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A study of common fixed points that belong to zeros of a certain given function with applications

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Abstract. In this paper, we establish some point of φ-coincidence and common φ-fixed point results for two self-mappings defined on a metric space via extended $\mathcal{C}_\mathcal{G}$-simulation functions. By giving an example we show that the obtained results are a proper extension of several well-known results in the existing literature. As applications of our results, we deduce some results in partial metric spaces besides proving an existence and uniqueness result on the solution of system of integral equations.

Keywords: point of φ-coincidence, common φ-fixed point, extended $\mathcal{C}_\mathcal{G}$-simulation functions, metric space, partial metric space.

1 Introduction

In 2015, Khojasteh et al. [16] introduced the notion of simulation functions and employ it to unify several fixed point results in the existence literature including Banach contraction principle. Thereafter, several authors studied and extended this notion enlarging such class of auxiliary functions. In this regard, in 2017, Roldán and Samet [10] bring in the concept of an extended simulation function and proved some fixed point results utilizing

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there extended notion. One year later, Liu et al. [19] obtained a new generalization of simulation functions using the class of $\mathcal{C}$-function (the class of $\mathcal{C}$-functions initiated by Ansari [2] in 2014) called $\mathcal{C}_{\mathcal{G}}$-simulation functions. In [31], the author successively extended the fixed point results from the metric setting to the partial metric setting. In [29], the ordered approach was involved to fixed point results. In [30], the author used the fixed point result to solve a first-order periodic differential problem.

Very recently, Chanda et al. [6] bring in the notion of extended $\mathcal{C}_{\mathcal{G}}$-simulation functions, which generalized several notions such as simulation functions, extended simulation functions and $\mathcal{C}_{\mathcal{G}}$-simulation functions.

On the other hand, the notion of φ-fixed point (a fixed point that belongs to the zero set of a given function $\varphi : X \to [0, \infty)$) was introduced by Jleli and Samet [12] to establish some φ-fixed point theorems on a metric space (X, d), which has been used to deduce some fixed point results on partial metric space (X, p).

For more details, we refer the reader to [3, 7, 8, 11, 13–15, 17, 18, 20, 25–28] and references cited therein.

Motivated by the above research work, in this paper, we use the idea of extended $\mathcal{C}_{\mathcal{G}}$-simulation functions to study the existence and uniqueness of point of φ-coincidence and common φ-fixed point for two self-mappings defined on complete metric and partial metric spaces. The obtained results extend and generalize several results as shown in the following diagram:

$$\begin{tikzcd}
\boxed{\text{Our Result}} \arrow[r, "reduce"] \arrow[dd, "reduce"] & \boxed{\text{Radenović et al. [24]}} \arrow[r, "reduce"] & \boxed{\text{Khojasteh et al. [16]}} \arrow[d, "reduce"] \\
 & & \boxed{\text{Banach [4]}} \\
\boxed{\text{Rolden et al. [10]}} \arrow[r, "reduce"] & \boxed{\text{Karapinar et al. [14]}} \arrow[r, "reduce"] & \boxed{\text{Jileli et al. [12]}} \arrow[u, "reduce"]
\end{tikzcd}$$

2 Preliminaries

With a view to have a self-contained presentation, we collect the relevant background material (basic notions, definitions, and fundamental results) starting with the definition of simulation functions, which runs as follows.

Definition 1. (See [16].) A simulation function is a mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ satisfying the following conditions:

(ζ1) $\zeta(0, 0) = 0$,

(ζ2) $\zeta(t, s) < s - t$ for all $t, s > 0$,

(ζ3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Roldan et al. [9] modified Definition 1 in order to enlarge the class of simulation functions by sharping the condition $(\zeta 3)$ as follows:

($\zeta 3'$) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Several examples of simulation functions can be found in [16]. Let us denote by $\mathcal{Z}$ the class of all simulation functions.

Roldán and Samet [10] extended the notion of simulation functions as under.

Definition 2. (See [10].) A function $\xi : [0,\infty)^2 \to \mathbb{R}$ is said to be an extended simulation function if the following conditions hold:

($\xi 1$) $\xi(t,s) < s - t$ for all $t, s > 0$,
($\xi 2$) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$ and $s_n > l$, $n \in \mathbb{N}_0$, then $\limsup_{n\to\infty} \xi(t_n, s_n) < 0$,
($\xi 3$) for any sequence $\{t_n\}$ in $(0,\infty)$, if $\lim_{n\to\infty} t_n = l \geqslant 0$ and $\xi(t_n, l) \geqslant 0$, $n \in \mathbb{N}_0$, then $l = 0$.

Proposition 1. (*See* [10, *Ex.* 2.6].) *Every simulation function is an extended simulation function, but the converse is not true in general.*

For basic examples and more details about extended simulation functions, we refer the reader to [10]. The family of all extended simulation functions will be denoted by $\varepsilon_{\mathcal{Z}}$.

Ansari [2] introduced the family of $\mathcal{C}$-class functions as below.

Definition 3. (See [2].) A continuous function $\mathcal{G} : [0,\infty)^2 \to \mathbb{R}$ is said to be a $\mathcal{C}$-class function if it satisfies the following conditions (for all $t, s \in [0,\infty)$):

(i) $\mathcal{G}(s,t) \leqslant s$,
(ii) $\mathcal{G}(s,t) = s$ implies that either $t = 0$ or $s = 0$.

The family of all $\mathcal{C}$-class functions will be denoted by $\mathcal{C}$.

Definition 4. (See [19].) A function $\mathcal{G} : [0,\infty)^2 \to \mathbb{R}$ has a property $\mathcal{C}_{\mathcal{G}}$ if there exists a constant $\mathcal{C}_{\mathcal{G}} \geqslant 0$ such that

($\mathcal{G}1$) $\mathcal{G}(s,t) > \mathcal{C}_{\mathcal{G}}$ implies $s > t$,
($\mathcal{G}2$) $\mathcal{G}(t,t) \leqslant \mathcal{C}_{\mathcal{G}}$ for all $t \in [0,\infty)$.

Liu et al. [19] defined $\mathcal{C}_{\mathcal{G}}$-simulation functions as follows.

Definition 5. (See [19].) A function $\zeta^* : [0,\infty)^2 \to \mathbb{R}$ is said to be a $\mathcal{C}_{\mathcal{G}}$-simulation function if the following conditions are satisfied:

(ζ^*1) $\zeta^*(0,0) = 0$,
(ζ^*2) $\zeta^*(t,s) < \mathcal{G}(s,t)$ for all $t, s > 0$, where $\mathcal{G} \in \mathcal{C}$ with the property $\mathcal{C}_{\mathcal{G}}$,
(ζ^*3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta^*(t_n, s_n) < \mathcal{C}_{\mathcal{G}}$.

