



On solutions of some delay Volterra integral problems on a half-line

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Abstract. In this paper, we study the existence of a.e. monotonic solutions of some general delay integral problems for both fractional and integer orders in the space of Lebesgue integrable functions on the interval $\mathbb{R}^+ = [0, \infty)$ and in the space of locally integrable functions $L_1^{loc}(\mathbb{R}^+)$. In particular, the uniqueness of solutions for considered problems is obtained.

Keywords: superposition operator, Carathéodory conditions, measure of noncompactness, delay integral equations, Darbo fixed point theorem.

1 Introduction

In this paper, we investigate some delay integral or integro-differential problems for both integer or fractional orders. For most of papers devoted to study such problems either on a finite interval or on a half-line, the expected solutions are continuous (on $C([0, \infty))$, cf. [29]) or continuous and bounded (on $BC([0, \infty))$, cf. [4, 9, 27, 28]). However, for integral problems, it is much more natural if solutions are not so regular and they are only integrable. This approach is not sufficiently investigated, and it requires to investigate operators acting on different spaces together with different qualitative indices in such spaces. We concentrate on delay integral equations.

Some particular problems are widely studied, but they are neither unified nor obtained for common and general assumptions. Delay integral or differential equations are quite frequent in mathematical biology, medicine, and physics. Let us recall some of our motivations. Many of the models of machining operations fall into the class of autonomous

delay differential equations of the form

$$\dot{x} = f(x(t), x(t-h)),$$

where the problem is studied in finite-dimensional spaces and $h > 0$ (cf. [24]).

In [15] the authors formulated a model to explain the observed periodic outbreaks of certain infectious diseases with periodic contact rate that varies seasonally, and it was also studied in [16, 32–34]. This model can also be interpreted as an equation describing the growth of a population when the birth rate varies seasonally. Then this model takes the form

$$x'(t) = f(t, (x(t)) - f(t - \tau, x(t - \tau))),$$

and is usually studied in its integral form.

However, many physical and biological models have been successfully described by delayed differential or integral problems with discontinuous functions, such as electric, pneumatic, and hydraulic networks (see [2, 11, 20]). For instance, in [10] the authors focused on the discontinuity solutions of problem with delays and proved some discontinuity properties for delay differential equations

$$\begin{aligned} y'(t) &= f(t, y(t), y(\alpha(t, y(t))))), \quad t \in [0, T], \\ y(t) &= \varphi(t), \quad t \in [a, 0], \quad \text{where } a = \inf_{t \geq 0} \alpha(t, y(t)) \leq 0. \end{aligned}$$

Note that we propose to study such problems with possibly general initial-value functions φ , not necessarily continuous. In particular, we will study the delay fractional integral problem

$$\begin{aligned} x(t) &= h(t) + m(t) \cdot g(t, x(t - \tau)) \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s - \tau)) ds, \quad t \in \mathbb{R}^+, \\ x(t) &= \varphi(t), \quad t \in [-\tau, 0], \quad 0 < \alpha < 1. \end{aligned} \tag{1}$$

All the problems mentioned above are extensions of earlier results, so we will try to extend and unify many of them. To achieve our goal, let us consider first the delay integral problem of Volterra–Hammerstein type

$$\begin{aligned} x(t) &= h(t) + m(t) \cdot g(t, x(t - \tau)) \\ &+ \int_0^t k(t, s) f(s, x(s - \tau)) ds, \quad t \in \mathbb{R}^+, \\ x(t) &= \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{2}$$

still being a general form of many previous problems considered, for instance, in [5, 7, 16, 17, 25]. It should be stressed that we will investigate these problems on unbounded

interval with integrable solutions as well as locally integrable ones. The above mentioned results are investigated either in the one of cases discussed here or even only on compact intervals. We will consider here the case of finite delay, and the the initial function describing the past will be defined on an interval $[\tau, 0]$.

More precisely, in this paper, we study the existence and the uniqueness of a.e. monotonic solutions of problem (2). We will also unify some known results being particular cases of (2), and we will extend some of them from bounded interval to unbounded one in the space $L_1(\mathbb{R}^+)$ or $L_1^{loc}(\mathbb{R}^+)$. As applications, we discuss the solvability of the integro-differential problems of fractional order (1). Proofs are based on operators form for considered problems with operators acting on appropriate function spaces and with the use of some special measures of noncompactness on these spaces together with the Darbo fixed point theorem. In particular, we need to investigate different properties of operators and different properties of considered subsets of functions paces. Thus proofs will be different than that for continuous solutions and for results about integrable solutions based on the weak topology argument (cf. [5, 8]).

