\bigg[\frac{1}{(\|A\| + \alpha)p}\big(\mathrm{e}^{(\|A\|+\alpha)p\tau_1} - 1\big)\bigg]^{1/p} \|L_f\|_{L^q(J,\mathbb{R}^+)}\|\nu - \gamma\|_{PC} \\ &\quad\leqslant MM_2\|\nu - \gamma\|_{PC}.\end{aligned}$$

Thus,

$$\begin{aligned}&\big\|(\mathcal{F}_1\nu)(t) - (\mathcal{F}_1\gamma)(t)\big\| \\ &\quad\leqslant \int_{0}^{t} \big\|\mathcal{Y}(t, s)\big\|\|C\|\|u_\nu(s) - u_\gamma(s)\|\,\mathrm{d}s\end{aligned}$$

$$\leqslant \left(\prod_{j=1}^{h} \left(\|D_j\| + 1 \right) \right) \int_0^{\tau_1} \mathrm{e}^{(\|A\|+\alpha)(\tau_1 - s)}\, \mathrm{d}s\, \|C\| M M_2 \|\nu - \gamma\|_{PC}$$
$$\leqslant \frac{a\|C\| M M_2}{\|A\| + \alpha} \left(\mathrm{e}^{(\|A\|+\alpha)\tau_1} - 1 \right) \|\nu - \gamma\|_{PC},$$

so we obtain

$$\|\mathcal{F}_1\nu - \mathcal{F}_1\gamma\|_{PC} \leqslant T\|\nu - \gamma\|_{PC}, \quad T = \frac{a\|C\| M M_2}{\|A\| + \alpha} \left(\mathrm{e}^{(\|A\|+\alpha)\tau_1} - 1 \right).$$

From (11) we have $T < 1$, so $\mathcal{F}_1$ is a contraction.

Step 3. We claim that $\mathcal{F}_2 : \mathcal{B}_r \to PC$ is a compact and continuous operator.

Let $\nu_n \in \mathcal{B}_r$ with $\nu_n \to \nu$ in $\mathcal{B}_r$. Using (H2), we have $f(s, \nu_n(s)) \to f(s, \nu(s))$ in PC, and thus, using the Lebesgue dominated convergence theorem, we have

$$\left\| (\mathcal{F}_2\nu_n)(t) - (\mathcal{F}_2\nu)(t) \right\|$$
$$\leqslant \int_0^t \left\| \mathcal{Y}(t, s) \right\| \left\| f\left(s, \nu_n(s)\right) - f\left(s, \nu(s)\right) \right\| \mathrm{d}s$$
$$\leqslant \left(\prod_{j=1}^{h} \left(\|D_j\| + 1 \right) \right) \int_0^t \mathrm{e}^{(\|A\|+\alpha)(t-s)} \left\| f\left(s, \nu_n(s)\right) - f\left(s, \nu(s)\right) \right\| \mathrm{d}s$$
$$\to 0 \quad \text{as } n \to 0,$$

which implies that $\mathcal{F}_2$ is continuous on $\mathcal{B}_r$.

To check the compactness of $\mathcal{F}_2 : \mathcal{B}_r \to PC$, we prove that $\mathcal{F}_2(\mathcal{B}_r)$ is equicontinuous and uniformly bounded. In fact, for any $\nu \in \mathcal{B}_r$, $t_k < t \leqslant t + d \leqslant t_{k+1}$, $k = 0, 1, \ldots, h$,

$$(\mathcal{F}_2\nu)(t + d) - (\mathcal{F}_2\nu)(t)$$
$$= \int_0^{t+d} \mathcal{Y}(t + d, s) f\left(s, \nu(s)\right) \mathrm{d}s - \int_0^t \mathcal{Y}(t, s) f\left(s, \nu(s)\right) \mathrm{d}s$$
$$= \int_t^{t+d} \mathcal{Y}(t + d,\ s) f\left(s, \nu(s)\right) \mathrm{d}s$$
$$+ \int_0^t \mathrm{e}^{A(t+d-s)} \left(\mathcal{X}(t + d,\ s + \vartheta) - \mathcal{X}(t,\ s + \vartheta) \right) f\left(s, \nu(s)\right) \mathrm{d}s$$
$$+ \int_0^t \left(\mathrm{e}^{A(t+d-s)} - \mathrm{e}^{A(t-s)} \right) \mathcal{X}(t,\ s + \vartheta) f\left(s, \nu(s)\right) \mathrm{d}s.$$

