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Global well-posedness of solutions for the epitaxy thin film growth model*

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Abstract. In this paper, we consider the global well-posedness of solutions for the initial-boundary value problems of the epitaxy growth model. We first construct the local smooth solution, then by combining some a priori estimates, continuity argument, the local smooth solutions are extended step by step to all $t > 0$, provided that the initial datums sufficiently small and the smooth nonlinear functions satisfy certain local growth conditions.

Keywords: lobal well-posedness, local existence, uniqueness, epitaxy thin film growth model.

1 Introduction

Recently, there have been several experimental studies exhibiting a novel type of the epitaxial growth of nanoscale thin films. A major reason for this interest is that compositions like $YBa_2Cu_3O_{7-\delta}$ (YBCO) are expected to be high-temperature superconducting and could be used in the design of semiconductors [10]. Due to stringent tolerances of filter characteristics, the YBCO films must be highly uniform in thickness and texture [17]. The process of growing a thin film layer may be extremely complex and the development of experimental and mathematical tools for their study remains a focal point of physical research.

We begin by sketching the lines along which the epitaxial thin film growth model is derived. Due to Zangwill [24], for a spatial variable $x = (x_1, x_2)$ in the domain $\Omega = [0, L]^2$, the function $u(x, t)$ denotes the height of a film in epitaxial growth obeys

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a conservation law,

$$\partial_t u(x,t) = -\operatorname{div} J\bigl(\nabla u(x,t)\bigr) + \eta(x,t), \tag{1}$$

where $J(\nabla h)$ comprises all processes, which move atoms along the surface, and η denotes some Gaussian noise. On a purely phenomenological basis, we may write

$$J(\nabla u) = A_1\nabla u + A_2\nabla\bigl(\nabla^2 i\bigr) + A_3|\nabla u|^2\nabla u + A_4\nabla|\nabla u|^2, \tag{2}$$

with constants A_1, A_2, A_3, A_4 in the growth law (1). It is easy to see that the surface mass current has been expanded in a power series involving the surface slope ∇u and various power and derivatives thereof.

Combining (1) and (2) together, dropping the noise term $\eta(x,t)$, we obtain the following fourth-order nonlinear evolution equation:

$$\partial_t u + A_1\Delta u + A_2\Delta^2 u + A_3\nabla\cdot\bigl(|\nabla u|^2\nabla u\bigr) + A_4\Delta|\nabla u|^2 = 0. \tag{3}$$

The spatial derivatives in (3) have the following physical interpretations:

- $A_1\Delta u$: diffusion due to evaporation-condensation [3, 16],
- $A_2\Delta^2 u$: capillarity-driven surface diffusion [8, 16],
- $A_3\nabla\cdot(|\nabla u|^2\nabla u)$: (upward) hopping of atoms [10, 20],
- $A_4\Delta|\nabla u|^2$: equilibration of the inhomogeneous concentration of the diffusing particles on the surface (known as the coarsening process) [1, 18].

Combining the resulting nonlinear terms with the second-order diffusion term yields

$$\partial_t u + A_2\Delta^2 u + A_1\nabla\cdot\left(\frac{A_3}{A_1}|\nabla u|^2 + 1\right)\nabla u + A_4\Delta|\nabla u|^2 = 0,$$

and it turns out that the case $A_1 > 0$ and $A_3 < 0$ is the one of interest [10, 17]. After relabeling of constants, we obtain the equation with positive coefficients α, β, γ, κ,

$$\partial_t u + \alpha\Delta^2 u - \beta\nabla\cdot\bigl(|\nabla u|^2\nabla u\bigr) + \gamma\Delta u + \kappa\Delta|\nabla u|^2 = 0. \tag{4}$$

It would be specially mentioned that Ortiz et al. [17] modified (4) in several respects. In particular, the authors showed that $A_4 = 0$ if Onsager's reciprocity relations hold. Hence, (4) becomes

$$\partial_t u + \alpha\Delta^2 u - \beta\nabla\cdot\bigl(|\nabla u|^2\nabla u\bigr) + \gamma\Delta u = 0.$$

In addition, from a mathematical point of view, King, Stein and Winkler [10] generalized the term involving second-order diffusion, considered the existence, uniqueness and regularity of solutions for following fourth-order evolution equation in $\Omega \subset \mathbb{R}^N$:

$$\partial_t u + \alpha\Delta^2 u - \beta\nabla\cdot\bigl(f(\nabla u)\bigr) = 0, \tag{5}$$

together with some assumptions on the nonlinear function $f(s)$. The authors also characterized the existence of nontrivial equilibria in terms of the size of the underlying domain.

Latterly, the problems of stability, long time behavior and other properties of solutions for the initial boundary value problem of (5) have been studied by various authors (see e.g. Liu [15], Zhao, Zhang and Liu [27], Kohn and Yan [11], Li and Liu [12], Zhang and Zhu [25], Li, Yin and Jin [14], Zhao and Liu [26]). It is particularly important to note that the global well-posedness for the Cauchy problem of Eq. (5) has caused wide public concern over the recent years. Li, Qiao and Tang [13] proved the global well-posedness when the initial datum $u_0 \in H^{\frac{N}{2}}(\mathbb{R}^N)$, where $f(s) = |s|^2 s - s$ and $N \leqslant 3$. Fan and Zhou [2] considered the global well-posedness when $u_0 \in H^4(\mathbb{R}^4)$. For the case $N \geqslant 5$, Fan, Alsaedi, Hayat and Zhou [4] also established some regularity criteria of strong solutions.

