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Combating unemployment through skill development

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Abstract. In this paper, we propose and analyze a nonlinear mathematical model to study the effect of skill development on unemployment. We assume that government promulgates different levels of skill development programs for unemployed persons through which two different categories of skilled persons, namely, the low-skilled and the highly-skilled persons, are coming out and the highly-skilled persons are able to create vacancies. The model is studied using stability theory of nonlinear differential equations. We find analytically that there exists a unique positive equilibrium point of the proposed model system under some conditions. Also, the resulting equilibrium is locally as well as globally stable under certain conditions. The effective use of implemented policies to control unemployment by providing skills to unemployed persons and the new vacancies created by highly-skilled persons are identified by using optimal control analysis. Finally, numerical simulation is carried out to support analytical findings.

Keywords: mathematical model, unemployment, highly-skilled persons and low-skilled persons, employed persons, stability, simulation.

1 Introduction

Unemployment is not only a serious threat for developing countries but also for developed countries. Unemployment refers to situation for both educated and uneducated persons in which they are actively looking and available for work but unable to find work to do. World Employment and Social Outlook report of International Labor Organization reveals that there were 172 million people worldwide without job in 2018 [21]. Unemployment may cause economic, social as well as political problems for a country.

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Unemployed individuals are susceptible to poor diet, cardiovascular disease, depression, drug addiction, suicide and other psychological and mental health problems [4, 14]. In spite of taken several transformative, legislative as well as administrative measures by policy makers of respective countries, employment situation is not improving. Even the employment condition in many developing countries, especially, the countries of Sub-Saharan Africa has become worse [2]. Causes of unemployment varies from country to country. Slow economic growth rate, increasing population, faulty planning, lack of job-oriented education system are some of the main problems, which contribute to the growth of unemployment in developing countries to huge proportions. Besides these factors, one of the main reasons of unemployment for most of the countries in present century is the unskilled and outdated skilled workforce [1, 3, 19]. Pedagogy in many developing countries such as India is academic-centric and unpractical. About 80 percent of Indian graduates do not get employment due to lack of employable skill [23]. There prevails a huge gap between demands of industries and skill level of youths. To bridge this gap, skill development may play a decisive role. The Global Employment Trends 2013 report also suggests that governments should tackle unemployment with new skills and training initiatives.

Prior to the industrial revolution skills for work were largely imparted at the work-places, but the situation is changed now. Over the past few decades, skills have been offered in school at secondary and tertiary levels through various technical, vocational and professional education programs. In the past, overpopulation of youths of a country used to be a curse for it, now it may become boon if they are skilled. Henley [7] and Sackey et al. [20] suggested that with the help of some professional education and specialized skill programs, unemployed young graduates may accept to become entrepreneurs and may offer job opportunities to other job seekers. By employing other job seekers these entrepreneurs can play a crucial role for social-economic development of a country by reduction of unemployment, poverty alleviation, reduction in rural–urban migration, reduction of crime rate, raising the standard of living, etc.

Although unemployment is a social problem yet, it must be dealt scientifically to pave the way for devising effective measures of its amelioration.

Motivated by the paper of Nikolopoulos and Tzanetis [15], Misra and Singh [10] have proposed and analyzed a nonlinear mathematical model to study the problem of unemployment. In this paper, they have considered three variables, namely, the number of unemployed persons, temporarily employed persons and regularly employed persons. Further, in 2013, Misra and Singh [11] have developed a mathematical model to analyze the effect of delay in creating new vacancies on control of unemployment. They found that unemployment may be controlled by creating new vacancies in proportion to unemployed persons without any delay. Munoli and Gani [13] have done the optimal control analysis for their proposed unemployment model using Pontryagin's maximum principle. After implementing some controls, they found that effective policies of government to create new vacancies play very significant role to reduce unemployment. Pathan and Bhathawala [16] have studied the role of self employment on controlling unemployment. In this paper, they have found that unemployment may be controlled to some extent if the unemployed persons start their independent business. Further, Pathan and Bhathaawala [17] have seen the effect of action taken by government and private sector to control unemployment without delay and emphasis that government and private sector must generate higher number of new vacancies in this regard. Misra et al. [12] proposed a nonlinear mathematical model to assess the effect of developing skills of unemployed persons to control the unemployment. In this study, it is shown that avenues for skill development play an important role to control unemployment. Taking the concept from Misra and Singh [10, 11], Harding and Neamtu [6] proposed a continuously distributed delay mathematical model to study the problem of unemployment in a country, where both natives and migrant unemployed persons search for jobs.

The aim of the present paper is to study the role of skill development to control unemployment with the help of a mathematical model. In addition to variables U, the number of unemployed persons, and R, the number of employed persons, we have introduced some new variables: S_l, the number of low-skilled persons, and S_h, the number of highly-skilled persons due to skill development programs. A variable V, which shows the number of vacancies created by highly-skilled persons is also introduced in our present model. We also investigate the role of highly-skilled persons equipped with entrepreneurship skills in job creation. We further explore for the optimal strategies, which minimize the number of unemployed persons by developing skills of the unemployed persons along with the cost associated with the creation of skilled persons and the vacancies generated by the highly-skilled persons.

2 Mathematical model

We consider a region in which at any time t, the number of unemployed and the employed persons are $U(t)$ and $R(t)$, respectively. Let the government promulgates skill development programs through courses, workshops, seminaries, etc. for unemployed persons, which remove the disconnection between demand and supply of skilled manpower. Since there are variations in skill levels of the unemployed persons after getting skills, therefore, in the model formulation, two categories of skilled persons, namely, the low-skilled persons $S_l(t)$ and the highly-skilled persons $S_h(t)$ are assumed. We further assume that highly-skilled persons are equipped with entrepreneurship skills and are able to create new employment opportunities. Therefore, an additional state variable $V(t)$, which represents the number of vacancies created by highly-skilled persons at any time t, has been considered.

In the process of modeling, we have assumed that all new entrants to the unemployed class are fully qualified (i.e., graduate or postgraduate) and competent to do any job, and it is assumed that the new entrants are coming in the unemployed class with a constant rate A and are migrating to other region in search of employment. It is also assumed that the total number of available regular vacancies is limited and is assumed to be constant. We further assume that highly-skilled persons are getting quickly employed with a constant rate b.

Now, at any time t, the number of available vacancies are $(R_a + V - bS_h - R)$, where R_a is the total number of available regular vacancies. Therefore, the rate of movement of persons from the unemployed class to the employed class will be jointly proportional to U and the total number of available vacancies $(R_a + V - bS_h - R)$ [15]. Also, the rate of low-skilled unemployed persons getting jobs will be jointly proportional to S_l and $(R_a + V - bS_h - R)$.