For basic examples of $\mathcal{C}_{\mathcal{G}}$-simulation functions, we refer the reader to [19]. Let us denote by $\mathcal{Z}_{\mathcal{G}}$ the family of all $\mathcal{C}_{\mathcal{G}}$-simulation functions.

Chanda et al. [6] extended the notion of $\mathcal{C}_{\mathcal{G}}$-simulation functions as under.

Definition 6. (See [6].) A function $\theta : [0,\infty)^2 \to \mathbb{R}$ is said to be an extended $\mathcal{C}_\mathcal{G}$-simulation function if the following conditions are hold:

($\theta 1$) $\theta(t,s) < \mathcal{G}(s,t)$ for all $t, s > 0$, where $\mathcal{G} \in \mathcal{C}$ with the property $\mathcal{C}_\mathcal{G}$,
($\theta 2$) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$ and $s_n > l$, $n \in \mathbb{N}_0$, then $\limsup_{n\to\infty} \theta(t_n, s_n) < \mathcal{C}_\mathcal{G}$,
($\theta 3$) for any sequence $\{t_n\}$ in $(0,\infty)$, if $\lim_{n\to\infty} t_n = l \geqslant 0$ and $\theta(t_n, l) \geqslant \mathcal{C}_\mathcal{G}$, $n \in \mathbb{N}_0$, then $l = 0$.

Let us denote by $\varepsilon_{(\mathcal{Z},\mathcal{G})}$ the family of all extended $\mathcal{C}_\mathcal{G}$-simulation functions.

Remark 1. Every simulation function, $\mathcal{C}_\mathcal{G}$-simulation function, an extended simulation function is an extended $\mathcal{C}_\mathcal{G}$-simulation function (see [6, Props. 3.3, 3.4 and 3.5]). The converse is not true in general (see Example 1).

In support of Remark 1, the following example is given in [6].

Example 1. Let $\theta : [0,\infty)^2 \to \mathbb{R}$ be a function defined by

$$\theta(t,s) = \begin{cases} 1 - \frac{t}{2} & \text{if } s = 0, \\ \frac{ks}{1+t} & \text{if } s > 0 \end{cases}$$

for all $t, s \in [0,\infty)$, $k \in [0,1)$, and let $\mathcal{G}(s,t) = s/(1+t)$ with $\mathcal{C}_\mathcal{G} = 1$. Then $\theta \in \varepsilon_{(\mathcal{Z},\mathcal{G})}$, but θ does not belong to $\mathcal{Z}$, $\mathcal{Z}_\mathcal{G}$, and $\varepsilon_\mathcal{Z}$.

In the present paper, X is a nonempty set, and the following notions are used:

- $\mathrm{Fix}(T) = \{x \in X\colon Tx = x\}$,
- $\mathrm{Pcoin}(T,S) = \{x \in X\colon x = Tv = Sv$ for some $v \in X\}$,
- $\mathrm{Com}(T,S) = \{x \in X\colon x = Tx = Sx\}$,
- $Z_\varphi = \{x \in X\colon \varphi(x) = 0,$ where $\varphi : X \to [0,\infty)$ is a given function$\}$.

Now, we present the notion of φ-fixed point, which runs as follows.

Definition 7. (See [12].) Let T be a self-mapping on X and $\varphi : X \to [0,\infty)$ a given function. An element $x \in X$ is said to be φ-fixed point of T if and only if it is a fixed point of T and $\varphi(x) = 0$, that is, $x \in \mathrm{Fix}(T) \cap Z_\varphi$.

Let T and S be two self-mappings defined on X.

- A sequence $\{x_n\} \subseteq X$ is called a *Picard–Jungck sequence* of T and S based on x_0 if $Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$.
- T and S are said to be weakly compatible if they are commute at their coincidence points, that is, $TSx = STx$ for all $x \in X$ such that $Tx = Sx$.

Proposition 2. (*See* [1].) *Let T and S be two weakly compatible self-mappings defined on X. If T and S have a unique point of coincidence u, then u is a unique common fixed point of T and S.*

Let $\mathcal{F}$ be the set of all functions $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following conditions for all $a, b, c \in [0,\infty)$:

(F1) $\max\{a,b\} \leqslant F(a,b,c)$,
(F2) $F(a,0,0) = a$,
(F3) F is continuous.

The following functions $F : [0,\infty)^3 \to [0,\infty)$ belong to $\mathcal{F}$:

1. $F(a,b,c) = a + b + c$,
2. $F(a,b,c) = \max\{a,b\} + c$,
3. $F(a,b,c) = (a+b)(c+1)^n$, $n = 0, 1, 2, \dots$.

3 Main results

At the beginning of this section, we define the notions of point of φ-coincidence and common φ-fixed point of the self-mappings T and S defined on a nonempty set X.

Definition 8. Let S and T be two self-mapping on X, and let $\varphi : X \to [0,\infty)$ be a given function. An element z in X is said to be

- point of φ-coincidence of T and S if and only if it is a point of coincidence of T and S and $\varphi(z) = 0$, that is, $z \in \operatorname{Pcoin}(T,S) \cap Z_\varphi$;
- common φ-fixed point of T and S if and only if it is a common fixed point of T and S such that $\varphi(z) = 0$, that is, $z \in \operatorname{Com}(T,S) \cap Z_\varphi$.

Now, we prove the following proposition.

Proposition 3. *Let T and S be two weakly compatible self-mappings defined on X. Suppose that T and S have unique point of φ-coincidence u, then u is a unique common φ-fixed point of T and S.*

Proof. Suppose that u is a unique point of φ-coincidence of the mappings T and S, that is, u is a unique point of coincidence of T and S with $\varphi(u) = 0$. Then it follows from Proposition 2 that u is a unique common fixed point of the mappings T and S and hence a unique common φ-fixed point (as $\varphi(u) = 0$). □

Let (X,d) be a metric space. For a given three functions $F \in \mathcal{F}$, $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$, and $\varphi : X \to [0,\infty)$, we consider the self-mappings $T, S : X \to X$ that satisfy the following contractive condition:

$$\theta\big(F\big(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\big), M_F^\varphi(x,y)\big) \geqslant \mathcal{C}_\mathcal{G} \tag{1}$$

for all $x, y \in X$ such that $Sx \neq Sy$, where

$$M_F^\varphi(x,y) = \max\big\{F\big(d(Sx,Sy),\varphi(Sx),\varphi(Sy)\big), F\big(d(Sx,Tx),\varphi(Sx),\varphi(Tx)\big),\\ F\big(d(Sy,Ty),\varphi(Sy),\varphi(Ty)\big)\big\}.$$

Before formulating our main results, we prove some auxiliary results as under.