Remark 1. Although it will not be used in the proofs of the main results of this paper, one would like to point out that Eq. (5) can be represented as the gradient flow of the following energy functional

$$E(u) = \int_{\Omega} \left(\frac{1}{2} |\Delta u|^2 - F(\nabla u) \right) \mathrm{d}x,$$

which means

$$\partial_t u = -\frac{\delta E}{\delta u} = -\nabla \cdot \bigl(\nabla \Delta u - f(\nabla u)\bigr),$$

where $F(s) = \int_0^s f(y) dy$. This fact was employed in [2, 10, 23] to study the thin film equation.

In 2015, Sandjo, Moutari and Gningue [19] studied the well-posedness of Eq. (5) together with Neumann boundary value condition. Applying Kato's method, the authors established the existence, uniqueness and regularity of solutions in space $C^0([0, T]; L^{\frac{N\alpha}{2-\alpha}}(\Omega))$, provided that $f(s) = |s|^\alpha s$, $1 < \alpha < 2$, the $L^{\frac{N\alpha}{2-\alpha}}(\Omega)$-norm of initial data is sufficiently small and the dimensional $N \geqslant 2$. The authors pointed out that if $u(x, t)$ is a smooth solution to Eq. (5) in $\mathbb{R}^N$, then for each $\lambda > 0$,

$$u_\lambda(x, t) = \lambda^{\frac{2}{\alpha} - 1} u\bigl(\lambda x, \lambda^4 t\bigr)$$

also solves (5) unless we consider the initial condition and the following scaling identity:

$$\bigl\| \nabla^k u_\lambda(\cdot, t) \bigr\|_{L^p(\mathbb{R}^N)} = \lambda^{\frac{2}{\alpha} + k - 1 - \frac{N}{p}} \bigl\| \nabla^k u\bigl(\cdot, \lambda^4 t\bigr) \bigr\|_{L^p(\mathbb{R}^N)}.$$

Hence, p is the critical exponent if it satisfies

$$\frac{N}{p} = \frac{2}{\alpha} + k - 1. \tag{6}$$

Sandjo et al. [19] focus on the case $k = 0$ with $p = \frac{\alpha N}{2-\alpha}$. In order to make p meaningful, the authors supposed that $1 < \alpha < 2$ and $N \geqslant 2$. By analyzing Sandjo et al.'s results, we find that there are two interesting problem need to be investigated: Can we drop the restriction $1 < \alpha < 2$? Can we consider the global well-posedness without the restrict on the dimensional? Our answer is "Yes". We can establish the small initial data global well-posedness result for any $\alpha \geqslant \frac{2}{N}$.

The purpose of this paper is to solve the above problems. We consider the small initial data global existence and uniqueness of solutions for the following initial-boundary value problem:

$$\begin{aligned} &\partial_t u + \Delta^2 u - \nabla \cdot \left(|\nabla u|^\alpha \nabla u\right) = 0, \\ &\partial_\nu u|_{\partial\Omega} = \partial_\nu \Delta u|_{\partial\Omega} = 0, \\ &u(x,0) = \varphi(x). \end{aligned} \tag{7}$$

where $x \in \Omega$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\partial\Omega$ denotes the boundary of Ω, ν denotes the unit outer vector normal to Ω and the positive constant $\alpha \geqslant \frac{2}{N}$. Let $k = 1$ in (6), we easily obtain $p = \frac{N\alpha}{2}$, and the space $\dot{W}^{1,\frac{N\alpha}{2}}$ is the critical space to Eq. $(7)_1$. Therefore, we can consider the small initial data global well-posedness for problem (7) in $\dot{W}^{1,\frac{N\alpha}{2}}$. The only restrictive condition we need is $\frac{N\alpha}{2} \geqslant 1$, that is $\alpha \geqslant \frac{2}{N}$.

Remark 2. Let $k \in \mathbb{N}$ and $p \in [1,\infty]$. The Sobolev space $\dot{W}^{k,p}(\Omega)$ is defined as

$$\dot{W}^{k,p}(\Omega) = \left\{u\colon\ D^k u \in L^p(\Omega)\right\}$$

with the norm

$$\|u\|_{\dot{W}^{k,p}(\Omega)} := \left(\int_\Omega \left|D^k u\right|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

which can be seen as the seminorm of $\|u\|_{W^{k,p}(\Omega)}$. In addition, $\dot{H}^k(\Omega) := \dot{W}^{k,2}(\Omega)$.

We are now able to state the main results established in this paper.

Theorem 1. *Suppose that $N\alpha \geqslant 2$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^4 boundary and $u(0) = \varphi(x) \in \dot{W}^{1,\frac{N\alpha}{2}}(\Omega)$. Then, for problem* (7)*, the following statements hold true:*

(i) *There exists a $T > 0$ and a unique mild solution $u \in C^0([0,T]; \dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$ such that*

$$\sup_{0\leqslant t\leqslant T} \left\{\left\|u(t)\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}} + t^{\frac{1}{2(\alpha+1)}}\left\|u(t)\right\|_{\dot{W}^{1,\frac{N\alpha(\alpha+1)}{2}}}\right\} < \infty.$$

(ii) *If $\|\varphi\|_{\dot{W}^{1,\frac{N\alpha}{2}}}$ is sufficiently small, the solution can be extended to a global one*:

$$u(t) \in C^0\left([0,\infty); \dot{W}^{1,\frac{N\alpha}{2}}(\Omega)\right).$$

Remark 3. The main purpose of this paper is to study the global well-posedness of solutions for problem (7). The stability condition in theoretical way for problem (7) can be found in [26, 27] and the reference therein.