Let

$$S_1(t) = \int_t^{t+d} \mathcal{Y}(t+d,\, s) f\bigl(s, \nu(s)\bigr)\,\mathrm{d}s,$$

$$S_2(t) = \int_0^{t} \mathrm{e}^{A(t+d-s)}\bigl(\mathcal{X}(t+d,\, s+\vartheta) - \mathcal{X}(t,\, s+\vartheta)\bigr) f\bigl(s, \nu(s)\bigr)\,\mathrm{d}s,$$

$$S_3(t) = \int_0^{t} \bigl(\mathrm{e}^{A(t+d-s)} - \mathrm{e}^{A(t-s)}\bigr)\mathcal{X}(t,\, s+\vartheta) f\bigl(s, \nu(s)\bigr)\,\mathrm{d}s = \int_0^{t} \bigl(\mathrm{e}^{Ad} - I\bigr)\mathcal{Y}(t,s) f\bigl(s, \nu(s)\bigr)\,\mathrm{d}s,$$

where I is the identity matrix.

From above we see that

$$\bigl\|(\mathcal{F}_2\nu)(t+d) - (\mathcal{F}_2\nu)(t)\bigr\| \leqslant \bigl\|S_1(t)\bigr\| + \bigl\|S_2(t)\bigr\| + \bigl\|S_3(t)\bigr\|.$$

Now, we only need to check $\|S_i(t)\| \to 0$ as $d \to 0$, $i = 1, 2, 3$. Clearly,

$$\begin{aligned}
\bigl\|S_1(t)\bigr\| &\leqslant \int_t^{t+d} \bigl\|\mathcal{Y}(t+d,\, s)\bigr\| \bigl\|f\bigl(s, \nu(s)\bigr)\bigr\|\,\mathrm{d}s \\
&\leqslant \int_t^{t+d} \biggl(\prod_{s<t_j\leqslant t+d} \bigl(\|D_j\| + 1\bigr)\biggr) \mathrm{e}^{(\|A\|+\alpha)(t+d-s)} \bigl(L_f(s)\bigl\|\nu(s)\bigr\| + \bigl\|f(s,0)\bigr\|\bigr)\,\mathrm{d}s \\
&\leqslant \biggl[\frac{1}{(\|A\|+\alpha)p}\bigl(\mathrm{e}^{(\|A\|+\alpha)pd} - 1\bigr)\biggr]^{1/p} \|L_f\|_{L^q(J,\mathbb{R}^+)} \|\nu\|_{PC} \\
&\quad + \frac{R_f}{\|A\|+\alpha}\bigl(\mathrm{e}^{(\|A\|+\alpha)d} - 1\bigr) \\
&\to 0 \quad \text{as } d \to 0,
\end{aligned}$$

