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Feedback control modulation for controlling chaotic maps

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Abstract. Based on existing feedback control methods such as OGY and Pyragas, alternative new schemes are proposed for stabilization of unstable periodic orbits of chaotic and hyperchaotic dynamical systems by suitable modulation of a control parameter. Their performances are improved with respect to: (i) robustness, (ii) rate of convergences, (iii) reduction of waiting time, (iv) reduction of noise sensitivity. These features are analytically investigated, the achievements are rigorously proved and supported by numerical simulations. The proposed methods result successful for stabilizing unstable periodic orbits in some classical discrete maps like 1-D logistic and standard 2-D Hénon, but also in the hyperchaotic generalized n-D Hénon-like maps.

Keywords: chaos control, proportional feedback control, delayed feedback control, parameter modulation, control of hyperchaos.

1 Introduction

Chaotic behavior is a very interesting nonlinear phenomena, but in many situations, it is desirable to be avoided, for example, when it restricts the operating range of electronic or mechanics devices. Moreover, this goal should be achieved with the only help of tiny perturbations properly chosen [2].

The consagrated idea of Ott, Grebogi, and Yorke [24] consists on turning the presence of chaos into an advantage. Indeed, the system may be stabilized on a particular unstable periodic orbit (UPO) embedded in a strange attractor by applying a small time-dependent feedback perturbation to some accessible parameter or variable system. The periodic orbit is preserved, but its stability is modified keeping the trajectory to stay close to the UPO. This control strategy is known as the OGY method.

A simple proportional feedback (SPF) control method basically consists on a perturbation proportional to the difference between the current system value and an unstable

fixed point (or one of the UPO's component), being the OGY method [24] a particular case of it. It is well known that it can control chaos for some 1-D maps [2]. Its usefulness arises when controlling chaos in highly dissipative systems; as there are cases in which it loses validity because Poincaré sections changes in each iteration, a recursive proportional feedback (RPF) has been proposed [36]. Both SPF and RPF methods require the exact knowledge of the UPO to be stabilized and the linearized dynamics about it. This is not always at one's disposal in real-world implementations. Moreover, the OGY method results very sensitive to nonlinearities and fluctuations of external noise, mainly for large orbit periods. Nevertheless, the OGY method holds good, and extended versions of it are still studied (for example, in [22], a two-controlling parameter extension is applied on a coupled 2-D chaotic maps). With the aim to overcome the limitations of the OGY method, Pyragas introduced a self-controlling delayed feedback [27], which, as the OGY, does not modify the original UPO, but it does not depend explicitly on it. Its discrete-time version results in a perturbation, which is proportional to the difference between the current system value and a previous one. This delayed feedback control (DFC) successfully controls chaotic behavior in a variety of experiments [6] (and its references). However, the stabilization capability of the DFC may be weaker than the OGY's. The domain of system parameters for which stabilization can be achieved via DFC is limited, namely, the method fails for highly UPO's [32]. The Pyragas method is renamed in [32] as "time-delay autosynchronization" (TDAS). Its extended versions (referred as ETDAS and EDFC) uses information from many previous states improving on stabilization objectives achievement. Recently published works aim to the efficient choice of the control parameters for the EDFC implementation. In [1], the parameter design without accurate information about the system is dealt with by employing a data-driven pole placement method. Moreover, in [4], a fixed point stabilization by means of EDFC methodology is warranted under a condition on the set of the eigenvalues of the Jacobian matrix of the map at the fixed point by rigorous mathematical approach although the extension of its results to a m-period UPO stabilization seems to remain an open problem. On the other hand, any UPO with an odd number of real Floquet multipliers greater than one (or with an odd number of real positive Floquet exponents) can never be stabilized by TDAS/ETDAS method. This restriction was pointed out in [34], and it is known as the *odd number limitation* (ONL). Later, ONL was also stated for stabilization of UPO's through ETDAS [33]. Several alternative methods, as the oscillating approach [20], the adaptive version [26], the delayed feedback with periodic control gain [16], the delayed feedback control methods based on predictive state [21, 35], the learning control of time-delay chaotic systems [11], the act-and-wait concept [8], and the observed-based-delayed feedback control [12], have been proposed to overcome this drawback. More references embracing the discrete and continuous-time cases may be found in [17].

Assume a dynamical system given by a continuously differentiable function f:

$$x_{k+1} = f(x_k, r), \tag{1}$$

where r is a scalar parameter, and that for $r = r_0$, the system develops chaotic behavior having an infinite number of UPOs embedded in a strange attractor, $\mathcal{A}$, within its basin of

attraction, $\mathcal{B}$. The dynamics in $\mathcal{A}$ is ergodic, meaning that for almost all initial condition in $\mathcal{B}$, a system trajectory visits any small neighborhood of every point of the UPO's.

A widely used technique for controlling chaotic behavior is modulation: by judiciously varying control parameters any system trajectory resulting from a randomly chosen initial condition, x_0, is driven to a given m-period UPO, $\boldsymbol{p} \equiv \{p_1, p_2, \ldots, p_m\}$. Suppose that r can be finely tuned in a small-range around r_0, namely, $r \in (r_0 - \delta,\, r_0 + \delta)$, $0 < \delta \ll 1$. Then the objective is to stabilize system (1) at $\boldsymbol{p}$ by feedback control modulation. Namely, the (control) parameter is affected by a control $u_k = u(x_k)$, so,

$$x_{k+1} = f(x_k,\, r_0 + u_k) \tag{2}$$

under the requirement that the adjusted parameter remains within a range for which the system is chaotic in the absence of perturbations.

Published works on controlling chaos by modulation are mostly based on experimental or numerical arguments, while analytical approaches usually concentrate on a fixed point stabilization problem [31]. DFC and SPF controls applied to 1-D maps are confronted in [6] by means of linear stability analysis and by implementation in an analog electric circuit. On the other hand, analytical approaches consider only additive control, and features related to control of chaos objectives are disregarded [16, 18, 27].

Both OGY and Pyragas methods have been extensively studied and analyzed in the literature on the subject like [3] and also revisited in [29, 30] among others, while several modifications on them have been introduced throughly more than twenty years ago. The present work relies on modifications of these control methods by modulation techniques improving some aspects on their performances related to robustness, rate of convergences, reduction in the waiting time, reduction in noise sensitivity, among others. Moreover, all of these features are rigorously proved. The approach to stabilize any UPO of the chaotic map (1) by modulation of the parameter r is investigated both analytically and numerically. The possibility of relaxing some requirements of the OGY and Pyragas methods is explored, and alternative types of SPF and DFC methods are built. Different from the original OGY method, the SPF method does not require exact knowledge on the linearization data of the UPO to stabilize. Based on these ideas, modifications to the original Pyragas method [27] are introduced even in the extended version presented in [32], and alternative types of DFC and EDFC methods are also proposed. In all the cases, the convergence properties of the control strategies are proven taking into account control bounds as well as control performance issues of the considered schemes.

The article is organized as follows. Throughout the work, the problem is stated for the general case of Eq. (1). The achievements of the proposed methods when applied to it are rigorously analyzed. By simplicity they are developed for 1-D maps at first. The generalized versions of the SPF and DFC methods, which involve a kind of switching controller, are introduced in Sections 2 and 3, respectively, and illustrated with the celebrated *logistic map*. In Section 4, the issue of their extension to the n-dimensional case is established, and some meeting points with previously developed proposals are considered. Numerical simulations using the 2-D *Hénon map* illustrate the goals. In Section 5, an ad-hoc reformulation is designed to control the hyperchaotic n-D *generalized Hénon-like maps*. Finally, Section 6 contains the concluding remarks and lines of future research.

2 Controlling chaos by SPF modulation

2.1 The OGY method

This method outlined in [24] is briefly reviewed here. Assume that at time k, the trajectory falls close to the component p_i. The linearized dynamics of (1) about p_i and r_0 is

$$x_{k+1} - p_{i+1} = f_x(p_i, r_0)(x_k - p_i) + f_r(p_i, r_0)(r_k - r_0). \tag{3}$$

To force the system towards $\boldsymbol{p}$, it is settled $x_{k+1} - p_{i+1} = 0$, $u_k^i = (\Delta r)_k = r_k - r_0$, so from (3) it follows

$$u_k^i = -\frac{f_x(p_i, r_0)}{f_r(p_i, r_0)}(x_k - p_i) = \alpha_i(x_k - p_i). \tag{4}$$

Equation (4) holds only when $|x_k - p_i| \leqslant \varepsilon \ll 1$, hence, the required parameter perturbation, $(\Delta r)_k$, is small, and the maximum parameter perturbation, δ, is proportional to ε (with factor α_i). When the trajectory is outside the ε-neighborhood of p_i, the perturbation is not applied, and the system evolves at its nominal chaotic parameter r_0. Note that by ergodicity the control is eventually activated. The lapse while the control is off is known as *waiting time*. Then, for given $\varepsilon, \delta > 0$, the control algorithm is

$$u_k^i = \begin{cases} \alpha_i(x_k - p_i), & |x_k - p_i| \leqslant \varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Note that $\boldsymbol{p}$ is preserved by a parameter perturbation of SPF type, i.e., $u_k^i \propto |x_k - p_i|$.