The main idea to prove Theorems 1 is to treat problem (7) as semilinear evolution equations of the following form:

$$u_t + \mathcal{A}u = F(u), \qquad u(0) = \varphi,$$

where the operator $\mathcal{A} := \Delta^2$, and

$$F(u) := \nabla \cdot \big(|\nabla u|^\alpha \nabla u\big).$$

Then, problem (7) can be studied via the corresponding integral equation

$$u(t) = \mathrm{e}^{-t\mathcal{A}}\varphi + \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} F(u)\,\mathrm{d}s. \tag{8}$$

Following Kato et al. [5,9], Wiegner [22], Giga et al. [6,7] and Sandjo et al. [19], we construct a solution in $C^0([0,T), \dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$ to problem (8) by successive approximations. This approximation is such that the sequence $\{R_j\}$ defined by

$$R_j := K_j^1 + K_j^2 = \sup_{0\leqslant t\leqslant T} \Big\{ \big\|u_j(t)\big\|_{\dot{W}^{1,\frac{N\alpha}{2}}} + t^{\frac{1}{2(\alpha+1)}} \big\|u_j(t)\big\|_{\dot{W}^{1,\frac{N\alpha(\alpha+1)}{2}}} \Big\}$$

for problem (7) is bounded. In order to establish the later result, we show that R_j satisfy the recursive relation

$$R_{j+1} \leqslant R_0 + C R_j^{\alpha+1}.$$

Thus, if $u(0) = \varphi$ has small $\dot{W}^{1,\frac{N\alpha}{2}}$-norm, then this recursive relation is uniformly bounded, i.e. there exists a $R > 0$ such that for all $j \geqslant 0$, $R_j \leqslant R$. This estimate enables us to use a standard argument to show that there exists a unfirmly converging sequence $\{u_j\}$ whose limit is a solution to problem (7) in $\dot{W}^{1,\frac{N\alpha}{2}}(\Omega)$.

The remaining parts of the present papers are organized as follows. We begin by giving some notations and introduce some preliminary results. Then, in Section 3, we establish the global well-posedness result for problem (7), provided that $\|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}}$ is sufficiently small. Finally, the last section illustrates the qualitative behavior of the constructed approximate solution to problem (7) through some numerical simulations.

2 Preliminaries

We denote $A \lesssim B$, the estimate $A \leqslant cB$, where $c > 0$ is an absolute constant.

The following auxiliary lemma can be proved by induction.

Lemma 1. (*See* [19].) *Suppose that* $\alpha, \lambda > 0$ *and* $\{b_m\}$ *is a nonnegative sequence such that for all* $m \in \mathbb{N}$*,*

$$b_m \leqslant b_0 + \lambda b_{m-1}^{1+\alpha}.$$

Let $2\lambda(2b_0)^\alpha < 1$*. Then, for all* $m \in \mathbb{N}$*, we have*

$$b_m \leqslant \frac{b_0}{1 - \lambda(2b_0)^\alpha}.$$

The well-know Weierstrass M-test result gives us the uniform convergence of the aforementioned sequence (u_j).

Lemma 2. (*See* [19].) *Suppose that X is a Banach space equipped with the norm $\|\cdot\|_X$. Assume that $(u_j)_{j\in\mathbb{N}}$ is a sequence of continuous functions from $[0,T]$ in X and $(M_j)_{j\in\mathbb{N}}$ is a sequence of nonnegative real numbers for which $\sum_{j=0}^{\infty} M_j < \infty$, and for each $j \in \mathbb{N}$,*

$$\|u_j(t)\|_X \leqslant M_j \quad \forall t \in [0,T],$$

where $0 < T \leqslant \infty$. Therefore, we have

$$\sum_{j=0}^{\infty} u_j \in C^0\big([0,T];X\big),$$

that is, (u_j) converges uniformly on $[0,T]$.

The following lemma is the generalized result of Lemma A.2 of [19]. Since the proof is similar to the proof of Lemma A.2 of [19], we omit it here.

Lemma 3. (*See* [19].) *Suppose that $T(t)_{t\geqslant 0}$ is a C_0-semigroup defined in a Banach space X. Assume that for all $t \geqslant 0$, we have $\|T(t)\| \leqslant L$, $L > 0$. Let $0 < h < t_2$. Give a sequence $(\psi_j)_{j\in\mathbb{N}}$ defined by $\psi_j\colon\ s \in \mathbb{R}_+ \longmapsto \psi_j(s) \in X$ locally integrable, we have*

$$\forall j \in \mathbb{N}, \quad \lim_{h\to 0^+} \int_0^{t_2-h} \big\|\big(T(h) - \mathbf{1}\big)\psi_j(s)\big\|_X \,\mathrm{d}s = 0,$$

where $\mathbf{1}$ denotes the identity operator on X.

The following remark can be found in [19].

Remark 4. (See [19].) Let $T > 0$ and $0 < t + 1 < t_2 < T$. Given $\delta \in (0,1)$, we have

$$\lim_{|t_2-t_1|\to 0^+} \int_{t_1}^{t_2} (t_2 - s)^{-\frac{\delta}{2}} s^{-1+\frac{\delta}{2}} \,\mathrm{d}s = 0.$$

Weisler [21] established the following result, which contains an abstract statement needed for the proof of our existence result.

Lemma 4. *Suppose that Ω is a bounded domain of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. Let $\mathcal{A}$ be a infinitesimal generator of a C_0 semigroup on Ω with domain $D(\mathcal{A})$ continuously embedding in $W^{m,p}(\Omega)$ and codomain $L^q(\Omega)$. Assume that $\mathcal{A}$ generates an analytic semigroup $\mathrm{e}^{-t\mathcal{A}}$. Then the semigroup $\mathrm{e}^{-t\mathcal{A}} : L^p(\Omega) \to L^q(\Omega)$ is a bounded linear operator whenever $1 < p \leqslant q < \infty$ and $t > 0$. In addition, for any $T > 0$, there exists a positive constant C, which depending only on p and q, such that for any nonnegative integer $j < 6$,*

$$\big\|\nabla^j \mathrm{e}^{-t\mathcal{A}}\big\|_{\mathcal{L}(L^p,L^q)} \leqslant C(p,q) t^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q})-\frac{j}{m}} \quad \forall t \in (0,T].$$

3 Proof of Theorem 1

In this section, we first use the method of successive approximation to prove the existence and uniqueness of local solutions for problem (7).