$$\begin{aligned}
\bigl\|S_2(t)\bigr\| &\leqslant \int_0^{t} \bigl\|\mathrm{e}^{A(t+d-s)}\bigr\| \bigl\|\mathcal{X}(t+d,\, s+\vartheta) - \mathcal{X}(t,\, s+\vartheta)\bigr\| \bigl\|f\bigl(s, \nu(s)\bigr)\bigr\|\,\mathrm{d}s \\
&\leqslant \mathrm{e}^{\|A\|\tau_1} \int_0^{t} \biggl\| \mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t+d-\vartheta-s)} - \mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-\vartheta-s)} \\
&\quad + \sum_{s<t_j\leqslant t} D_j\bigl(\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t+d-\vartheta-t_j)} - \mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-\vartheta-t_j)}\bigr)\mathcal{X}(t_j,\, s+\vartheta)\biggr\| \\
&\quad \times \bigl(L_f(s)\bigl\|\nu(s)\bigr\| + \bigl\|f(s,0)\bigr\|\bigr)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathrm{e}^{\|A\|\tau_1}\Biggl\{ \int\limits_0^t \bigl\|\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t+d-\vartheta-s)} - \mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-\vartheta-s)} \\
&\quad \times \bigl\|\bigl(L_f(s)\bigl\|\nu(s)\bigr\| + \bigl\|f(s,0)\bigr\|\bigr)\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{h} \|D_j\|\bigl\|\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t+d-\vartheta-t_j)} - \mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-\vartheta-t_j)}\bigr\| \\
&\quad \times \Biggl(\int\limits_0^t \bigl\|\mathcal{X}(t_j,\, s+\vartheta)\bigr\|\bigl\|L_f(s)\bigl\|\nu(s)\bigr\|\,\mathrm{d}s + \int\limits_0^t \bigl\|\mathcal{X}(t_j,\, s+\vartheta)\bigr\|\bigl\|f(s,0)\bigr\|\,\mathrm{d}s \Biggr) \Biggr\} \\
&\leqslant \mathrm{e}^{\|A\|\tau_1}\Biggl\{ \|L_f\|_{L^q(J,\mathbb{R}^+)} r \Biggl(\int\limits_0^t \bigl\|\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t+d-\vartheta-s)} - \mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-\vartheta-s)}\bigr\|^p\,\mathrm{d}s \Biggr)^{1/p} \\
&\quad + R_f \int\limits_0^t \bigl\|\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t+d-\vartheta-s)} - \mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-\vartheta-s)}\bigr\|\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{h} \|D_j\|\bigl\|\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t+d-\vartheta-t_j)} - \mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(t-\vartheta-t_j)}\bigr\| \\
&\quad \times \Biggl[\|L_f\|_{L^q(J,\mathbb{R}^+)} r \Biggl(\int\limits_0^t \bigl\|\mathcal{X}(t_j,\, s+\vartheta)\bigr\|^p\,\mathrm{d}s \Biggr)^{1/p} + R_f \int\limits_0^t \bigl\|\mathcal{X}(t_j,\, s+\vartheta)\bigr\|\,\mathrm{d}s \Biggr] \Biggr\}.
\end{aligned}
$$

By the continuity of $\mathcal{E}_{\vartheta_1,\ldots,\vartheta_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n(\cdot)}$ we have $\|S_2\| \to 0$ as $d \to 0$. Also,

$$
\begin{aligned}
\bigl\|S_3(t)\bigr\| &\leqslant \int\limits_0^t \bigl\|\mathrm{e}^{Ad} - I\bigr\|\bigl\|\mathcal{Y}(t,s)\bigr\|\bigl\|f\bigl(s,\nu(s)\bigr)\bigr\|\,\mathrm{d}s \\
&\leqslant \bigl\|\mathrm{e}^{Ad} - I\bigr\| \int\limits_0^t \Biggl(\prod_{s<t_j\leqslant t} \bigl(\|D_j\|+1\bigr) \Biggr) \mathrm{e}^{(\|A\|+\alpha)(t-s)} \\
&\quad \times \bigl(L_f(s)\bigl\|\nu(s)\bigr\| + \bigl\|f(s,0)\bigr\|\bigr)\,\mathrm{d}s \\
&\leqslant \bigl\|\mathrm{e}^{Ad} - I\bigr\| a \Biggl(\biggl[\frac{1}{(\|A\|+\alpha)p} \bigl(\mathrm{e}^{(\|A\|+\alpha)p\tau_1} - 1\bigr) \biggr]^{1/p} \|L_f\|_{L^q(J,\mathbb{R}^+)} r \\
&\quad + \frac{R_f}{\|A\|+\alpha} \bigl(\mathrm{e}^{(\|A\|+\alpha)\tau_1} - 1\bigr) \Biggr) \to 0 \quad \text{as } d \to 0.
\end{aligned}
$$

As a result, we immediately obtain that

$$
\bigl\|(\mathcal{F}_2\nu)(t+d) - (\mathcal{F}_2\nu)(t)\bigr\| \to 0 \quad \text{as } d \to 0
$$

for all $\nu \in \mathcal{B}_r$. Therefore, $\mathcal{F}_2(\mathcal{B}_r)$ is equicontinuous in PC.