2 Preliminaries

Let \mathbb{R} be the field of real numbers, and let \mathbb{R}^+ be the interval $[0, \infty)$. Denote by $L_1 = L_1(\mathbb{R}^+)$ the Banach space of all real functions defined and Lebesgue integrable on \mathbb{R}^+ endowed with the norm

$$\|x\|_{L_1(\mathbb{R}^+)} = \int_0^\infty |x(t)| dt$$

and by $L_1^{loc}(\mathbb{R}^+)$ the Fréchet space of all locally integrable spaces on \mathbb{R}^+ endowed with a family of seminorms $\|x\|_T = \int_0^T |x(t)| dt$ ($T > 0$). A nonempty set A in $L_1^{loc}(\mathbb{R}^+)$ is bounded if it is bounded in every seminorm, i.e., $\sup_{T>0} \{\|\chi_{[0,T]}x\|_T : x \in A\} < \infty$.

Definition 1. (See [1].) Assume that a function $f(t, x) = f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, i.e., it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in \mathbb{R}^+$. Then for every function x being measurable on \mathbb{R}^+ , we may assign

$$F_f(x)(t) = f(t, x(t)), \quad t \in \mathbb{R}^+.$$

The operator F_f defined in such a way is called the superposition (Nemytskii) operator generated by the function f .

Acting and continuity conditions in the case of σ -finite measure spaces were proved by Appell and Zabreiko:

Theorem 1. (See [1, Thm. 3.1].) Suppose that f satisfies the Carathéodory conditions. The superposition operator F_f maps continuously the space L_1 into L_1 if and only if

$$|f(t, x)| \leq a(t) + b|x|$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, where $a \in L_1$ and $b \geq 0$. Moreover, this operator $F_f : L_1 \rightarrow L_1$ is continuous.

Let $S = S(I)$ denotes the set of measurable (in Lebesgue sense) functions on an interval I . Identifying the functions equal almost everywhere the set S furnished with the metric

$$d(x, y) = \inf_{a>0} [a + \text{meas}\{s: |x(s) - y(s)| \geq a\}],$$

we obtain a complete metric space. Moreover, the convergence in measure on I is equivalent to the convergence with respect to the metric d in [35, Prop. 2.14].

For σ -finite subsets of \mathbb{R} , we say that the sequence x_n is convergent in finite measure to x if it is convergent in measure on each set T of finite measure. The compactness in such a space is called “compactness in measure”. The following property will be useful in our investigation.

Theorem 2. (See [12, Thm. 2.3].) *Let X be a bounded subset of L_1 consisting of functions, which are a.e. nonincreasing (or a.e. nondecreasing) on the half-line \mathbb{R}^+ . Then X is compact in measure in L_1 .*

Now, let us recall the concept of measure of noncompactness. Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ , and the symbol B_r stands for the closed ball with radius r and centered at θ . Denote by M_E the family of all nonempty and bounded subsets of E and by N_E its subfamily consisting of all relatively compact sets. The symbols \overline{X} , \overline{X}^W stand for the closure, and the weak closure of a set X , respectively, and the symbol $\text{conv } X$ will denote the convex closed hull of a set X .

Definition 2. (See [6].) A mapping $\mu : M_E \rightarrow [0, \infty)$ is called a regular measure of noncompactness in E if it satisfies the following conditions:

- (i) $\mu(X) = 0 \Leftrightarrow X \in N_E$.
- (ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
- (iii) $\mu(\overline{X}) = \mu(\text{conv } X) \leq \mu(X)$.
- (iv) $\mu(\lambda X) = |\lambda| \mu(X)$ for $\lambda \in \mathbb{R}$.
- (v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
- (vi) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.
- (vii) If X_n is a sequence of nonempty, bounded, closed subsets of E , $X_n = \overline{X_n}^W$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$, and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

An important example of such mappings is the following.

Definition 3. (See [3].) Let X be a nonempty and bounded subset of E . The Hausdorff measure of noncompactness $\chi(X)$ is defined as

$$\chi(X) = \inf\{r > 0: \text{there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + B_r\}.$$

We need to construct a measure of noncompactness in L_1 . For nonempty and bounded subset X of the space L_1 , let

$$\mu(X) = c(X) + d(X) \tag{3}$$

(cf. [8]), where

$$\begin{aligned}
 c(X) &= \lim_{\varepsilon \rightarrow 0} \sup_{x \in X} \left\{ \sup_D \left\{ \int_D |x(t)| dt : D \subset \mathbb{R}^+, \text{meas } D \leq \varepsilon \right\} \right\} \\
 &= \lim_{\varepsilon \rightarrow 0} \sup_{\substack{D \subset \mathbb{R}^+ \\ \text{meas } D \leq \varepsilon}} \sup_{x \in X} \|x \cdot \chi_D\|_1
 \end{aligned} \tag{4}$$

is a cocalled measure of uniform integrability (cf. [5]), and

$$d(X) = \lim_{T \rightarrow \infty} \sup \left\{ \int_T^\infty |x(t)| dt : x \in X \right\}. \tag{5}$$

Immediately, we get the following.