Lemma 1. *Let T and S be two self-mappings defined on a metric space (X, d). Assume that there exist three functions $F \in \mathcal{F}$, $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$, and $\varphi : X \to [0, \infty)$ such that* (1) *holds. If $\{x_n\}_{n\in\mathbb{N}_0}$ is a Picard–Jungck sequence of the pair (S, T) based on $x_0 \in X$ such that $Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N}$, then*

(i) $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = \lim_{n\to\infty} \varphi(Sx_n) = 0$,
(ii) $\{Sx_n\}$ *is a Cauchy sequence.*

Proof. Let $x_0 \in X$ be an arbitrary point and $\{x_n\}_{n\in\mathbb{N}_0}$ be the Picard–Jungck sequence of the pair (T, S) based on x_0, that is, $Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N}_0$. Assume that $Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N}$.

(i) In view of (F1), we have

$$F\big(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\big) \geqslant d(Sx_n, Sx_{n+1}) > 0 \text{ for all } n \in \mathbb{N}.$$

Now, we show that $M_F^{\varphi}(x_n, x_{n+1}) > 0$. For simplicity, let $a_n = F(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1}))$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} M_F^{\varphi}(x_n, x_{n+1}) &= \max\big\{F\big(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\big), \\ &\qquad F\big(d(Sx_n, Tx_n), \varphi(Sx_n)\varphi(Tx_n)\big), \\ &\qquad F\big(d(Sx_{n+1}, Tx_{n+1}), \varphi(Sx_{n+1}), \varphi(Tx_{n+1})\big)\big\} \\ &= \max\big\{F\big(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\big), \\ &\qquad F\big(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\big), \\ &\qquad F\big(d(Sx_{n+1}, Sx_{n+2}), \varphi(Sx_{n+1}), \varphi(Sx_{n+2})\big)\big\} \\ &= \max\{a_n, a_n, a_{n+1}\} = \max\{a_n, a_{n+1}\} > 0. \end{aligned}$$

Setting $x = x_n$ and $y = x_{n+1}$ for all $n \in \mathbb{N}$ in (1) and utilizing (θ1), we get

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} &\leqslant \theta\big(F\big(d(Tx_n, Tx_{n+1}), \varphi(Tx_n), \varphi(Tx_{n+1})\big), M_F^{\varphi}(x_n, x_{n+1})\big) \\ &= \theta\big(F\big(d(Sx_{n+1}, Sx_{n+2}), \varphi(Sx_{n+1}), \varphi(Sx_{n+2})\big), M_F^{\varphi}(x_n, x_{n+1})\big) \\ &= \theta\big(a_{n+1}, \max\{a_n, a_{n+1}\}\big) < \mathcal{G}\big(\max\{a_n, a_{n+1}\}, a_{n+1}\big), \end{aligned}$$

which follows from ($\mathcal{G}$1) that $a_{n+1} < \max\{a_n, a_{n+1}\}$, that is, $a_{n+1} < a_n$ for all $n \in \mathbb{N}$. This implies that the sequence of real numbers $\{a_n\}$ is decreasing and bounded below by zero. Therefore, there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} a_n = r.$$

Our claim is $r = 0$. On the contrary, suppose that $r > 0$ and consider two sequences

$$t_n = F\big(d(Sx_{n+1}, Sx_{n+2}), \varphi(Sx_{n+1}), \varphi(Sx_{n+2})\big) = a_{n+1}$$

and

$$s_n = M_F^{\varphi}(x_n, x_{n+1}) = \max\{a_n, a_{n+1}\} = a_n$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = r$. As $\{a_n\}$ is strictly decreasing, then $r < a_n = s_n$ for all $n \in \mathbb{N}$, and hence, condition ($\theta 2$) implies that

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leqslant \limsup_{n\to\infty} \theta\bigl(F\bigl(d(Sx_{n+1}, Sx_{n+2}), \varphi(Sx_{n+1}), \varphi(Sx_{n+2})\bigr), M_F^{\varphi}(x_n, x_{n+1})\bigr) \\ &< \mathcal{C}_{\mathcal{G}},\end{aligned}$$

which is a contradiction. So, we conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} F\bigl(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\bigr) = 0. \tag{2}$$

Using condition (F1), we have

$$0 < d(Sx_n, Sx_{n+1}) \leqslant F\bigl(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\bigr)$$

and

$$0 < \varphi(Sx_n) \leqslant F\bigl(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\bigr).$$

Letting $n \to \infty$ in the above two inequalities and using (2), we deduce that

$$\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = \lim_{n\to\infty} \varphi(Sx_n) = 0.$$

(ii) Let us assume that the sequence $\{Sx_n\}$ is not Cauchy. Then (due to Lemma 13 of [5]) there exist $\epsilon > 0$ and two subsequences $\{Sx_{n_k}\}$ and $\{Sx_{m_k}\}$ of $\{Sx_n\}$ with $m_k > n_k \geqslant k$ for all $k \in \mathbb{N}$ such that

$$d(Sx_{n_k}, Sx_{m_k}) > \epsilon \quad \text{and} \quad d(Sx_{n_k}, Sx_{m_k-1}) \leqslant \epsilon \tag{3}$$

with

$$\lim_{n\to\infty} d(Sx_{n_k}, Sx_{m_k}) = \lim_{n\to\infty} d(Sx_{n_k+1}, Sx_{m_k+1}) = \epsilon. \tag{4}$$

Let $t_k = F(d(Sx_{n_k+1}, Sx_{m_k+1}), \varphi(Sx_{n_k+1}), \varphi(Sx_{m_k+1}))$ and $s_k = M_F^{\varphi}(x_{n_k}, x_{m_k})$ for all $k \in \mathbb{N}$. Using (3), (4), (F2), part (i) of Lemma 1 and the continuity of F, one easily can show that

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = F(\epsilon, 0, 0) = \epsilon = l > 0.$$

Making use of (F1), we have

$$\begin{aligned}s_k = M_F^{\varphi}(x_{n_k}, x_{m_k}) &\geqslant F\bigl(d(Sx_{n_k}, Sx_{m_k}), \varphi(Sx_{n_k}), \varphi(Sx_{m_k})\bigr) \\ &\geqslant d(Sx_{n_k}, Sx_{m_k}) > \epsilon = l\end{aligned}$$

for all $k \in \mathbb{N}$. Applying ($\theta 2$), we obtain

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leqslant \limsup_{n\to\infty} \theta\bigl(F\bigl(d(Sx_{n_{k+1}}, Sx_{m_{k+1}}), \varphi(Sx_{n_{k+1}}), \varphi(Sx_{m_{k+1}})\bigr), M_F^{\varphi}(x_{n_k}, x_{m_k})\bigr) \\ &< \mathcal{C}_{\mathcal{G}},\end{aligned}$$

which is a contradiction. Hence, we must have that $\{Sx_n\}$ is a Cauchy sequence. $\square$