For system (7), we choose $\mathcal{A} := \Delta^2$. The solution to the integral equation

$$u(t) = \mathrm{e}^{-t\mathcal{A}} u_0 + \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} \nabla \cdot \big(|\nabla u|^\alpha \nabla u\big)\, \mathrm{d}s \tag{9}$$

can be given by

$$u_{j+1}(t) = u_0(t) + G(u_j)(t), \quad t \geqslant 0,\ j \geqslant 0,$$

together with

$$u_0(t) = \mathrm{e}^{-t\mathcal{A}} \varphi$$

and

$$G(u)(t) = \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} \nabla \cdot \big(|\nabla u|^\alpha \nabla u\big)(s)\, \mathrm{d}s,$$

where $u_0 \in \dot{W}^{1,\frac{N\alpha}{2}}(\Omega)$. Let $T > 0$ and

$$K_j^1 := \sup_{0<t\leqslant T} \big\|u_j(t)\big\|_{\dot{W}^{1,\frac{N\alpha}{2}}}, \qquad K_j^2 := \sup_{0<t\leqslant T} t^{\frac{1}{2(\alpha+1)}} \big\|u_j(t)\big\|_{\dot{W}^{1,\frac{N\alpha(\alpha+1)}{2}}},$$

$$R_j := K_j^1 + K_j^2.$$

The a priori estimates for K_j^1, K_j^2 and R_j can be derived as

Lemma 5. *For every $j \in \mathbb{N}$, the following inequalities hold*:

$$K_{j+1}^1 \lesssim K_0^1 + \big(K_j^2\big)^{\alpha+1}, \qquad K_{j+1}^2 \lesssim K_0^1 + \big(K_j^2\big)^{\alpha+1}.$$

Moreover, we also have the following recursive inequality:

$$R_{j+1} \lesssim R_0 + R_j^{\alpha+1}, \quad R_0 \lesssim \|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}}.$$

Proof. It follows from the integral equation (9) and Hölder's inequality that

$$\begin{aligned}
&\big\|u_{j+1}(t)\big\|_{\dot{W}^{1,\frac{N\alpha}{2}}} \\
&\quad\leqslant \|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}} + \int_0^t \big\|\mathrm{e}^{-(t-s)\mathcal{A}} \nabla \cdot \big(|\nabla u_j|^\alpha \nabla u_j\big)\big\|_{\dot{W}^{1,\frac{N\alpha}{2}}}\, \mathrm{d}s \\
&\quad\leqslant K_0^1 + \int_0^t \big\|D^2 \mathrm{e}^{-(t-s)\mathcal{A}} |\nabla u_j|^\alpha \nabla u_j\big\|_{L^{\frac{N\alpha}{2}}}\, \mathrm{d}s \\
&\quad\leqslant K_0^1 + \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla u_j\|_{L^{\frac{N\alpha(\alpha+1)}{2}}}^{\alpha+1}\, \mathrm{d}s
\end{aligned}$$

$$\lesssim K_0^1 + \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \left(s^{\frac{1}{2(\alpha+1)}} \|\nabla u_j\|_{L^{\frac{N(\alpha+1)\alpha}{2}}} \right)^{\alpha+1} \mathrm{d}s$$
$$\lesssim K_0^1 + \left(K_j^2\right)^{\alpha+1}$$

and

$$t^{\frac{1}{2(\alpha+1)}} \|u_{j+1}\|_{\dot{W}^{1,\frac{N\alpha(\alpha+1)}{2}}}$$
$$\leqslant t^{\frac{1}{2(\alpha+1)}} \left\|\nabla u_0(t)\right\|_{L^{\frac{N\alpha(\alpha+1)}{2}}} + t^{\frac{1}{2(\alpha+1)}}$$
$$\times \int_0^t \left\|\mathrm{e}^{-(t-s)\mathcal{A}} \nabla\nabla \cdot \left(|\nabla u_j|^2 \nabla u_j\right)\right\|_{L^{\frac{N\alpha(\alpha+1)}{2}}} \mathrm{d}s$$
$$\leqslant K_0^2 + t^{\frac{1}{2(\alpha+1)}} \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{4}(\frac{2}{N\alpha}-\frac{2}{N\alpha(\alpha+1)})} \|\nabla u_j\|^{\alpha+1}_{L^{\frac{N(\alpha+1)\alpha}{2}}} \mathrm{d}s$$
$$\lesssim K_0^2 + t^{\frac{1}{2(\alpha+1)}} \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2(\alpha+1)}} s^{-\frac{1}{2}} \left(s^{\frac{1}{2(\alpha+1)}} \|\nabla u_j\|_{L^{\frac{N(\alpha+1)\alpha}{2}}} \right)^{\alpha+1} \mathrm{d}s$$
$$\lesssim K_0^1 + \left(K_j^2\right)^{\alpha+1}.$$

Combining the above two inequalities together gives $R_{j+1} \leqslant R_0 + cR_j^{\alpha+1}$. It remains to prove that $R_0 \lesssim \|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}}$. Let $0 < t \leqslant T$, then

$$t^{\frac{1}{2(\alpha+1)}} \left\|\nabla u_0(t)\right\|_{L^{\frac{N\alpha(\alpha+1)}{2}}} \lesssim t^{\frac{1}{2(\alpha+1)}-\frac{N}{4}(\frac{2}{N\alpha}-\frac{2}{N\alpha(\alpha+1)})} \|\nabla u_0\|_{L^{\frac{N\alpha}{2}}}$$
$$\lesssim \|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}}.$$

Hence, the proof is completed. □

Based on Lemma 5, we immediately obtain the a priori estimate for the approximating sequence. Now, choosing the norm $\|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}}$ sufficiently small, we deduce the following result.