Lemma 1. *The measure μ is a measure of noncompactness if restricted to the family of subsets being compact in measure in L_1 .*

Proof. In [8, Lemma 2], it is proved that μ is a measure of weak noncompactness in L_1 (see also [8, Thm. 4]). But [18, Thm. 1] implies that for any bounded subset $X \subset L_1$ being additionally compact in measure, this quantity is equal to the Hausdorff measure of (strong) noncompactness (see [6]). \square

Moreover, measures $\chi(x)$ and $\mu(x)$ are equivalent:

Theorem 3. (See [5, Thm. 5].) *Let X be a nonempty, bounded, and compact in measure subset of L_1 . Then*

$$\chi(x) \leq \mu(x) \leq 2\chi(x).$$

Let us recall some fixed point theorems. In the proof of the existence of solutions of considered problems in L_1 , the following Darbo fixed point theorem will be useful:

Theorem 4. (See [6, Thm. 3.1].) *Let Q be a nonempty, bounded, closed, and convex subset of E , and let $H : Q \rightarrow Q$ be a continuous transformation, which is a contraction with respect to the measure of noncompactness μ , i.e., there exists $k \in [0, 1)$ such that*

$$\mu(H(X)) \leq k\mu(X)$$

for any nonempty subset X of E . Then H has at least one fixed point in the set Q .

Let E_1 be a Fréchet space, and let its topology is defined by a family of seminorms $(\|\cdot\|_n)$. By μ_n ($n \in \mathbb{N}$) denote the family of measures of noncompactness related to this family of seminorms. For instance, it could be done like in the Hausdorff measure of noncompactness, i.e.,

$$\mu_n(X) = \inf \{ r > 0 : \text{there exists a finite subset } Y \text{ of } E \text{ such that } x \subset Y + B_r^n \},$$

where $B_r^n = \{x \in E_1 : \|x\|_n < r\}$. A detailed study of the case $L_1^{\text{loc}}(\mathbb{R}^+)$ can be found, for instance, in [30].

Because the space $L_1^{\text{loc}}\mathbb{R}^+$ is Fréchet, but not a Banach space, we need the following version of the Darbo fixed point theorem, which is sufficient for our investigation.

Theorem 5. (See [21].) Let Q be a nonempty, bounded, closed, and convex subset of a Fréchet space E_1 , and let $H : Q \rightarrow Q$ be a continuous transformation such that for the family of measures of noncompactness, $(\mu_n)_{n \in \mathbb{N}}$, i.e., there exist constants $k_n \in [0, 1)$ ($n \in \mathbb{N}$) such that

$$\mu_n(H(X)) \leq k_n \mu_n(X)$$

for any nonempty bounded subset X of E_1 and $n \in \mathbb{N}$. Then H has at least one fixed point in the set Q .

Next, we give short notes about fractional operators. We will restrict our attention to the case of L_1 because for L_1^{loc} , they will result directly.

Definition 4. (See [26].) Let $f \in L_1$ and $\alpha \in \mathbb{R}^+$. The Riemann–Liouville (RL) fractional integral of the function f of order α is defined as

$$I_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds, \quad \alpha > 0, \quad a \leq t \leq b,$$

where $\Gamma(\alpha)$ is the Euler gamma function.

Lemma 2. (See [25, 31].) If $f \in L_1$ and $\alpha \in (0, 1)$, then

- (a) The operator I_a^α maps L_1 into itself continuously.
- (b) The operator I_a^α maps the monotonic nondecreasing function into functions of the same type.

3 Main results

First, let us consider problem (2). We will rewrite it in an operator form. The key difference between the case of integrable solutions and continuous ones is the action of operators on appropriate function spaces. It will not be surprising to assume that the expected solution observed in the past should have the same properties as in the future, so we will assume that $\varphi \in L_1([-\tau, 0])$ is a.e. nonincreasing and positive.

3.1 Existence of integrable solution

Rewrite problem (2) in the operator form

$$\begin{aligned} x(t) &= (H(x))(t) = h(t) + m(t)(F_g x_\tau)(t) + K(F_f x_\tau)(t), \\ x(t) &= \varphi(t), \quad t \in [-\tau, 0), \end{aligned} \tag{6}$$

where $x_\tau = x(t - \tau)$, $\tau < t$, $Kx(t) = \int_0^t k(t, s)x(s) \, ds$, and F_f, F_g be the superposition operators generated by f and g , respectively. Note that if φ is integrable, then for any integrable function x , a function x_τ is integrable too.

First, we will prove the existence of solutions in L_1 . Consider the following assumptions.

- (i) Let $m, h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a.e. nonincreasing functions, where m is a bounded function with $\sup_{t \in \mathbb{R}^+} |m(t)| \leq M$ and $h \in L_1$. Moreover, let $\varphi \in L_1([-\tau, 0])$ be a.e. nonincreasing and positive.
- (ii) Assume that the functions $f, g : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy Carathéodory conditions. Moreover, $f(t, x) \geq 0, g(t, x) \geq 0$ for $x \geq 0$, and f, g are a.e. nonincreasing with respect to t and nondecreasing with respect to x .
- (iii) There are positive integrable functions $a_i \in L_1$ and constants $b_i \geq 0$ ($i = 1, 2$) such that $|f(t, x)| \leq a_1(t) + b_1|x|, |g(t, x)| \leq a_2(t) + b_2|x|$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.
- (iv) $k : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies Carathéodory conditions such that the linear operator $K : L_1 \rightarrow L_1$ is continuous and maps the set of a.e. nonincreasing and positive functions into functions of the same type.
- (v) $b_2M + b_1\|K\|_{L_1} < 1/2$, where $\|K\|_{L_1}$ is an operator norm $\|K\|_{L_1 \rightarrow L_1}$.