2.2 A more flexible OGY-based scheme

The control in Eq. (5) is turned on only at the end of each oscillation becoming very sensitive to nonlinearities and to fluctuation of external noises, mainly for large values of the orbit's period, m. Namely, for every i, different control gain and waiting time come out, yielding to different performances for each u_k^i.

In order to improve these features, a sort of "switching control" is first proposed, which works as the OGY but applying the perturbation u_k^i for all $1 \leqslant i \leqslant m$, i.e., each time the trajectory is close to any p_i (a similar modification appears in [14] as part of a numerical implementation). As an immediate consequence, a net reduction on waiting time is obtained. The control algorithm is

$$u_k = \begin{cases} \alpha_i(x_k - p_i), & |x_k - p_i| \leqslant \varepsilon,\ 1 \leqslant i \leqslant m, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

provided that $\varepsilon < |p_i - p_j|/2$ for all $i \neq j$. As a second consequence, this strategy displays a notably better performance in presence of external noise. This fact is illustrated when applied to the well-known *logistic map* defined by

$$x_{k+1} = r x_k (1 - x_k), \tag{7}$$

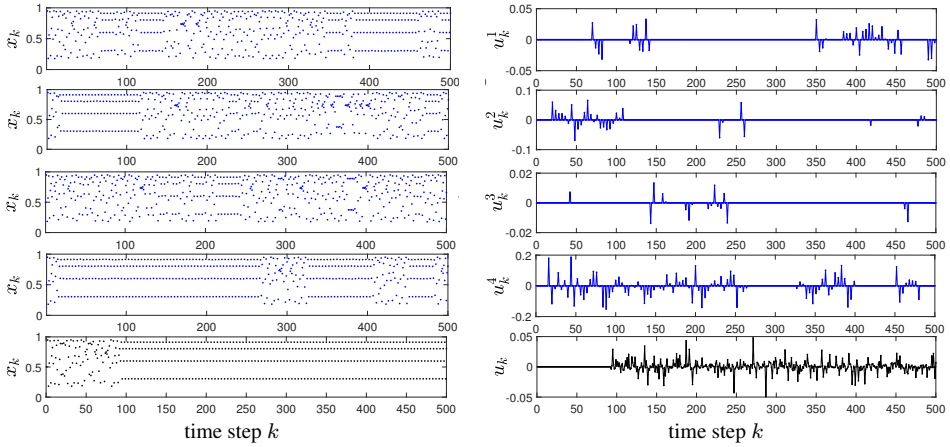


Figure 1. States x_k and controls u_k^i, $i = 1, \ldots, 4$, of Eq. (5) with $\varepsilon = 0.005$ applied to stabilize 4-UPO of Eq. (7) for $r_0 = 3.8$, $\{p_1 \approx 0.3,\ p_2 \approx 0.8,\ p_3 \approx 0.6,\ p_4 \approx 0.91\}$ under the effect of additive noise modeled by $5 \cdot 10^{-4}\sigma_k$, $\sigma_k \sim N(0, 1)$, $x_0 = 0.5$, $\varepsilon = 0.005$ (blue). The same, but applying control (6) (black).

where $x \in [0, 1]$, and r is a control parameter. For $r_\infty \approx 3.569 < r \leqslant 4$, the map presents one chaotic attractor with $\mathcal{B} = [0, 1]$ (except for small windows of periodicity). For fixed r_0 in that range, the parameter modulation is stated through the following controlled system:

$$x_{k+1} = [r_0 + u_k]x_k(1 - x_k). \tag{8}$$

For the map (7), $f_x(p_i, r_0) = r_0(1 - 2p_i)$, $f_r(p_i, r_0) = p_i\,(1 - p_i)$, so $\alpha_i = r_0(2p_i - 1)/(p_i(1 - p_i))$. The condition $r_0 + |u_k| \leqslant 4$ makes the dynamics remain globally bounded, so, for $\varepsilon > 0$, $|\alpha_i\,\varepsilon| \leqslant \delta \leqslant 4 - r_0$, or $\varepsilon \leqslant \delta/|\alpha_i|$ for all $1 \leqslant i \leqslant m$. Figure 1 illustrates the comparison of applying the controls of Eqs. (5) and (6) about a 4-UPO of the map (7) for $r_0 = 3.8$. Even in the presence of noise, the shortening of the waiting time and the reduction in the control bound may be appreciated. Note that, once activated, the control stays on.

2.3 Improving SPF modulation

Here SPF modulation is selected from a set of control laws built by replacing each fixed gain α_i with a coefficient β_i adequately chosen:

$$u_k = \begin{cases} \beta_i\,(x_k - p_i), & |x_k - p_i| \leqslant \varepsilon,\ 1 \leqslant i \leqslant m, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

provided that $\varepsilon < |p_i - p_j|/2$ for all $i \neq j$. This dynamics also preserves the m-UPO. Once the control (9) is activated, Eq. (2) becomes $x_{k+1} = f_{\beta_i}(x_k)$, where

$$f_{\beta_i}(x) \equiv f\big(x,\, r_0 + \beta_i(x - p_i)\big), \tag{10}$$

The linear stability criterion for p_i to be an asymptotically stable (a.s.) point, states the condition for β_i. By applying chain rule, if $f_r(p_i, r_0) \neq 0$, there exists a range $(\beta_i^{\inf}, \beta_i^{\sup})$ of β_i values such that

$$\left|f'_{\beta_i}(p_i)\right| < 1, \quad 1 \leqslant i \leqslant m. \tag{11}$$

Under the stated conditions, convergence is formally proven.

Proposition 1. *Let the controlled system* (2) *and* (9) *with* $\beta_i \in (\beta_i^{\inf}, \beta_i^{\sup})$, $1 \leqslant i \leqslant m$, *for which Eq.* (11) *is valid and* $\beta \equiv \max_{1\leqslant i\leqslant m}\{|\beta_i|\}$. *Given* $\delta > 0$, *there exists* ε_0, $0 < \varepsilon_0 < \min_{i\neq j}\{|p_i - p_j|/2; \delta/\beta\}$ *such that for all* ε, $0 < \varepsilon < \varepsilon_0$, *and for almost every initial condition* $x_0 \in \mathcal{B}$, *it verifies* $|u_k| \leqslant \delta$ *for all* k, *and* $(x_k)_{k\geqslant 1}$ *converges to the* m*-periodic orbit* $\{p_1, \ldots, p_m\}$.

Proof. Let $\varepsilon'_0 \equiv \min_{i\neq j}\{|p_i - p_j|/2\}$. For all i, there exists $\sigma > 0$ such that $|f'_{\beta_i}(p_i)| < \sigma < 1$, and for every i, there exists $0 < \varepsilon_i < \varepsilon'_0$ such that $|f'_{\beta_i}(x)| \leqslant \sigma$ for $x \in (p_i - \varepsilon_i, p_i + \varepsilon_i)$. Fixing $\varepsilon < \varepsilon_0 \equiv \min_{1\leqslant i\leqslant m} \varepsilon_i$, for almost every $x_0 \in \mathcal{B}$, the ergodicity of the uncontrolled system guarantees the existence of $k_0 = k(x_0, \varepsilon)$ such that $|x_{k_0} - p_i| \leqslant \varepsilon$ for some i, then

$$|x_{k_0+1} - p_{i+1}| = \left|f_{\beta_i}(x_{k_0}) - f_{\beta_i}(p_i)\right| = \left|f'_{\beta_i}(\xi_i)\right||x_{k_0} - p_i| \leqslant \sigma\varepsilon < \varepsilon$$

for $\xi_i \in (p_i - \varepsilon_i,\, p_i + \varepsilon_i)$ $(p_i \equiv p_{i\,(\mathrm{mod\ m})})$. Then by recursion it yields to

$$|x_{k_0+n} - p_{i+n}| \leqslant \sigma^n\varepsilon < \varepsilon \quad \forall n \geqslant 1,$$

and the thesis follows. □

The existence of a range $(\beta_i^{\inf}, \beta_i^{\sup})$ states the robustness of the method of Eq. (9), which includes Eq. (6) (note that for $\alpha_i = (\beta_i^{\inf} + \beta_i^{\sup})/2$, $f'_{\alpha_i}(p_i) = 0$). By means of (9) the objective is fulfilled with smaller values for the control gain, or else, for the same control effort, δ, a greater ε is allowed improving the waiting time to active the control. These facts are illustrated for the logistic map (7) for which Eq. (10) becomes

$$f_{\beta_i}(x) = \left[r_0 + \beta_i(x - p_i)\right] x\,(1 - x)$$

with $\beta_i^{\inf} = -(1 + r_0(1 - 2p_i))/(p_i(1 - p_i))$ and $\beta_i^{\sup} = (1 - r_0(1 - 2p_i))/(p_i(1 - p_i))$. For $r_0 = 3.8$, β_i must verify $|\beta_i\varepsilon| \leqslant \delta \leqslant 0.2$ for all i to ensure the desired bound on the control effort and a globally bounded dynamics. As an example, the performances of controls of Eqs. (6) and (9) are compared when applied to stabilize the unstable fixed point $p = 1 - 1/r_0 \approx 0.736$. In Fig. 2(a), it is appreciated both: the neat reduction of control effort by changing the control gain α by β, and an increase in the convergence time once the control is turned on. Figure 2(b) shows, for the similar control effort, the reduction in the waiting time explained above.