Lemma 2. *Let T and S be two self-mappings defined on a metric space (X,d). Assume that there exist three functions $F \in \mathcal{F}$, $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$, and $\varphi : X \to [0,\infty)$ such that* (1) *holds. Then the point of φ-coincidence of T and S is unique, provided it exists.*

Proof. For the seek of contradiction, we suppose that T and S have two distinct points of φ-coincidence u and v, that is, $Sw = Tw = u \neq v = Tz = Sz$ for some $z, w \in X$ and $\varphi(u) = \varphi(v) = 0$. In view of (F2), we have

$$F\bigl(d(Tz,Tw),\varphi(Tz),\varphi(Tw)\bigr) = F\bigl(d(v,u),0,0\bigr) = d(v,u) > 0.$$

Again, in view of (F2), we also have $M_F^{\varphi}(z,w) > 0$, in fact,

$$\begin{aligned} M_F^{\varphi}(z,w) &= \max\bigl\{F\bigl(d(Sz,Sw),\varphi(Sz),\varphi(Sw)\bigr), F\bigl(d(Sz,Tz),\varphi(Sz),\varphi(Tz)\bigr),\\ &\qquad F\bigl(d(Sw,Tw),\varphi(Sw),\varphi(Tw)\bigr)\bigr\}\\ &= \max\bigl\{F\bigl(d(v,u),0,0\bigr), F(0,0,0), F(0,0,0)\bigr\}\\ &= F\bigl(d(v,u),0,0\bigr) = d(v,u) > 0. \end{aligned}$$

Setting $x = z$ and $y = w$ in (1) and utilizing (θ1) and ($\mathcal{G}$2), we get

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} &\leqslant \theta\bigl(F\bigl(d(Tz,Tw),\varphi(Tz),\varphi(Tw)\bigr), M_F^{\varphi}(z,w)\bigr)\\ &= \theta\bigl(d(v,u),d(v,u)\bigr) < \mathcal{G}\bigl(d(v,u),d(v,u)\bigr) \leqslant \mathcal{C}_{\mathcal{G}}, \end{aligned}$$

which is a contradiction. Hence, the point of φ-coincidence of T and S is unique. □

Now, we are equipped to state and prove our main results starting with the following one.

Theorem 1. *Let T and S be two self-mappings defined on a metric space (X,d). Suppose that there exists a Picard–Jungck sequence $\{x_n\}_{n\in\mathbb{N}_0}$ of T and S, and the following conditions are satisfied*:

(i) *there exist $F \in \mathcal{F}$, $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$ and a lower semicontinuous function $\varphi : X \to [0,\infty)$ such that* (1) *holds,*
(ii) (SX,d) (*or* (TX,d)) *is complete.*

Then

(a) $\mathrm{Pcoin}(T,S) \subseteq Z_{\varphi}$, *and the pair (T,S) has a unique point of φ-coincidence.*
(b) $\mathrm{Com}(T,S) \subseteq Z_{\varphi}$. *Moreover, if (T,S) is weakly compatible pair, then it has a unique common φ-fixed point.*

Proof. (a) Firstly, we show that $\mathrm{Pcoin}(T,S) \subseteq Z_{\varphi}$. To do so, let $u \in \mathrm{Pcoin}(T,S)$, that is, $u = Tz = Sz$ for some $z \in X$. Since

$$\begin{aligned} M_F^{\varphi}(z,z) &= \max\bigl\{F\bigl(d(Sz,Sz),\varphi(Sz),\varphi(Sz)\bigr), F\bigl(d(Sz,Tz),\varphi(Sz),\varphi(Tz)\bigr),\\ &\qquad F\bigl(d(Sz,Tz),\varphi(Sz),\varphi(Tz)\bigr)\bigr\}\\ &= \max\bigl\{F\bigl(0,\varphi(u),\varphi(u)\bigr), F\bigl(0,\varphi(u),\varphi(u)\bigr), F\bigl(0,\varphi(u),\varphi(u)\bigr)\bigr\}\\ &= F\bigl(0,\varphi(u),\varphi(u)\bigr), \end{aligned}$$

therefore, on using (1) with $x = y = z$, we get

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leqslant \theta\bigl(F\bigl(d(Tz,Tz),\varphi(Tz),\varphi(Tz)\bigr),M_F^{\varphi}(z,z)\bigr)\\ &= \theta\bigl(F\bigl(0,\varphi(u),\varphi(u)\bigr),F\bigl(0,\varphi(u),\varphi(u)\bigr)\bigr).\end{aligned} \tag{5}$$

Now, we claim that $F(0,\varphi(u),\varphi(u)) = 0$. On contrary, let $F(0,\varphi(u),\varphi(u)) > 0$. In view of (5), ($\theta 1$), and ($\mathcal{G}2$), we have

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leqslant \theta\bigl(F\bigl(0,\varphi(u),\varphi(u)\bigr),F\bigl(0,\varphi(u),\varphi(u)\bigr)\bigr)\\ &< \mathcal{G}\bigl(F\bigl(0,\varphi(u),\varphi(u)\bigr),F\bigl(0,\varphi(u),\varphi(u)\bigr)\bigr) \leqslant \mathcal{C}_{\mathcal{G}},\end{aligned}$$

a contradiction. Therefore, we must have $F(0,\varphi(u),\varphi(u)) = 0$. Now, employing $(F1)$, we obtain

$$\varphi(u) \leqslant \max\bigl\{0,\varphi(u)\bigr\} \leqslant F\bigl(0,\varphi(u),\varphi(u)\bigr) = 0,$$

which implies that $\varphi(u) = 0$, and hence, $\mathrm{Pcoin}(T,S) \subseteq Z_{\varphi}$.