Let us recall that some sufficient conditions for the acting and continuity conditions in (iv) can be found in [22] (a full description is unknown). Some conditions guaranteeing preservation of monotonicity of functions by K exactly on \mathbb{R}^+ can be found in [23, Sect. 4]. As this paper is not easily accessible, let us recall that criterion:

Proposition 1. (See [23].) *The operator K with the kernel $k(t, s)$ being locally integrable with respect to s on \mathbb{R}^+ for each fixed t preserves the monotonicity of functions from $L_1^{loc}(\mathbb{R}^+)$ if and only if*

$$\int_0^b k(t_1, s) \, ds \geq \int_0^b k(t_2, s) \, ds \tag{7}$$

for $t_1 < t_2, t_1, t_2 \in [0, T]$ and for any $b \geq 0$.

Our main result is the following.

Theorem 6. *Let assumptions (i)–(v) be satisfied. Then problem (2) has at least one solution $x \in L_1$, which is additionally a.e. nonincreasing function on \mathbb{R}^+ .*

Proof. We need to investigate acting, continuity, and contraction conditions for all operators describing equation (6).

By assumptions (ii), (iii) and due to Theorem 1, we can conclude that F_f, F_g map L_1 into itself continuously. By assumption (iv) the operator $K : L_1 \rightarrow L_1$ is continuous, and then $KF_f : L_1 \rightarrow L_1$ and is continuous too. For a given $x \in L_1$, by assumption (i) we can deduce that $H(x)$ belongs to L_1 and H is continuous. Then

$$\begin{aligned} \|H(x)\|_{L_1} &= \int_0^\infty \left| h(t) + m(t) \cdot g(t, x(t - \tau)) + \int_0^t k(t, s)f(s, x(s - \tau)) \, ds \right| dt \\ &\leq \int_0^\infty |h(t)| \, dt + \int_0^\infty |m(t)| \cdot |g(t, x(t - \tau))| \, dt \end{aligned}$$

$$\begin{aligned}
 & + \int_0^\infty \int_0^t |k(t, s)| |f(s, x(s - \tau))| \, ds \, dt \\
 & \leq \|h\|_{L_1} + M \int_0^\infty (a_2(t) + b_2|x(t - \tau)|) \, dt \\
 & \quad + \int_0^\infty \int_s^\infty |k(t, s)| (a_1(s) + b_1|x(s - \tau)|) \, dt \, ds \\
 & \leq \|h\|_{L_1} + M\|a_2\|_{L_1} + b_2M \int_0^\tau |x(t - \tau)| \, dt + b_2M \int_\tau^\infty |x(t - \tau)| \, dt \\
 & \quad + \|K\|_{L_1}\|a_1\|_{L_1} + b_1\|K\|_{L_1} \int_0^\tau |x(s - \tau)| \, ds \\
 & \quad + b_1\|K\|_{L_1} \int_\tau^\infty |x(s - \tau)| \, ds.
 \end{aligned}$$

Put $t - \tau = u$, so $du = dt$ and then

$$\begin{aligned}
 \|H(x)\|_{L_1} & \leq \|h\|_{L_1} + M\|a_2\|_{L_1} + b_2M \int_{-\tau}^0 |x(u)| \, du + b_2M \int_0^\infty |x(u)| \, du \\
 & \quad + \|K\|_{L_1}\|a_1\|_{L_1} + b_1\|K\|_{L_1} \int_{-\tau}^0 |x(u)| \, du + b_1\|K\|_{L_1} \int_0^\infty |x(u)| \, du \\
 & \leq \|h\|_{L_1} + M\|a_2\|_{L_1} + \|K\|_{L_1}\|a_1\|_{L_1} + b_2M \int_{-\tau}^0 |\varphi(u)| \, du + b_2M\|x\|_{L_1} \\
 & \quad + b_1\|K\|_{L_1} \int_{-\tau}^0 |\varphi(u)| \, du + b_1\|K\|_{L_1}\|x\|_{L_1} \\
 & \leq \|h\|_{L_1} + M\|a_2\|_{L_1} + \|K\|_{L_1}\|a_1\|_{L_1} + \|\varphi\|_{L_1([-\tau, 0])} (b_2M + b_1\|K\|_{L_1}) \\
 & \quad + (b_2M + b_1\|K\|_{L_1})\|x\|_{L_1}.
 \end{aligned}$$