Remark 1. Verification of ε-nearness becomes superfluous once k_0 is detected.

Remark 2. The choosing of β_i implies (but it is not equivalent to!):

$$\left|f'_{\beta_1}(p_1) f'_{\beta_2}(p_2) \cdots f'_{\beta_m}(p_m)\right| < 1. \tag{12}$$

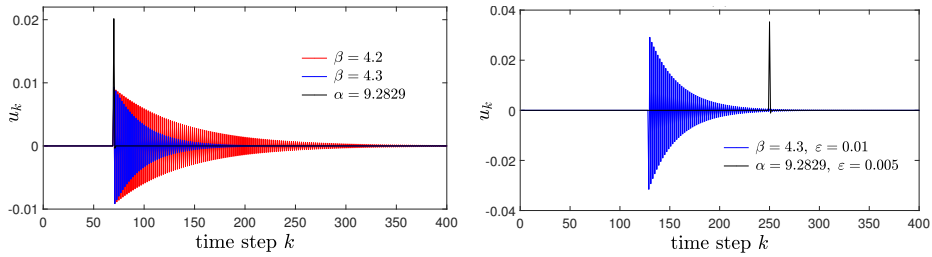


Figure 2. (a) Performance of controls (6) and (9) to stabilize Eq. (7) with $r_0 = 3.8$ about the fixed point $p \approx 0.736$, $x_0 = 0.94$, $\varepsilon = 0.005$, $\alpha = 9.2829$ (black), $\beta = 4.2$ (red), $\beta = 4.3$ (blue) ($\beta^{\inf} \approx 4.126$, $\beta^{\sup} \approx 14.44$). (b) Idem (a), but keeping the control effort and varying the ε values: $\alpha = 9.2829$ and $\varepsilon = 0.005$ (black), $\beta = 4.3$ and $\varepsilon = 0.01$ (blue), $x_0 = 0.5$.

Hence, a generalization of (9), which also includes (5), results in

$$u_k = \begin{cases} 0, & k < k_0, \\ \beta_{(i+k-k_0)\,(\mathrm{mod\ m})}\big(x_k - p_{(i+k-k_0)\,(\mathrm{mod\ m})}\big), & k \geqslant k_0, \end{cases} \tag{13}$$

where $k_0 = k_0(x_0, \varepsilon) \equiv \min\{k \geqslant 0\colon |x_k - p_i| \leqslant \varepsilon \text{ for some } i,\ 1 \leqslant i \leqslant m\}$ for $\varepsilon < \min_{i \neq j}\{|p_i - p_j|/2\}$.

Note that the existence of k_0 is warranted by system ergodicity.

Proposition 2. *Let the controlled system* (2) *and* (13) *with* β_j, $1 \leqslant j \leqslant m$, *for which Eq.* (12) *is valid and* $\beta \equiv \max_{1 \leqslant j \leqslant m} |\beta_j|$. *Given* $\delta > 0$, *there exists* ε_0, $0 < \varepsilon_0 < \min_{i \neq j}\{|p_i - p_j|/2; \delta/\beta\}$ *such that for all* ε, $0 < \varepsilon < \varepsilon_0$, *and for almost every initial condition* $x_0 \in \mathcal{B}$, *it verifies* $|u_k| < \delta$, *and* $(x_k)_{k \geqslant 1}$ *converges to the* m*-periodic orbit* $\{p_1, \ldots, p_m\}$.

Proof. Let $\varepsilon_0' \equiv \min_{i \neq j}\{|p_i - p_j|/2\}$, and let $0 < \sigma < 1$ such that

$$\left|f_{\beta_1}'(p_1) f_{\beta_2}'(p_2) \cdots f_{\beta_m}'(p_m)\right| < \sigma < 1$$

equivalent to

$$\left|f_{\beta_i}'(p_1) f_{\beta_2}'\big(f_{\beta_1}(p_1)\big) \cdots f_{\beta_m}'\big(f_{\beta_{m-1}}\big(\cdots\big(f_{\beta_1}(p_1)\big)\big)\big)\right| < \sigma.$$

Therefore, there exists ε_1, $0 < \varepsilon_1 < \varepsilon_0'$, such that

$$\left|f_{\beta_i}'(x) f_{\beta_2}'\big(f_{\beta_1}(x)\big) \cdots f_{\beta_m}'\big(f_{\beta_{m-1}}\big(\cdots\big(f_{\beta_1}(x)\big)\big)\big)\right| \leqslant \sigma$$

for $x \in (p_1 - \varepsilon_1, p_1 + \varepsilon_1)$. Proceeding in the same way for $p_2, \ldots, p_m$, the $\varepsilon_2, \ldots, \varepsilon_m$ values arise. Let $\varepsilon_0 \equiv \min_{1 \leqslant j \leqslant m} \varepsilon_j$, and fix $\varepsilon < \varepsilon_0$. Without loss of generality, it is assumed that the condition $|x_k - p_i| \leqslant \varepsilon$ is verified, for the first time, by $i = 1$. Then

$$\begin{aligned} |x_{k_0+m} - p_1| &= \left|f_{\beta_m} \circ f_{\beta_{m-1}} \circ \cdots \circ f_{\beta_1}(x_k) - f_{\beta_m} \circ f_{\beta_{m-1}} \circ \cdots \circ f_{\beta_1}(p_1)\right| \\ &= \left|(f_{\beta_m} \circ f_{\beta_{m-1}} \circ \cdots \circ f_{\beta_1})'(\xi_1)\right| |x_k - p_1| \leqslant \sigma\varepsilon < \varepsilon \end{aligned}$$

for $\xi_1 \in (p_1 - \varepsilon_1,\, p_1 + \varepsilon_1)$, and so following:

$$|x_{k_0+\ell m} - p_1| \leqslant \sigma^\ell \varepsilon < \varepsilon, \quad \ell \geqslant 1.$$

By the same arguments it is obtained that $|x_{k_0+j+m} - p_{j+1}| \leqslant \sigma\varepsilon < \varepsilon$ and $|x_{k_0+j+\ell m} - p_{j+1}| \leqslant \sigma^\ell \varepsilon < \varepsilon$ for $1 \leqslant j \leqslant m-1$. Hence, control bounds and convergence to the periodic orbit follow. □

Remark 3. Condition (11) relies on the existence of the range $(\beta_i^{\inf}, \beta_i^{\sup})$ for each i. Looking for a single common β value, as referred in [3, Sect. 5.3.2], assumes a nonempty intersection of these intervals. However, this may not be the case in many examples as it is easily seen for the same 4-period UPO of Fig. 1 for which the ranges for β_1 and β_2 are $(-11.0716, -2.325)$ and $(8.2897, 20.9639)$, respectively.

Remark 4. The problem of finding a control depending just on one gain control for stabilizing an m-UPO is left unsolvable in [25]. In the terms of this work, it requires to find a value β such that Eq. (12) is satisfied with $\beta_i = \beta$ for all i involving a nonlinear inequality whose solution is not warranted in any case.