In light of the above facts, the dynamics of the model may be governed by the following set of nonlinear differential equations:

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}t} &= A - k_1 U(R_a + V - bS_h - R) - \beta U - \delta_1 U + \gamma R,\\
\frac{\mathrm{d}S_l}{\mathrm{d}t} &= p\beta U - k_2 S_l(R_a + V - bS_h - R) - \delta_2 S_l,\\
\frac{\mathrm{d}S_h}{\mathrm{d}t} &= (1-p)\beta U - bS_h - \delta_3 S_h,\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= (k_1 U + k_2 S_l)(R_a + V - bS_h - R) - \delta_4 R - \gamma R,\\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \phi S_h - \phi_0 V.
\end{aligned} \tag{1}$$

In the model system (1), the proportionality constants k_1 and k_2 show the rates by which the unemployed and the low-skilled persons are getting jobs, respectively. β is the rate through, which unemployed persons are getting skilled. p denotes the fraction of skilled persons going to low-skilled class, where $0 < p < 1$. δ_i's $(i = 1, 2, 3)$ are migration or natural death rate coefficients in their respective classes, while δ_4 represents the rate of retirement or death of employed persons. The constant γ represents the rate of persons being fired from the job. ϕ is the rate of vacancies created by highly-skilled persons, and ϕ_0 is the rate of diminution of V due to recession, closing of firm, etc. All the above constants are assumed to be positive.

3 Equilibrium analysis

We assume that the highly-skilled persons are providing jobs rather than getting employed. Therefore, we take $\phi/\phi_0 - b > 0$, then the nonnegative equilibrium of the model system (1) may be obtained by solving the following set of algebraic equations:

$$A - k_1 U(R_a + V - bS_h - R) - \beta U - \delta_1 U + \gamma R = 0, \tag{2}$$
$$p\beta U - k_2 S_l(R_a + V - bS_h - R) - \delta_2 S_l = 0, \tag{3}$$
$$(1-p)\beta U - bS_h - \delta_3 S_h = 0, \tag{4}$$
$$(k_1 U + k_2 S_l)(R_a + V - bS_h - R) - \delta_4 R - \gamma R = 0, \tag{5}$$
$$\phi S_h - \phi_0 V = 0. \tag{6}$$

Adding equations (2), (3) and (5), we get

$$A - \big\{\delta_1 + \beta(1-p)\big\}U - \delta_2 S_l - \delta_4 R = 0. \tag{7}$$

From equations (4), (6) and (7) we have

$$S_h = \frac{(1-p)\beta U}{\delta_3+b}, \qquad V = \frac{\phi}{\phi_0}\frac{(1-p)\beta U}{\delta_3+b}, \qquad R = \frac{A-\{\delta_1+\beta(1-p)\}U-\delta_2 S_l}{\delta_4}. \quad (8)$$

Now from (8) we obtain

$$R_a + V - bS_h - R = a_1 U + a_2 S_l - a_3, \quad (9)$$

where

$$a_1 = \left(\frac{\phi}{\phi_0} - b\right)\frac{(1-p)\beta}{\delta_3+b} + \frac{\delta_1+\beta(1-p)}{\delta_4}, \qquad a_2 = \frac{\delta_2}{\delta_4}, \qquad a_3 = \left(\frac{A}{\delta_4} - R_a\right).$$

Using equation (9) in (2) and (3), we get the following equations of conics in variables U and S_l:

$$\alpha_1 U^2 + \alpha_2 S_l U + \alpha_3 U + \alpha_4 S_l - \alpha_5 = 0, \quad (10)$$

$$\beta_1 S_l^2 + \beta_2 S_l U - \beta_3 S_l - \beta_4 U = 0, \quad (11)$$

where $\alpha_1 = a_1 k_1$, $\alpha_2 = a_2 k_1$, $\alpha_3 = \gamma\{\delta_1+\beta(1-p)\}/\delta_4 + (\delta_1+\beta) - a_3 k_1$, $\alpha_4 = a_2\gamma$, $\alpha_5 = A(1+\gamma/\delta_4)$, $\beta_1 = a_2 k_2$, $\beta_2 = a_1 k_2$, $\beta_3 = a_3 k_2 - \delta_2$, $\beta_4 = p\beta$. From (11) we have

$$U = \frac{\beta_1 S_l^2 - \beta_3 S_l}{\beta_4 - \beta_2 S_l}. \quad (12)$$

Now from (10) and (12) we get

$$\gamma_1 S_l^3 + \gamma_2 S_l^2 + \gamma_3 S_l + \gamma_4 = 0, \quad (13)$$

where

$$\gamma_1 = \alpha_2(\beta_2\beta_3 - \beta_1\beta_4) + \beta_2(\alpha_3\beta_1 - \alpha_4\beta_2),$$
$$\gamma_2 = \beta_2(\alpha_5\beta_2 - \alpha_3\beta_3) + \beta_3(\alpha_2\beta_4 - \alpha_1\beta_3) + \beta_4(2\alpha_4\beta_2 - \alpha_3\beta_1),$$
$$\gamma_3 = \beta_4(\alpha_3\beta_3 - \alpha_4\beta_4 - 2\alpha_5\beta_2), \qquad \gamma_4 = \alpha_5\beta_4^2.$$

Proposition 1. *If*

$$\alpha_3 > a_1\gamma \quad \text{and} \quad a_3 k_2(1-p) > \delta_2, \quad (14)$$

then Eq. (13) *has a unique positive solution* $S_l^* \in (\beta_4/\beta_2, \beta_3/\beta_1)$. *Further, the nonnegative equilibrium* E^* *has the components*

$$U^* = \frac{S_l^*(\beta_1 S_l^* - \beta_3)}{\beta_4 - \beta_2 S_l^*}, \qquad S_h^* = \frac{(1-p)\beta U^*}{\delta_3+b},$$

$$R^* = \frac{A - \left\{\delta_1 + \beta(1-p)\right\}U^* - \delta_2 S_l^*}{\delta_4}, \qquad V^* = \frac{\phi}{\phi_0} S_h^*.$$

Proof. See Appendix A. □

Hence, the model system (1) has only one nonnegative equilibrium $E^*(U^*, S_l^*, S_h^*, R^*, V^*)$ under the conditions given by (14).

4 Stability analysis

To study the local stability behaviour of the nonnegative equilibrium E^*, we will find the sign of eigenvalues of the corresponding Jacobian matrix. In this regard, the Jacobian matrix J^* about the equilibrium point E^* for the model system (1) is given as follows:

$$J^* = \begin{pmatrix} -k_1q_1 - \beta - \delta_1 & 0 & bk_1U^* & k_1U^* + \gamma & -k_1U^* \\ \beta p & -k_2q_1 - \delta_2 & bk_2S_l^* & k_2S_l^* & -k_2S_l^* \\ \beta(1-p) & 0 & -b - \delta_3 & 0 & 0 \\ k_1q_1 & k_2q_1 & -bq_2 & -q_2 - \delta_4 - \gamma & q_2 \\ 0 & 0 & \phi & 0 & -\phi_0 \end{pmatrix},$$

where $q_1 = R_a + V^* - bS_h^* - R^*$, $q_2 = k_1U^* + k_2S_l^*$.