From the above estimate we deduce that the function $H(x)$ is bounded on \mathbb{R}^+ , thus $H : L_1 \rightarrow L_1$. Moreover, we get

$$\begin{aligned}
 \|H(x)\|_{L_1} & \leq \|h\|_{L_1} + M\|a_2\|_{L_1} + \|K\|_{L_1}\|a_1\|_{L_1} \\
 & \quad + \|\varphi\|_{L_1([-\tau, 0])} (b_2M + b_1\|K\|_{L_1}) + (b_2M + b_1\|K\|_{L_1})r \\
 & = r,
 \end{aligned}$$

where

$$r = \frac{\|h\|_{L_1} + M\|a_2\|_{L_1} + \|K\|_{L_1}\|a_1\|_{L_1} + \|\varphi\|_{L_1([- \tau, 0])}(b_2M + b_1\|K\|_{L_1})}{1 - (b_2M + b_1\|K\|_{L_1})} > 0.$$

The inequality obtained above infer that the operator H maps the ball B_r into itself, i.e., $H : B_r \rightarrow B_r$ and is continuous.

Further, let Q_r denote the subset of B_r consisting of all function being a.e. nonincreasing and positive on \mathbb{R}^+ . The set Q_r is nonempty, bounded, closed, convex, and compact in measure in view of Theorem 2 (cf. [13], for instance).

Now, we will show that H preserves the monotonicity and positivity of functions from Q_r . Take an arbitrary $x \in Q_r$, then $x(t)$ is a.e. nonincreasing and positive on \mathbb{R}^+ , and consequently, f, g are also of the same type virtue of assumption (ii). By assumption (i) m, h are a.e. nonincreasing and positive functions on \mathbb{R}^+ , and from assumption (iv) the operator K maps a.e. nonincreasing and positive functions into functions of the same type.

Thus, we can deduce that $(H(x))$ is also a.e. nonincreasing and positive on \mathbb{R}^+ . This fact gives that $H : Q_r \rightarrow Q_r$ and is continuous.

Next, to prove that H is a contraction, we will assume that $\emptyset \neq X \subset Q_r$, and let a constant $\varepsilon > 0$ be arbitrary, but fixed.

Then for an arbitrary $x \in X \subset Q_r$ and for any measurable set $D \subset \mathbb{R}^+$, $\text{meas } D \leq \varepsilon$, we obtain

$$\begin{aligned} \int_D |(H(x))(t)| dt &\leq \int_D |h(t)| dt + \int_D |m(t)||g(t, x(t - \tau))| dt \\ &\quad + \int_D \int_0^t |k(t, s)||f(s, x(s - \tau))| ds dt \\ &\leq \int_D |h(t)| dt + \int_D |m(t)||g(t, x(t))| dt \\ &\quad + \int_D \int_0^t |k(t, s)||f(s, x(s))| ds dt \\ &\leq \|h\|_{L_1(D)} + M \int_D (a_2(t) + b_2|x(t)|) dt \\ &\quad + \|K\|_{L_1(D)} \int_D (a_1(s) + b_1|x(s)|) ds \\ &= \|h\|_{L_1(D)} + M\|a_2\|_{L_1(D)} + \|K\|_{L_1(D)}\|a_1\|_{L_1(D)} \\ &\quad + (b_2M + b_1\|K\|_{L_1(D)}) \int_D |x(s)| ds, \end{aligned}$$

where the symbol $\|K\|_{L_1(D)}$ denotes the norm of the operator K acting from the space $L_1(D)$ into itself. Recall that D is arbitrary, so the set $D \cap [0, \tau]$ need not be empty. Then the above estimation require the use of assumption (i) as for such points we get $t - \tau \in [-\tau, 0]$, on this interval, we have $x(s) = \varphi(s)$. Hence we keep the expected properties of a.e. monotonicity and positivity of functions.

It suffice to prove that H is a contraction with respect to some regular measure of noncompactness. Since $h, a_i \in L_1$, $i = 1, 2$, then we have the equality

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup \left\{ \|h\|_{L_1(D)} + M\|a_2\|_{L_1(D)} + \|K\|_{L_1(D)}\|a_1\|_{L_1(D)} : D \subset \mathbb{R}^+, \text{meas } D \leq \varepsilon \right\} \right\} = 0.$$

As $\|K\|_{L_1(D)} \leq \|K\|_{L_1}$, from definition 4 it follows that

$$c(H(X)) \leq (b_2M + b_1\|K\|_{L_1})c(X). \quad (8)$$

For any $T > 0$, we have the following estimate:

$$\int_T^\infty |(H(x))(t)| dt \leq \|h\|_{L_1(T)} + M\|a_2\|_{L_1(T)} + \|K\|_{L_1(T)}\|a_1\|_{L_1(T)} + (b_2M + b_1\|K\|_{L_1(T)}) \int_T^\infty |x(u)| du,$$

where the symbol $\|\cdot\|_{L_1(T)}$ denotes the operator norm acting from the space $L_1[T, \infty)$ into itself. Because $T \rightarrow \infty$, by the definition 5 we get

$$d(H(X)) \leq (b_2M + b_1\|K\|_{L_1(T)})d(X). \quad (9)$$

By combining (8) and (9) and by applying definition 3 we have

$$\mu(H(X)) \leq (b_2M + b_1\|K\|_{L_1})\mu(X).$$

Since $X \subset Q_r$ and we know that Q_r is compact in measure, by Theorem 3 we have

$$\chi(H(X)) \leq 2(b_2M + b_1\|K\|_{L_1})\chi(X).$$

Using all properties of Q_r , by assumption (iv) $2(b_2M + b_1\|K\|_{L_1}) < 1$, then we can apply Theorem 4, which completes the proof. \square