3 Controlling chaos by DFC modulation

3.1 The Pyragas method

For stabilizing the logistic map (7) to its unstable fixed point, $p = 1 - 1/r_0$, additive forcing in the form of one time-delay linear perturbation, $\gamma(x_k - x_{k-1})$, was early introduced in [27]. Here DFC modulation is stated as follows:

$$u_k = \begin{cases} \gamma(x_k - x_{k-1}), & [|x_k - p|^2 + |x_{k-1} - p|^2]^{1/2} \leqslant \frac{\varepsilon}{\sqrt{2}}, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

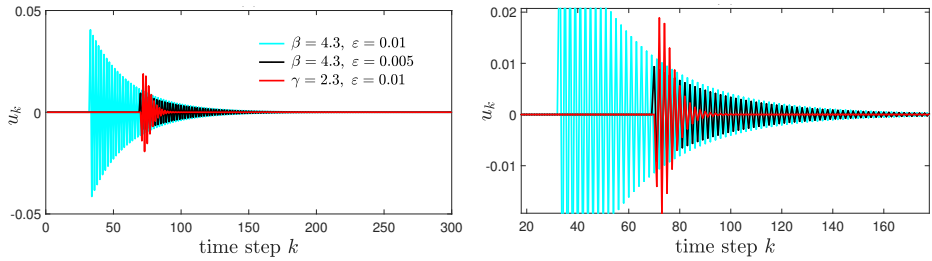
Control (14) vanishes when system (2) state attains p, then the fixed point is preserved. For the logistic map, the controlled system (8) and (14) yields to the two-dimensional dynamical system

$$\begin{aligned} x_{k+1} &= \big[r_0 + \gamma(x_k - y_k)\big] x_k(1 - x_k), \\ y_{k+1} &= x_k, \end{aligned}$$

which has $\mathbf{P} \equiv [p, p]$ as a fixed point. The Jacobian matrix at $\mathbf{P}$ is a (2×2)-*companion matrix*

$$\mathbf{J}(\mathbf{P}) = \begin{pmatrix} a_{11} & a_{12} \\ 1 & 0 \end{pmatrix},$$

where $a_{11} = (\gamma/r_0)(1 - 1/r_0) + 2 - r_0$ and $a_{12} = -(\gamma/r_0)(1 - 1/r_0)$. The necessary and sufficient conditions for $\mathbf{P}$ to be a.s. [13] yields to a range $(\gamma^{\inf}, \gamma^{\sup})$ with $\gamma^{\inf} = r_0^2(r_0 - 3)/(2(r_0 - 1))$ and $\gamma^{\sup} = r_0^2/(r_0 - 1)$. Fixing a γ in this range and choosing an adequate ε to assure the control effort to be bounded by δ, the convergence of the trajectories to p is obtained. The resulting control performance of applying (14) may be comparable to (or even better than) the one obtained by applying (9) if adequate coefficients are chosen as shown in Fig. 3.



(a) (b)

Figure 3. (a) Performance of the controls (9) and (14) to stabilize Eq. (7) with $r_0 = 3.8$ about the fixed point $p \approx 0.736$, $x_0 = 0.94$, $y_0 = 0.45$. (b) Enlargement of subfigure (a).

3.2 Improving DFC modulation

The idea of "switching control" that depends on m-different control gains adequately chosen, introduced in Section 2 for improving SPF modulation, is here applied to improve DFC modulation. The proposal of an extension of (14) to a m-UPO, $\boldsymbol{p}$, of (1) is to set a "switching" control, as follows:

$$u_k = \begin{cases} \gamma_i(x_k - x_{k-m}), & [\sum_{j=0}^{m} |x_{k-j} - p_{(i-j)\,(\mathrm{mod\ m})}|^2]^{1/2} \\ & \quad \leqslant \frac{\varepsilon}{\sqrt{2}},\ 1 \leqslant i \leqslant m, \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

if $0 < \varepsilon < \|\mathbf{P}_i - \mathbf{P}_j\|/\sqrt{2}$ for all $i \neq j$, being $\mathbf{P}_i \equiv [p_i, p_{i-1}, \ldots, p_1, p_m, p_{m-1}, \ldots, p_i]$, $1 \leqslant i \leqslant m$.

System (2) and (15) yields to a $(m+1)$-dimensional system (with m switches) with $[\mathrm{x}^1, \ldots, \mathrm{x}^{m+1}]$ as its state variables:

$$\begin{aligned} &\mathrm{x}_{k+1}^1 = f\big(\mathrm{x}_k^1,\, r_0 + \gamma_i\,(\mathrm{x}_k^1 - \mathrm{x}_k^{m+1})\big), \quad \big\|\big[\mathrm{x}_k^1, \ldots, \mathrm{x}_k^{m+1}\big] - \mathbf{P}_i\big\| \leqslant \frac{\varepsilon}{\sqrt{2}}, \\ &\mathrm{x}_{k+1}^2 = \mathrm{x}_k^1, \quad \mathrm{x}_{k+1}^3 = \mathrm{x}_k^2, \quad \ldots, \quad \mathrm{x}_{k+1}^{m+1} = \mathrm{x}_k^m. \end{aligned} \tag{16}$$

Note that $\{P_1, \ldots, \mathbf{P}_m\}$ is m-UPO of the free system ($\gamma_i = 0$ for all i), and it is preserved when (15) is applied. The Jacobian matrix of system (16) at $\mathbf{P}_i$ is a $(m+1) \times (m+1)$–companion matrix given by

$$\mathbf{J}_i = \begin{pmatrix} a_{11}^{(i)} & 0 & \cdots & 0 & a_{1(m+1)}^{(i)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where $a_{11}^{(i)} = f_x(p_i, r_0) + \gamma_i f_r(p_i, r_0)$ and $a_{1(m+1)}^{(i)} = -\gamma_i f_r(p_i, r_0)$.

Unfortunately, in most cases for $m > 1$, there is no γ_i that warranties all of $\mathbf{J}_i$ eigenvalues to be of modulus less than one, and so, achieving UPO stabilization by this way. Even it is not possible for the 2-UPO of the logistic map.

The conditions for the orbit to be a.s. involve the product of the matrices, $\prod_{i=1}^{m} \mathbf{J}_i$. It should be found γ_i making this product have all eigenvalues of modulus less than one. This is the equivalent of condition in Eq. (12) for the control (13). Moreover, it reduces to the (only) Jacobian matrix of the system for the fixed point stabilization (Eq. (14)).

As each $\mathbf{J}_i$ is a companion matrix, the characteristic polynomial, $\chi(\lambda)$, of the product $\prod_{i=1}^{m} \mathbf{J}_i$ is obtained by using tools from [10], and it results

$$\chi(\lambda) = -\lambda^{m+1} + \prod_{i=1}^{m} \left(a_{1(m+1)}^{(i)} + \lambda a_{11}^{(i)}\right). \tag{17}$$

Therefore, the control is proposed as

$$u_k = \begin{cases} 0, & k < k_0, \\ \gamma_{(i+k-k_0)\,(\mathrm{mod\ m})}(x_k - x_{k-m}), & k \geqslant k_0, \end{cases} \tag{18}$$

if $0 < \varepsilon < \|\mathbf{P}_i - \mathbf{P}_j\|/\sqrt{2}$ for all $i \neq j$, being $k_0 = k_0(x_0, \varepsilon) \equiv \min\{k \geqslant m\colon \|[x_{k+j}, \ldots, x_{k+j-m}] - \mathbf{P}_{i+j}\| \leqslant \varepsilon/\sqrt{2}$ for some $i, 1 \leqslant i \leqslant m\}$. Existence of k_0 is a consequence of ergodicity. Its actual computation requires certain information on UPO's location, but this does not violate the main feature of DFC method, i.e., exact knowledge of the UPO is not necessary.

Introducing $\mathbf{X}_k \equiv [\mathrm{x}_k^1, \mathrm{x}_k^2, \ldots, \mathrm{x}_k^{m+1}]^\top$ and $F_{\gamma_i}(\mathbf{X}_k) \equiv [f(\mathrm{x}_k^1,\, r_0 + \gamma_i(\mathrm{x}_k^1 - \mathrm{x}_k^{m+1})), \ldots, \mathrm{x}_k^m]^\top$, the controlled system (2) and (18) is equivalent to

$$\mathbf{X}_{k+1} = F_{\gamma_{(i+k-k_0)\,(\mathrm{mod\ m})}}(\mathbf{X}_k) \quad \forall k \geqslant k_0,$$

which has $\{\mathbf{P}_i\}_{1 \leqslant i \leqslant m}$ as m-UPO. Note that $\mathrm{D}_{\mathbf{X}} F_{\gamma_i}(\mathbf{P}_i) = \mathbf{J}_i$ and $k_0 \equiv \min\{k \geqslant m\colon \|\mathbf{X}_k - \mathbf{P}_i\| \leqslant \varepsilon/\sqrt{2}\}$. Next, following the same steps that in the proof of Proposition 2, the success of the proposed scheme is proved.