Now the eigenvalues of the Jacobian matrix J^* are given by the following characteristic equation:

$$\lambda^5 + A_1\lambda^4 + A_2\lambda^3 + A_3\lambda^2 + A_4\lambda + A_5 = 0,$$

where

$$\begin{aligned} &A_1 = \phi_0 + a_{11},\ A_2 = a_{11}\phi_0 + a_{21} + a_{22} - a_{23}, \\ &A_3 = (a_{21} + a_{22})\phi_0 + a_{31} + a_{32} + a_{33} + a_{34} - a_{35}, \\ &A_4 = (a_{31} + a_{32} + a_{33})\phi_0 + a_{41} + a_{42} + a_{43} - a_{44}, \\ &A_5 = a_{51} + a_{52} + a_{53} + a_{54} \end{aligned} \tag{15}$$

with

$$\begin{aligned} a_{11} &= (k_1 + k_2)q_1 + q_2 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \beta + b + \gamma, \\ a_{21} &= (k_2q_1 + \delta_2)(k_1U^* + \delta_4 + \gamma) + (\delta_1 + \delta_3 + \beta + b)(k_1U^* + \gamma) + \delta_2k_2S_l^*, \\ a_{22} &= (k_2S_l^* + k_2q_1 + \delta_2 + \delta_4)(k_1q_1 + \delta_1 + \delta_3 + \beta + b) + (\delta_3 + b)(k_1q_1 + \delta_1 + \beta), \\ a_{23} &= (1-p)\beta bk_1U^*, \\ a_{31} &= (\delta_3 + b)\bigl\{(\delta_1 + \beta)(k_1U^* + \gamma) + (k_2S_l^* + k_2q_1 + \delta_2 + \delta_4)(k_1q_1 + \delta_1 + \beta)\bigr\}, \\ a_{32} &= k_1q_1\bigl\{\delta_2k_2S_l^* + \delta_4(k_2q_1 + \delta_2) + (\delta_1 + \delta_3 + b)(k_1U^* + \gamma)\bigr\}, \\ a_{33} &= (1-p)\beta k_2q_1(k_1U^* + \gamma) + (\delta_1 + \delta_3 + \beta + b)(k_2q_1 + \delta_2q_2 + \delta_2\delta_4 + \delta_2\gamma), \\ a_{34} &= (1-p)\beta(\phi k_1U^* + bk_2\gamma S_l^*), \quad a_{35} = a_{23}(k_2q_1 + \delta_2 + \delta_4 + \phi), \\ a_{41} &= (1-p)\beta k_2q_1(k_1U^* + \gamma) + (k_2q_1 + \delta_2)\bigl\{\delta_4(k_1q_1 + \beta) + \delta_1(\delta_4 + \gamma)\bigr\}, \\ a_{42} &= k_1k_2q_1(\delta_1U^* + \delta_2S_l^*) + \delta_2q_2(\delta_1 + \beta) + \beta\gamma\delta_2, \\ a_{43} &= \phi(1-p)\beta k_1U^*(k_2q_1 + \delta_2 + \delta_4) + bk_2\gamma S_l^*(\delta_2 + \phi_0), \\ a_{44} &= bk_1U^*(\delta_4 + \phi_0)(k_2q_1 + \delta_2) + \phi_0b\delta_4k_1U^* + \phi(1-p)\beta\gamma k_2S_l^*, \\ a_{51} &= (1-p)\beta\delta_4k_1U^*(\phi - \phi_0b)(k_2q_1 + \delta_2) + \phi_0(1-p)\beta b\delta_2\gamma k_2S_l^*, \\ a_{52} &= \phi_0\beta b\delta_2k_1U^* + \delta_2\beta k_2S_l^*(\phi_0b + \phi p\gamma - \phi\gamma) + \phi_0\delta_2q_2(\delta_1b + \delta_1\delta_3 + \delta_3\beta), \\ a_{53} &= \phi_0(\delta_3 + b)\bigl\{(1-p)\beta k_2q_1(k_1U^* + \gamma) + \gamma\delta_1(k_2q_1 + \delta_2) + \beta\gamma\delta_2\bigr\}, \\ a_{54} &= \phi_0(\delta_3 + b)\bigl\{\delta_4(k_2q_1 + \delta_2)(k_1q_1 + \delta_1 + \beta) + k_1k_2q_1(\delta_1U^* + \delta_2S_l^*)\bigr\}. \end{aligned}$$

Here all the a_{ij}'s $(i, j = 1, 2, 3, 4, 5)$ are positive, therefore, from (15) we have $A_1 > 0$ and $A_5 > 0$. Now, by applying the Routh–Hurwitz criterion [8], we have the following theorem.

Theorem 1. *The equilibrium E^* is locally asymptotically stable, provided the following conditions are satisfied*:

$$\begin{aligned} &A_1A_2 - A_3 > 0, \quad (A_1A_2 - A_3)A_3 - A_1^2A_4 > 0, \\ &(A_3A_4 - A_2A_5)(A_1A_2 - A_3) - (A_1A_4 - A_5)^2 > 0, \end{aligned} \tag{16}$$

where A_i's $(i = 1, 2, 3, 4, 5)$ are defined as above.

Lemma 1. *The set*

$$\Omega = \left\{ (U, S_l, S_h, R, V) \in \mathbb{R}^5 \colon 0 < U + S_l + S_h + R \leqslant \frac{A}{\delta},\ 0 \leqslant V \leqslant \frac{\phi A}{\phi_0 \delta} \right\},$$

where $\delta = \min\{\delta_1, \delta_2, \delta_3 + b, \delta_4\}$, is a region of attraction for the model system (1)*, and it attracts all solution initiating in the interior of the nonnegative octant.*

Proof. See Appendix B. □

In the following theorem, the global stability analysis of $E^*(U^*, S_l^*, S_h^*, R^*, V^*)$ is investigated.

Theorem 2. *The equilibrium E^* is globally stable inside the region of attraction Ω if the following conditions are satisfied*:

$$p\beta(1-p)k_1U^* < \frac{2}{15}(\delta_1 + \beta)(\delta_3 + b), \tag{17}$$

$$k_1q_1(k_1U^* + \gamma) < \frac{1}{9}(\delta_1 + \beta)(\delta_4 + \gamma), \tag{18}$$

$$k_2^2q_1S_l^* < \frac{2}{15}\delta_2(\delta_4 + \gamma), \tag{19}$$

where $q_1 = (R_a + V^* - bS_h^* - R^*)$.

Proof. See Appendix **C**. □

5 Optimal control analysis

In this section, we try to search for the strategies, which minimize the number of unemployed persons minimizing the cost functions. It is considered that the rate p and ϕ are not constant but will work as control parameters and are Lebesgue measurable functions on finite time interval $[0, t_f]$, namely, $p(t)$ and $\phi(t)$. With these assumptions, we rewrite the model system (1) as

$$
\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}t} &= A - k_1U(R_a + V - bS_h - R) - \beta U - \delta_1 U + \gamma R,\\
\frac{\mathrm{d}S_l}{\mathrm{d}t} &= p(t)\beta U - k_2 S_l(R_a + V - bS_h - R) - \delta_2 S_l,\\
\frac{\mathrm{d}S_h}{\mathrm{d}t} &= \bigl(1 - p(t)\bigr)\beta U - bS_h - \delta_3 S_h,\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= (k_1U + k_2S_l)(R_a + V - bS_h - R) - \delta_4 R - \gamma R,\\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \phi(t)S_h - \phi_0 V.
\end{aligned}
\tag{20}
$$

The objective function to be minimize is

$$
J = \int_0^{t_f} \bigl[BU(t) + Cp^2(t) + D\phi^2(t)\bigr]\,\mathrm{d}t \tag{21}
$$

subjected to state system (20) and the nonnegative initial data B, C and D, which denote weight constants for the unemployment, their efficiency to get skilled and the efficiency of highly-skilled persons to create vacancies, respectively. The term $[Cp^2(t) + D\phi^2(t)]$ describes the implementing efficiency.