Secondly, we show that T and S have a point of φ-coincidence. Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}_{n\in\mathbb{N}}$ be the Picard–Jungck sequence of T and S based at x_0, that is, $Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. If $d(Sx_{n_0}, Sx_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a coincidence point of T and S. Therefore, T and S have a point of coincidence and hence a point of φ-coincidence (as $\mathrm{Pcoin}(T,S) \subseteq Z_{\varphi}$), which is unique (due to Lemma 2). Now, suppose that $d(Sx_n, Sx_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then by Lemma 1, the sequence $\{Sx_n\}$ is Cauchy. Assume that (SX,d) is complete, then there exists $u = Sz \in SX$ (for some $z \in X$) such that

$$\lim_{n\to\infty} d(Sx_n, Sz) = 0. \tag{6}$$

Since φ is lower semicontinuous, therefore, in view of (6) and part (i) of Lemma 1, we have

$$0 \leqslant \varphi(Sz) = \varphi(u) \leqslant \liminf_{n\to\infty} \varphi(Sx_n) = 0,$$

which implies that

$$\varphi(Sz) = \varphi(u) = 0. \tag{7}$$

Now, we prove that u is a point of φ-coincidence. On contrary, assume that u is not a point of φ-coincidence for (T,S). We distinguish the following two cases:

Case 1. Assume that $u = Sz \neq Tz$ for all $z \in X$. Let $l = F(d(Sz,Tz),0,\varphi(Tz))$ and $t_n = F(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz))$ for all $n \in \mathbb{N}_0$. Then, in view of (F1), we have

$$l = F\bigl(d(Sz,Tz),0,\varphi(Tz)\bigr) \geqslant d(Sz,Tz) > 0. \tag{8}$$

Using the continuity of F, (6), and part (i) of Lemma 1, we have

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} F\bigl(d(Sx_{n+1},Tz),\varphi(Sx_{n+1}),\varphi(Tz)\bigr) = l.$$

Observe that

$$\begin{aligned} M_F^{\varphi}(x_n, z) &= \max\bigl\{F\bigl(d(Sx_n, Sz), \varphi(Sx_n), \varphi(Sz)\bigr), \\ &\qquad F\bigl(d(Sx_n, Tx_n), \varphi(Sx_n), \varphi(Tx_n)\bigr), \\ &\qquad F\bigl(d(Sz, Tz), \varphi(Sz), \varphi(Tz)\bigr)\bigr\} \\ &= \max\bigl\{F\bigl(d(Sx_n, Sz), \varphi(Sx_n), 0\bigr), \\ &\qquad F\bigl(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\bigr), \\ &\qquad F\bigl(d(Sz, Tz), 0, \varphi(Tz)\bigr)\bigr\}. \end{aligned}$$

Owing to the continuity of F, we get

$$\begin{aligned} \lim_{n\to\infty} F\bigl(d(Sx_n, Sz), \varphi(Sx_n), 0\bigr) = F(0,0,0) = 0, \\ \lim_{n\to\infty} F\bigl(d(Sx_n, Sx_{n+1}), \varphi(Sx_n), \varphi(Sx_{n+1})\bigr) = F(0,0,0) = 0. \end{aligned}$$

As a consequence, we can find $n_0 \in \mathbb{N}$ such that

$$M_F^{\varphi}(x_n, z) = F\bigl(d(Sz, Tz), 0, \varphi(Tz)\bigr) = l \quad \text{for all } n \geqslant n_0. \tag{9}$$

Therefore, using (1), (9), and ($\theta 3$), we obtain (for all $n \in \mathbb{N}$ with $n \geqslant n_0$)

$$\begin{aligned} &\theta(t_n, l) = \theta\bigl(F\bigl(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)\bigr), M_F^{\varphi}(x_n, z)\bigr) \geqslant \mathcal{C}_{\mathcal{G}} \\ &\quad\Longrightarrow \quad l = 0 \quad \Longrightarrow \quad F\bigl(d(Sz, Tz), 0, \varphi(Tz)\bigr) = 0, \end{aligned}$$

which contradicts (8). Therefore, u must be a point of φ-coincidence of the pair (T, S).

Case 2. Assume that $\varphi(u) \neq 0$. This assumption contradicts Eq. (7). Therefore, again u must be a point of φ-coincidence of the pair (T, S).

Similarly, if we assume that (TX, d) is complete, then we again reach to a contradiction. Therefore, these contradictions in all cases show that u is a point of φ-coincidence of T and S, which is unique (due to Lemma 2).

(b) Following a similar argument used in part (a), one can easily prove that $\mathrm{Com}(T, S) \subseteq Z_{\varphi}$. Now, as T and S are weakly compatible mappings, in view of Lemma 2 and Proposition 3, the mappings T and S have a unique common φ-fixed point. This completes the proof. □

For a given three functions $F \in \mathcal{F}$, $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$, and $\varphi : X \to [0, \infty)$, let the contractive condition (1) in Theorem 1 be replaced by the following one:

$$\theta\bigl(F\bigl(d(Tx, Ty), \varphi(Tx), \varphi(Ty)\bigr), F\bigl(d(Sx, Sy), \varphi(Sx), \varphi(Sy)\bigr)\bigr) \geqslant \mathcal{C}_{\mathcal{G}} \tag{10}$$

for all $x, y \in X$ such that $Sx \neq Sy$. Then the proof of the following theorem is similar and much easier than that in the proof of Theorem 1, so the proof is omitted. Notice that there is no direct relation between these theorems as the extended $\mathcal{C}_{\mathcal{G}}$-simulation function need not be monotone in its second argument.

Theorem 2. *Let T and S be two self-mappings defined on a metric space (X,d). Assume that there exists a Picard–Jungck sequence $\{x_n\}_{n\in\mathbb{N}}$ of T and S, and the following conditions are satisfied*:

(i) *there exist $F\in\mathcal{F}$, $\zeta\in\mathcal{E}_{(\mathcal{Z},\mathcal{G})}$ and a lower semicontinuous function $\varphi: X\to[0,\infty)$ such that* (10) *holds,*

(ii) (SX,d) (*or* (TX,d)) *is complete.*

Then

(a) $\mathrm{Pcoin}(T,S)\subseteq Z_\varphi$, *and the pair (T,S) has a unique point of φ-coincidence.*

(b) $\mathrm{Com}(T,S)\subseteq Z_\varphi$. *Moreover, if (T,S) is weakly compatible pair, then it has a unique common φ-fixed point.*

The following example shows that Theorem 1 is a genuine extension of [24, Thm. 2.2] and [10, Thm. 3.1].