Proposition 3. *Let the controlled system* (2) *and* (18) *with* γ_j, $1 \leqslant j \leqslant m$, *such that all the roots of* $\chi(\lambda)$ *are of modulus less than one, and* $\gamma \equiv \max_{1 \leqslant i \leqslant m} |\gamma_j|$. *Given* $\delta > 0$, *there exists* ε_0, $0 < \varepsilon_0 < \min_{i \neq j}\{\|\mathbf{P}_i - \mathbf{P}_j\|/\sqrt{2}; \delta/\gamma\}$ *such that for all* ε, $0 < \varepsilon < \varepsilon_0$, *and for almost every initial condition* $x_0 \in \mathcal{B}$, *it verifies* $|u_k| \leqslant \delta$, *and that* $(x_k)_{k \geqslant 1}$ *converges to the* m*-UPO* $\{p_1, \ldots, p_m\}$.

Proof. The roots of $\chi(\lambda)$ are of modulus less than one, so $\|\prod_{i=1}^{m} \mathbf{J}_i\| < 1$ with the matrix norm induced by the Euclidean one. Let $\varepsilon_0' \equiv \min_{i \neq j}\{\|\mathbf{P}_i - \mathbf{P}_j\|/\sqrt{2}\}$ and $0 < \sigma < 1$ such that $\|\prod_{i=1}^{m} \mathbf{J}_i\| < \sigma < 1$, which is the same as

$$\left\|\mathrm{D}_{\mathbf{X}} F_{\gamma_1}(\mathbf{P}_1) \cdots \mathrm{D}_{\mathbf{X}} F_{\gamma_m}\left(F_{\gamma_{m-1}}\left(\cdots\left(F_{\gamma_1}(\mathbf{P}_1)\right)\right)\right)\right\| < \sigma.$$

Then there exists ε_1, $0 < \varepsilon_1 < \varepsilon_0'$, such that

$$\left\|\mathrm{D}_{\mathbf{X}} F_{\gamma_1}(\mathbf{X}) \mathrm{D}_{\mathbf{X}} F_{\gamma_2}\left(F_{\gamma_1}(\mathbf{X})\right) \cdots \mathrm{D}_{\mathbf{X}} F_{\gamma_m}\left(F_{\gamma_{m-1}}\left(\cdots\left(F_{\gamma_1}(\mathbf{X})\right)\right)\right)\right\| \leqslant \sigma$$

for $\mathbf{X}$, $\|\mathbf{X}-\mathbf{P}_1\| \leqslant \varepsilon_1/\sqrt{2}$. Proceeding analogously for $\mathbf{P}_2,\dots,\mathbf{P}_m$, it is obtained $\varepsilon_2,\dots,\varepsilon_m$. Let $\varepsilon_0 \equiv \min_{1\leqslant i\leqslant m}\varepsilon_i$, and fix $\varepsilon < \varepsilon_0$. Without loss of generality, it is assumed that condition $\|\mathbf{X}_{k_0}-\mathbf{P}_1\| \leqslant \varepsilon/\sqrt{2}$ is verified, for the first time, by $i=1$. Then

$$\begin{aligned}\|\mathbf{X}_{k_0+m}-\mathbf{P}_1\| &= \big\|F_{\gamma_m}\circ\cdots\circ F_{\gamma_1}(\mathbf{X}_{k_0}) - F_{\gamma_m}\circ\cdots\circ F_{\gamma_1}(\mathbf{P}_1)\big\| \\ &\leqslant \big\|\mathrm{D}_{\mathbf{X}}\big(F_{\gamma_m}\circ\cdots\circ F_{\gamma_1}\big)(\xi_1)\big\|\|\mathbf{X}_{k_0}-\mathbf{P}_1\| \leqslant \sigma\,\frac{\varepsilon}{\sqrt{2}} < \frac{\varepsilon}{\sqrt{2}},\end{aligned}$$

where ξ_1 lies in the segment joining $\mathbf{P}_1$ and $\mathbf{X}_{k_0}$. By the same arguments it is obtained

$$\big\|\mathbf{X}_{k_0+j+\ell m}-\mathbf{P}_{j+1}\big\| \leqslant \sigma^{\ell}\frac{\varepsilon}{\sqrt{2}} < \frac{\varepsilon}{\sqrt{2}}, \quad j=1,\dots,m-1,\ \ell\geqslant 1. \tag{19}$$

Hence, the convergence of $(\mathbf{X}_k)_{k\geqslant 1}$ to the m-UPO $\{\mathbf{P}_i\}_{1\leqslant i\leqslant m}$ follows, and so the convergence of $(x_k)_{k\geqslant 1}$ to the m-UPO $\{p_1,\dots,p_m\}$. From Eq. (19) it also results

$$|x_{k_0+j+\ell m}-x_{k_0+j+m(\ell-1)}| < \varepsilon, \quad j=1,\dots,m-1,\ \ell\geqslant 1,$$

and so, the control bounds are obtained. □

The successfulness of the controlled system (2) and (18) hinges on finding a range of γ_i's for orbit stability, i.e., for the roots of $\chi(\lambda)$ in Eq. (17) to be within the unit circle. It is easy to check that $\prod_{j=1}^m f_x(p_j,r_0)\leqslant 1$ is a necessary condition for it, so, the ONL remains for this modified DFC. The use of Jury test (as carried out in [18]) should be useful, but, even out of ONL, it is neither possible to obtain the γ_i's explicitly nor ensure a range finding in the general case. However, a first attempt to look for a solution is setting $\gamma_i\neq 0$ for a fixed i and $\gamma_j=0$ for all $j\neq i$ in order to simplify the equation. In this way, $\lambda=0$ results a $(m-1)$-multiplicity root of $\chi(\lambda)$, and the other two are the roots of a quadratic function $\chi(\lambda)=\lambda^{m-1}(-\lambda^2+A\lambda+B_i)$, where $A\equiv\prod_{j=1}^m a_{11}^{(j)}$, $B_i\equiv a_{1(m+1)}^{(i)}C_i$, and $C_i\equiv\prod_{j\neq i}a_{11}^{(j)}$. From the conditions on A and B_i so that $|\lambda|<1$ [13] it is deduced that a range $(\gamma_i^{\inf},\gamma_i^{\sup})$ exists if and only if $-3<\prod_{j=1}^m f_x(p_j,r_0)<1$. This strategy is illustrated in the logistic map (Eq. (7)) for the stabilization of m-UPO, $\{p_1,\dots,p_m\}$. The last condition becomes

$$\left|1+r_0^m\prod_{j=1}^m(1-2p_j)\right| < 2. \tag{20}$$

The range $(\gamma_i^{\inf},\gamma_i^{\sup})$ is settled on

$$\gamma_i^{\inf}=\begin{cases}\frac{-1-r(1-2p_i)C_i}{2p_i(1-p_i)C_i}, & C_i>0,\\ \frac{1}{p_i(1-p_i)C_i}, & C_i<0,\end{cases}\qquad \gamma_i^{\sup}=\begin{cases}\frac{1}{p_i(1-p_i)C_i}, & C_i>0,\\ \frac{-1-r(1-2p_i)C_i}{2p_i(1-p_i)C_i}, & C_i<0.\end{cases} \tag{21}$$

For $r_0=3.62$ and 4-UPO, $p_1\approx 0.5522$, $p_2\approx 0.8951$, $p_3\approx 0.3398$, $p_4\approx 0.8121$, condition (20) is accomplished. Taking, for instance, $\gamma_1=\gamma_2=\gamma_3=0$, a range for γ_4 is obtained from (21). Figure 4 shows this UPO being stabilized by control (18) for initial

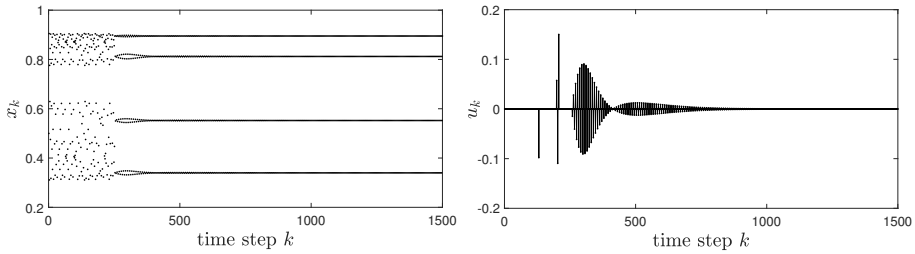


Figure 4. States x_k and controls u_k of Eq. (18), with $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\gamma_4 = 4.7997$, $\varepsilon = 0.05$, applied to stabilize 4-UPO of Eq. (7) for $r_0 = 3.62$.