Corollary 1. *This theorem is also true for solutions in $L_p(\mathbb{R}^+)$ ($p > 1$) with a suitable set of modified assumptions assuring acting and continuity conditions for considered operators (cf. [14]).*

We are interested in studying delay integral problem, but in view of our proof, we can also formally generalize our main theorem by considering functional integral problem:

$$\begin{aligned}
 x(t) &= h(t) + m(t) \cdot g(t, x(t - \tau)) \\
 &\quad + \int_0^t k(t, s) f(s, x(\psi(s))) \, ds, \quad t \in \mathbb{R}^+, \\
 x(t) &= \varphi(t), \quad t \in [-\tau, 0).
 \end{aligned}
 \tag{10}$$

Corollary 2. *Under the assumption of Theorem 6, there exists solution $x \in L_1$, which is additionally a.e. nonincreasing function on \mathbb{R}^+ of the functional integral problem (10), provided the functional delay $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is increasing, absolutely continuous, and there is a constant $B > 0$ such that $\psi'(t) \geq B$ for a.e. $t \in \mathbb{R}^+$ and when assumption (v) is modified to the form*

$$(v') \quad b_2 M + b_1 \|K\|_{L_1} / B < 1/2.$$

3.2 Locally integrable solutions

Let us present some results for solutions of the considered problem being only locally integrable. On the one hand, it will weaken the assumptions, but on the other hand, we need to proceed in a Fréchet space $L_1^{loc}(\mathbb{R}^+)$. Now, we describe the differences between sets of assumptions in two considered cases, and we will emphasize on differences between them.

Let us present a set of modified assumptions:

- (i1) Let $m, h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a.e. nonincreasing functions, where m is a measurable essentially bounded function with $\text{ess sup}_{t \in \mathbb{R}^+} |m(t)| \leq M$ and $h \in L_1^{loc}(\mathbb{R}^+)$. Moreover, let $\varphi \in L_1([-\tau, 0])$ be a.e. nonincreasing and positive.
- (ii1) Assume that the functions $f, g : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy Carathéodory conditions. Moreover, $f(t, x) \geq 0, g(t, x) \geq 0$ for $x \geq 0$, and f, g are a.e. nonincreasing with respect to both variables t and x , separately.
- (iii1) There are positive integrable functions $a_i \in L_1^{loc}(\mathbb{R}^+)$ and measurable essentially bounded functions $b_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+ (i = 1, 2)$ such that $|f(t, x)| \leq a_1(t) + b_1(t)|x|, |g(t, x)| \leq a_2(t) + b_2(t)|x|$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.
- (iv1) $k : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies Carathéodory conditions such that the linear operator $K : L_1^{loc}(\mathbb{R}^+) \rightarrow L_1^{loc}(\mathbb{R}^+)$ is continuous and maps the set of a.e. nonincreasing and positive functions into functions of the same type.
- (v1) For any $T > 0$, the following inequality holds true: $b_2(T)M + b_1(T)\|K\|_T < 1/2, b_i(T) = \text{ess sup}_{t \in [0, T]} b_i(t) (i = 1, 2)$, and $\|K\|_T$ denotes the operator norm $\|K\|_{L_1([0, T]) \rightarrow L_1([0, T])}$.

Theorem 7. *Let assumptions (i1)–(v1) be satisfied. Then problem (2) has at least one locally integrable solution $x \in L_1^{loc}(\mathbb{R}^+)$, which is additionally a.e. nonincreasing function on \mathbb{R}^+ .*

Sketch of the proof. We fix an arbitrary $T > 0$ (or even T/τ , in fact), and then we follow the lines of the proof of Theorem 6 with some necessary changes. All estimations should be done on interval $[0, T]$, so they are correct. Note that for a fixed $T > 0$, assumption (v1) is the same as (v), so any additional changes in the proof are not necessary.

Clearly, Theorem 1 should be replaced by a result in $L_1^{\text{loc}}(\mathbb{R}^+)$; cf. [30, Thm. 4.2] and [30, Lemma 4.3]. Note that the set Q_r consists of functions not necessarily integrable on \mathbb{R}^+ , but in this case, it is compact in finite measure, and the remaining part holds true on every finite subinterval $[0, T] \subset \mathbb{R}^+$.