Next, repeating the above computations, we have

$$\begin{aligned}
&\|(\mathcal{F}_2\nu)(t)\| \\
&\quad\leqslant \int_0^t \|\mathcal{Y}(t,s)\|\|f(s,\nu(s))\|\,\mathrm{d}s \\
&\quad\leqslant \int_0^t \Bigl(\prod_{s<t_j\leqslant t}(\|D_j\|+1)\Bigr)\mathrm{e}^{(\|A\|+\alpha)(t-s)}\bigl(L_f(s)\|\nu(s)\|+\|f(s,0)\|\bigr)\,\mathrm{d}s \\
&\quad\leqslant a\biggl[\frac{1}{(\|A\|+\alpha)p}\bigl(\mathrm{e}^{(\|A\|+\alpha)p\tau_1}-1\bigr)\biggr]^{1/p}\|L_f\|_{L^q(J,\mathbb{R}^+)}r \\
&\qquad+\frac{aR_f}{\|A\|+\alpha}\bigl(\mathrm{e}^{(\|A\|+\alpha)\tau_1}-1\bigr).
\end{aligned}$$

Hence, $\mathcal{F}_2(\mathcal{B}_r)$ is uniformly bounded. From Theorem 1, $\mathcal{F}_2(\mathcal{B}_r)$ is relatively compact in PC. Thus, $\mathcal{F}_2:\mathcal{B}_r\to PC$ is a compact and continuous operator.

Furthermore, using Theorem 3, $\mathcal{F}$ has a fixed point ν on $\mathcal{B}_r$. Obviously, ν is a solution of system (1) satisfying $\nu(\tau_1)=\nu_{\tau_1}$. The boundary condition $\nu(t)=\psi(t)$, $-\vartheta\leqslant t\leqslant 0$ holds from (4). The proof is complete. □

4 Numerical examples

Example 1. Consider the following semilinear impulsive multi-delay differential controlled system:

$$\begin{aligned}
\nu'(t) &= A\nu(t)+\sum_{m=1}^{2}B_m\nu(t-\vartheta_m) \\
&\qquad + f\bigl(t,\nu(t)\bigr)+Cu(t), \quad t\in J,\ t\notin\mathscr{T}, \\
\Delta\nu(t_i) &= \begin{pmatrix}0.2 & 0\\ 0 & 0.2\end{pmatrix}\nu(t_i), \quad t_i\in\mathscr{T}, \\
\nu(t) &= (3,4)^{\mathrm{T}}, \quad -0.3\leqslant t\leqslant 0,
\end{aligned}\tag{13}$$

and set $J=[0,0.6]$, $\tau_1=0.6$. $\vartheta_1=0.3$, $\vartheta_2=0.2$. Then $\vartheta=\max\{\vartheta_1,\vartheta_2\}=0.3$ and $\mathscr{T}=\{0.35,0.7,1.05,\dots\}$,

$$A=\begin{pmatrix}-2 & -1\\ 0 & -3\end{pmatrix},\qquad B_1=\begin{pmatrix}0.2 & 0.1\\ 0 & 0.3\end{pmatrix},\qquad B_2=\begin{pmatrix}0.3 & -0.1\\ 0 & 0.2\end{pmatrix},$$
$$C=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad f\bigl(t,\nu(t)\bigr)=\begin{pmatrix}-0.06t\nu_1(t)\\ 0.04t\nu_2(t)\end{pmatrix}.$$

Note A, B_m, D_i are mutually permutable for $m = 1, 2$, $i = 1, 2, 3, \dots$, and

$$\begin{aligned}
W_{0.3,0.2}[0, 0.6] &= \int_0^{0.6} \mathcal{Y}(0.6, s) C C^{\mathrm{T}} \mathcal{Y}^{\mathrm{T}}(0.6, s) \,\mathrm{d}s \\
&= \int_0^{0.6} \mathrm{e}^{A(0.6-s)} \mathcal{X}(0.6, s+0.3) C C^{\mathrm{T}} \mathcal{X}^{\mathrm{T}}(0.6, s+0.3) \mathrm{e}^{A^{\mathrm{T}}(0.6-s)} \,\mathrm{d}s \\
&= W_1 + W_2 + W_3 + W_4 + W_5 + W_6,
\end{aligned}$$