Our aim is to seek an optimal control pair (p^*, ϕ^*) such that

$$
J(p^*, \phi^*) = \min_{(p,\phi)\in X} J\bigl(p, \phi\bigr),
$$

where $X = \{(p, \phi)\colon$ p and ϕ are measurable, $p_{\min} < p(t) < p_{\max},\ 0 < \phi(t) < \phi_{\max}$ for $t \in [0, t_f]\}$ is the control set.

5.1 Existence of an optimal control

Theorem 3. *There exists an optimal control pair $(p^*, \phi^*) \in X$ such that $J(p^*, \phi^*) = \min_{(p,\phi)\in X} J(p, \phi)$ subject to system* (20) *with initial data.*

Proof. We prove the existence of an optimal control with the help of a result of Fleming and Rishel [5]. The existence of solution of system (20) is implied by the boundedness of its solution [9]. We note that the state variables as well as control variables are nonnegative. The objective functional in $p(t)$ and $\phi(t)$ also satisfy necessary convexity conditions. The control set X is convex and closed. Further, the boundedness of the solution of the state system (20) shows the compactness, which is needed for the existence of the optimal control. The integrand in the objective functional (21) is convex on the control set X. Also, it can be easily seen that there exists constants c_1, $c_2 > 0$ and $\rho > 1$ such that

$$
BU(t) + Cp^2(t) + D\phi^2(t) \leqslant c_1 + c_2\bigl(\bigl|p(t)\bigr|^2 + \bigl|\phi(t)\bigr|^2\bigr)^{\rho/2},
$$

where c_1 is dependent on the upper bound of U, and $c_2 = \sup(C, D)$. This proves the existence of the optimal control. $\square$

5.2 Characterization of optimal controls

Using the Pontryagin's maximum principle [18], we derive the necessary condition that an optimal solution must satisfy. This principle changes the minimizing objective functional problem subject to the system in to minimizing pointwise problem either the Lagrangian or the Hamiltonian with respect to control variables. The Hamiltonian can be defined as follows:

$$\begin{aligned} &H(U, S_l, S_h, R, V, p, \phi, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \\ &\quad = BU(t) + Cp^2(t) + D\phi^2(t) \\ &\qquad + \mu_1\{A - k_1U(R_a + V - bS_h - R) + \beta U - \delta_1 U + \gamma R\} \\ &\qquad + \mu_2\{p\beta U - k_2S_l(R_a + V - bS_h - R) - \delta_2 S_l\} \\ &\qquad + \mu_3\{(1-p)\beta U - bS_h - \delta_3 S_h\} \\ &\qquad + \mu_4\{(k_1U + k_2S_l)(R_a + V - bS_h - R) - \delta_4 R - \gamma R\} \\ &\qquad + \mu_5\{\phi S_h - \phi_0 V\}, \end{aligned} \tag{22}$$

where μ_i ($i = 1, 2, 3, 4, 5$) are the associated adjoint variables or co-state variables for the states U, S_l, S_h, R and V, respectively.

The differential equations governing the adjoint variables are obtained by taking the partial derivatives of the Hamiltonian (22) with respect to the associated co-state variables, which are as follows:

$$\begin{aligned} \dot{\mu}_1 &= -\frac{\partial H}{\partial U} \\ &= -B + \mu_1\{k_1U(R_a + V - bS_h - R) + \beta + \delta_1\} - \mu_2 p\beta - \mu_3(1-p)\beta \\ &\quad - \mu_4\{k_1U(R_a + V - bS_h - R)\}, \\ \dot{\mu}_2 &= -\frac{\partial H}{\partial S_l} = \mu_2\{k_2(R_a + V - bS_h - R) + \delta_2\} - \mu_4\{k_2(R_a + V - bS_h - R)\}, \\ \dot{\mu}_3 &= -\frac{\partial H}{\partial S_h} = -\mu_1 k_1 bU - \mu_2 k_2 bS_l + \mu_3(\delta_3 + b) + \mu_4 b(k_1U + k_2S_l) - \mu_5\phi, \\ \dot{\mu}_4 &= -\frac{\partial H}{\partial R} = -\mu_1(k_1U + \gamma) - \mu_2 k_2 S_l + \mu_4 b(k_1U + k_2S_l + \delta_4 + \gamma), \\ \dot{\mu}_5 &= -\frac{\partial H}{\partial V} = \mu_1 k_1 U + \mu_2 k_2 S_l - \mu_4(k_1U + k_2S_l) + \mu_5\phi_0. \end{aligned}$$

Since the state variables are not assigned at the final time t_f, so the transversality conditions are $\mu_1(t_f) = \mu_2(t_f) = \mu_3(t_f) = \mu_4(t_f) = \mu_5(t_f) = 0$. For characterizing the optimal control variables p^*, ϕ^*, we use the optimality conditions

$$\frac{\partial H}{\partial p} = 0, \qquad \frac{\partial H}{\partial \phi} = 0$$

at $p = p^*$, $\phi = \phi^*$ on the set $\{t \in [0, t_f]\colon p_{\min} < p_t < p_{\max},\ 0 < \phi(t) < \phi_{\max}\}$.

Therefore, we have

$$p^* = \frac{\beta U(\mu_3 - \mu_2)}{2C}, \qquad \phi^* = -\frac{\mu_5}{2D} S_h$$

on the interior of the control set. Considering for the bound constraints for the controls, we have the following characterization of p^* and ϕ^*:

$$p^* = \begin{cases} p_{\min} & \text{if } \frac{\beta U(\mu_3-\mu_2)}{2C} \leqslant p_{\min}, \\ \frac{\beta U(\mu_3-\mu_2)}{2C} & \text{if } \frac{\beta U(\mu_3-\mu_2)}{2C} \leqslant p_{\max}, \\ p_{\max} & \text{if } \frac{\beta U(\mu_3-\mu_2)}{2C} \geqslant p_{\max}, \end{cases} \qquad \phi^* = \begin{cases} 0 & \text{if } -\frac{\mu_5 S_h}{2D} \leqslant 0, \\ -\frac{\mu_5 S_h}{2D} & \text{if } 0 < -\frac{\mu_5 S_h}{2D} < \phi_{\max}, \\ \phi_{\max} & \text{if } -\frac{\mu_5 S_h}{2D} \geqslant \phi_{\max}. \end{cases}$$

Thus, we have the following theorem regarding the characterization of optimal control.