Lemma 6. *Assume that* $\|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}} \leqslant \epsilon$, *where* ϵ *is a positive constant. Then*

$$R_j \leqslant 2R_0 \quad \forall j \geqslant 1.$$

Proof. Applying Lemmas 1 and 5, if $2c(2R_0)^\alpha < 1$, we derive that

$$R_j \leqslant \frac{R_0}{1 - c(2R_0)^\alpha}.$$

But the latter inequality is satisfied if we choose $\|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}} \leqslant \epsilon$, where ϵ is a sufficiently small positive number. Then the proof is completed. □

To construct the contraction mapping, we introduce the sequences

$$\varpi_j(t) = u_j(t) - u_{j-1}(t), \quad \varpi_0 = u_0$$

and the corresponding quantities

$$\tilde{K}_j^1 := \sup_{0<t\leqslant T} \|\varpi_j(t)\|_{\dot{W}^{1,\frac{N\alpha}{2}}}, \qquad \tilde{K}_j^2 := \sup_{0<t\leqslant T} t^{\frac{1}{2(\alpha+1)}} \|\nabla\varpi_j(t)\|_{\dot{W}^{1,\frac{N\alpha(\alpha+1)}{2}}}.$$

Define

$$\tilde{R}_j := \tilde{K}_j^1 + \tilde{K}_j^2.$$

Direct calculations show that

$$\begin{aligned}\varpi_{j+1}(t) &= G(v_j)(t) - G(v_{j-1})(t)\\ &= \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} \nabla\cdot\big(|\nabla u_j|^\alpha \nabla u_j - |\nabla u_{j-1}|^\alpha \nabla u_{j-1}\big)\,\mathrm{d}s.\end{aligned}$$

Hence, we obtain the following lemma.

Lemma 7. *There exists a positive constant c such that if* $\|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}} \leqslant \epsilon$, *we have*

$$\tilde{R}_{j+1} \leqslant c\tilde{R}_j R_0^\alpha \leqslant c\big(R_0^\alpha\big)^j \quad \forall j \in \mathbb{N}. \tag{10}$$

Proof. We derive the a priori estimates for $\tilde{K}_j^1$ and $\tilde{K}_j^2$. Note that

$$\begin{aligned}\tilde{K}_{j+1}^1 &\leqslant \Big\|\int_0^t D^2\mathrm{e}^{-(t-s)\mathcal{A}}\big[|\nabla u_j|^\alpha\nabla u_j - |\nabla u_{j-1}|^\alpha \nabla u_{j-1}\big]\,\mathrm{d}s\Big\|_{L^{\frac{N\alpha}{2}}}\\ &\lesssim \int_0^t (t-s)^{-1/2}\|\nabla\varpi_j\|_{L^{\frac{N\alpha(\alpha+1)}{2}}}\big(\|\nabla u_j\|^\alpha_{L^{\frac{N\alpha(\alpha+1)}{2}}} + \|\nabla u_{j-1}\|^\alpha_{L^{\frac{N\alpha(\alpha+1)}{2}}}\big)\,\mathrm{d}s\\ &\lesssim \int_0^t (t-s)^{-1/2}s^{-1/2}\big(s^{\frac{1}{2(\alpha+1)}}\|\nabla\varpi_j\|_{L^{\frac{N\alpha(\alpha+1)}{2}}}\big)\\ &\qquad\times\big[s^{\frac{\alpha}{2(\alpha+1)}}(\|\nabla u_j\|^\alpha_{L^{\frac{N\alpha(\alpha+1)}{2}}} + \|\nabla u_{j-1}\|^\alpha_{L^{\frac{N\alpha(\alpha+1)}{2}}})\big]\,\mathrm{d}s\\ &\lesssim \tilde{K}_j^2(K_j^2)^\alpha \lesssim \tilde{K}_j^2 R_0^\alpha.\end{aligned}$$

In addition, we also have

$$\begin{aligned}\tilde{K}_{j+1}^2 &\leqslant t^{\frac{1}{2(\alpha+1)}}\bigg\|\int_0^t D^2\mathrm{e}^{-(t-s)\mathcal{A}}\big[|\nabla u_j|^\alpha\nabla u_j - |\nabla u_{j-1}|^\alpha\nabla u_{j-1}\big]\,\mathrm{d}s\bigg\|_{L^{\frac{N\alpha(\alpha+1)}{2}}}\\ &\lesssim t^{\frac{1}{2(\alpha+1)}}\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2}(\frac{2}{N\alpha}-\frac{2}{N\alpha(\alpha+1)})}\|\nabla\varpi_j\|_{L^{\frac{N\alpha(\alpha+1)}{2}}}\\ &\qquad\times\big(\|\nabla u_j\|^\alpha_{L^{\frac{N\alpha(\alpha+1)}{2}}} + \|\nabla u_{j-1}\|^\alpha_{L^{\frac{N\alpha(\alpha+1)}{2}}}\big)\,\mathrm{d}s\end{aligned}$$

$$\lesssim t^{\frac{1}{2(\alpha+1)}} \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2(\alpha+1)}} s^{-1/2} \big(s^{\frac{1}{2(\alpha+1)}} \|\nabla \varpi_j\|_{L^{\frac{N\alpha(\alpha+1)}{2}}}\big) \times \big[s^{\frac{\alpha}{2(\alpha+1)}} \big(\|\nabla u_j\|^\alpha_{L^{\frac{N\alpha(\alpha+1)}{2}}} + \|\nabla u_{j-1}\|^\alpha_{L^{\frac{N\alpha(\alpha+1)}{2}}}\big)\big] \,\mathrm{d}s$$
$$\lesssim \tilde{K}_j^2 (K_j^2)^\alpha \lesssim \tilde{K}_j^2 R_0^\alpha.$$