where

$$\begin{aligned}
W_1 = \int_0^{0.05} &\mathrm{e}^{A(0.6-s)} \biggl[I + \widetilde{B}_1(0.2-s) + \widetilde{B}_2(0.3-s) + \frac{\widetilde{B}_2^2}{2}(0.1-s)^2 \\
&+ D_1 \bigl[I + \widetilde{B}_2(0.05-s) \bigr] \biggr] C C^{\mathrm{T}} \biggl[I + \widetilde{B}_1^{\mathrm{T}}(0.2-s) + \widetilde{B}_2^{\mathrm{T}}(0.3-s) \\
&+ \frac{(\widetilde{B}_2^{\mathrm{T}})^2}{2}(0.1-s)^2 + \bigl[I + \widetilde{B}_2^{\mathrm{T}}(0.05-s) \bigr] D_1^{\mathrm{T}} \biggr] \mathrm{e}^{A^{\mathrm{T}}(0.6-s)} \,\mathrm{d}s, \\
W_2 = \int_{0.05}^{0.1} &\mathrm{e}^{A(0.6-s)} \biggl[I + \widetilde{B}_1(0.2-s) + \widetilde{B}_2(0.3-s) + \frac{\widetilde{B}_2^2}{2}(0.1-s)^2 + D_1 \biggr] \\
&\times C C^{\mathrm{T}} \biggl[I + \widetilde{B}_1^{\mathrm{T}}(0.2-s) + \widetilde{B}_2^{\mathrm{T}}(0.3-s) + \frac{(\widetilde{B}_2^{\mathrm{T}})^2}{2}(0.1-s)^2 + D_1^{\mathrm{T}} \biggr] \\
&\times \mathrm{e}^{A^{\mathrm{T}}(0.6-s)} \,\mathrm{d}s, \\
W_3 = \int_{0.1}^{0.2} &\mathrm{e}^{A(0.6-s)} \bigl[I + \widetilde{B}_1(0.2-s) + \widetilde{B}_2(0.3-s) + D_1 \bigr] \\
&\times C C^{\mathrm{T}} \bigl[I + \widetilde{B}_1^{\mathrm{T}}(0.2-s) + \widetilde{B}_2^{\mathrm{T}}(0.3-s) + D_1^{\mathrm{T}} \bigr] \mathrm{e}^{A^{\mathrm{T}}(0.6-s)} \,\mathrm{d}s, \\
W_4 = \int_{0.2}^{0.25} &\mathrm{e}^{A(0.6-s)} \bigl[I + \widetilde{B}_2(0.3-s) + D_1 \bigr] \\
&\times C C^{\mathrm{T}} \bigl[I + \widetilde{B}_2^{\mathrm{T}}(0.3-s) + D_1^{\mathrm{T}} \bigr] \mathrm{e}^{A^{\mathrm{T}}(0.6-s)} \,\mathrm{d}s, \\
W_5 = \int_{0.25}^{0.3} &\mathrm{e}^{A(0.6-s)} \bigl[I + \widetilde{B}_2(0.3-s) \bigr] C C^{\mathrm{T}} \bigl[I + \widetilde{B}_2^{\mathrm{T}}(0.3-s) \bigr] \mathrm{e}^{A^{\mathrm{T}}(0.6-s)} \,\mathrm{d}s, \\
W_6 = \int_{0.3}^{0.5} &\mathrm{e}^{A(0.6-s)} C C^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}(0.6-s)} \,\mathrm{d}s.
\end{aligned}$$

Specifically,

$$W_1 = \begin{pmatrix} 0.0432 & 0.0427 \\ 0.0427 & 0.1010 \end{pmatrix}, \qquad W_2 = \begin{pmatrix} 0.0449 & 0.0446 \\ 0.0446 & 0.0955 \end{pmatrix},$$
$$W_3 = \begin{pmatrix} 0.0955 & 0.0954 \\ 0.0954 & 0.1769 \end{pmatrix}, \qquad W_4 = \begin{pmatrix} 0.0523 & 0.0525 \\ 0.0525 & 0.0838 \end{pmatrix},$$
$$W_5 = \begin{pmatrix} 0.0405 & 0.0405 \\ 0.0405 & 0.0583 \end{pmatrix}, \qquad W_6 = \begin{pmatrix} 0.2272 & 0.2272 \\ 0.2272 & 0.2639 \end{pmatrix},$$

then

$$W_{0.3,0.2}[0, 0.6] = \begin{pmatrix} 0.5036 & 0.5029 \\ 0.5029 & 0.7794 \end{pmatrix}, \qquad M = \sqrt{\|W^{-1}_{0.3,0.2}[0, 0.6]\|} = 3.0308.$$