Theorem 4. *The optimal control variables p^*, ϕ^*, which minimizes the objective functional J subjected to system* (20) *over the control set X is characterized by*

$$p^*(t) = \max\left\{\min\left(\frac{\beta U(\mu_3 - \mu_2)}{2C}, p_{\max}\right), p_{\min}\right\}, \tag{23}$$

$$\phi^*(t) = \max\left\{\min\left(-\frac{\mu_5 S_h}{2D}, \phi_{\max}\right), 0\right\}. \tag{24}$$

The optimality system consists of the state system and adjoint system together with the characterization of optimal control pair, initial conditions and transversality conditions. Thus we have the optimality system as follows:

$$\begin{aligned}
\dot{U} &= A - k_1 U(R_a + V - bS_h - R) - \beta U - \delta_1 U + \gamma R, \\
\dot{S}_l &= p^*(t)\beta U - k_2 S_l (R_a + V - bS_h - R) - \delta_2 S_l, \\
\dot{S}_h &= \bigl(1 - p^*(t)\bigr)\beta U - bS_h - \delta_3 S_h, \\
\dot{R} &= (k_1 U + k_2 S_l)(R_a + V - bS_h - R) - \delta_4 R - \gamma R, \\
\dot{V} &= \phi^*(t) S_h - \phi_0 V, \\
\dot{\mu}_1 &= -B + \mu_1\bigl\{k_1 U(R_a + V - bS_h - R) + \beta + \delta_1\bigr\} - \mu_2 p^*(t)\beta \\
&\quad - \mu_3(1 - p^*(t))\beta - \mu_4\bigl\{k_1 U(R_a + V - bS_h - R)\bigr\}, \\
\dot{\mu}_2 &= \mu_2\bigl\{k_2(R_a + V - bS_h - R) + \delta_2\bigr\} - \mu_4\bigl\{k_2(R_a + V - bS_h - R)\bigr\}, \\
\dot{\mu}_3 &= -\mu_1 k_1 bU - \mu_2 k_2 bS_l + \mu_3(\delta_3 + b) + \mu_4 b(k_1 U + k_2 S_l) - \mu_5 \phi^*(t), \\
\dot{\mu}_4 &= -\mu_1(k_1 U + \gamma) - \mu_2 k_2 S_l + \mu_4 b(k_1 U + k_2 S_l + \delta_4 + \gamma), \\
\dot{\mu}_5 &= \mu_1 k_1 U + \mu_2 k_2 S_l - \mu_4(k_1 U + k_2 S_l) + \mu_5 \phi_0,
\end{aligned}$$

where $p^*(t)$ and $\phi^*(t)$ are given by equations (23) and (24), respectively, with $U(0) = 0$, $S_l(0) = S_{l0}$, $S_h(0) = S_{h0}$, $R(0) = R_0$, $V(0) = V_0$ and $\mu_i(t_f) = 0$ for $i = 1, 2, 3, 4, 5$.

6 Numerical simulations

Since the model is nonlinear, it is very perplexing to obtain an analytical solution. Therefore, numerical simulation has been carried out with the help of MATLAB to support the analytical findings. Normalized value of A in thousand is 7720 persons/year, which is calculated from the data of Labour Force of India from 1990–2015 [22]. Further, we have chosen the following set of parameter values in the model system (1): $R_a = 24000$, $k_1 = 0.000005$, $k_2 = 0.000006$, $\beta = 0.16$, $p = 0.6$, $\delta_1 = 1.8$, $\delta_2 = 0.08$, $\delta_3 = 0.3$, $\delta_4 = 0.13$, $\phi = 0.021$, $\phi_0 = 0.008$, $b = 0.25$, $\gamma = 0.0009$.

We may verify that above set of parameter values satisfy the conditions given in Eq. (14). The equilibrium values of E^* in view of the above set of data are as follows: $U^* = 3745.512524$, $S_l^* = 1774.049711$, $S_h^* = 436.8414581$, $R^* = 4588.005148$, $V^* = 1144.083828$.

The eigenvalues of the variational matrix corresponding to the equilibrium point $E^*(U^*, S_l^*, S_h^*, R^*, V^*)$ for the model system (1) are -2.063434541, -0.5497511403, -0.1373930641, -0.2245922897 and -0.008019126535. Since all the eigenvalues of the variational matrix are negative, so the equilibrium point E^* is locally asymptotically stable. Also, for the above set of parameters, the local stability conditions (16) as well as global stability conditions (17), (18) and (19) are satisfied. Figures 1 and 2 shows the global stability in U, S_h, S_l and U, R, V-spaces, respectively. We can observe that the trajectories are converging at a fix point for any initial starts in the region of attraction in respective spaces.

Figure 3 shows the variations of unemployed persons U and employed persons R for different values of k_1. It is clear from the figure that as k_1 increases, the number of unemployed persons decreases, whereas the number of employed persons increases. The effects on unemployed persons, low-skilled persons, highly-skilled persons and employed persons for different values β are presented in Fig. 4. From this figure we see that the unemployed persons decreases as the rate of the skill development of unemployed persons increases, whereas number of low-skilled, highly-skilled persons and employed persons

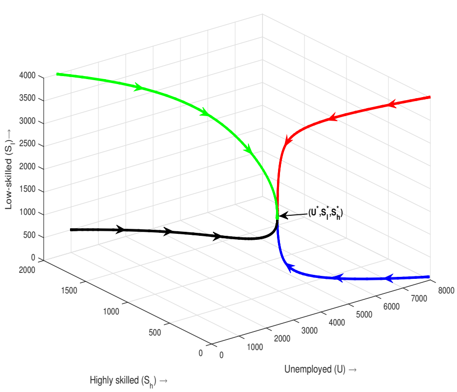

Figure 1. Global stability in U, S_h, S_l-space.

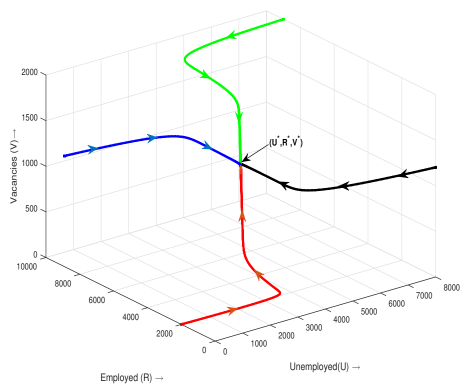

Figure 2. Global stability in U, R, V-space.

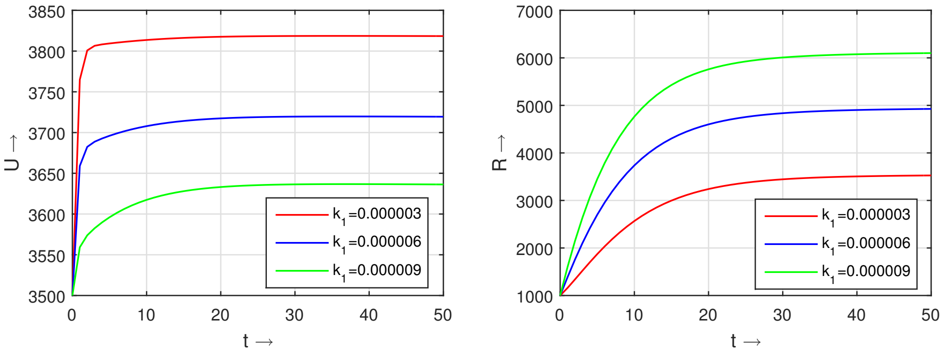

Figure 3. Variation of U and R with respect to time t for different values of k_1.