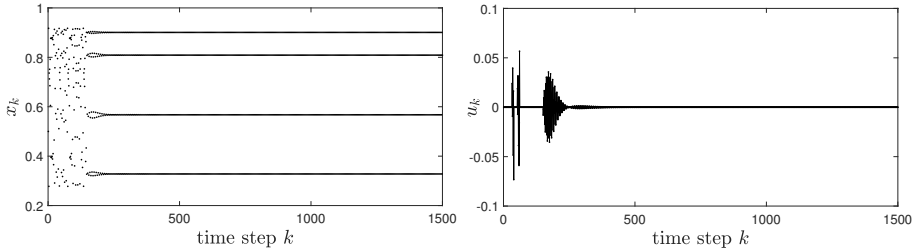


Figure 5. States x_k and controls u_k of Eq. (18), with $\gamma_1 = 0.4$, $\gamma_2 = 4.97156$, $\gamma_3 = -0.598$, $\gamma_4 = 2.09$, $\varepsilon = 0.05$, applied to stabilize 4-UPO of Eq. (7) for $r_0 = 3.67$.

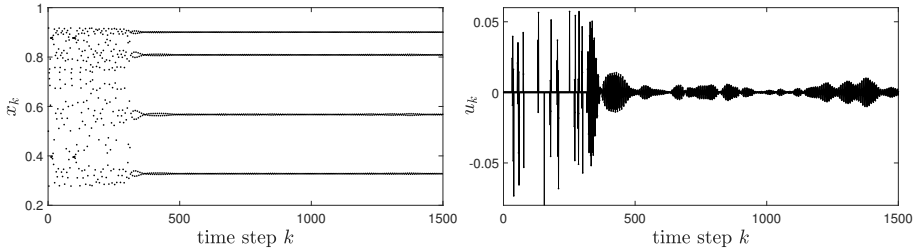


Figure 6. States x_k and controls u_k of Eq. (18), with $\gamma_1 = 0.4$, $\gamma_2 = 4.97156$, $\gamma_3 = -0.598$, $\gamma_4 = 2.09$, $\varepsilon = 0.05$, applied to stabilize 4-UPO of Eq. (7) for $r_0 = 3.67$ under the effect of additive noise modeled by $2 \times 10^{-5}\sigma_k$, $\sigma_k \sim N(0, 1)$.

condition $x_0 = 0.5$ and $\varepsilon = 0.05$. The performance of the control u_k is shown in the same figure. This resource does not work for $r_0 > 3.625$ because the Floquet multiplier of the 4-UPO is smaller than -3, and Eq. (20) is not accomplished anymore. Even so, 4-tuples $[\gamma_1, \gamma_2, \gamma_3, \gamma_4]$ can be found for all the roots of $\chi(\lambda)$ to be within the unit circle. Indeed, by means of detailed heuristic search on the coefficient values of $\chi(\lambda)$ a solution is obtained. Figure 5 shows an example where $r_0 = 3.67$. It is worth noting that, in this case, the control strategy (18) overcomes the Pyragas method, which fails above $r \approx 3.62$ (see [32, p. 50]). Namely, by control (18) the 4-UPO is stabilized for an enlargement of r_0 values: $3.625 \leqslant r_0 \leqslant 3.67$. Moreover, the control (18) also works successfully even in presence of noise as it is exhibited in Fig. 6.

3.3 Improving EDFC modulation

The extension of the Pyragas method to stabilize the system in a m-UPO, known as "extended time-delay autosynchronization system" (ETDAS) or "extended delayed feedback control" (EDFC), is defined as

$$u_k = \gamma(x_k - x_{k-m}) + Ru_{k-m} \tag{22}$$

with $0 \leqslant R < 1$, which reduces to TDAS for $R = 0$. It was introduced to stabilize higher instabilities. Indeed, going back to the previous example, it is reported in [32] that with $R = 0.5$, stabilization of the 4-UPO of the logistic map can be maintained up to the parameter value $r_0 \approx 3.75$.

Based on control (18), a kind of extension like (22) is proposed to stabilize a system in a m-UPO that consists in replacing it by

$$u_k = \begin{cases} 0, & k < k_0, \\ \gamma_{(i+k-k_0)\,(\mathrm{mod}\ \mathrm{m})}(x_k - x_{k-m}) + Ru_{k-m}, & k \geqslant k_0. \end{cases} \tag{23}$$

System (2) and (23) yields to a $2m$-dimensional system (with m switches) with $[\mathrm{x}^1, \ldots, \mathrm{x}^m, \mathrm{u}^1, \ldots, \mathrm{u}^m] \equiv [x_k, \ldots, x_{k-m+1}, u_k, \ldots, u_{k-m+1}]$ as its state variables. For example, for the logistic map, the $m = 4$ case is given by

$$\begin{aligned} &\mathrm{x}^1_{k+1} = \big(r_0 + \mathrm{u}^1_{\mathrm{k}}\big)\mathrm{x}^1_k\big(1 - \mathrm{x}^1_k\big), \quad \big\|\big[\mathrm{x}^1_k, \mathrm{x}^2_k, \mathrm{x}^3_k, \mathrm{x}^4_k\big] - \widetilde{\mathbf{P}}_i\big\| \leqslant \frac{\varepsilon}{\sqrt{2}}, \\ &\mathrm{x}^2_{k+1} = \mathrm{x}^1_k, \quad \mathrm{x}^3_{k+1} = \mathrm{x}^2_k, \quad \mathrm{x}^4_{k+1} = \mathrm{x}^3_k, \\ &\mathrm{u}^1_{k+1} = \gamma_i\big(\big(r_0 + \mathrm{u}^1_{\mathrm{k}}\big)\mathrm{x}^1_k\big(1 - \mathrm{x}^1_k\big) - \mathrm{x}^4_k\big) + R\mathrm{u}^4_{\mathrm{k}}, \\ &\mathrm{u}^2_{k+1} = \mathrm{u}^1_k, \quad \mathrm{u}^3_{k+1} = \mathrm{u}^2_k, \quad \mathrm{u}^4_{k+1} = \mathrm{u}^3_k. \end{aligned} \tag{24}$$

Note that the 4-UPO of the free system ($\gamma_i = 0$ for all i)

$$\begin{aligned} &\widetilde{\mathbf{P}}_1 \equiv [p_1, p_4, p_3, p_2, 0, 0, 0, 0], \qquad \widetilde{\mathbf{P}}_2 \equiv [p_2, p_1, p_4, p_3, 0, 0, 0, 0], \\ &\widetilde{\mathbf{P}}_3 \equiv [p_3, p_2, p_1, p_4, 0, 0, 0, 0], \qquad \widetilde{\mathbf{P}}_4 \equiv [p_4, p_3, p_2, p_1, 0, 0, 0, 0] \end{aligned}$$

is preserved when (23) is applied. The Jacobian matrix of system (24) at $\widetilde{\mathbf{P}}_i$ is, in this case, a 8×8 matrix given by

$$\mathbf{J}_i = \begin{pmatrix} a(i) & 0 & 0 & 0 & b(i) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ c(i) & 0 & 0 & -\gamma_i & d(i) & 0 & 0 & R \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

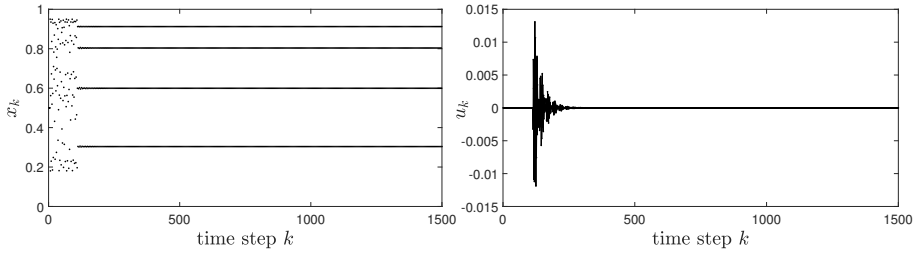


Figure 7. States x_k and controls u_k of Eq. (23), with $\gamma_1 = 3.50293$, $\gamma_2 = 1.38$, $\gamma_3 = 7.49498$, $\gamma_4 = -1.181$, $R = 0.3$, $\varepsilon = 0.05$, applied to stabilize 4-UPO of Eq. (7) for $r_0 = 3.8$.

where $a(i) \equiv r(1-2p_i)$, $b(i) \equiv p_i(1-p_i)$, $c(i) \equiv \gamma_i r(1-2p_i)$ and $d(i) \equiv \gamma_i p_i(1-p_i)$. As $c(i) = \gamma_i a(i)$, and $d(i) = \gamma_i b(i)$, the characteristic polynomial is given by

$$\chi(\lambda) = \lambda^3\big[\lambda^5 - \big(a(i) + \gamma_i b(i)\big)\lambda^4 - R\lambda + a(i)R + \gamma_i b(i)\big].$$

Applying the heuristic algorithm to search on the coefficient values of $\chi(\lambda)$, 4-tuples, $[\gamma_1, \gamma_2, \gamma_3, \gamma_4]$ are found, and the 4-UPO for $3.67 < r_0 \leqslant 3.8$ can be stabilized. Thus, the range of parameter values for which the 4-cycle-logistic map is stabilized, is enlarged. Figure 7 illustrates the orbit stabilization and the control performance for $r_0 = 3.8$.