Further, for any $\nu, \mu \in \mathbb{R}^2$,

$$\begin{aligned} \big\|f(t,\nu) - f(t,\mu)\big\| &= \max\big\{-0.06t|\nu_1 - \mu_1|, 0.04t|\nu_2 - \mu_2|\big\} \\ &\leqslant 0.06t \max\big\{|\nu_1 - \mu_1|, |\nu_2 - \mu_2|\big\} \\ &= 0.06t\|\nu - \mu\|. \end{aligned}$$

Note $L_f(t) = 0.06t$ and let $p = q = 2$, $\|L_f\|_{L^2(J,\mathbb{R}^+)} = (\int_0^{0.6} (0.06s)^2 \,\mathrm{d}s)^{1/2} = 0.0161$. Note

$$\begin{aligned} \|\widetilde{B}_1\| &= \|\mathrm{e}^{-A\vartheta_1} B_1\| = \left\| \begin{pmatrix} 0.3644 & 0.3735 \\ 0 & 0.7379 \end{pmatrix} \right\| \leqslant \alpha_1 \mathrm{e}^{0.3\alpha_1}, \\ &\text{choose } \alpha_1 = 0.61384, \\ \|\widetilde{B}_2\| &= \|\mathrm{e}^{-A\vartheta_2} B_2\| = \left\| \begin{pmatrix} 0.4475 & -0.0831 \\ 0 & 0.3644 \end{pmatrix} \right\| \leqslant \alpha_2 \mathrm{e}^{0.2\alpha_2}, \\ &\text{choose } \alpha_2 = 0.4820, \end{aligned}$$

$\alpha = \alpha_1 + \alpha_2 = 1.09584$, $\|A\| + \alpha = 4.09584$, $a = \prod_{j=1}^{h} (\|D_j\| + 1) = 1.2$.

As a result,

$$\begin{aligned} M_2 &= a\left[\frac{1}{(\|A\| + \alpha)p}\left(\mathrm{e}^{(\|A\|+\alpha)p\tau_1} - 1\right)\right]^{1/p} \|L_f\|_{L^q(J,\mathbb{R}^+)} = 0.0785, \\ M_2 &\left[1 + \frac{a\|C\|M}{\|A\| + \alpha}\left(\mathrm{e}^{(\|A\|+\alpha)\tau_1} - 1\right)\right] \\ &= 0.0785 \times \left[1 + \frac{1.2 \times 3.0308}{4.09584}\left(\mathrm{e}^{4.09584 \times 0.6} - 1\right)\right] = 0.8226 < 1. \end{aligned}$$

Thus all the conditions of Theorem 3 are satisfied, so (13) is relatively controllable on $[0, 0.6]$; see Fig. 1.

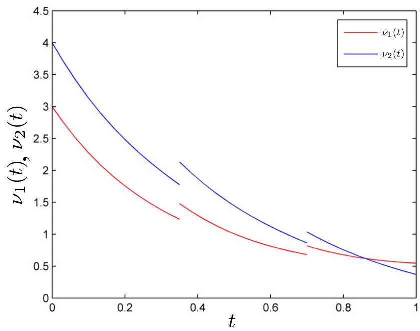

Figure 1. The state trajectories of $\nu(t)$ in $[0,1]$ when $u = [0.8t, 0.9t]^{\mathrm{T}}$ in Example 1.

Example 2. In Example 1, let $f(t,\nu(t)) = \mathbf{0}$, $t \in [0,0.6]$. Note $W_{0.3,0.2}[0,0.6]$ is a nonsingular matrix. From Theorem 2 we know that the linear multi-delay system is relatively controllable. Furthermore, one can get

$$\begin{aligned}\eta &= \nu_{\tau_1} - \mathcal{Y}(\tau_1,-\vartheta)\psi(-\vartheta)\\ &\quad - \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1,s)\bigl[\psi'(s) - A\psi(s)\bigr]\,\mathrm{d}s + \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(\tau_1,\, s+\vartheta_m)\psi(s)\,\mathrm{d}s\\ &= \nu_{\tau_1} - \mathcal{Y}(0.6,-0.3)\psi(-0.3)\\ &\quad - \int_{-0.3}^{0} \mathcal{Y}(0.6,s)\bigl[\psi'(s) - A\psi(s)\bigr]\,\mathrm{d}s + B_2 \int_{-0.3}^{-0.2} \mathcal{Y}(0.6,s+0.2)\psi(s)\,\mathrm{d}s,\end{aligned}$$