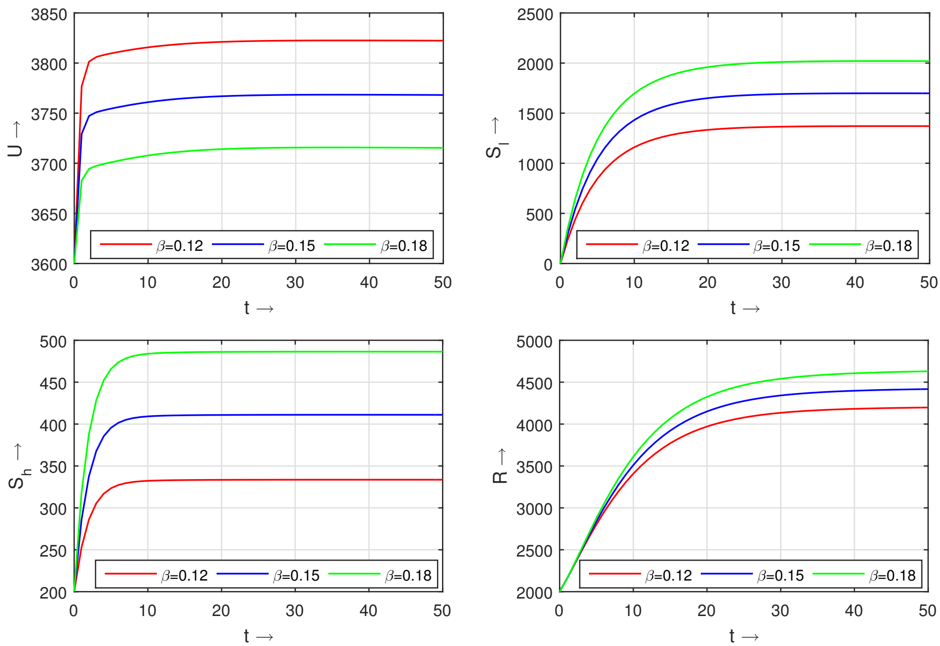

Figure 4. Variation of U, S_l, S_h and R with respect to time t for different values of β.

increases. Figure 5 shows effects of different values of p on unemployed persons, low-skilled persons, highly-skilled persons and on vacancies created by highly-skilled persons. We may note that the number of unemployed and low skilled persons decreases as p decreases, whereas the number of highly-skilled persons and the number of vacancies created by highly-skilled persons increases. Finally, Fig. 6 shows that on increasing the value of ϕ, the number of unemployed and low-skilled persons decreases, whereas the number of employed persons and the number of vacancies V created by highly-skilled persons increases.

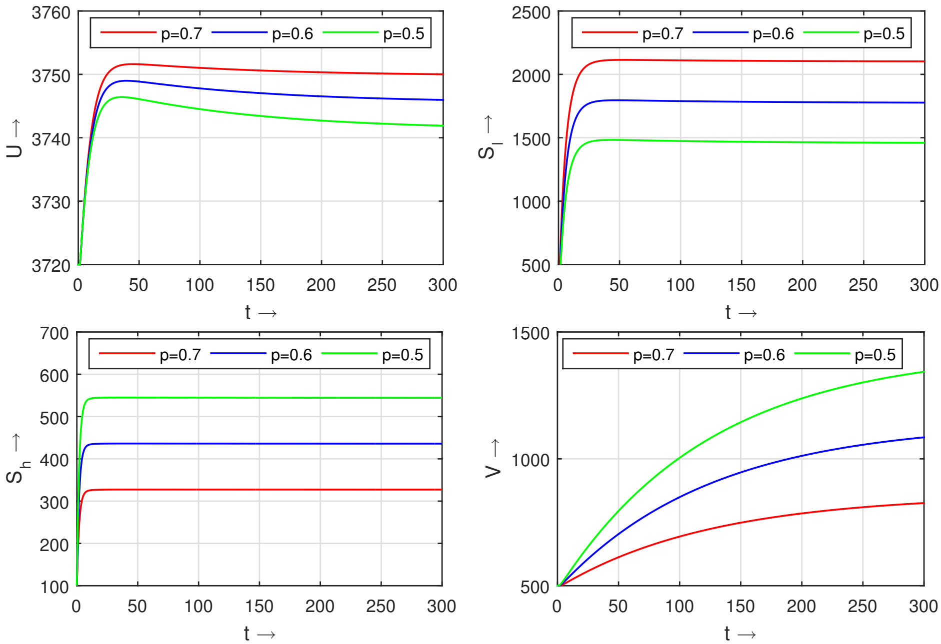

Figure 5. Variation of U, S_l, S_h and V with respect to time t for different values of p.

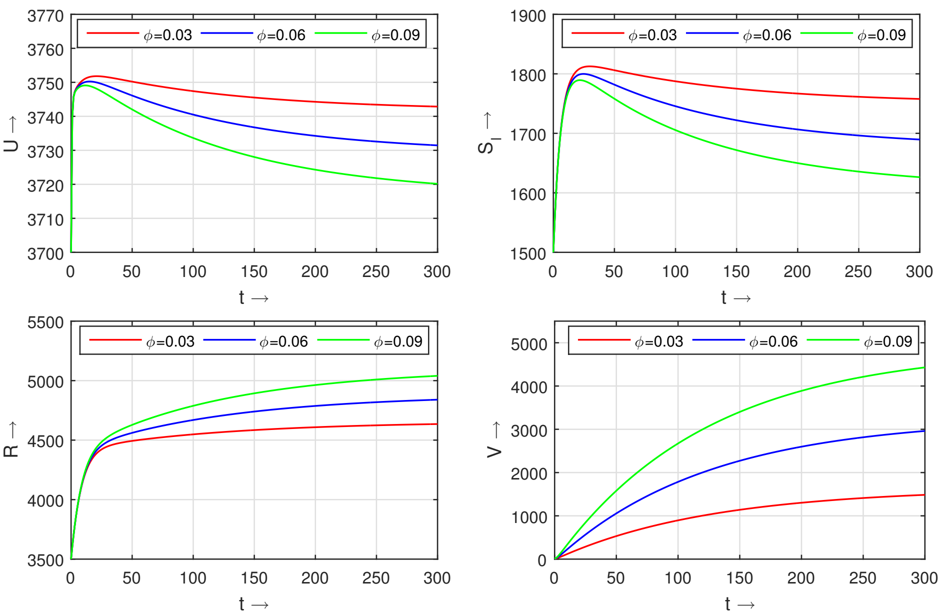

Figure 6. Variation of U, S_l, R and V with respect to time t for different values of ϕ.

Further, this section deals with the numerical simulation of the optimality system for the above set of parameter values. We have used the MATLAB routine bvp4c with the relative error tolerance on the residuals of 10^{-5} and the absolute error tolerance of 10^{-3}. We have taken the initials conditions $U(0) = 3745.512524$, $S(0) = 1774.049711$, $S_h(0) = 436.8414581$, $R(0) = 4588.005148$ and $V(0) = 1144.083828$. In view of S_l and S_h in the model system (20), we consider the minimum and maximum value of p. We take $p_{\min} = 0.3$, $p_{\max} = 0.8$, $\phi_{\max} = 0.15$, and the final time t_f is taken to be 50 years. Two different combinations of weight factors has been taken to depict different strategies in different cases. Numerical simulations are performed when both controls on creating skilled persons and on creating new vacancies are optimized.