Example 2. Consider the metric space (l^∞,d), where l^∞ is the space of all bounded sequences of complex numbers, and d is defined by

$$d(x,y)=\sup_{i\in\mathbb{N}}\bigl|x(i)-y(i)\bigr|\quad\text{for all } x,y\in l^\infty.$$

Let $X=\{e_0,e_i,\, i\in\mathbb{N}\}$, where e_0 is the zero sequence, and e_i is the sequence whose ith term equals to 4 and all other terms are zeros. It is clear that the pair (X,d) is a complete metric space. Define two mappings $T,S: X\to X$ by

$$Tx=\begin{cases} e_0 & \text{if } x=e_0,e_1,\\ e_2 & \text{otherwise}\end{cases}\qquad\text{and}\qquad Sx=\begin{cases} e_0 & \text{if } x=e_0,\\ e_{i+1} & \text{if } x=e_i.\end{cases}$$

First, we show that [24, Thm. 2.3] is not applicable in this example. In fact, on contrary, assume that there exists $\zeta^*\in\mathcal{Z}_\mathcal{G}$ such that $\zeta^*(d(Tx,Ty),d(Sx,Sy))\geqslant\mathcal{C}_\mathcal{G}$ for all $x,y\in X$ such that $Sx\neq Sy$ with $\mathcal{C}_\mathcal{G}\geqslant 0$. Then, taking $x=e_0$, $y=e_2$ and using (ζ^*2) and $(\mathcal{G}2)$, we obtain

$$\mathcal{C}_\mathcal{G}\leqslant\zeta^*\bigl(d(Te_0,Te_2),d(Se_0,Se_2)\bigr)=\zeta^*(4,4)<\mathcal{G}(4,4)\leqslant\mathcal{C}_\mathcal{G},$$

which is a contradiction. This contradiction ensures that there is no $\zeta^*\in\mathcal{Z}_\mathcal{G}$ such that $\zeta^*(d(Tx,Ty),d(Sx,Sy))\geqslant\mathcal{C}_\mathcal{G}$. Therefore, [24, Thm. 2.3] is not applicable.

Now, to show the applicability of Theorem 1, we define two essential functions $\varphi: X\to[0,\infty]$ and $F:[0,\infty)\to[0,\infty)$ by

$$\varphi(x)=\begin{cases} 0 & \text{if } x=e_0,\\ 10 & \text{otherwise}\end{cases}\qquad\text{and}\qquad F(a,b,c)=a+b+c\quad\text{for all } a,b,c\in[0,\infty).$$

It is easy to see that $F\in\mathcal{F}$, and φ is a lower semicontinuous function.

Now, consider the extended $\mathcal{C}_\mathcal{G}$-simulation function $\theta:[0,\infty)^2\to\mathbb{R}$ given by

$$\theta(t,s)=\begin{cases} 1-\frac{t}{2} & \text{if } s=0,\\ \frac{ks}{1+t} & \text{if } s>0\end{cases}\qquad\text{with } \mathcal{C}_\mathcal{G}=1 \text{ and } k=\frac{7}{8}.$$

We have to prove that the contractive condition (1) holds for all $x, y \in X$ such that $x \neq y$. For this purpose, we consider three cases:

Case 1. If $x = e_0$ and $y = e_1$, then $F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) = 0$ and $M_F^{\varphi}(x,y) = 14$, and hence, we have

$$\begin{aligned}&\theta\big(F\big(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\big), M_F^{\varphi}(x,y)\big)\\&\quad= \frac{kM_F^{\varphi}(x,y)}{1+F(d(Tx,Ty),\varphi(Tx),\varphi(Ty))} = \frac{49}{4} \geqslant 1 = \mathcal{C}_{\mathcal{G}}.\end{aligned}$$

Case 2. If $x \in \{e_0, e_1\}$ and $y \in X - \{e_0, e_1\}$, then $F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) = 14$ and $M_F^{\varphi}(x,y) = 20$, and hence, we have

$$\begin{aligned}&\theta\big(F\big(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\big), M_F^{\varphi}(x,y)\big)\\&\quad= \frac{kM_F^{\varphi}(x,y)}{1+F(d(Tx,Ty),\varphi(Tx),\varphi(Ty))} = \frac{7}{6} \geqslant 1 = \mathcal{C}_{\mathcal{G}}.\end{aligned}$$

Case 3. If $x, y \in X - \{e_0, e_1\}$, then $F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) = 20$ and $M_F^{\varphi}(x,y) = 24$, and hence, we have

$$\begin{aligned}&\theta\big(F\big(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\big), M_F^{\varphi}(x,y)\big)\\&\quad= \frac{kM_F^{\varphi}(x,y)}{1+F(d(Tx,Ty),\varphi(Tx),\varphi(Ty))} = 1 = \mathcal{C}_{\mathcal{G}}.\end{aligned}$$

Therefore, in all cases, the contractive condition (1) is satisfied. Also, observe that T and S are weakly compatible and TX is complete subspace of X. Hence, all the hypotheses of Theorem 1 are satisfied, and consequently, the mappings T and S have a unique common fixed point (namely, $x = e_0$).

As consequences of our newly proved results, we deduce several corollaries, which can be viewed as generalizations of various results in the existing literature.

Putting $S = I_X$, the identity mapping on X, in Theorems 1 and 2 and taking to the account that every $\mathcal{C}_{\mathcal{G}}$-simulation function is an extended $\mathcal{C}_{\mathcal{G}}$-simulation function, we deduce the following two corollaries, which seem to be new to the existing literature.

Corollary 1. *Let T be a self-mapping defined on a metric space (X, d). Suppose that there exist $F \in \mathcal{F}$, $\theta \in \mathcal{Z}_{\mathcal{G}}$ (or $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$), and a lower semicontinuous function $\varphi : X \to [0, \infty)$ such that*

$$\theta\big(F\big(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\big), N_F^{\varphi}(x,y)\big) \geqslant \mathcal{C}_{\mathcal{G}} \quad \textit{for all } x, y \in X,$$

where

$$\begin{aligned}N_F^{\varphi}(x,y) = \max\big\{&F\big(d(x,y),\varphi(x),\varphi(y)\big), F\big(d(x,Tx),\varphi(x),\varphi(Tx)\big),\\&F\big(d(y,Ty),\varphi(y),\varphi(Ty)\big)\big\}.\end{aligned}$$

Then $\mathrm{Fix}(T) \subseteq Z_{\varphi}$ *and T has a unique φ-fixed point.*

Corollary 2. *Let T be a self-mapping on a metric space (X,d). Suppose that there exist $F \in \mathcal{F}$, $\theta \in \mathcal{Z}_{\mathcal{G}}$ (or $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$), and a lower semicontinuous function $\varphi : X \to [0,\infty)$ such that*

$$\theta\bigl(F\bigl(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\bigr),F\bigl(d(x,y),\varphi(x),\varphi(y)\bigr)\bigr) \geqslant \mathcal{C}_{\mathcal{G}} \quad \textit{for all } x,y \in X.$$

Then $\mathrm{Fix}(T) \subseteq Z_\varphi$, *and T has a unique φ-fixed point.*

Since every simulation function (also, extended simulation function) is an extended $\mathcal{C}_{\mathcal{G}}$-simulation function, then from Theorems 1 and 2 we deduce the following two corollaries, which also seem to be new to the existing literature.