and then the control function is given by

$$\begin{aligned}u(t) &= C^{\mathrm{T}}\mathcal{Y}^{\mathrm{T}}(\tau_1,t)W^{-1}_{\vartheta_1,\dots,\vartheta_n}[0,\tau_1]\eta\\ &= C^{\mathrm{T}}\mathcal{X}^{\mathrm{T}}(0.6,t+0.3)\mathrm{e}^{A^{\mathrm{T}}(0.6-t)}W^{-1}_{0.3,0.2}[0,0.6]\eta\\ &= \begin{cases} C^{\mathrm{T}}[I + \widetilde{B}_1^{\mathrm{T}}(0.2-t) + \widetilde{B}_2^{\mathrm{T}}(0.3-t) + \frac{(\widetilde{B}_2^{\mathrm{T}})^2}{2}(0.1-t)^2\\ \quad +[I+\widetilde{B}_2^{\mathrm{T}}(0.05-t)]D_1^{\mathrm{T}}]\mathrm{e}^{A^{\mathrm{T}}(0.6-t)}W^{-1}_{0.3,0.2}[0,0.6]\eta, \quad 0 \leqslant t \leqslant 0.05,\\ C^{\mathrm{T}}[I + \widetilde{B}_1^{\mathrm{T}}(0.2-t) + \widetilde{B}_2^{\mathrm{T}}(0.3-t) + \frac{(\widetilde{B}_2^{\mathrm{T}})^2}{2}(0.1-t)^2 + D_1^{\mathrm{T}}]\\ \quad \times\mathrm{e}^{A^{\mathrm{T}}(0.6-t)}W^{-1}_{0.3,0.2}[0,0.6]\eta, \quad 0.05 < t \leqslant 0.1,\\ C^{\mathrm{T}}[I + \widetilde{B}_1^{\mathrm{T}}(0.2-t) + \widetilde{B}_2^{\mathrm{T}}(0.3-t) + D_1^{\mathrm{T}}]\mathrm{e}^{A^{\mathrm{T}}(0.6-t)}W^{-1}_{0.3,0.2}[0,0.6]\eta,\\ \quad 0.1 < t \leqslant 0.2,\\ C^{\mathrm{T}}[I + \widetilde{B}_2^{\mathrm{T}}(0.3-t) + D_1^{\mathrm{T}}]\mathrm{e}^{A^{\mathrm{T}}(0.6-t)}W^{-1}_{0.3,0.2}[0,0.6]\eta, \quad 0.2 < t \leqslant 0.25,\\ C^{\mathrm{T}}[I + \widetilde{B}_2^{\mathrm{T}}(0.3-t)]\mathrm{e}^{A^{\mathrm{T}}(0.6-t)}W^{-1}_{0.3,0.2}[0,0.6]\eta, \quad 0.25 < t \leqslant 0.3,\\ C^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}(0.6-t)}W^{-1}_{0.3,0.2}[0,0.6]\eta, \quad 0.3 < t \leqslant 0.5,\\ \Theta, \quad 0.5 < t \leqslant 0.6. \end{cases}\end{aligned}$$

5 Conclusion

In this paper the relative controllability of impulsive multi-delay differential systems in finite-dimensional space is considered. In [24] the authors construct the index of impulsive multi delay matrix and give the explicit solution of linear impulsive multi delay differential equations. Based on the expression of the solution of linear impulsive multi delay differential equations, necessary and sufficient conditions for the relative controllability of linear systems and the Gramian criteria are given. In Theorem 3, using Krasnoselskii fixed point theorem, we give a sufficient condition for the controllability of semilinear systems.

In Theorem 2 the control function is given, but it is not necessarily optimal, and we hope in the future to study the optimal control problem of impulsive multi-delay differential equations. In Theorem 3, we require the operator $\mathcal{F}_2$ to be compact, and we hope to study controllability under noncompact conditions in the future.

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