4 On the extension to n-dimensional systems

This section points out the required adjustments for applying the developed strategies in the n-dimensional case, and even more, they are compared to interesting background in the literature.

For SPF modulation of Section 2.3, Eq. (9) remains the same save that control gain is $\beta_i^\top \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$, and the distance is given by the Euclidean norm in $\mathbb{R}^n$. The Jacobian of the controlled system at each $\mathbf{p}_i$, $1 \leqslant i \leqslant m$, is given by

$$\mathrm{D}_{\mathbf{x}} f_{\beta_i}(\mathbf{p}_i) = \mathbf{A}_{\mathbf{x}} + \mathbf{A}_r\, \beta_i^\top,$$

where $\mathbf{A}_{\mathbf{x}} = \mathbf{A}_{\mathbf{x}}^{(i)} \equiv \mathrm{D}_{\mathbf{x}} f(\mathbf{p}_i, r_0)$ and $\mathbf{A}_r = \mathbf{A}_r^{(i)} \equiv \mathrm{D}_r f(\mathbf{p}_i, r_0)$ are $n \times n$ and $n \times 1$ matrices, respectively.

The insertion of a switching controller as part of a SPF method appears in previous bibliography. Indeed, the multipoint OGY formula (see [3, Sect. 3.2.6]) may be interpreted as an extension of Eq. (6) to the n-dimensional case. In turn, some modifications to overcome its limitations have been developed [3]. A two-level control method is proposed in [5], where the distance between the trajectory and the UPO is minimized in each step. The authors claim that its implementation on UPO's requires a number of switchings greater than m to obtain successful control. In [25], it is proposed to choose feedback gains β_i such that the whole of the eigenvalues of $\mathbf{A}_{\mathbf{x}}^{(i)} + \mathbf{A}_r^{(i)}\beta_i^\top$ be of modulus less

than one. Note that this criterion may fail to detect parameters β_i for which the Jacobean matrix of the composition has the whole of its eigenvalues within the unit circle.

The stability condition stated in Eq. (11) becomes $\|\mathrm{D}_{\mathbf{x}} f_{\beta_i}(p_i, r_0)\| < 1$ for the n-dimensional case so forcing the trajectory to be closer to the UPO. This is verified if, for all i, $\mathbf{A}_{\mathbf{x}} + \mathbf{A}_r\, \beta_i^{\top}$ have the maximum singular value of modulus less than one [15], which is accomplished if $\mathbf{A}_{\mathbf{x}} + \mathbf{A}_r \beta_i^{\top}$ has a basis of eigenvectors and the whole of its eigenvalues within the unit circle. If $\operatorname{rank}\{\mathbf{A}_r, \mathbf{A}_{\mathbf{x}}\mathbf{A}_r, \ldots, \mathbf{A}_{\mathbf{x}}^{n-1}\mathbf{A}_r\} = n$, pole-placement method [23] is a useful tool to obtain $\beta_i^{\top} \in \mathbb{R}^n$ satisfying this condition. Then Proposition 1 is straightforward generalized to the n-dimensional case. On the other hand, the generalized version of Proposition 2 to the n-dimensional case requires the product $\prod_{i=1}^{m} \mathbf{A}_{\mathbf{x}}^{(i)} + \mathbf{A}_r^{(i)} \beta_i^{\top}$ to have the whole of its eigenvalues within the unit circle, which is a weaker condition, but it yields to nonlinear equations in β_i. Defining $\beta = \max_{1 \leqslant i \leqslant m} \|\beta_i\|$, the results of Section 2 are accordingly valid in the n-dimensional case.

Similarly, in the DFC modulation defined in Section 3.2, a vector $\gamma_i^{\top} \in \mathbb{R}^n$ becomes the control gain for the n-dimensional version of Eq. (15). The stability criterion involves the product $\mathbf{J} \equiv \prod_{i=1}^{m} \mathbf{J}^{(i)}$ being $\mathbf{J}^{(i)}$ a $n(m+1) \times n(m+1)$ matrix given by

$$\mathbf{J}^{(i)} \equiv \begin{pmatrix} \mathbf{A}_{11}^{(i)} & \mathbf{0}_n & \cdots & \mathbf{0}_n & \mathbf{A}_{1(m+1)}^{(i)} \\ \mathbf{I}_n & \mathbf{0}_n & \cdots & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{I}_n & \cdots & \mathbf{0}_n & \mathbf{0}_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_n & \mathbf{0}_n & \cdots & \mathbf{I}_n & \mathbf{0}_n, \end{pmatrix}$$

where $\mathbf{A}_{11}^{(i)} \equiv \mathrm{D}_{\mathbf{x}} f(\mathbf{p}_i, r_0) + \mathrm{D}_r f(\mathbf{p}_i, r_0)\, \gamma_i^{\top}$, $\mathbf{A}_{1\,(m+1)}^{(i)} \equiv -\mathrm{D}_r f(\mathbf{p}_i)\, \gamma_i^{\top}$ are both $n \times n$ matrices, $\mathbf{I}_n$ is the $(n \times n)$-identity matrix and $\mathbf{0}_n$ is the $(n \times n)$-null matrix.

In the stability condition of DFC studied in [19], for the multidimensional case, the system is affected by an additive feedback control that depends only on a constant gain, γ. The insertion of m different control gains $\{\gamma_i\}_{1 \leqslant i \leqslant m}$ in Section 3.2 improves the proposals in [18, 19, 32], where $\gamma_i = \gamma$ for all i, and which hinges on the eventual scenario, where nonempty overlapping γ_i-ranges comes out. In addition, the proof of stability outlined in the present work avoids the computation of the product of the matrices $\mathbf{J}_i$, and the characteristic polynomial of Eq. (17) is obtained by a more straightforward calculation.

The EDFC algorithm of Eq. (22) may also be formulated for the n-dimensional case. Its stability analysis yields to a $2mn$-dimensional system (with m switches). As in the SPF modulation, controllability tools become useful to obtain adequate control gains. Defining $\gamma \equiv \max_{1 \leqslant i \leqslant m} \|\gamma_i\|$, the arguments on the validity of these methods are also straightforward generalized from Section 3, although a full description of them becomes tiresome.

In most cases, the parameter r affects only one component of the system given by f, say f_j. Then, when looking for $\beta_i \in \mathbb{R}$ (respectively γ_i) in these situations, a conductive

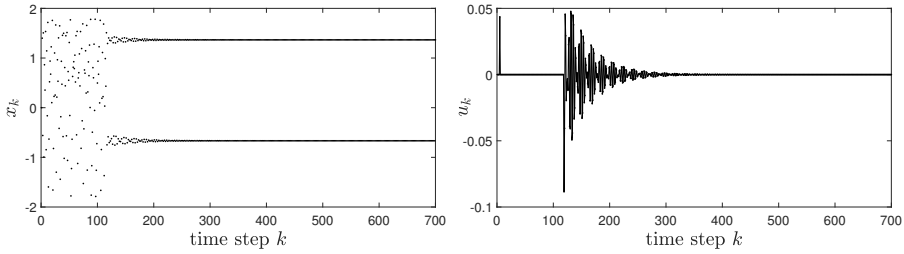


Figure 8. States x_k of system (26) and controls u_k of Eq. (15), with $\gamma_1 = 0.4098$, $\gamma_2 = -0.2498$, $x_0 = y_0 = z_0 = 0$, $\varepsilon = 0.01$, applied to stabilize the 2-UPO $\{\mathbf{p}_1, \mathbf{p}_2\}$ with $\mathbf{p}_1 \equiv [x^*, y^*]$ and $\mathbf{p}_2 \equiv [y^*, x^*]$ of Eq. (26) for $a_0 = 1.4$, $b = 0.3$ and $x^* = 1.3661$, $y^* = -0.6661$.