Corollary 3. *Let T and S be two self-mappings defined on a metric space (X,d). Assume that there exists a Picard–Jungck sequence $\{x_n\}_{n\in\mathbb{N}}$ of T and S, and the following conditions are satisfied*:

(i) *there exist $F \in \mathcal{F}$, $\theta \in \mathcal{Z}$ (or $\theta \in \mathcal{E}_{\mathcal{Z}}$) and a lower semicontinuous function $\varphi : X \to [0,\infty)$ such that*

$$\theta\bigl(F\bigl(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\bigr),M_F^\varphi(x,y)\bigr) \geqslant 0 \quad \textit{for all } x,y \in X,$$

where

$$\begin{aligned} M_F^\varphi(x,y) = \max\bigl\{&F\bigl(d(Sx,Sy),\varphi(Sx),\varphi(Sy)\bigr),\\ &F\bigl(d(Sx,Tx),\varphi(Sx),\varphi(Tx)\bigr),\\ &F\bigl(d(Sy,Ty),\varphi(Sy),\varphi(Ty)\bigr)\bigr\},\end{aligned}$$

(ii) (SX,d) (*or* (TX,d)) *is complete.*

Then

(a) $\mathrm{Pcoin}(T,S) \subseteq Z_\varphi$, *and the pair (T,S) has a unique point of φ-coincidence.*

(b) $\mathrm{Com}(T,S) \subseteq Z_\varphi$. *Moreover, if (T,S) is weakly compatible pair, then it has a unique common φ-fixed point.*

Corollary 4. *Let T and S be two self-mappings defined on a metric space (X,d). Assume that there exists a Picard–Jungck sequence $\{x_n\}_{n\in\mathbb{N}}$ of T and S, and the following conditions are satisfied*:

(i) *there exist $F \in \mathcal{F}$, $\theta \in \mathcal{Z}$ (or $\theta \in \mathcal{E}_{\mathcal{Z}}$), and a lower semicontinuous function $\varphi : X \to [0,\infty)$ such that*

$$\begin{aligned}&\theta\bigl(F\bigl(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\bigr),F\bigl(d(Sx,Sy),\varphi(Sx),\varphi(Sy)\bigr)\bigr)\\ &\quad\geqslant 0 \quad \textit{for all } x,y \in X,\end{aligned}$$

(ii) (SX,d) (*or* (TX,d)) *is complete.*

Then

(a) $\mathrm{Pcoin}(T,S) \subseteq Z_\varphi$, *and the pair (T,S) has a unique point of φ-coincidence.*

(b) $\mathrm{Com}(T,S) \subseteq Z_\varphi$. *Moreover, if (T,S) is weakly compatible pair, then it has a unique common φ-fixed point.*

4 Applications

In this section, we employ our main results obtained in metric spaces (Theorems 1 and 2) to deduce some related results in partial metric spaces besides proving an existence and uniqueness result on the solution of system of functional equations.

4.1 Application to partial metric spaces

In 1994, Matthews [21] introduced the notion of partial metric spaces as below.

Definition 9. (See [21].) Let X be a nonempty set. A partial metric is a mapping $p : X \times X \to [0, \infty)$ satisfying the following conditions:

(P1) $p(x, x) = p(y, y) = p(x, y) \Leftrightarrow x = y$,
(P2) $p(x, x) \leqslant p(x, y)$,
(P3) $p(x, y) = p(y, x)$,
(P4) $p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$

for all $x, y, z \in X$. The pair (X, p) is called a partial metric space.

Observe that, in the setting of partial metric spaces, the distance from a point to itself need not to be zero.

In the following definition, we present some well-known basic notions related to partial metric spaces.

Definition 10. (See [21].) Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in X is called convergent and converges to x in X if $p(x, x) = \lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(x_n, x)$.
(ii) A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) A partial metric space (X, p) is called a complete partial metric space if every Cauchy sequence in X converges to a point x in X such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

For a partial metric p on a nonempty set X, the function $d_p : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad \text{for all } x, y \in X \tag{11}$$

remains a standard metric on X.

Lemma 3. (*see* [21, 23].) *Let (X, p) be a partial metric space. Then*

(i) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p).*
(ii) *If the metric space (X, d_p) is complete, then the partial metric space (X, p) is also complete and vice versa. Furthermore, $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.*

Lemma 4. (*see* [22].) *Let (X, p) be a partial metric space, and let $\varphi : X \to [0, \infty)$ a function defined by $\varphi(x) = p(x, x)$ for all $x \in X$. Then φ is lower semicontinuous in (X, d_p).*

From Theorem 1 we deduce the following fixed point result in the setting of partial metric spaces.

Theorem 3. *Let T and S be two self-mappings defined on a partial metric space (X, p). Assume that there exists a Picard–Jungck sequence $\{x_n\}_{n\in\mathbb{N}}$ of T and S, and the following conditions are satisfied*:

(i) *there exists a function $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$ such that*

$$\begin{aligned}&\theta\bigl(p(Tx, Ty), \max\bigl\{p(Sx, Sy), p(Sx, Tx), p(Sy, Ty)\bigr\}\bigr)\\ &\quad\geqslant \mathcal{C}_{\mathcal{G}} \quad \text{for all } x, y \in X,\end{aligned} \tag{12}$$

(ii) (SX, p) (*or* (TX, p)) *is complete.*

Then T and S have a unique point of coincidence u. Moreover, if T and S are weakly compatible, then u is a unique common fixed point with $p(u, u) = 0$.

Proof. Consider the metric d^* on X defined as

$$d^* = \frac{d_p}{2}, \tag{13}$$

where d_p is given in (11). Due to Lemma 3, (X, d^*) forms a complete metric space. Define two functions $F : [0, \infty)^3 \to [0, \infty)$ and $\varphi : X \to [0, \infty)$ by

$$F(a, b, c) = a + b + c \quad \text{and} \quad \varphi(u) = \frac{p(u, u)}{2}. \tag{14}$$

Observe that φ is lower semicontinuous (due to Lemma 4) and $F \in \mathcal{F}$.