Combining the above two inequalities together gives

$$\tilde{R}_{j+1} \lesssim \tilde{R}_j R_0^\alpha.$$

Then we obtain (10) and complete the proof. □

The above lemma means that the sequence $\tilde{R}_j$ is summable, provided that R_0 sufficiently small. It is easy to see that $u_j = u_0 + \sum_{k=1}^j \varpi_k$ is a Cauchy sequence in $C^0([0,T];\dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$ and converges to some solution $u \in C^0([0,T];\dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$ of the integral equation (9). In the following, we state the conditions under which the sequence $\{u_j\}$ converges.

Lemma 8. *There exists a positive constant ϵ, and if*

$$R_0 = \max\Big\{\sup_{0<t\leqslant T} \|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}} + t^{\frac{1}{2(\alpha+1)}} \|u_0\|_{\dot{W}^{1,\frac{N\alpha(\alpha+1)}{2}}}\Big\} < \epsilon,$$

then the sequence $(u_j) \subset C^0([0,T];\dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$ converges uniformly.

Proof. We need to verify that the assumptions of Weierstrass M-Test are satisfied.

As [19], we prove that $u_j(t)$ is continuous on $(0,T]$. Suppose that $T(t) = \mathrm{e}^{-(t-s)\mathcal{A}}$, $t_1, t_2 \in (0,T]$ and $u_0 \in \dot{W}^{1,\frac{N\alpha}{2}}(\Omega)$. Observe that

$$\begin{aligned}
&u_j(t_2) - u_j(t_1) \\
&\quad = T(t_1)\big[T(t_2-t_1) - \mathbf{1}\big]u_0 + \int_{t_1}^{t_2} T(t_2-s)\nabla\cdot\big(\big|\nabla u_{j-1}(s)\big|^2 \nabla u_{j-1}(s)\big)\,\mathrm{d}s \\
&\qquad + \int_0^{t_1} \big[T(t_2-s) - T(t_1-s)\big]\nabla\cdot\big(\big|\nabla u_{j-1}(s)\big|^\alpha \nabla u_{j-1}(s)\big)\,\mathrm{d}s \\
&\quad = T(t_1)\big[T(t_2-t_1) - \mathbf{1}\big]u_0 \\
&\qquad + \int_0^{t_1} T(t_1-s)\big[T(t_2-t_1) - \mathbf{1}\big]\nabla\cdot\big(\big|\nabla u_{j-1}(s)\big|^\alpha \nabla u_{j-1}(s)\big)\,\mathrm{d}s \\
&\qquad + \int_{t_1}^{t_2} T(t_2-s)\nabla\cdot\big(\big|\nabla u_{j-1}(s)\big|^\alpha \nabla u_{j-1}(s)\big)\,\mathrm{d}s \\
&\quad =: J_1 + J_2 + J_3,
\end{aligned}$$

where $\mathbf{1}$ denotes the identity operator on $\dot{W}^{1,\frac{N\alpha}{2}}(\Omega)$. In addition, let $\|T(t)\|_{\dot{H}^1} \leqslant L$ on $[0,T]$, $L > 0$. We obtain

$$\left\|T(t_1)\left[T(t_2-t_1)-\mathbf{1}\right]u_0\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}} \leqslant L\left\|\left(T(t_2-t_1)-\mathbf{1}\right)u_0\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}}.$$

Applying the strong continuity of the semigroup $T(t) = \mathrm{e}^{t\mathcal{A}}$ on $\dot{W}^{1,\frac{N\alpha}{2}}(\Omega)$, we obtain that the limit of this term vanishes as $|t_2 - t_1| \to 0^+$. For J_2, we have

$$\begin{aligned}&\left\|\int_0^{t_1} T(t_1-s)\left[T(t_2-t_1)-\mathbf{1}\right]\nabla\cdot\left(\left|\nabla u_{j-1}(s)\right|^{\alpha}\nabla u_{j-1}(s)\right)\mathrm{d}s\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}}\\&\quad\leqslant L\int_0^{t_1}\left\|\left[T(t_2-t_1)-\mathbf{1}\right]\nabla\cdot\left(\left|\nabla u_{j-1}(s)\right|^{\alpha}\nabla u_{j-1}(s)\right)\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}}\mathrm{d}s.\end{aligned}$$

Set $h = t_2 - t_1$ and $\psi_j(s) = \nabla\cdot(|\nabla u_{j-1}(s)|^{\alpha}\nabla u_{j-1}(s))$, then

$$\begin{aligned}&\left\|\left[T(t_2-t_1)-\mathbf{1}\right]\nabla\cdot\left(\left|\nabla u_{j-1}(s)\right|^{2}\nabla u_{j-1}(s)\right)\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}}\mathrm{d}s\\&\quad=\int_0^{t_2-h}\left\|\left(T(h)-\mathbf{1}_p\right)\psi_j(s)\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}}\mathrm{d}s.\end{aligned}$$

By Lemma 3 we obtain that the limit of J_2 vanishes as $|t_2 - t_1| \to 0^+$. Moreover, for J_3, we have

$$\begin{aligned}&\int_{t_1}^{t_2}\left\|T(t_2-s)\nabla\cdot\left(\left|\nabla u_{j-1}(s)\right|^{\alpha}\nabla u_{j-1}(s)\right)\right\|_{W^{1,\frac{N\alpha}{2}}}\mathrm{d}s\\&\quad\leqslant cM\int_{t_1}^{t_2}(t_2-s)^{-1/2}s^{-1/2}\,\mathrm{d}s,\qquad M = t^{\frac{1}{2(\alpha+1)}}\|\nabla u_j\|^{\alpha}_{\dot{W}^{1,\frac{N\alpha(\alpha+1)}{2}}}.\end{aligned}$$

On the basis of Remark 4, the limit of J_3 vanishes as $|t_2 - t_1| \to 0^+$.