Moreover, a contraction condition on every $[0, T]$ holds true for a measure μ_T , and finally, fixed point Theorem 5 should be applied. \square

Remark 1. The result presented in Theorem 7 remains true if we replace assumptions about nonincreasing functions by appropriate conditions with nondecreasing ones. However, as globally integrable functions cannot be a.e. increasing on \mathbb{R}^+ , this remark cannot be applied for the case of Theorem 6.

3.3 Uniqueness of solutions

We are able to discuss the uniqueness of solution of problem (2) in the case of integrable solutions (the case of locally integrable solutions can be studied in a similar manner).

Theorem 8. *Let assumptions of Theorem 6 be satisfied, but instead of assumption (iii), consider the following condition holds:*

- (vi) *There exist constants $b_i \geq 0$ and positive functions $a_i \in L_1$, $i = 1, 2$, such that $|f(t, x) - f(t, y)| \leq b_1|x - y|$, $|g(t, x) - g(t, y)| \leq b_2|x - y|$, $x, y \in Q_r$, and $|f(t, 0)| \leq a_1(t)$, $|g(t, 0)| \leq a_2(t)$, where Q_r is defined in Theorem 6.*

Then problem (2) has a unique solution in Q_r .

Proof. From assumption (vi) we have

$$\begin{aligned} \left| |f(t, x)| - |f(t, 0)| \right| &\leq |f(t, x) - f(t, 0)| \leq b_1|x| \\ \implies |f(t, x)| &\leq |f(t, 0)| + b_1|x| \leq a_1(t) + b_1|x|. \end{aligned}$$

Similarly, $|g(t, x)| \leq a_2(t) + b_2|x|$. Thus, all assumptions of Theorem 6 be satisfied, then problem (2) has at least one solution $x \in L_1$.

To prove the uniqueness of a solution of problem (2), suppose that x, y be any two different solutions of problem (2), and then we have

$$\begin{aligned} \|x - y\|_{L_1} = &\left\| h(t) + m(t) \cdot g(t, x(t - \tau)) + \int_0^t k(t, s)f(s, x(s - \tau)) \, ds \right. \\ &\left. - h(t) - m(t) \cdot g(t, y(t - \tau)) - \int_0^t k(t, s)f(s, y(s - \tau)) \, ds \right\|_{L_1} \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^\infty |m(t)| |g(t, x(t-\tau)) - g(t, y(t-\tau))| dt \\
 &\quad + \int_0^\infty \int_0^t |k(t, s)| |f(s, x(s-\tau)) - f(s, y(s-\tau))| ds dt \\
 &\leq Mb_2 \int_0^\infty |x(t-\tau) - y(t-\tau)| dt \\
 &\quad + b_1 \int_0^\infty \int_s^\infty |k(t, s)| |x(s-\tau) - y(s-\tau)| dt ds \\
 &\leq Mb_2 \int_0^\tau |x(t-\tau) - y(t-\tau)| dt + Mb_2 \int_\tau^\infty |x(t-\tau) - y(t-\tau)| dt \\
 &\quad + b_1 \|K\|_{L_1} \int_0^\tau |x(s-\tau) - y(s-\tau)| ds \\
 &\quad + b_1 \|K\|_{L_1} \int_\tau^\infty |x(s-\tau) - y(s-\tau)| ds.
 \end{aligned}$$

As $x(t) = y(t) = \varphi(t)$ on $[-\tau, 0)$, letting $u = t - \tau$, we have

$$\begin{aligned}
 \|x - y\|_{L_1} &\leq Mb_2 \int_{-\tau}^0 |x(u) - y(u)| du + Mb_2 \int_0^\infty |x(u) - y(u)| du \\
 &\quad + b_1 \|K\|_{L_1} \int_{-\tau}^0 |x(u) - y(u)| du + b_1 \|K\|_{L_1} \int_0^\infty |x(u) - y(u)| du \\
 &= (Mb_2 + b_1 \|K\|_{L_1}) \|x - y\|_{L_1}.
 \end{aligned}$$

Therefore,

$$(1 - (Mb_2 + b_1 \|K\|_{L_1})) \|x - y\|_{L_1} \leq 0,$$

which implies that $\|x - y\|_{L_1} = 0 \Rightarrow x = y$ a.e., which completes the proof. □

4 Applications

As fractional integral equations are special forms of a general results of Hammerstein–Volterra equations with a convolutions kernel K , then the continuity property is dependent

on it. It is worthwhile to note that in a particular case of Riemann–Liouville fractional integral operators, i.e., with the kernel

$$k(t, s) = \frac{1}{\Gamma(\alpha)}(t - s)^{\alpha-1}\chi_{[a,t]}(s),$$

for $\alpha = 1$, it is not continuous from $L_1([0, T])$ into $L_\infty([0, T])$ ([19, Remark 4.1.2]) and discontinuous as an operator from $L_1([0, T])$ into itself ([19, Remark 4.1.1]). But in a considered case $0 < \alpha < 1$, it is continuous (see Lemma 2).