Now, using (13) and (14) in (12), we get

$$\theta\bigl(F\bigl(d^*(Tx, Ty), \varphi(Tx), \varphi(Ty)\bigr), M_F^\varphi(x, y)\bigr) \geqslant \mathcal{C}_{\mathcal{G}} \quad \text{for all } u, v \in X,$$

where

$$\begin{aligned}M_F^\varphi(x, y) = \max\bigl\{&F\bigl(d^*(Sx, Sy), \varphi(Sx), \varphi(Sy)\bigr),\\ &F\bigl(d^*(Sx, Tx), \varphi(Sx), \varphi(Tx)\bigr),\\ &F\bigl(d^*(Sy, Ty), \varphi(Sy), \varphi(Ty)\bigr)\bigr\}.\end{aligned}$$

Therefore, all the hypotheses of Theorem 1 are satisfied, and hence, the result follows, which completes the proof. □

Similarly, from Theorem 2 we deduce the following related result in partial metric spaces.

Theorem 4. *Let T and S be two self-mappings defined on partial metric space (X,p). Assume that there exists a Picard–Jungck sequence $\{x_n\}_{n\in\mathbb{N}}$ of T and S, and the following conditions are satisfied*:

(i) *there exists a function $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$ such that*

$$\theta\big(p(Tx,Ty),p(Sx,Sy)\big) \geqslant \mathcal{C}_{\mathcal{G}} \quad \textit{for all } x,y \in X,$$

(ii) (SX,p) (*or* (TX,p)) *is complete.*

Then T and S have a unique point of coincidence u. Moreover, if T and S are weakly compatible, then u is a unique common fixed point with $p(u,u)=0$.

Proof. The proof follows on the similar lines of proof of Theorem 3. □

Taking $S = I_X$, the identity mapping on X, in Theorems 3 and 4 and taking to the account that every $\mathcal{C}_{\mathcal{G}}$-simulation function is an extended $\mathcal{C}_{\mathcal{G}}$-simulation function, we deduce the following two corollaries, which seem to be new to the existing literature.

Corollary 5. *Let T be a self-mapping defined on a partial metric space (X,p). Suppose that there exists $\theta \in \mathcal{Z}_{\mathcal{G}}$ (or $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$) such that*

$$\theta\big(p(Tx,Ty),\max\big\{p(x,y),p(x,Tx),p(y,Ty)\big\}\big) \geqslant \mathcal{C}_{\mathcal{G}} \quad \textit{for all } x,y \in X.$$

Then T has a unique fixed point u with $p(u,u)=0$.

Corollary 6. *Let T be a self-mapping defined on a partial metric space (X,p). Suppose that there exists $\theta \in \mathcal{Z}_{\mathcal{G}}$ (or $\theta \in \mathcal{E}_{(\mathcal{Z},\mathcal{G})}$) such that*

$$\theta\big(p(Tx,Ty),p(x,y)\big) \geqslant \mathcal{C}_{\mathcal{G}} \quad \textit{for all } x,y \in X.$$

Then T has a unique fixed point u with $p(u,u)=0$.

4.2 Application to system of integral equations

In this section, to highlight the applicability of Theorem 2, we investigated the existence and uniqueness of a common solution of the following system of integral equations:

$$u(t) = f(t) + \int_0^t G\big(t,s,u(s)\big)\,\mathrm{d}s, \quad t \in [0,1], \tag{15}$$

$$v(t) = g(t) + \int_0^t Q\big(t,s,v(s)\big)\,\mathrm{d}s, \quad t \in [0,1], \tag{16}$$

where $Q,G : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ and $f,g : [0,1] \to \mathbb{R}$ are given functions. Let $X = C([0,1],\mathbb{R})$ denotes the set of all real valued continuous functions defined on $[0,1]$.

For any arbitrary $u \in X$, define a norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Let X be endowed with the metric

$$d(u, v) = \|u - v\| = \sup_{t\in[0,1]} \bigl|u(t) - v(t)\bigr|.$$

Then $(C([0,1], \mathbb{R}), \|\cdot\|)$ is a Banach space.

Now, we are equipped to state and prove our result in this section as under.

Theorem 5. *Consider the system of Eqs.* (15) *and* (16)*. Assume that the following conditions are satisfied*:

(i) *G, Q, f, and g are continuous functions,*

(ii) *$T, S : X \to X$ are two mappings defined by*

$$Tu(t) = f(t) + \int_0^t G\bigl(t, s, u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$

$$Su(t) = g(t) + \int_0^t Q\bigl(t, s, u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$

with the property that $TSu = STu$ for all $u \in X$ such that $Tu = Su$,

(iii) *for all $u, v \in X$ and $t, s \in [0,1]$, we have*

$$\bigl|G(t, s, u) - G(t, s, v)\bigr| \leqslant \frac{|Su - Sv|}{\|Su - Sv\| + 1}.$$

Then the system of the integral equations (15) *and* (16) *have a unique common solution.*

Proof. For all $u, v \in X$, we have

$$\begin{aligned}
&\bigl|T\bigl(u(t)\bigr) - T\bigl(v(t)\bigr)\bigr| \\
&\quad= \left|\int_0^t \bigl(Q\bigl(t, s, u(s)\bigr) - G\bigl(t, s, v(s)\bigr)\bigr)\,\mathrm{d}s\right| \\
&\quad\leqslant \int_0^t \bigl|G\bigl(t, s, u(s)\bigr) - G\bigl(t, s, v(s)\bigr)\bigr|\,\mathrm{d}s \leqslant \int_0^t \frac{|Su - Sv|}{\|Su - Sv\| + 1}\,\mathrm{d}s \\
&\quad\leqslant \frac{1}{\|Su - Sv\| + 1} \int_0^t \sup_{t\in[0,1]} |Su - Sv|\,\mathrm{d}s = \frac{\|Su - Sv\|}{\|Su - Sv\| + 1} t,
\end{aligned}$$

which on taking supremum leads to

$$d(Tu, Tv) = \|Tu - Tv\| \leqslant \frac{\|Su - Sv\|}{\|Su - Sv\| + 1} = \frac{d(Su, Sv)}{d(Su, Sv) + 1}$$

or

$$\frac{d(Su, Sv)}{d(Su, Sv) + 1} - d(Tu, Tv) \geqslant 0.$$

Now, we define two essential functions F and φ as

$$F(a, b, c) = a + b + c \quad \text{for all } a, b, c \in [0, \infty)$$

and

$$\varphi(x) = 0 \quad \text{for all } x \in X.$$

Hence, the above inequality can be written as

$$\frac{F(d(Su, Sv), \varphi(Su), \varphi(Sv))}{F(d(Su, Sv), \varphi(Su), \varphi(Sv)) + 1} - F\bigl(d(Tu, Tv), \varphi(Tu), \varphi(Tv)\bigr) \geqslant 0.$$

Thus, the contractive condition (10) is satisfied with $\theta(t, s) = s/(s + 1) - t$ and $\mathcal{C}_{\mathcal{G}} = 0$. Therefore, all the hypotheses of Theorem 2 are satisfied. Hence, the result is established. □

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