Case 1. In this case, we discuss the optimal strategy when $B = 1$, $C = 3$, $D = 150$. In Fig. 7, first plot shows the number of unemployed persons when control options are present and absent. It can be observed that the number of unemployed persons decreases when the control options are applied. Further, the second and third plots of Fig. 7 shows that the control p is at lower bound till final time and the control ϕ dropped from its maximum to minimum value after time $t = 42$ years.

Case 2. Here we take the weight constant $D = 500$, and the values of B and C are same as in Case 1. Now, in this case, the profiles of the number of unemployed persons and optimal control strategies are shown in Fig. 8. From this figure we observe that in comparison to the first case, control option ϕ is very low and the number of unemployed persons settles to a higher level at final time. Thus, the reduction in the number of unemployed persons is least in this case.

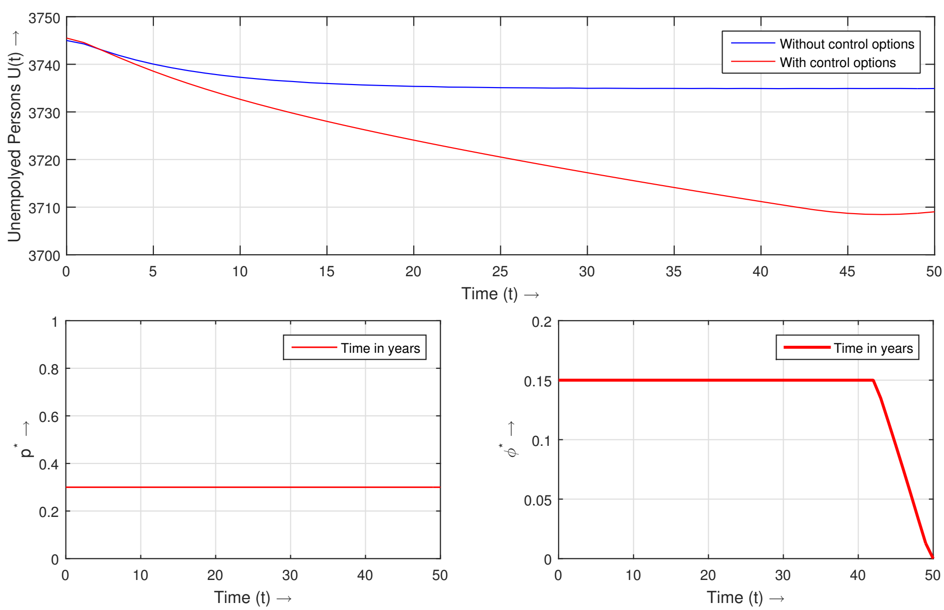

Figure 7. Effect of control profile on unemployed persons for Case 1 ($B = 1$, $C = 3$, $D = 150$).

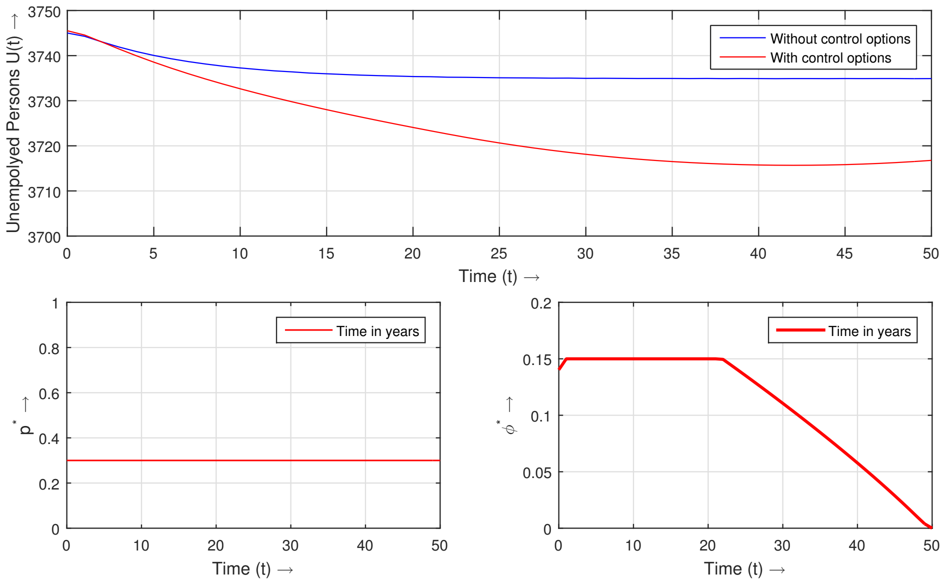

Figure 8. Effect of control profile on unemployed persons for Case 2 ($B = 1$, $C = 3$, $D = 500$).

Comparing the above two cases, we can note that as the cost associated with ϕ (i.e., the weight constant D) increases, the reduction in the number of unemployed persons decreases.

7 Conclusion

In this paper, we have proposed and analyzed a nonlinear mathematical model on unemployment with skill-development of unemployed persons taking into account. At any time t, we have considered five variables: the number of unemployed individuals U, the number of low-skilled individuals S_l, the number of highly-skilled individuals S_h, the number of employed individuals R and the number of vacancies created by highly-skilled persons V. The model exhibits a unique positive equilibrium point under some conditions. Further, we have shown that the equilibrium point is locally asymptotically as well as globally stable under certain conditions on the parameters. The modeling analysis shows that as the rate of skill development of unemployed persons increases, the number of unemployed persons decreases, whereas the employed persons increases. Also, this paper points out the importance of highly-skilled persons as a potential engine to curb the problem of unemployment by employing other job seekers. It can be observed from Fig. 5 that vacancies created by highly-skilled persons increase due to increase in number of highly-skilled persons leading to a decrease in the number of unemployed persons. The optimal control analysis for the proposed model system also have been performed using Pontryagin's maximum principle to identify the optimal strategies, which minimize the number of unemployed persons along with the cost associated with control options in different cases. We have found that the optimal policies to develop skills of unemployed persons and the vacancies created by highly-skilled persons play an important role to curb the problem of unemployment. However, if the cost of creating skilled persons increases, the optimal reduction in the number of unemployed persons decreases. Therefore, governments must focus on developing skills of unemployed persons in a cost-effective way.

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Appendix A: The proof of Proposition 1

Let the function

$$f(S_l) = \gamma_1 S_l^3 + \gamma_2 S_l^2 + \gamma_3 S_l + \gamma_4. \tag{A.1}$$

We may verify that γ_1 is positive for $\alpha_3 > a_1\gamma$. Also, we find that β_3 and $(\beta_2\beta_3 - \beta_1\beta_4)$ are positive for $a_3k_2(1-p) > \delta_2$. Now from Eq. (A.1) we may easily find the facts that:

(i) $f(0) = \gamma_4 > 0$,
(ii) $f(\beta_4/\beta_2) = -(\beta_4^2/\beta_2^2)(\alpha_1\beta_3^2 + \alpha_2\beta_1(\beta_4^2/\beta_2) < 0$,
(iii) $f(\beta_3/\beta_1) = \{A + \gamma(R_a + \delta_2/k_2)\}(\beta_2\beta_3/\beta_1 - \beta_4)^2 + (1 - p\beta)(\alpha_3\beta_3^2/\beta_1) > 0$.