Despite that the lemma presented below seems to be known, we are unable to find its proof, so let us prove it (it is also claimed in Lemma 2):

Lemma 3. *For any $0 < \alpha < 1$, the kernel k of the Riemann–Liouville fractional integral operator satisfies condition (7), so our result applies also for the fractional problems of any order α .*

Proof. Let $t_1 < t_2, t_1, t_2 \in [0, T]$. Recall that in the kernel, we apply the characteristic function, so the limits of integration depend on the choice b . As

$$\int_x^y (t - s)^{\alpha-1} ds = \frac{1}{\alpha}((t - x)^\alpha - (t - y)^\alpha),$$

then, in view of arbitrariness of $b \geq 0$, we need to consider three cases.

1. Let $b > t_2$. Then

$$\Delta = \int_0^b k(t_1, s) ds - \int_0^b k(t_2, s) ds = \frac{1}{\alpha}(t_2^\alpha - t_1^\alpha) > 0.$$

2. Let $0 < t_1 < b < t_2$. Then

$$\Delta = \int_0^b k(t_1, s) ds - \int_0^b k(t_2, s) ds = \frac{1}{\alpha}(t_2^\alpha - t_1^\alpha - (t_2 - b)^\alpha).$$

But $t_2 - b < t_2 - t_1$ and then $\Delta > (t_2^\alpha - t_1^\alpha - (t_2 - t_1)^\alpha)/\alpha$.

We need to use the fact that for any $0 < \alpha < 1$, a function $g(t) = t^\alpha$ is concave.

For arbitrary points $x_1, x_2 \in [0, T]$, we have $g((x_1 + x_2)/2) \geq (g(x_1) + g(x_2))/2$. Put $x_1 = t_1, x_2 = t_2 - t_1$. Then $(x_1 + x_2)/2 = t_2/2$ and $g((x_1 + x_2)/2) = 2^{-\alpha}t_2^\alpha$. Consequently, $2 \cdot g((x_1 + x_2)/2) = 2^{1-\alpha} \cdot t_2^\alpha > t_2^\alpha$.

Therefore $t_2^\alpha > t_1^\alpha + (t_2 - t_1)^\alpha$ and $\Delta > 0$.

3. Let $0 < b < t_1 < t_2$. Then

$$\Delta = \int_0^b k(t_1, s) ds - \int_0^b k(t_2, s) ds = \frac{1}{\alpha}([t_2^\alpha - t_1^\alpha] - [(t_2 - b)^\alpha - (t_1 - b)^\alpha]).$$

As in this case $t_2^\alpha - t_1^\alpha > 0$ and $(t_2 - b)^\alpha - (t_1 - b)^\alpha > 0$, we get the thesis. □

Consequently, we get an existence result for the fractional problem (1):

Theorem 9. *Let assumptions of Theorem 7 be satisfied. Then for any $0 < \alpha < 1$, the fractional integral problem (1) has at least one locally integrable solution on \mathbb{R}^+ .*

The uniqueness results can be also obtained in the same manner.

5 Example

We give an example to illustrate the applicability of our assumptions.

Consider the following delay integral problem for $t \in \mathbb{R}^+$:

$$x(t) = h(t) + e^{-t} \frac{1 + x(t - \tau)}{(t + 19)^2} + \int_0^t \frac{1}{t^2 + s^2} \left(\frac{1}{(s + 2)^3} + \frac{1}{20} x(s - \tau) \right) ds, \tag{11}$$

$$x(t) = x_0, \quad t \in [-\tau, 0).$$

It is clear that problem (11) is a particular case of problem (2), where $\varphi(t) = x_0 = \text{const}$, $m(t) = e^{-t}$, $g(t, x) = (1 + x(t - \tau))/(t + 19)^2$, $k(t, s) = 1/(t^2 + s^2)$, $f(t, x) = 1/(t + 2)^3 + (x(t - \tau))/20$, and

$$h(t) = \begin{cases} 0, & t \text{ is rational,} \\ \frac{\pi}{2} - \arctan t, & t \text{ is irrational.} \end{cases}$$

One can easily check that

- (a1) $\sup_{t \in \mathbb{R}^+} m(t) = \sup_{t \in \mathbb{R}^+} e^{-t} \leq 1 = M$;
- (a2) $|f(t, x)| \leq 1/(t + 2)^3 + |x|/20$ and $|g(t, x)| \leq 1/(t + 19)^2 + |x|/20$ with $b_1 = b_2 = 1/20$;
- (a3) $\int_0^\infty k(t, s) dt = \int_0^\infty 1/(t^2 + s^2) dt = (1/s) \arctan(t/s)|_0^\infty \leq \pi/2$;
- (a4) $(Mb_2 + b_1 \|K\|_{L_1}) \leq 1/20 + 1/20 \cdot \pi/2 < 1/2$.

Thus, all assumptions of Theorem 6 are satisfied. Then problem (11) has at least one integrable solution a.e. nonincreasing on \mathbb{R}^+ . Clearly, this problem cannot have continuous solutions.

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