simplification can be made by setting all of its components equal to zero, except the jth one. In turn, for DFC strategy, it means that less delayed states are introduced, yielding to a decrease on the controlled system dimension (namely, on the $\mathbf{J}$ matrix dimension). Moreover, the $(m+1)$-multiplication effect on the controlled system is avoided in these particular situations. This is the case of the two-dimensional *Hénon map*, where the second component of the state variable is just the delayed first one [7]:

$$\begin{aligned} x_{k+1} &= a - x_k^2 + by_k, \\ y_{k+1} &= x_k. \end{aligned} \tag{25}$$

It is well known that for $a = a_0 = 1.4$ and $b = 0.3$, the map posses a one-piece chaotic attractor. This map may be stabilized on its 2-UPO within the attractor by tuning the parameter a around the nominal value a_0. This 2-UPO is given by $\{\mathbf{p}_1, \mathbf{p}_2\}$ with $\mathbf{p}_1 \equiv [x^*, y^*]$ and $\mathbf{p}_2 \equiv [y^*, x^*]$, being $x^*(a) = (1 - b + \sqrt{4a - 3(1-b)^2}/2$ and $y^*(a) = (1 - b - \sqrt{4a - 3(1-b)^2})/2$. The implementation of DFC modulation on (25) (confront with Eq. (16)) yields to the system

$$\begin{aligned} &x_{k+1} = \big[a_0 + \gamma_i(x_k - z_k)\big] - x_k^2 + by_k, \\ &y_{k+1} = x_k, \quad z_{k+1} = y_k, \\ &\big\|[x_k, y_k, z_k] - \mathbf{P}_i\big\| \leqslant \frac{\varepsilon}{\sqrt{2}}, \quad i = 1, 2, \end{aligned} \tag{26}$$

where $\mathbf{P}_1 \equiv [x^*, y^*, x^*]$ and $\mathbf{P}_2 \equiv [y^*, x^*, y^*]$ (here $j = 1$ and $\gamma_i \equiv [\gamma_i, 0, 0]$, $i = 1, 2$). The stability condition is obtained from the product-matrix $\mathbf{J} = \mathbf{J}_1\mathbf{J}_2$, where

$$\mathbf{J}_i = \begin{pmatrix} -2x^* - \gamma_i & b & -\gamma_i \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad i = 1, 2,$$

by looking for γ_1 and γ_2 that make the whole eigenvalues of $\mathbf{J}$ be of modulus less than one. Figure 8 shows the 2-UPO stabilization by this strategy. The performance of the control $u_k = \gamma_i(x_k - z_k)$, $i = 1, 2$, is appreciated in the same figure.

5 Controlling hyperchaotic maps. A case study

This methodology results useful for controlling hyperchaos. The applicability of stabilization methods on hyperchaotic systems is pointed out in [9]. Here the successfulness of a DFC-type algorithm for the stabilization of hyperchaotic *generalized Hénon maps* at a fixed point is analyzed. The n-dimensional system is defined [9, 28]:

$$\begin{aligned} x_{1,k+1} &= a - x_{n-1,k}^2 - bx_{n,k}, \\ x_{2,k+1} &= x_{1,k}, \\ &\dots, \\ x_{n,k+1} &= x_{n-1,k}, \end{aligned} \tag{27}$$

where $x_{i,k}$ represents the state i at time k for $i = 1, \dots, n$. Fixing the parameter value b, the map is hyperchaotic for certain parameter range of a since the number of positive Lyapunov exponents is $n - 1$ [28]. It has two fixed points, $\mathbf{x}_{1,2}^* \equiv x_{1,2}^*[1, \dots, 1] \in \mathbb{R}^n$, with $x_{1,2}^* = (-b-1 \pm \sqrt{(b+1)^2 + 4a})/2$, being $\mathbf{x}_1^*$ within the hyperchaotic attractor. To stabilize (27) on it, an ad-hoc reformulation of control (18) is proposed involving $n - 1$ parameters γ_i. The $n = 4$ case is taken for illustration. When control is activated, the system results

$$\begin{aligned} &x_{k+1} = \big[a + \gamma_1(x_k - y_k) + \gamma_2(y_k - z_k) + \gamma_3(z_k - w_k)\big] - z_k^2 - bw_k, \\ &y_{k+1} = x_k, \qquad z_{k+1} = y_k, \qquad w_{k+1} = z_k, \\ &\big\|[x_k, y_k, z_k, w_k] - \mathbf{x}_1^*\big\| \leqslant \frac{\varepsilon}{\sqrt{2}}. \end{aligned} \tag{28}$$

Note that although the number of control parameters is augmented, the algorithm remains not invasive, and the system dimension is not affected. By working out algebraic features of the characteristic polynomial, adequate sets of parameters are found. Formulation and procedure in the general n-case come out straightly. Figure 9 shows stabilization of (27) at the fixed point $\mathbf{x}_1^*$ for $a_0 = 1.76$ and $b = 0.1$. Note that the bigger size of ε and the number of iteration k, needed for the system to achieve the controlled regime, is according to the bigger size of the embedded dimension n. The performance of the control, $u_k = \gamma_1(x_k - y_k) + \gamma_2(y_k - z_k) + \gamma_3(z_k - w_k)$, is displayed in the same figure.

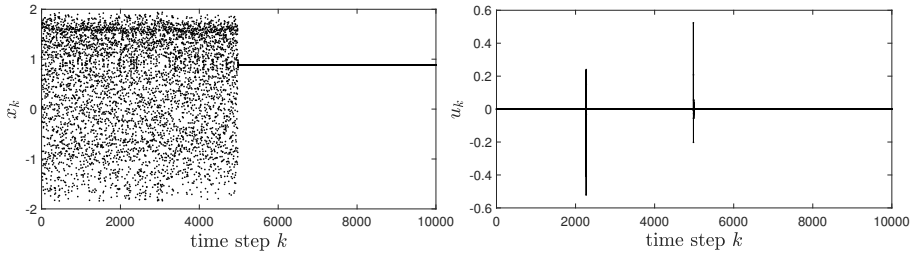


Figure 9. States x_k and controls u_k of Eq. (28), with $\gamma_1 = -1.3896$, $\gamma_2 = -1.8723$, $\gamma_3 = -0.1$, $x_0 = y_0 = z_0 = w_0 = 0.8$, $\varepsilon = 0.1$, applied to stabilize the saddle fixed point $\mathbf{x}^* \equiv x^*[1, 1, 1, 1]$ of Eq. (27) for $a_0 = 1.76$, $b = 0.1$ and $x^* = 0.88614$.

6 Conclusions and discussion

It is well known that the advantage of any DFC method over SPF methods is just that the full knowledge of the UPO to be stabilized is not needed, instead, limitations on convergence issues arise. In this work, both SPF and DFC methods have been revisited, and, based on them, new modifications yielding to a "switching" SPF- and DFC-type strategies have been proposed. Throughout the paper, the introduction of each modification has been fully argued, and the comparison to similar proposals previously published have been mentioned in detail. The convergence of the improved version has been rigorously proved and confronted to the original method without claiming their optimality over all other conceivable approaches. Additionally, control performance aspects, usually disregarded in published works dealing with DFC methods, have been also taken care of.

Each one of the developed control strategies consists of two stages depending on various control parameters. For a m-UPO, the activated control law depends on m control gains, so more data must be stocked for increasing m, but this helps to find a successful control strategy. Under stated hypothesis, UPO stabilization is assured by choosing each control gain within a certain range of values. Among them, it is plausible to look for those values, which make the control strategy verify required conditions on waiting time and control magnitude bound. The right choice of the parameter ε is crucial, and a bound on it is stated for which convergence and desired control magnitude are guaranteed.

Even though the improved DFC does not overcome the ONL, it does work successfully when applied to an UPO presenting high instability with negative Floquet multipliers (and for which DFC fails) if adequate control gains γ_i are found. The insertion of m control gains γ_i together with the parameter control R into the EDFC method is also interesting: for a given UPO, the modified EDFC should require a smaller R than the original EDFC and so yielding to a greater speed of convergence.

The implementation of the proposed control strategies to stabilize any n-dimensional discrete-time systems in any m-UPO is also stated. For generalized Hénon-like systems, an ad-hoc control modulation is designed and applied to stabilize the hyperchaotic motion on a fixed point within its attractor. The hyperchaotic behavior is harder to be controlled than the classical chaos because more than one of the Lyapunov exponents are positive, which means more instability directions for dynamics to spread across the whole phase-space. This fact is critical in the n-D generalized Hénon maps case where the number of such instabilities is just $n-1$ [28], yielding a longer waiting time to achieve the controlled regime as shown in Fig. 9.

The feasibility and applicability of the these novel control modulation methods are all validated by the good simulation results.

Future work aims to introduce this kind of modifications into control schemes based on oscillating or periodic gain control or on predictive states. As the ONL restriction is overcome by these methods, improvements on their convergence and control parameter features result an attracting task.

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