The above points (i) and (ii) together implies that there exists a positive root $(S_l = S_{l*})$ of Eq. (A.1) in the interval $(0, \beta_4/\beta_2)$. Further, the points (ii) and (iii) imply that another positive root $(S_l = S_l^*)$ of Eq. (A.1) lies in the interval $(\beta_4/\beta_2, \beta_3/\beta_1)$.

Now from Eq. (12) we get the positive value of U if S_l lies in the interval $(\beta_4/\beta_2, \beta_3/\beta_1)$. Thus after knowing the positive values of $S_l = S_l^*$ and $U = U^*$, we get the positive values of $S_h = S_h^*$, $R = R^*$ and $V = V^*$ with the help of Eq. (8).

Appendix B: The proof of Lemma 1

Adding the first four equations of model system (1) we obtain,

$$\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{\mathrm{d}S_l}{\mathrm{d}t} + \frac{\mathrm{d}S_h}{\mathrm{d}t} + \frac{\mathrm{d}R}{\mathrm{d}t} = A - \delta_1 U - \delta_2 S_l - (b + \delta_3)S_h - \delta_4 R,$$

which gives

$$\frac{d(U + S_l + S_h + R)}{\mathrm{d}t} \leqslant A - \delta(U + S_l + S_h + R),$$

where $\delta = \min\{\delta_1, \delta_2, (\delta_3 + b), \delta_4\}$, this implies that

$$\lim_{t\to\infty} \sup(U + S_l + S_h + R) \leqslant \frac{A}{\delta}. \tag{B.1}$$

Again from last equation of model system (1) and using (B.1), we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant \frac{\phi A}{\delta} - \phi_0 V, \quad \text{and hence} \quad \lim_{t\to\infty} \sup V \leqslant \frac{\phi A}{\phi_0 \delta}.$$

This establishes our lemma.

Appendix C: The proof of Theorem 2

The proof of theorem is based on the Lyapunov direct method. For this, we consider the following positive definite function about E^*:

$$W_0 = \frac{1}{2}(U-U^*)^2 + \frac{1}{2}n_1(S_l-S_l^*)^2 + \frac{1}{2}n_2(S_h-S_h^*)^2 + \frac{1}{2}n_3(R-R^*)^2 + \frac{1}{2}n_4(V-V^*)^2, \tag{C.1}$$

where n_1, n_2, n_3 and n_4 are some positive constants to be picked out appropriately. On differentiating equation (C.1) with respect to t along the solution of system (1), and after some algebraic manipulation, we get

$$\begin{aligned}
\frac{\mathrm{d}W_0}{\mathrm{d}t} &= -\big\{k_1(R_a+V-bS_h-R)+\beta+\delta_1\big\}(U-U^*)^2 \\
&\quad - n_1\big\{k_2(R_a+V-bS_h-R)+\delta_2\big\}(S_l-S_l^*)^2 - n_2(\delta_3+b)(S_h-S_h^*)^2 \\
&\quad - n_3\big\{k_1U+k_2S_l+\delta_4+\gamma\big\}(R-R^*)^2 - n_4\phi_0(V-V^*)^2 \\
&\quad + (k_1U^*+\gamma+n_3k_1q_1)(U-U^*)(R-R^*) - k_1U^*(U-U^*)(V-V^*) \\
&\quad + n_1p\beta(U-U^*)(S_l-S_l^*) + n_3(k_1U+k_2S_l)(R-R^*)(V-V^*) \\
&\quad + n_4\phi(S_h-S_h^*)(V-V^*) + \{n_1k_2S_l^*+n_3k_2q_1\}(S_l-S_l^*)(R-R^*) \\
&\quad + \{k_1b+n_2\beta(1-p)\}(U-U^*)(S_h-S_h^*) - n_1k_2S_l^*(S_l-S_l^*)(V-V^*) \\
&\quad + n_1k_2bS_l^*(S_l-S_l^*)(S_h-S_h^*) - n_3b(k_1U+k_2S_l)(S_h-S_h^*)(R-R^*).
\end{aligned}$$

Now $\mathrm{d}W_0/\mathrm{d}t$ will be negative definite inside the region of attraction Ω, provided the following inequalities are satisfied:

$$(k_1U^*+\gamma)^2 < \frac{1}{9}n_3(\delta_1+\beta)(\delta_4+\gamma), \tag{C.2}$$

$$n_3(k_1q_1)^2 < \frac{1}{9}(\delta_1+\beta)(\delta_4+\gamma), \tag{C.3}$$

$$(k_1bU^*)^2 < \frac{2}{15}n_2(\delta_1+\beta)(\delta_3+b), \tag{C.4}$$

$$n_1(p\beta)^2 < \frac{2}{15}\delta_2(\delta_1+\beta), \tag{C.5}$$

$$n_1(k_2S_l^*)^2 < \frac{2}{15}n_3\delta_2(\delta_4+\gamma), \tag{C.6}$$

$$n_3(k_2q_1)^2 < \frac{2}{15}n_1\delta_2(\delta_4+\gamma), \tag{C.7}$$

$$n_1(k_2bS_l^*)^2 < \frac{4}{25}n_2\delta_2(\delta_3+b), \tag{C.8}$$

$$n_2\big((1-p)\beta\big)^2 < \frac{2}{15}(\delta_1+\beta)(\delta_3+b), \tag{C.9}$$

$$n_3(k_1U+k_2S_l)^2 < \frac{1}{6}n_4\phi_0(\delta_4+\gamma), \tag{C.10}$$

$$n_3 b^2 (k_1 U + k_2 S_l)^2 < \frac{2}{15} n_2 (\delta_3 + b)(\delta_4 + \gamma), \tag{C.11}$$

$$n_4 \phi^2 < \frac{1}{5} n_2 \phi_0 (\delta_3 + b), \tag{C.12}$$

$$(k_1 U^*)^2 < \frac{1}{6} n_4 \phi_0 (\delta_1 + \beta), \tag{C.13}$$

$$n_1 (k_2 S_l^*)^2 < \frac{1}{5} n_4 \phi_0 \delta_2. \tag{C.14}$$

From inequalities (C.4) and (C.9) we may choose $n_2 > 0$, provided condition (17) holds. Now from the inequalities (C.2), (C.3), (C.10), (C.11) and (C.13) we may choose $n_3 > 0$ as

$$0 < n_3 < \min\left\{ \frac{n_4 \phi_0 (\delta_4 + \gamma)}{6(k_1 U + k_2 S_l)^2}, \frac{2 n_2 (\delta_3 + b)(\delta_4 + \gamma)}{15 b^2 (k_1 U + k_2 S_l)^2} \right\}$$

such that condition (18) is satisfied.

Now from inequalities (C.5), (C.6), (C.7), (C.8), (C.12) and (C.14) we may choose $n_1 > 0$ as

$$0 < n_1 < \min\left\{ \frac{2}{15} \frac{\delta_2 (\delta_1 + \beta)}{(p\beta)^2}, \frac{4}{25} \frac{n_2 \phi_0^2 \delta_2 (\delta_3 + b)}{(k_2 S_l^*)^2} \right\}$$

such that condition (19) is satisfied.

Thus, $\mathrm{d}W_0/\mathrm{d}t$ will be negative definite inside the region of attraction Ω, provided conditions (17), (18) and (19) are satisfied. Hence, the model system (1) is globally stable.

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