Continuity up to $t = 0$ follows from the fact

$$\begin{aligned}&\left\|u(t)-u(0)\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}}\\&\quad\leqslant\left\|\mathrm{e}^{-t\mathcal{A}}-\mathbf{1}\right\|_{L^\infty}\|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}}+\int_0^t\left\|T(t-s)\nabla\cdot\left(|\nabla u|^2\nabla u\right)\right\|_{\dot{W}^{1,\frac{N\alpha}{2}}}\mathrm{d}s\\&\quad\leqslant\left\|\mathrm{e}^{-t\mathcal{A}}-\mathbf{1}\right\|_{L^\infty}\|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}}+cM\int_0^t(t-s)^{-1/2}s^{-1/2}\,\mathrm{d}s.\end{aligned}$$

Hence, $u_j \in C([0,T];\dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$.

Secondly, we prove the boundedness of $u_j(t)$ for $0 < t \leqslant T$. As we observe before, $u_j(t) = \sum_{k=0}^{j} \varpi_k(t)$. Then

$$\|u_j(t)\|_{\dot{W}^{1,\frac{N\alpha}{2}}} \leqslant \sum_{k=0}^{j} M_k, \quad M_k = \sup_{0<t\leqslant T} \|\varpi_k\|_{\dot{W}^{1,\frac{N\alpha}{2}}},$$

and the assumption of the proposition ensures that the sequence M_k is summable. Hence, we can use Weierstrass M-Test and complete the proof. □

After obtain Lemmas 5–8, we are now able to sketch the proof of the global well-posedness of problem (7).

Proof of Theorem 1. First, we prove part (i) of Theorem 1 on the local well-posedness of solutions. There are three steps for us to carry out the proof.

Step 1 (*Well-definition of* v_j). The a priori estimates of Lemma 5 show that $u_j(t)$ is well-defined for $j \geqslant 0$ as element of $\dot{W}^{1,\frac{N\alpha}{2}}(\Omega)$.

Step 2 (*Existence*). By using Weierstrass M-Test and Lemma 7, since $\sum_{k=1}^{j}(R_0^\alpha)^k$ is a convergent geometric series, we obtain that if R_0 is sufficiently small, the sequence $\tilde{K}_j^1$ is summable. Applying Lemma 8, $u_j = u_0 + \sum_{k=1}^{j} \varpi_k$ is a Cauchy sequence in $C^0([0,T]; \dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$ and converges to some solution $u \in C^0([0,T]; \dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$ of the integral equation

$$u(t) = \mathrm{e}^{-t\mathcal{A}}\varphi + \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} \nabla \cdot \big(|\nabla u|^\alpha \nabla u\big)\,\mathrm{d}s.$$

The continuity in time for $t \in [0,T]$ follows from standard results of Lemma 8.

Step 3 (*Uniqueness*). The uniqueness of the solution follows straightforward from the fact that (u_j) is a Cauchy sequence in the Banach space $C^0([0,T]; \dot{W}^{1,\frac{N\alpha}{2}}(\Omega))$.

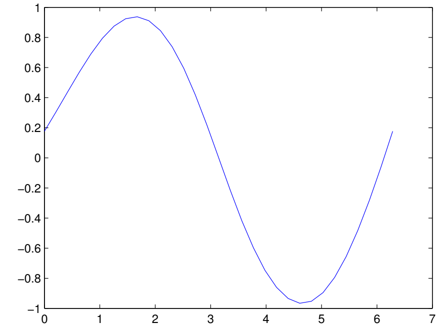
Second, we prove that this unique local solution can be extended to the unique global one. On the basis of Lemma 2, we know that Lemma 8 is still holds even if $T = \infty$. In fact, Lemma 6 implies that $\|u_j(t)\|_{\dot{W}^{1,\frac{N\alpha}{2}}}$ is bounded in j, provided that $\|u_0\|_{\dot{W}^{1,\frac{N\alpha}{2}}}$ sufficiently small for all $t \in (0,T)$ even if $T = \infty$. Then the unique local solution can be extended to a unique global one. □

4 Numerical experiments

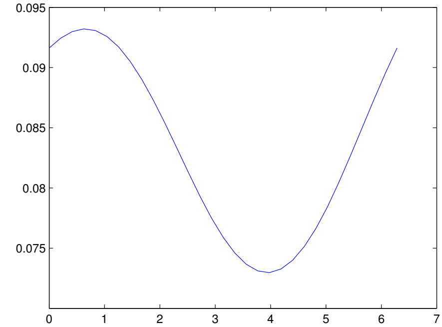
We give some numerical simulations to illustrate the dynamics of the height of thin film described by (7) for some sample initial conditions.

Firstly, we find $u(x,t) \in \Omega \times (0,T)$ such that

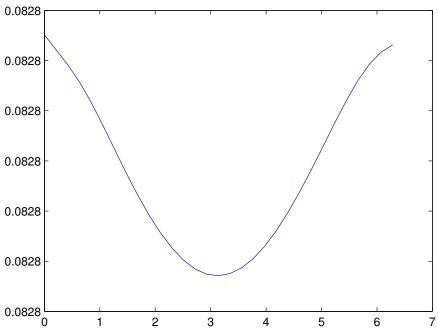
$$\begin{aligned} &\partial_t u + \partial_{xxxx} u = \partial_x\big(f(\partial_x u)\big),\\ &\partial_x u|_{\partial\Omega} = \partial_{xxx} u|_{\partial\Omega} = 0, \\ &u(x,0) = \sin x, \end{aligned} \tag{11}$$



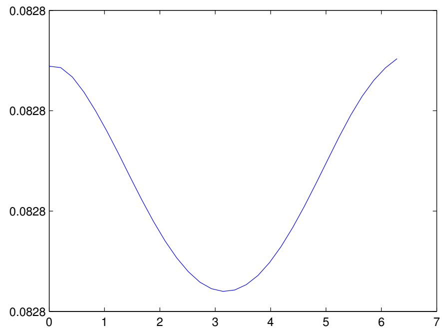
(a) $t = 0.1$ s



(b) $t = 10$ s



(c) $t = 100$ s



(d) $t = 1000$ s

Figure 1. Approximation solution for problem (11).

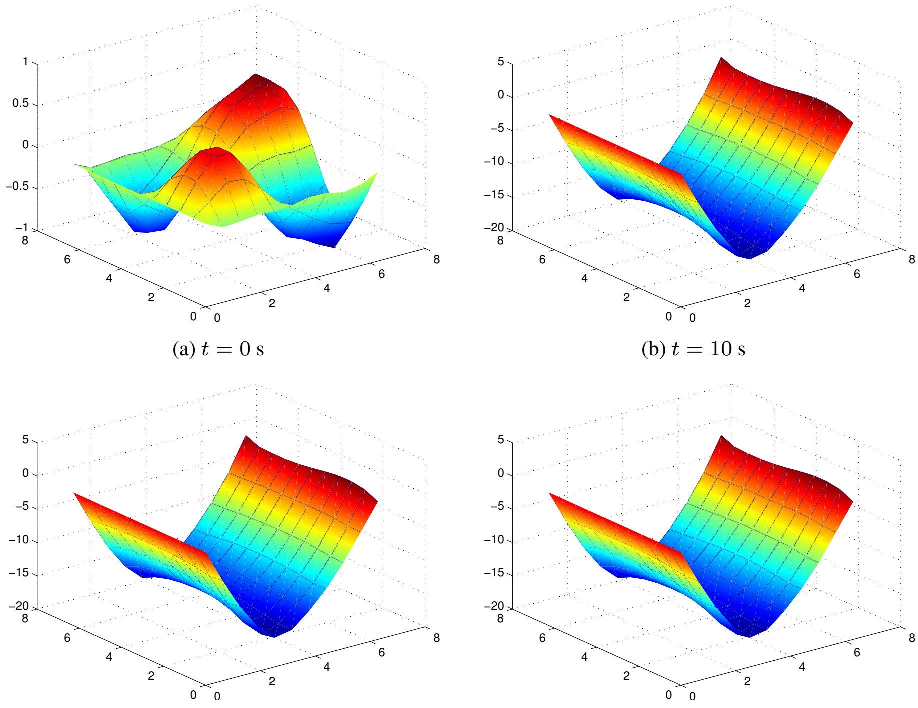
where $\Omega = [0, 2\pi]$, $f(s) = |s|^3 s$. By using the finite difference method and Matlab the space subdivision step size is $2\pi/20$, and the time subdivision step size is $1/10000$, respectively, the evolution of the film height $u(x,t)$ at different time points, namely, at the initial stage, i.e. $t = 0.1$ second, $t = 10$ second, $t = 100$ second and $t = 1000$ second, are depicted in Fig. 1.

Secondly, we find $u(x, y, t) \in \Omega \times (0, T)$ such that

$$\begin{aligned} &\partial_t u + \Delta^2 u - \nabla \cdot \big(|\nabla u|^2 \nabla u\big) = 0, \\ &\partial_\nu u|_{\partial\Omega} = \partial_\nu \Delta u|_{\partial\Omega} = 0, \\ &u(x, 0) = \sin x \sin y, \end{aligned} \tag{12}$$

where $\Omega = [0, 2\pi]^2$. Applying the finite difference method and Matlab, the space subdivision step size and the time subdivision step size are also $2\pi/20$ and $1/10000$, respectively. The evolution of the film height $u(x, y, t)$ at different time points, namely, at the initial stage, i.e. $t = 0$ second, $t = 10$ second, $t = 100$ second and $t = 1000$ second, are depicted in Fig. 2.

The numerical solutions describe well some experimentally observed phenomena, which characterize the growth of thin film such as rapid grain coarsening process,



(a) $t = 0$ s (b) $t = 10$ s

(c) $t = 100$ s (d) $t = 1000$ s

Figure 2. Approximation solution for problem (12).

a medium term island growth process and eventually a thickness growth process. Those figures illustrate the qualitative behavior of the approximate solutions of (11) and (12). On the basis of the above figures, we find out that the solutions to problems (11) and (12) tend to be stable as long as the time goes on, which means that the results on global existence of solutions for the epitaxy thin film growth model are reasonable.

5 Conclusion

We study a continuum model of YBCO film growth, which accounts for nucleation and the transition to island growth, as well as for the subsequent roughening and coarsening of the surface profile. This model is phenomenological in nature and is based on a formal expansion of the surface mass current in a power series involving the surface slope ∇u and various powers and derivatives. This continuum model is known to simulate experimentally observed dynamics very well. In this paper, we prove the local well-posedness of solutions for problem (7) and study the global well-posedness under the condition that $\|\varphi\|_{\dot{W}^{1,\frac{N\alpha}{2}}}$ is sufficiently small. This can be seen as the first step of our study on the epitaxy thin film growth model. We will study the long time behavior and numerical approximation of solutions for such problem in the future.

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